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Research article

Linearized $L1$-Galerkin method for variable order time-fractional Schrödinger equation with unconditional convergence

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Abstract: The nonlinear Schrödinger equation with a nonlocal operator plays an important role in quantum mechanics, and the time-fractional Schrödinger problems have been widely studied for the case of constant exponents. In this paper, we propose a linearized unconditionally convergent $L1$-Galerkin method to solve the variable-exponent fractional Schrödinger equations. The optimal error convergence of the fully discrete scheme is proved without any time-space step restriction condition, even when incorporating the influence of the nonlocal operator in the temporal direction. The proof relies critically on the Sobolev embedding theorem combined with the inverse inequality. The discrete fractional Grönwall inequality is also used to obtain the error estimates. Numerical experiments are given to verify our theoretical results.

Keywords: variable order time-fractional Schrödinger equation; Sobolev embedding theorem; interpolation inequalities; unconditional convergence; finite element method

1. Introduction

This paper numerically deals with the following nonlinear variable order time-fractional Schrödinger equation (VOTFSE) in multiple dimensions:

$$\begin{cases} \mathrm{i}\,{}_0^C D_t^{\alpha(t)} u + \Delta u + |u|^2 u = 0, & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(t, \mathbf{x}) = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, T], \end{cases} \tag{1.1}$$

where i denotes the imaginary unit; $\Omega \in \mathcal{R}^d$ with d being the dimension; and ${}_0^CD_t^{\alpha(t)}$ is the variable order Caputo fractional differential operator defined by [1],

$$ {}_0^CD_t^{\alpha(t)}\phi = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{\phi'(s)}{(t-s)^{\alpha(t)}}\mathrm{d}s, \quad 0 < \alpha(t) < 1. \tag{1.2} $$

Time-fractional Schrödinger equations have been considered in various fields, including physical [2], chemical [3], and biological situations [4,5]. It can be used to describe the laws of fluids' motion in fluid dynamics and can also explain the impact of in-process interactions arising from the propagation of free particles in physics [6, 7].

Recent studies have demonstrated that anomalous diffusion processes often exhibit time- or space-dependent rate dynamics, challenging traditional constant-order models. This finding has drawn researchers' attention to applying variable-order (VO) fractional problems to describe these physical phenomena [8–10]. Smako and Ross [11] first introduced the definition of a fractional operator and established some properties of the integral with VO. Later, Lorenzo and Hartley [12] aimed at the concept of VO integration and differentiation and introduced several candidate definitions in terms of different order memories. Sun et al. [13] discussed the application of VO in terms of simulating the anomalous diffusion processes in many systems. To investigate this problem systematically and explore the connections between variable-order and constant-order fractional operators, Wang and Zheng [14] discussed the well-posedness of VO time-fractional diffusion equations and the regularity of the solutions. Their works provide theoretical supports for solving VO fractional equations numerically.

Considering the complexities of the VO fractional derivatives, analytical solutions are difficult to obtain, with which several numerical explorations arise. Lin et al. [15] proposed a new explicit finite difference method to solve a VO fractional diffusion equation and proved the convergence results under the assumption that the nonlinear term satisfies a Lipschitz condition. Moreover, Zhao et al. [16] via a Mid-point formula to discrete the fractional derivative operator with VO, and they derived a second order scheme. Zheng and Wang [17] studied the effects on convergence results from $\alpha(t)$ and proved that applying the finite element methods to VO time-fractional diffusion equations (TFDEs) achieved the optimal convergent order on uniform meshes with $\alpha(0) = \alpha'(0) = 0$. They also considered a VO space-fractional equation under the condition that the VO reduces to an integer value at the boundary. Zaky et al. [18] studied the VOTFDEs with fixed delay, and the VO fractional derivatives were discreted by L1−scheme. Collocation methods for this problem were subsequently developed in [19]. Later, Wei and Yang [20] conducted a finite difference local discontinuous Galerkin method to approximate the linear VOTFDEs, proving the optimal error estimate and the stability analysis for the fully discrete scheme. Du et al. [21] developed a second-order finite difference method in the temporal direction, achieved high accuracy in space with applying alternating direction implicit (ADI) and compact ADI skills to approximate space derivative. Jia et al. proposed a fast algorithm [22] and Huang et al. [23] constructed a super convergent scheme to solve linear VOTFEs. Existing studies primarily focus on linear formulations or Lipschitz-constrained VOTFDEs, while few studies address nonlinear cases or VOTFDEs. In [24], the Jacobi collocation method was used for VO space-time Schrödinger equation and they proved that the scheme had high accuracy in one dimension, which could be extended to solve the two dimensional case. Atangana and Cloot [25] derived a Crank−Nicolson scheme to solve the space-fractional VO Schrödinger equation and found the stability and error estimate order to be $O(\tau + h^2)$.

The primary objective of this paper is to construct an unconditional convergent linearized $L1$-Galerkin discrete scheme for the nonlinear VOTFSE (1.1). As is widely known, nonlinear equations are typically addressed through two primary approaches: The fully implicit method, which involves solving a nonlinear system iteratively, and the linearized method, which relies on the boundedness of the numerical solution. Classical error analysis for these methods often requires restrictive time-step constraints.

To achieve our goals, two primary difficulties should be addressed: (1) The loss of the convolution structure in the VO differential operator, which is retained in constant-order fractional equations and essential for deriving error estimates; (2) the inherent complexity of the VO time-fractional Schrödinger equation, accounting for the influence of the imaginary part and the nonlinear term. To overcome these challenges, we apply a vital method which is also considered in [26, 27] with inter-order cases for analyzing unconditional convergence. This approach leverages the Sobolev embedding theorem and interpolation inequalities, diverging from the popular time−space splitting technique which was first introduced by Li and Sun [28, 29]. This time-space splitting technique has been applied to various Schrödinger equations [30–32], but is not employed here. Furthermore, we incorporate an extrapolation technique to avoid solving nonlinear systems. An iterative process is applied at the initial time step to ensure optimal error estimation.

The organization of this paper is as follows. In Section 2, a fully linearized $L1$-Galerkin scheme is performed to solve the VOTFSE. Moreover, we give some important lemmas in this section, which are necessary for our analysis. In Section 3, we prove our main results. Considering this two-step method, we start by presenting the proof of the case $n = 1$, and then derive the analysis of the unconditional convergent results in Theorem 2.1 by utilizing Sobolev's embedding theorem and the discrete energy technique. Some numerical experiments are carried out to investigate the numerical accuracy, reliability, and efficiency in Section 4. Finally, several conclusions are done in Section 5.

