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Research article

Positive periodic solution for a stochastic non-autonomous enterprise cluster model

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Abstract: This paper investigates the dynamical behaviors for a stochastic non-autonomous enterprise cluster model. We will analyze how the parameters of the system and white noise affect the dynamical properties of the system. Using Itô's formula, the comparison principle and inequality techniques, we study the existence, uniqueness, and extinction of nontrivial positive solutions. Particularly, we also study the existence of a stochastic positive periodic solution by using stochastic differential equation theory. Finally, two examples are introduced to verify the main results of this paper.

Keywords: positive periodic solution; existence; stochastic; enterprise cluster model
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1. Introduction

Enterprise clusters have a significant impact on the development of local and national economies. Due to the competitive relationships among various enterprises within the cluster, it is necessary to study these relationships in order to promote the development of the enterprise cluster. In [1], Tian and Nie considered the following autonomous competition model of two enterprises:

$$\begin{aligned} \frac{du_1}{dt} &= a_1u_1(t)[1 - \frac{u_1(t)}{k} - \frac{\alpha(u_2(t) - c_2)^2}{k}] \\ \frac{du_2}{dt} &= a_2u_2(t)[1 - \frac{u_2(t)}{k} + \frac{\beta(u_1(t) - c_1)^2}{k}], \end{aligned} \tag{1.1}$$

where u_1 and u_2 are the outputs of two enterprises, A and B, respectively; a_1 and a_2 present the intrinsic growth rates of two enterprises, respectively; k is the carrying capacity of the market under unlimited conditions; and α and β are the competitive coefficients of enterprises A and B, respectively; and c_1 and c_2 denote the initial production of enterprises A and B, respectively. In 2014, Liang, Xu and Tang [2,3] studied the following enterprise cluster model with multiple delays

$$\begin{aligned} \frac{du_1}{dt} &= u_1(t)[a_1 - b_{11}u_1(t-\tau_1) - b_{12}(u_2(t-\tau_2) - c_2)^2] \\ \frac{du_2}{dt} &= u_2(t)[a_2 - b_{21}u_2(t-\tau_3) + b_{22}(u_1(t-\tau_4) - c_1)^2], \end{aligned} \tag{1.2}$$

where τ_i $(i = 1, 2, 3, 4)$ are constant delays. Stability and the complex Hopf bifurcation phenomenon have been obtained for system (1.2).

In recent papers, many studies have incorporated various factors into the enterprise the cluster model. Such as variable environments [4]; the competition and cooperation model with impulse [5]; permanence, periodic solution, and global attractiveness [6]; and the competition and cooperation model with multiple feedback controls [7–9].

Due to the unpredictability of the external environment, system (1.1) is inevitably affected by random disturbances. For example, regarding the external competitive environment, the development of a company is inevitably influenced and constrained by factors such as technological innovation, economic policies, peer competition, and so on. Thus, it is reasonable to further incorporate environmental fluctuations into system (1.1), which may provide a deeper cognizance of the dynamics of the enterprise cluster model in stochastic environments. In the present paper, we introduce the white noise into system (1.1), expressed as follows:

$$\begin{aligned} du_1(t) &= a_1(t)u_1(t)[1 - \frac{u_1(t)}{k(t)} - \frac{\alpha(t)(u_2(t) - c_2(t))^2}{k(t)}]dt + \sigma_1(t)u_1(t)dB_1(t) \\ du_2(t) &= a_2(t)u_2(t)[1 - \frac{u_2(t)}{k(t)} + \frac{\beta(t)(u_1(t) - c_1(t))^2}{k(t)}]dt + \sigma_2(t)u_2(t)dB_2(t), \end{aligned} \tag{1.3}$$

where $B_i(t)$ $(i = 1, 2)$ is one-dimensional Brownian motion, $dB_i(t)$ $(i = 1, 2)$ is white noise, $\sigma_i^2(t) > 0$ $(i = 1, 2)$ is the intensity of white noise. The coefficients of the system are all positive and periodic.

The main innovations of this paper are listed as follows:

(1) There are few results for the stochastic non-autonomous enterprise cluster model. This study fills the gap in the aforementioned research and promotes the study of enterprise clusters.

(2) The construction of Lyapunov functions in this paper is innovative and provides new ideas for studying similar problems.

The remaining framework of this paper is organized as follows: Section 2 initially gives the preliminaries, including the necessary assumptions and some useful lemmas. In Section 3, we study the existence and asymptotic behaviors of stochastic positive periodic solutions for system (1.3). Section 4 gives two numerical examples for verifying our results. We draw some conclusions and discussions in Section 5.

2. Preliminaries

Let $\mathbb{R}$ be real numbers, $\mathbb{R}_+ = [0, +\infty)$, and $\mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2,\ x_1, x_2 > 0\}$. For $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$. Let $(\Theta, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a completed probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the standard normal conditions. In addition, if the number of factors is zero, then a product equals unity. Throughout this paper, we need the following assumption:

(H_1) The functions $a_1(t)$, $a_2(t)$, $\alpha(t)$, $\beta(t)$, $k(t)$, $c_1(t)$, $c_2(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ on $\mathbb{R}_+$ are all bounded positive periodic functions with a common period $\gamma > 0$.

If $f(t)$ is a $\gamma-$periodic function, we define

$$f^u = \max_{t\in[0,\gamma]} f(t),\ f^l = \min_{t\in[0,\gamma]} f(t),\ \langle f\rangle_\gamma = \frac{1}{\gamma}\int_0^\gamma f(t)dt.$$

We first give the definition of periodic Markove processes.

Definition 2.1 [10] A stochastic process $\xi(t)$ is said to be $\gamma-$periodic if its finite dimensional distributions are $\gamma-$periodic, i.e., there exist positive integer n and any moments of $t_1, t_2, \cdots, t_n$ such that the joint distributions of $\xi(t_1 + k\gamma), \cdots, \xi(t_n + k\gamma)$ do not depend on k, where $k \in \mathbb{Z}$.

Consider the following $n-$dimensional stochastic system by [11]

$$dx = f(t, x) + g(t, x)dB(t), \tag{2.1}$$

where $f(t, x) = (f_1(t, x), f_2(t, x), \cdots, f_n(t, x))$ is an $n-$dimensional vector value function, $(g(t, x))_{n\times m}$ is an $n \times m$ matrix function, and $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))$ is an $m-$dimensional standard Brownian motion on the probability space $(\Theta, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Define the following differential operator:

$$\mathcal{L}V(t, x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} f_i(t, x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} g_{ik}(t, x)g_{jk}(t, x)\frac{\partial^2 V}{\partial x_i \partial x_j},$$

where $V(t, x) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

Lemma 2.1 [12] Let $x(t)$ be a solution of system (2.1) and $V(t, x) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then,

$$dV(t, x) = \mathcal{L}V(t, x)dt + V_x(t, x)g(t, x)dB(t),$$

where $V_x(t, x) = (\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \cdots, \frac{\partial V(t,x)}{\partial x_n})$.

We give the following lemma, which describes a criterion for the existence of the periodic solution of a stochastic differential system.

Lemma 2.2 [13] Assume that the coefficients of system (2.1) are all continuous $\gamma-$periodic functions and system (2.1) has a global solution, and further suppose that there is a $\gamma-$periodic function $V(t, x) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ such that

(i) $\inf_{|x|>M} V(t, x) \to \infty$ as $M \to \infty$,

(ii) $\mathcal{L}V(t, x) \leq -1$ outside some compact set.

Then system (2.1) has a $\gamma-$periodic solution.

3. Main results

Before considering the dynamic properties of system (1.3), we should first study the existence conditions of solutions. Therefore, combining the practical significance of the system, we discuss the existence of global positive solutions to system (1.3) by using the random analysis technique. The following theorem can be obtained.

Theorem 3.1. Suppose that assumption (H_1) holds. There exists a unique solution $(u_1(t), u_2(t))$ of system (1.3) with initial value $(u_1(0), u_2(0)) \in \mathbb{R}^2_+$ on $t \geq 0$, and the solution will remain in $\mathbb{R}^2_+$ with probability 1, i.e., $(u_1(t), u_2(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$ almost surely.

Proof. In view of the fact that coefficients of system (1.3) are locally Lipschitz continuous for any given initial value $(u_1(0), u_2(0)) \in \mathbb{R}_+^2$, then system (1.3) has a unique local solution $(u_1(t), u_2(t))$ for $t \in [0, \tau_e)$, where τ_e is the explosion time (see [14]). To show this solution is global, we should prove that $\tau_e = +\infty$ almost surely. Let $n_0 > 0$ be sufficiently large such that $u_1(0), u_2(0) \in [\frac{1}{n_0}, n_0]$. For each $n \geq n_0$, define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{u_1(t), u_2(t)\} \leq \frac{1}{n} \text{ or } \max\{u_1(t), u_2(t)\} \geq n\right\}.$$

It is easy to see that τ_n is increasing as $n \to +\infty$. Let $\tau_\infty = \lim_{n\to+\infty} \tau_n$. Obviously, $\tau_\infty \leq \tau_e$ almost surely. If we prove that $\tau_\infty = \infty$, then $\tau_e = \infty$ and $(u_1(t), u_2(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$ almost surely. If $\tau_\infty \neq \infty$, then there is a pair of constants $L > 0$ and $\varepsilon \in (0, 1)$ such that $P(\tau_\infty \leq L) > \varepsilon$. Thus, there is an integer $n_1 \geq n_0$ such that

$$P(\tau_n \leq L) \geq \varepsilon \text{ for all } n \geq n_1. \tag{3.1}$$

Define a function $V(u_1(t), u_2(t)) \in C^2(\mathbb{R}_+^2, \mathbb{R}_+)$ as follows:

$$V(u_1(t), u_2(t)) = u_1 - a - a\ln\frac{u_1}{a} + u_2 - b - b\ln\frac{u_2}{b},$$