2. $L1$-Galerkin methods and main results

Let $\mathcal{T}_h$ be a conforming and shape regular simplicial triangulation or tetrahedra of Ω, and let $h = \max_{K\in\mathcal{T}_h}\{\text{diam } K\}$ be the mesh size, where the diam K means the diameter of any element K. We use V_h to denote the finite-dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree r ($r \geq 1$) on $\mathcal{T}_h$. Let N be a positive number, $t_k = k\tau, k = 0, \dots, N$ be the mesh points, and the time step $\tau = T/N$. For a sequence of functions $\{\omega^n\}$, we write

$$\delta_\tau\omega^n = \omega^n - \omega^{n-1}, \quad \hat{\omega}^n = \frac{3}{2}\omega^{n-1} - \frac{1}{2}\omega^{n-2}. \tag{2.1}$$

We introduce the inner product over the complex space

$$(u, v) = \int_\Omega uv^* \mathrm{d}\Omega,$$

where v^* means the conjugate of v. Denoting $\alpha_n := \alpha(t_n)$ and taking t_n in place of t in (1.2), $L1$

approximation to the VO Caputo fractional derivative leads to

$$\begin{aligned}
{}_0^C D_t^{\alpha_n}\phi(t_n) &= \frac{1}{\Gamma(1-\alpha_n)}\int_0^{t_n}\frac{\phi'(s)}{(t_n-s)^{\alpha_n}}\mathrm{d}s = \frac{1}{\Gamma(1-\alpha_n)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{\phi'(s)}{(t_n-s)^{\alpha_n}}\mathrm{d}s\\
&= \frac{1}{\tau\Gamma(1-\alpha_n)}\sum_{k=1}^{n}\delta_\tau\phi^k\int_{t_{k-1}}^{t_k}(t_n-s)^{-\alpha_n}\mathrm{d}s + Q^n\\
&= \frac{1}{\tau\Gamma(2-\alpha_n)}\sum_{k=1}^{n}\delta_\tau\phi^k[(t_n-t_{k-1})^{1-\alpha_n}-(t_n-t_k)^{1-\alpha_n}] + Q^n\\
&= \sum_{k=1}^{n}\delta_\tau\phi^k b_{n,k} + Q^n = \sum_{k=1}^{n-1}(b_{n,k}-b_{n,k+1})\phi^k + b_{n,n}\phi^n - b_{n,1}\phi^0 + Q^n\\
&:= D_\tau^{\alpha_n}\phi^n + Q^n,
\end{aligned}$$

where

$$b_{n,k} = \frac{(t_n-t_{k-1})^{1-\alpha_n}-(t_n-t_k)^{1-\alpha_n}}{\tau\Gamma(2-\alpha_n)}, \tag{2.2}$$

and Q^n means the local truncation error. If $\phi \in C^2([0,T];L^2(\Omega))$, the error satisfies([33,34]),

$$\|Q^n\|_{L^2} \le C\tau. \tag{2.3}$$

Noting that $\varphi(x) = x^{1-\alpha_n} - (x-1)^{1-\alpha_n}$ is a decreasing function when $x \ge 1$, we have

$$0 < b_{n,1} < b_{n,2} < \ldots < b_{n,n-1} < b_{n,n} = \frac{\tau^{-\alpha_n}}{\Gamma(2-\alpha_n)}. \tag{2.4}$$

Assume that

$$0 \le \alpha(t) \le \alpha_M := \max_{t\in[0,T]} \alpha(t) < 1.$$

Letting U_h^n be approximate to $u^n := u(x,t_n)$ and recalling the notations (2.1), we can propose a linearized $L1$-Galerkin finite element method. First, find $U_h^n \in V_h$, such that for any $v \in V_h$,

$$\mathrm{i}(D_\tau^{\alpha_n}U_h^n, v) + (\Delta U_h^n, v) + (|\hat{U}_h^n|^2 U_h^n, v) = 0, \quad n \ge 2. \tag{2.5}$$

At the initial step $n = 1$, we choose $U_h^0 = \Pi_h u^0$ and apply a linearized Euler method to arrive

$$\mathrm{i}(\frac{U_h^1 - U_h^0}{\mu}, v) + (\Delta U_h^1, v) + (|U_h^0|^2 U_h^1, v) = 0, \quad \forall v \in V_h, \tag{2.6}$$

where $\mu = \tau^{\alpha_1}\Gamma(2-\alpha_1)$.

Lemma 2.1. *Let U_h^m be the numerical solution of Eq* (2.5)*. We then have*

$$\|U_h^m\|_{L^2} \le \|U_h^0\|_{L^2}, \quad m = 1, 2, \ldots, N. \tag{2.7}$$

Proof. We will prove the result by applying mathematical induction. Considering the imaginary part of Eq (2.6) with $v = U_h^1$, it is obvious that

$$\|U_h^1\|_{L^2} \leq \|U_h^0\|_{L^2},$$

which shows that the result holds with $m = 1$. Suppose that Eq (2.7) is satisfied when $m \leq n - 1$. Then we prove the case $m = n$.

Taking $v = U_h^n$ in Eq (2.5) and extracting the imaginary part, we derive

$$\Re(D_\tau^{\alpha_n} U_h^n, U_h^n) = 0. \tag{2.8}$$

Considering the definition of a fractional derivative and Cauchy−Schwarz inequality, we can obtain the following from (2.8):

$$\begin{aligned}
b_{n,n}\|U_h^n\|_{L^2}^2 &= \sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})\Re(U_h^k, U_h^n) + b_{n,1}\Re(U_h^0, U_h^n) \\
&\leq \sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})\frac{\|U_h^k\|_{L^2}^2 + \|U_h^n\|_{L^2}^2}{2} + b_{n,1}\frac{\|U_h^0\|_{L^2}^2 + \|U_h^n\|_{L^2}^2}{2} \\
&= \frac{1}{2}\sum_{k=1}^{n-1}(b_{n,k+1}-b_{n,k})\|U_h^k\|_{L^2}^2 + \frac{1}{2}\sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})\|U_h^n\|_{L^2}^2 + b_{n,1}\frac{\|U_h^0\|_{L^2}^2 + \|U_h^n\|_{L^2}^2}{2} \\
&= \frac{1}{2}\sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})\|U_h^k\|_{L^2}^2 + \frac{1}{2}b_{n,n}\|U_h^n\|_{L^2}^2 + \frac{1}{2}b_{n,1}\|U_h^0\|_{L^2}^2.
\end{aligned}$$

If we recall the assumption of induction and inequality (2.4), it can be obtained that

$$\begin{aligned}
b_{n,n}\|U_h^n\|_{L^2}^2 &\leq \frac{1}{2}\sum_{k=1}^{n-1}(b_{n,k+1} - b_{n,k})\|U_h^0\|_{L^2}^2 + \frac{1}{2}b_{n,n}\|U_h^n\|_{L^2}^2 + \frac{1}{2}b_{n,1}\|U_h^0\|_{L^2}^2 \\
&= \frac{1}{2}(b_{n,n} - b_{n,1})\|U_h^0\|_{L^2}^2 + \frac{1}{2}b_{n,n}\|U_h^n\|_{L^2}^2 + \frac{1}{2}b_{n,1}\|U_h^0\|_{L^2}^2 \\
&= \frac{1}{2}b_{n,n}\|U_h^0\|_{L^2} + \frac{1}{2}b_{n,n}\|U_h^n\|_{L^2}^2.
\end{aligned} \tag{2.9}$$

Together with $b_{n,n} > 0$, we can obtain the following from (2.9):

$$\|U_h^n\|_{L^2} \leq \|U_h^0\|_{L^2}$$

and the mathematical induction is finished, which completes our proof. □

Lemma 2.2. *[35] Suppose that the non-negative sequences $\{\xi^n, \eta^n | n \geq 0\}$ and ϕ^n satisfy*

$$\phi^n D_\tau^{\alpha_n}\phi^n \leq \eta^n\phi^n + \xi^n, \tag{2.10}$$

then

$$\phi^n \leq \phi^0 + \max_{1\leq j\leq n}\{\Gamma(1-\alpha_j)t_j^{\alpha_j}\eta^j\} + \max_{1\leq j\leq n}\left\{\sqrt{\Gamma(1-\alpha_j)}t_j^{\alpha_j/2}\xi^j\right\}.$$

Now, we give the main result and provid its proof in next section.

Theorem 2.1. *Suppose that $u_0 \in H^{r+1}(\Omega)\bigcap H_0^1(\Omega)$ and the problem* (1.1) *has an unique solution $u \in C^2([0,T])$. Then, system* (2.5)–(2.6) *has a unique solution U_h^n, $n = 1,2,3,\dots,N$ and positive constants τ_0 and h_0 exist, such that when $\tau \le \tau_0$ and $h \le h_0$, , satisfying*

$$\|u^n - U_h^n\|_{L^2} \le C_0(\tau + h^{r+1}), \tag{2.11}$$

where $u^n = u(\cdot, t_n)$ and C_0 is a positive constant that is independent of τ and h.

Remark 1. *When the weak singular regularity at the initial time is considered, the corresponding unconditional convergent results can also be obtained by using some nonuniform meshes, such as the graded meshes.*

3. Proof of the main result

In this section, we present a rigorous analysis of Theorem 2.1.

3.1. Preliminaries

Let $R_h : H_0^1(\Omega) \to V_h$ be Ritz projection operator satisfying

$$(\nabla(u - R_h u), \nabla\omega) = 0, \quad \forall \omega \in V_h. \tag{3.1}$$

By classical finite element method theory [36], we can find that for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$,

$$\|v - R_h v\|_{L^2} + h\|\nabla(v - R_h v)\|_{L^2} \le C_\gamma h^s \|v\|_{H^s}, \quad 1 \le s \le r+1, \tag{3.2}$$

where C_γ denotes a positive constant dependent only only on the domain Ω. Besides, for any $v \in H_0^1(\Omega)$, we have the Sobolev embedding theorem and Cauchy−Schwarz inequality

$$\|v\|_{L^p} \le C(\|v\|_{L^2} + \|\nabla v\|_{L^2}), \quad 1 \le p \le 6, \tag{3.3}$$

$$\|v\|_{L^3}^2 \le \|v\|_{L^2}\|v\|_{L^6}. \tag{3.4}$$

For any $v \in V_h$, the following inverse inequality holds.

$$\|v\|_{L^p} \le C_\gamma h^{\frac{d}{p}-\frac{d}{q}}\|v\|_{L^p}, \quad 1 \le q \le p \le \infty, \ d = 2 \text{ or } 3. \tag{3.5}$$

Taking $t = t_n$ in the first equation of (1.1) and considering the weak form of the resulting equation, we obtain

$$\mathrm{i}(D_\tau^{\alpha_n} u^n, v) + (\Delta u^n, v) + (|\hat{u}^n|^2 u^n, v) = (P^n, v), \quad \forall v \in V_h, \ n \ge 2, \tag{3.6}$$

where

$$P^n = \mathrm{i}\Big(D_\tau^{\alpha_n} u^n - {}_0^C D_t^{\alpha_n}(u^n)\Big) + |\hat{u}^n|^2 u^n - |u^n|^2 u^n.$$

Moreover, u^1 solves

$$\mathrm{i}\frac{u^1 - u^0}{\mu} + \Delta u^1 + |u^0|^2 u^1 = P^1, \tag{3.7}$$

with

$$P^1 = \mathrm{i}\Big(\frac{u^1-u^0}{\mu} - {}_0^C D_t^{\alpha_1}(u^1)\Big) + |u^0|^2u^1 - |u^1|^2u^1.$$

The truncation error can be estimated as follows from (2.3):

$$\|P^n\|_{L^2} \le C\tau, \text{ for } 1 \le n \le N. \tag{3.8}$$

Subtracting (3.6) from (2.5), applying (3.1) and letting

$$u^n - U_h^n = u^n - R_hu^n + R_hu^n - U_h^n := u^n - R_hu^n + \theta_h^n, \quad n = 1, 2, \ldots, N,$$

we have the following for $n \ge 2$,

$$\mathrm{i}(D_\tau^{\alpha_n}\theta_h^n, v) + (\Delta\theta_h^n, v) + (|\hat{u}^n|^2u^n - |\hat{U}_h^n|^2U_h^n, v) = (P^n, v) - \mathrm{i}(D_\tau^{\alpha_n}(u^n - R_hu^n), v), \tag{3.9}$$

and when $n = 1$, the error equation gives

$$\mathrm{i}(\frac{\theta_h^1}{\mu}, v) - (\nabla\theta_h^1, \nabla v) + (|\hat{u}^0|^2u^1 - |\hat{U}_h^0|^2U_h^1, v) = (P^1, v) - \mathrm{i}(\frac{u^1 - R_hu^1}{\mu}, v). \tag{3.10}$$

Lemma 3.1. *If we suppose that U_h^1 is the solution of Eq (2.6) and $R_hu^0 = U_h^0$, then there exist two positive constants $\hat{\tau}$, $\hat{h}$, such that when $\tau \le \hat{\tau}$ and $h \le \hat{h}$, (3.10) can be estimated as*

$$\|\theta_h^1\|_{L^2} + \tau^{\alpha_1/2}\|\nabla\theta_h^1\|_{L^2} \le C_2(\tau + h^{r+1}), \tag{3.11}$$

where $C_2 > 0$ is independent of τ and h.

Proof. Substituting θ_h^1 for v and considering the imaginary part of Eq (3.10), we can obtain that

$$\frac{1}{\mu}\|\theta_h^1\|_{L^2}^2 + \Im(|u^0|^2u^1 - |U_h^0|^2U_h^1, \theta_h^1) = \Im(P^1, \theta_h^1) - \Re(\frac{u^1 - R_hu^1}{\mu}, \theta_h^1), \tag{3.12}$$

and

$$\begin{aligned}
&\Im(|u^0|^2u^1 - |U_h^0|^2U_h^1, \theta_h^1)\\
&\le |\Im(|u^0|^2u^1 - |U_h^0|^2u^1 + |U_h^0|^2u^1 - |U_h^0|^2U_h^1, \theta_h^1)|\\
&\le \Big||u^0|^2 - |U_h^0|^2\Big|\|u^1\|_{L^\infty}\|\theta_h^1\|_{L^2} + \||U_h^0|^2\|_{L^\infty}\|u^1 - U_h^1\|_{L^2}\|\theta_h^1\|_{L^2}\\
&\le C_k(\|u^0 - R_hu^0\|_{L^2}^2 + \frac{1}{4}\|\theta_h^1\|_{L^2}^2) + (\|u^1 - R_hu^1\|^2 + \frac{1}{4}\|\theta_h^1\|_{L^2}^2)\\
&\le (C_k^2 + C_k)h^{2(r+1)} + \frac{(C_k+1)}{4}\|\theta_h^1\|_{L^2}^2,
\end{aligned}$$

where C_k is a positive constant dependent only on the regularity of the exact solution. The Cauchy-Schwarz inequality and classical theory in finite element space are used in the last inequality. Together with (3.8), Eq (3.12) yields:

$$\begin{aligned}
\frac{1}{\mu}\|\theta_h^1\|_{L^2}^2 &\le (C_k^2 + C_k)h^{2(r+1)} + \frac{C_k+1}{4}\|\theta_h^1\|_{L^2}^2\\
&\quad + \|P^1\|_{L^2}^2 + \frac{1}{4}\|\theta_h^1\|_{L^2}^2 + \frac{1}{\mu}(u^1 - R_hu^1, \theta_h^1)\\
&\le (C_k^2 + 2C_k)h^{2(r+1)} + C^2\tau^2 + \frac{1+\mu+\mu(C_k+1)}{4\mu}\|\theta_h^1\|_{L^2}^2.
\end{aligned}$$

Taking $\hat{\tau}^{\alpha_1}\Gamma(2-\alpha_1) = \frac{3}{2}$, when $\tau \le \hat{\tau}$, the inequality above yields that

$$\|\theta_h^1\|_{L^2}^2 \le 4(C_k^2 + 2C_k)h^{2(r+1)} + 4C^2\tau^2.$$

Taking $v = \theta_h^1$ in (3.10) and extracting the real part of the resulting equation, the following formula is derived:

$$\|\nabla\theta_h^1\|_{L^2}^2 = \Re(|u^0|^2u^1 - |U_h^0|^2U_h^1, \theta_h^1) - \Re(P^1, \theta_h^1) + \Im(\frac{u^1 - R_hu^1}{\mu}, \theta_h^1). \tag{3.13}$$

Similarly, we can get

$$\|\nabla\theta_h^1\|_{L^2}^2 \le 4(C_k^2 + 2C_k)h^{2(r+1)} + 4C^2\tau^2 + \tau^{2-\alpha_1}. \tag{3.14}$$

The proof is finished. □

3.2. *Proof of Theorem 2.1*

We have obtained the convergent results when $n = 1$, as described in Lemma 3.1. In this subsection, our goal is to present unconditionally optimal error estimates of the fully discrete numerical schemes.

Taking $v = \theta_h^n$ and $R_1^n = |\hat{u}^n|^2u^n - |\hat{U}_h^n|^2U_h^n$ in (3.9), we obtain

$$\mathrm{i}(D_\tau^{\alpha_n}\theta_h^n, \theta_h^n) + (\Delta\theta_h^n, \theta_h^n) + (R_1^n, \theta_h^n) = (P^n, \theta_h^n) - \mathrm{i}(D_\tau^{\alpha_n}(u^n - R_hu^n), \theta_h^n). \tag{3.15}$$

Then extracting the imaginary part of (3.15) yields

$$\Re(D_\tau^{\alpha_n}\theta_h^n, \theta_h^n) + \Im(R_1^n, \theta_h^n) = \Im(P^n, \theta_h^n) - \Re(D_\tau^{\alpha_n}(u^n - R_hu^n), \theta_h^n). \tag{3.16}$$

Now, we estimate the second part on the left-hand side of (3.16). Using the Sobolev embedding theorem and Cauchy−Schwarz inequality, we arrive

$$\begin{aligned}
&\Im(R_1^n, \theta_h^n) = \Im(|\hat{u}^n|^2u^n - |\hat{U}_h^n|^2U_h^n, \theta_h^n)\\
=&\Im(|\hat{u}^n|^2(u^n - \hat{u}^n) + |\hat{u}^n|^2(\hat{u}^n - R_hu^n) + |\hat{u}^n|^2\theta_h^n + (|\hat{u}^n|^2 - |\hat{U}_h^n|^2)(R_hu^n - \theta_h^n), \theta_h^n)\\
\le&\left|(|\hat{u}^n|^2(u^n - \hat{u}^n), \theta_h^n)\right| + C_k\|u^n - R_hu^n\|_{L^2}\|\theta_h^n\|_{L^2} + C_k\|\theta_h^n\|_{L^2}^2\\
&+ \left|(\hat{u}^n(R_hu^n - \theta_h^n)(\hat{u}^n - R_h\hat{u}^n + \hat{\theta}_h^n)^*, \theta_h^n)\right|\\
&+ \left|((R_hu^n - \theta_h^n)(\hat{u}^n - R_h\hat{u}^n + \hat{\theta}_h^n)(R_h\hat{u}^n - \hat{\theta}_h^n)^*, \theta_h^n)\right|\\
\le&C_1(\|\hat{\theta}_h^n\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + C_2(\|\hat{\theta}_h^n\|_{L^3}^2\|\theta_h^n\|_{L^6}^2) + C_3(\tau + h^{r+1})^2.
\end{aligned} \tag{3.17}$$

Here we use the equality $|a|^2 - |b|^2 = a(a-b)^* + (a-b)b^*$ and

$$\begin{aligned}
|((\hat{\theta}_h^n)^*\theta_h^n, \theta_h^n)| \le \|\hat{\theta}_h^n\theta_h^n\|_{L^2}\|\theta_h^n\|_{L^2} &\le \frac{1}{2}(\|\hat{\theta}_h^n\|_{L^2}^2\|\theta_h^n\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2)\\
&\le \frac{1}{2}(\|\hat{\theta}_h^n\|_{L^3}^2\|\theta_h^n\|_{L^6}^2 + \|\theta_h^n\|_{L^2}^2),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
|((\hat{\theta}_h^n)^*\hat{\theta}_h^n, \theta_h^n)| \le \|\hat{\theta}_h^n\|_{L^2}^2\|\theta_h^n\|_{L^2} &\le \frac{1}{2}(\|\hat{\theta}_h^n\|_{L^2}^2\|\theta_h^n\|_{L^2}^2 + \|\hat{\theta}_h^n\|_{L^2}^2)\\
&\le \frac{1}{2}(\|\hat{\theta}_h^n\|_{L^3}^2\|\theta_h^n\|_{L^6}^2 + \|\hat{\theta}_h^n\|_{L^2}^2).
\end{aligned} \tag{3.19}$$

We then consider the estimation on the right-hand side of (3.16). On the basis of, the assumptions of regularity and classical theory in (3.2), we obtain

$$
\begin{aligned}
|\Re(D_\tau^{\alpha_n}(u^n - R_h u^n), \theta_h^n)| &\le \|D_\tau^{\alpha_n}(u^n - R_h u^n)\|_{L^2}\|\theta_h^n\|_{L^2} \\
&\le \|D_\tau^{\alpha_n} u^n\|_{H^{r+1}} h^{r+1}\|\theta_h^n\|_{L^2} \\
&\le \frac{1}{2}(\|\theta_h^n\|_{L^2}^2 + h^{2(r+1)}),
\end{aligned}
\tag{3.20}
$$

and

$$
|(P^n, \theta_h^n)| \le C(\tau^2 + \|\theta_h^n\|_{L^2}^2). \tag{3.21}
$$

Substituting (3.17)–(3.21) into (3.16), we can get

$$
\Re(D_\tau^{\alpha_n}\theta_h^n, \theta_h^n) \le C_1(\|\hat{\theta}_h^n\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + C_2(\|\hat{\theta}_h^n\|_{L^3}^2\|\theta_h^n\|_{L^6}^2) + C_3(\tau + h^{r+1})^2. \tag{3.22}
$$

Recalling the definition of $D_\tau^{\alpha_n}\theta_h^n$ and using the Cauchy−Schwarz inequality, we derive

$$
\begin{aligned}
\Re(D_\tau^{\alpha_n}\theta_h^n, \theta_h^n) &\ge \sum_{k=1}^{n-1}(b_{n,k} - b_{n,k+1})\frac{\|\theta_h^k\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2}{2} + b_{n,n}\|\theta_h^n\|_{L^2}^2 - b_{n,1}\frac{\|\theta_h^0\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2}{2} \\
&= \frac{1}{2}(b_{n,n}\|\theta_h^n\|_{L^2}^2 + \sum_{k=1}^{n-1}(b_{n,k} - b_{n,k+1})\|\theta_h^k\|_{L^2}^2 - b_{n,1}\|\theta_h^0\|_{L^2}^2 \\
&= \frac{1}{2}D_\tau^{\alpha_n}\|\theta_h^n\|_{L^2}^2.
\end{aligned}
\tag{3.23}
$$

For the proof of Theorem 2.1, it suffices to prove that τ_2 and h_2 exist, when $\tau \le \tau_2$, $h \le h_2$, the following holds

$$
\|\theta_h^n\|_{L^2} + \tau^{\alpha_n/2}\|\nabla\theta_h^n\|_{L^2} \le C_4(\tau + h^{r+1}), \tag{3.24}
$$

where C_4 is a constant independent of τ and h.

It follows from Lemma 3.1 that (3.24) holds when $m = 1$. Suppose that for $m = 2, 3, \ldots, n-1$, the conclusion (3.24) holds. Now we consider the situation $m = n$. First, we estimate the equation $\|\hat{\theta}_h^m\|_{L^3}^2\|\theta_h^m\|_{L^6}$. Then, the inequality is estimated in the following two cases.

Case A: $\tau \le h$

Applying the inverse inequality (3.5) and the assumption of mathematical induction, it yields

$$
\begin{aligned}
\|\hat{\theta}_h^m\|_{L^3} \le C_\gamma h^{-\frac{d}{6}}\|\hat{\theta}_h^m\|_{L^2} &\le C_\gamma h^{-\frac{d}{6}}(\frac{3}{2}\|\theta_h^{m-1}\|_{L^2} + \frac{1}{2}\|\theta_h^{m-2}\|_{L^2}) \\
&\le 2C_\gamma h^{2-\frac{d}{6}}, \quad m = 2, 3, \ldots, n,
\end{aligned}
\tag{3.25}
$$

and

$$
\|\theta_h^m\|_{L^6} \le C_\gamma h^{-\frac{d}{3}}\|\theta_h^m\|_{L^2}. \tag{3.26}
$$

Together with Lemma 3.1, we obtain

$$
\|\hat{\theta}_h^m\|_{L^3}\|\theta_h^m\|_{L^6} \le 2C_\gamma^2 h^{2-\frac{d}{2}}\|\theta_h^m\|_{L^2} \le \|\theta_h^m\|_{L^2}, \quad m \le n, \tag{3.27}
$$

where $h_3 = (2C_\gamma^2)^{\frac{2}{4-d}}$ and $h \le h_3$. Combining (3.23) with (3.25), (3.27) and inserting inequality (3.22), we obtain

$$D_\tau^{\alpha_n}\|\theta_h^n\|_{L^2}^2 \le C_1(\|\hat{\theta}_h^n\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + \|\theta_h^n\|_{L^2}^2 + C_3(\tau + h^{r+1})^2.$$

Applying the discrete fractional Grönwall inequality(2.10) leads to

$$\|\theta_h^n\|_{L^2} \le C(\tau + h^{r+1}). \tag{3.28}$$

Then, we give the estimation of $\|\nabla\theta_h^n\|_{L^2}$. First, we derive the estimation of $\|D_\tau^{\alpha_n}\theta_h^n\|_{L^2}$ from the definition of the VO Caputo fractional derivative:

$$\begin{aligned}\|D_\tau^{\alpha_n}\theta_h^n\|_{L^2} &\le \sum_{k=1}^{n-1}(b_{n,k} - b_{n,k+1})\max_{1\le j\le n-1}\|\theta_h^j\|_{L^2} + b_{n,n}\|\theta_h^n\|_{L^2} + b_{n,1}\|\theta_h^0\|_{L^2}\\ &\le (b_{n,1} - b_{n,n})\max_{1\le j\le n-1}\|\theta_h^j\|_{L^2} + b_{n,n}\|\theta_h^n\|_{L^2} + b_{n,1}\|\theta_h^0\|_{L^2}.\end{aligned} \tag{3.29}$$

Together with (2.4) and (3.28), we can obtain the following from (3.29):

$$\|D_\tau^{\alpha_n}\theta_h^n\|_{L^2} \le 4Cb_{n,n}(\tau + h^{r+1}) \le \frac{4C}{\Gamma(2-\alpha_n)}\tau^{-\alpha_n}(\tau + h^{r+1}). \tag{3.30}$$

Then, we take the real part of (3.15) to get

$$\begin{aligned}\|\nabla\theta_h^n\|_{L^2}^2 &= \Im(D_\tau^{\alpha_n}\theta_h^n, \theta_h^n) + \Re(R_1^n, \theta_h^n) - \Re(P^n, \theta_h^n) + \Im(D_\tau^{\alpha_n}(u^n - R_hu^n), \theta_h^n)\\ &\le (\|D_\tau^{\alpha_n}\theta_h^n\|_{L^2} + \|P^n\|_{L^2} + \|D_\tau^{\alpha_n}(u^n - R_hu^n)\|_{L^2})\|\theta_h^n\|_{L^2} + \Re(R_1^n, \theta_h^n)\\ &\le ((C_4\tau^{-\alpha_n} + C_5)(\tau + h^{r+1}) + Ch^{r+1})\|\theta_h^n\|_{L^2} + |(R_1^n, \theta_h^n)|.\end{aligned} \tag{3.31}$$

Replacing with (3.31), we can invert (3.31) into

$$\|\nabla\theta_h^n\|_{L^2}^2 \le C_6\tau^{-\alpha_n}(\tau + h^{r+1})\|\theta_h^n\|_{L^2} + C_1(\|\hat{\theta}_h^n\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + C_2(\|\hat{\theta}_h^n\|_{L^3}^2\|\theta_h^n\|_{L^6}^2). \tag{3.32}$$

Substituting (3.27) and (3.28) into the inequality yields

$$\begin{aligned}\tau^{\alpha_n}\|\nabla\theta_h^n\|_{L^2}^2 &\le C_6(\tau + h^{r+1})^2 + \tau^{\alpha_n}\|\hat{\theta}_h^n\|_{L^3}^2\|\theta_h^n\|_{L^6}^2\\ &\le (3C_1 + C_6 + \tau^{\alpha_n})(\tau + h^{r+1})^2 \le C_5(\tau + h^{r+1})^2.\end{aligned} \tag{3.33}$$

Thus, (3.24) holds for $m = n$.

Case B: $\tau > h$

By mathematical induction, we need to consider the case $m = n$. Applying Sobolev inequality (3.3), we obtain the following, for $m = 2, 3, \ldots, n$:

$$\|\hat{\theta}_h^m\|_{L^3}^2 \le \|\hat{\theta}_h^m\|_{L^2}\|\hat{\theta}_h^m\|_{L^6} \le C\|\hat{\theta}_h^m\|_{L^2}(\|\nabla\hat{\theta}_h^m\|_{L^2} + \|\hat{\theta}_h^m\|_{L^2}) \le C_4\tau^{2-\frac{\alpha_m}{2}}.$$

Together with inequality (3.3) we have

$$\begin{aligned}\|\hat{\theta}_h^m\|_{L^3}^2\|\theta_h^m\|_{L^6}^2 &\le C_4\tau^{2-\frac{\alpha_m}{2}}(\|\theta_h^m\|_{L^2}^2 + \|\nabla\theta_h^m\|_{L^2}^2)\\ &\le \frac{1}{2}\|\nabla\theta_h^m\|_{L^2}^2 + \|\theta_h^m\|_{L^2}^2,\end{aligned} \tag{3.34}$$

where $\tau \le \tau_3 = \min\{2C_4^{-\frac{2}{4-\alpha_m}}, 1\}$.

Substituting (3.34) into (3.32) and applying the Cauchy−Schwarz inequality, we have

$$\tau^{\alpha_n}\|\nabla\theta_h^n\|_{L^2}^2 \le C_6(\tau + h^{r+1})^2 + C_1(\|\hat{\theta}_h^n\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + \frac{\tau^{\alpha_n}}{2}\|\nabla\theta_h^n\|_{L^2}^2. \tag{3.35}$$

Together with (3.34) and (3.35), Eq (3.22) yields

$$\begin{aligned}\Re(D_\tau^{\alpha_n}\theta_h^n, \theta_h^n) &\le C_1(\|\theta_h^{n-1}\|_{L^2}^2 + \|\theta_h^{n-2}\|_{L^2}^2) + (C_1+1)\|\theta_h^n\|_{L^2}^2 + \tau^{\alpha_n}\|\nabla\theta_h^n\|_{L^2}^2 + C_3(\tau + h^{r+1})^2 \\ &\le (C_1 + \frac{9}{4})\|\theta_h^{n-1}\|_{L^2}^2 + (C_1 + \frac{1}{4})\|\theta_h^{n-2}\|_{L^2}^2 + (C_1+7)\|\theta_h^n\|_{L^2}^2 + 2C_4(\tau + h^{r+1})^2,\end{aligned}$$

Using Lemma 2.2 and inequality (3.23), we obtain

$$\|\theta_h^n\|_{L^2}^2 \le C_7(\tau + h^{r+1})^2, \tag{3.36}$$

which further implies that

$$\tau^{\alpha_n}\|\nabla\theta_h^n\|_{L^2}^2 \le C_8(\tau + h^{r+1})^2.$$

Taking

$$\tau_0 = \min\{\tau_1, \tau_2, \tau_3\}, \quad h_0 = \min\{h_1, h_2, h_3\},$$

we can see that when $\tau \le \tau_0$ and $h \le h_0$, conclusion (3.24) holds. We now turn to consider the case $\tau + h^{r+1} \ge C_\tau$. Applying the triangle inequality and Lemma 2.1, we have

$$\begin{aligned}\|u^n - U_h^n\|_{L^2} &\le \|u^n\|_{L^2} + \|U_h^n\|_{L^2} \le \|u^n\|_{L^2} + \|U_h^0\|_{L^2} \\ &\le \frac{\max_{1\le n\le N}\|u^n\|_{L^2} + \|U_h^0\|_{L^2}}{C_\tau}(\tau + h^{r+1}).\end{aligned}$$

Taking $C_0 = \max\{C_7, \frac{\max\limits_{1\le n\le N}\|u^n\|_{L^2} + \|U_h^0\|_{L^2}}{C_\tau}\}$, the proof of Theorem 2.1 is completed. □

4. Numerical experiments

In this section, we present two numerical examples to verify the unconditional convergence results of the full−discrete scheme. The finite element method (FEM) is implemented in space in the following examples. Here, we choose $\alpha(t) = \alpha(1) + (\alpha(0) - \alpha(1))(1 - t - \sin(2\pi(1-t))/(2\pi))$.

Example 1. *Consider the following cubic Schrödinger equation:*

$$\begin{cases} \mathrm{i}\,{}_0^C D_t^{\alpha(t)} u + \Delta u + |u|^2 u = g(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, 1], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, 1], \end{cases} \tag{4.1}$$

where $\Omega = [0, 1] \times [0, 1]$ *and* $g(x, t)$ *was chosen such that* (4.1) *has the following solution*

$$u(\mathbf{x}, t) = (1 + t^3)x^2 \sin(\pi x) y^2 \sin(\pi y).$$

Table 1. L^2-errors and convergence orders in the temporal direction for (Example 1)with $\alpha(1) = 0.9$.

	$\alpha(0) = 0$		$\alpha(0) = 0.5$		$\alpha(0) = 0.8$	
N	Errors	Orders	Errors	Orders	Errors	Orders
5	$9.36e-04$	$*$	$1.02e-03$	$*$	$1.07e-03$	$*$
10	$4.10e-04$	1.18	$4.10e-04$	1.31	$4.07e-04$	1.40
20	$1.99e-04$	1.06	$1.97e-04$	1.06	$1.97e-04$	1.05
40	$9.38e-05$	1.09	$9.31e-05$	1.08	$9.37e-05$	1.07

Table 2. Errors and convergence rates with $\alpha(0) = 0.4$ and $\alpha(1) = 0.6$ (Example 1).

	L−FEM		Q−FEM	
M	Errors	Orders	Errors	Orders
5	$2.93e-02$	$*$	$1.94e-03$	$*$
10	$8.71e-03$	1.75	$2.19e-04$	3.15
20	$2.28e-03$	1.93	$2.65e-05$	3.05
40	$5.77e-04$	1.98	$3.36e-06$	2.98

We compute the VOTFSE (4.1) by applying linear FEM (L-FEM) and quadratic FEM (Q-FEM). In order to test the convergence order in temporal direction, we set $M = 50$, where M implies the uniform triangular partition with $M + 1$ nodes and $N = 5, 10, 20, 40$, with $\alpha(0) = 0.0,\ 0.5,\ 0.8$, and $\alpha(1) = 0.9$. Table 1 reveals that the numerical scheme for solving VOTFSEs can achieve first order in time with different parameters. The spatial results are verified by taking $M = 5, 10, 20, 40$ with fixed $N = 1000$. Taking the VO differential operator into account, we choose $\alpha(0) = 0.4, \alpha(1) = 0.6$, which gives the results in Table 2. The unconditional results are displayed in Figure 1 with $\alpha(0) = 0.4, \alpha(1) = 0.6$. It is easy to find that for any fixed τ, the value of the scheme's error tends to a constant, which justifies our conclusion.

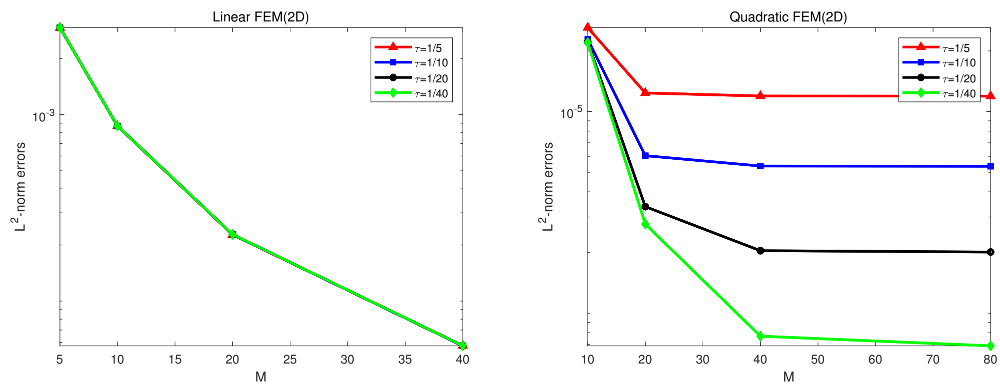

Figure 1. Two-dimensinal problem: L^2-errors of L-FEM (left) and Q-FEM (right).

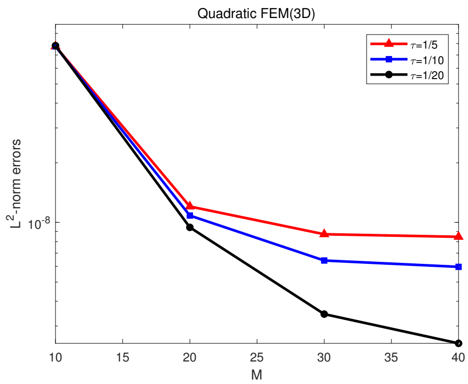

Figure 2. Three-dimensional problem: L^2-errors of Q-FEM.

Example 2. *The three-dimensional time-fractional Schrödinger equation is considered as follows:*

$$\begin{cases} \mathrm{i}\,{}_0^C D_t^{\alpha(t)} u + \Delta u + |u|^2 u = g(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, 1], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times (0, 1], \end{cases} \tag{4.2}$$

where $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ *and* g *is chosen correspondingly to the exact solution*

$$u(x, y, z, t) = (1 + t^3)x^2(1 - x)^3 y^2(1 - y)^3 z^2(1 - z)^3.$$

Table 3. L^2-errors and convergence orders with $N = M^2$ (Example 2).

M	Errors	Orders
5	$7.25e-06$	$*$
10	$2.24e-06$	1.58
20	$6.01e-07$	1.87
40	$1.68e-07$	1.97

In order to verify the convergence orders proposed in Theorem 2.1, the L-FEM is chosen in the fully discrete scheme (2.5) and (2.6) with $N = M^2$, $\alpha(0) = 0.4$ and $\alpha(1) = 0.6$. The results displayed in Table 3 give that the convergence rates can reach first order in time and second order in space, respectively. Figure 2 shows that the error tends to a constant for a fixed time step τ.

To estimate more examples, we define the errors in the L^2-norm

$$\|e_h^n\|_{L^2} = \Big(\int_\Omega |u^n - U_h^n|^2 d\Omega\Big)^{1/2}.$$

We test the convergence order of the scheme (2.5) and (2.6) with L-FEM and Q-FEM, respectively. Consider the following condition:

$$O\Big(\frac{\tau}{4} + (\frac{h}{2})^2\Big) = \frac{1}{4}O(\tau + h^2), \quad \text{and} \quad O\Big(\frac{\tau}{8} + (\frac{h}{2})^3\Big) = \frac{1}{8}O(\tau + h^3),$$

where the spatial convergence orders of u can be defined for a sufficiently small h and τ as follows

$$\text{order}_1 = \log_2\left(\frac{\|e_h^n\|_{L^2}(\tau,h)}{\|e_h^n\|_{L^2}(\tau/4,h/2)}\right), \quad \text{order}_2 = \log_2\left(\frac{\|e_h^n\|_{L^2}(\tau,h)}{\|e_h^n\|_{L^2}(\tau/8,h/2)}\right).$$

The numerical experiments are carried out using different time and space step sizes. First, we choose the spatial step sizes $h = 1/4,\ 1/8,\ 1/16,\ 1/32$. To verify the convergence order $O(\tau + h^2)$ of the schemes (2.5) and (2.6) with L-FEM in space, the time steps are set to $\tau = 1/4,\ 1/16,\ 1/64,\ 1/256$. The values in the column Order_1 of Table 4 indicate that the scheme achieves second-order convergence, which is consistent with our theoretical findings. Similarly, for the schemes with Q-FEM in space, the time steps selected $\tau = 1/4,\ 1/32,\ 1/256,\ 1/2048$ to examine the spatial convergence behavior. The results in Table 4 reveal that the schemes achieve third order in space.

Table 4. Spatial convergence rates with different values h in Example 2.

h	τ	$\|e_h^n\|_{L^2}$	Order_1	τ	$\|e_h^n\|_{L^2}$	Order_2
1/4	1/4	$9.50e-06$	*	1/4	$1.26e-06$	*
1/8	1/16	$3.56e-06$	1.42	1/32	$1.57e-07$	3.01
1/16	1/64	$1.01e-06$	1.81	1/256	$1.83e-08$	3.10
1/32	1/256	$2.63e-07$	1.95	1/2048	$2.22e-09$	3.04

Moreover, we calculate the exact numerical solutions in three dimensions via L-FEM and Q-FEM. From Figure 3, we can observe the efficiency of our numerical methods.

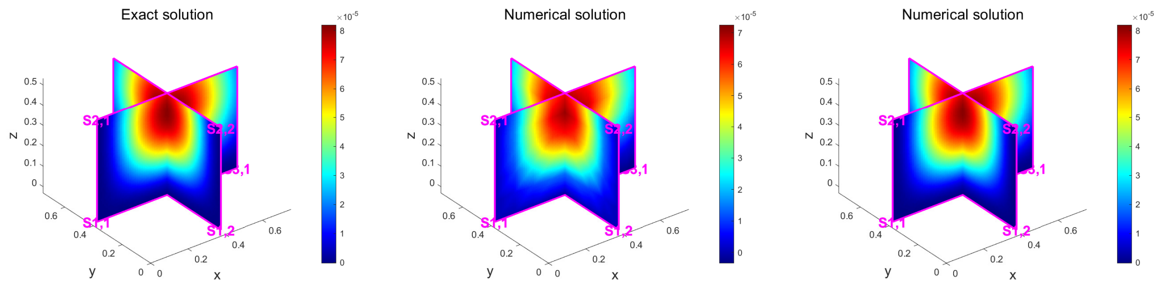

Figure 3. Exact and numerical solutions for Example 2 with L-FEM (middle) and Q-FEM (right) at $t = 1$.

Now, we extend our methods to VOTFSE with non-smooth solutions.

Example 3. *Consider equation (4.1) with the exact solution*

$$u(\mathbf{x},t) = (1 + t^{\sigma})\sin(\pi x)\sin(\pi y).$$

where $\Omega = [0,1]\times[0,1]$ *and* $0 < \sigma < 1$ *is a constant. The source term* $g(\mathbf{x},t)$ *is chosen such that (4.1) admits the non−smooth solution above.*

To illustrate the unconditional convergence, we solve the problem by applying Q-FEM with varying spatial step sizes for each fixed τ. The errors in the L^2-norm at time $T = 1$ are presented in Figure 4. We can see that for each fixed τ with different values of σ, the errors tend to be a constant, which demonstrates that time-step restriction is not necessary.

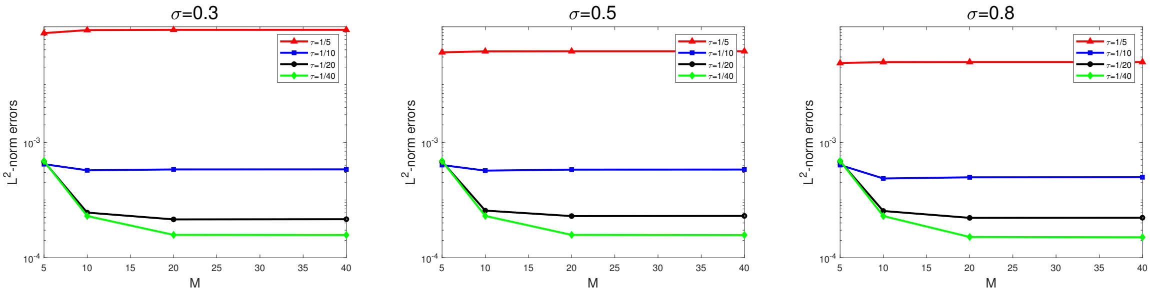

Figure 4. 2D problem: L^2-errors of Q-FEM with different σ.

5. Conclusions

This paper proposes a linearized unconditional convergence scheme to solve time−fractional Schrödinger equations with a variable order. Classical time−space splitting methods, while effective in many contexts, face challenges when dealing with variable exponents due to the inherent complexities of such nonlinearities. Here, we apply another approach with the Sobolev inequality to achieve the results, which provides a more adaptable framework for handling these structural difficulties. Numerical examples are provided to demonstrate our theoretical findings. We note that the unconditional convergence of the proposed scheme is developed on the basis uniform meshes. Hence the given examples are tested with smooth exact solutions to demonstrate the theoretical results. Actually, the proposed scheme could be extended to handle the problems with initial singularity using nonuniform meshes. The relevant unconditional convergence requires special treatment, which relies heavily on the properties of the meshes.

Author contributions

Boya Zhou: Conceptualization, Validation, Formal analysis, Writing—original draft preparation, Project administration; Shaohong Pan: Software, Validation, Formal analysis, Investigation, Data curation, Data curation, Visualization; Zhiwei Fang: Conceptualization, Data curation, Investigation, Validation, Methodology, Data curation, Writing—review and editing, Supervision; Min Li: Software, Resources, Visualization, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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