where a and b are positive constants to be determined later. From $x - 1 - \ln x \geq 0$ for $x > 0$, then V is nonnegative. Using It$\hat{o}$'s formula to system (1.3), we have

$$dV = \mathcal{L}Vdt + (1 - \frac{a}{u_1})\sigma_1 u_1 dB_1(t) + (1 - \frac{b}{u_2})\sigma_2 u_2 dB_2(t),$$

where

$$\begin{aligned}\mathcal{L}V = &(1 - \frac{a}{u_1})a_1(t)u_1[1 - \frac{u_1}{k(t)} - \frac{\alpha(t)(u_2 - c_2(t))^2}{k(t)}] + \frac{a\sigma_1^2}{2}\\ &+ (1 - \frac{b}{u_2})a_2(t)u_2[1 - \frac{u_2}{k(t)} + \frac{\beta(t)(u_1 - c_1(t))^2}{k(t)}] + \frac{b\sigma_2^2}{2}.\end{aligned}$$

Note that

$$\begin{aligned}&(1 - \frac{a}{u_1})a_1(t)u_1[1 - \frac{u_1}{k(t)} - \frac{\alpha(t)(u_2 - c_2(t))^2}{k(t)}]\\ &= a_1u_1 - \frac{a_1u_1^2}{k} - \frac{a_1\alpha u_1(u_2 - c_2)^2}{k} - aa_1 + \frac{aa_1u_1}{k} + \frac{aa_1\alpha(u_2 - c_2)^2}{k}\\ &\leq \left(a_1 + \frac{aa_1}{k}\right)u_1 - \frac{a_1u_1^2}{k} - \frac{a_1\alpha u_1u_2^2}{k} + \frac{2a_1\alpha c_2u_1u_2}{k} + \frac{aa_1\alpha u_2^2}{k} + \frac{aa_1\alpha c_2^2}{k} - \frac{2c_2aa_1\alpha u_2}{k}.\end{aligned} \tag{3.2}$$

Similar to the proof of (3.2), we have

$$\begin{aligned}&(1 - \frac{b}{u_2})a_2(t)u_2[1 - \frac{u_2}{k(t)} + \frac{\beta(t)(u_1 - c_1(t))^2}{k(t)}]\\ &\leq \left(a_2 + \frac{ba_2}{k}\right)u_2 - \frac{a_2u_2^2}{k} + \frac{a_2\beta c_1^2}{k} + \frac{a_2\beta u_2u_1^2}{k} - \frac{2c_1a_2\beta u_1}{k}\\ &- \frac{ba_2\beta u_1^2}{k} + \frac{2ba_2\beta c_1u_1}{k}.\end{aligned} \tag{3.3}$$

From Eqs (3.2) and (3.3), we have

$$\begin{aligned}
&(1-\frac{a}{u_1})a_1(t)u_1[1-\frac{u_1}{k(t)}-\frac{\alpha(t)(u_2-c_2(t))^2}{k(t)}]\\
&+(1-\frac{b}{u_2})a_2(t)u_2[1-\frac{u_2}{k(t)}+\frac{\beta(t)(u_1-c_1(t))^2}{k(t)}]\\
&\le\Big(a_1+\frac{aa_1}{k}-\frac{2c_1a_2\beta}{k}+\frac{2ba_2\beta c_1}{k}\Big)u_1+\Big(\frac{a_1\alpha c_2}{k}+\frac{3a_2^2\beta^2}{k^2}+3-\frac{a_1}{k}-\frac{ba_2\beta}{k}\Big)u_1^2\\
&+\Big(a_2+\frac{ba_2}{k}-\frac{2c_2aa_1\alpha}{k}\Big)u_2+\Big(\frac{a_1\alpha c_2}{k}+\frac{aa_1\alpha}{k}+\frac{3a_1^2\alpha^2}{k^2}+3-\frac{a_2}{k}\Big)u_2^2\\
&+\frac{aa_1\alpha c_2^2}{k}+\frac{a_2\beta c_1^2}{k}\\
&\le\Big(a_1^u+\frac{aa_1^u}{k^l}-\frac{2c_1^la_2^l\beta^l}{k^u}+\frac{2b^ua_2^u\beta^uc_1^u}{k^l}\Big)u_1+\Big(\frac{a_1^u\alpha^uc_2^u}{k^l}+\frac{3(a_2^2)^u(\beta^2)^u}{(k^2)^l}+3-\frac{a_1^l}{k^u}-\frac{ba_2^l\beta^l}{k^u}\Big)u_1^2\\
&+\Big(a_2^u+\frac{b^ua_2^u}{k^l}-\frac{2c_2^laa_1^l\alpha^l}{k^u}\Big)u_2+\Big(\frac{a_1^u\alpha^uc_2^u}{k^l}+\frac{aa_1^u\alpha^u}{k^l}+\frac{3(a_1^2)^u(\alpha^2)^u}{(k^2)^l}+3-\frac{a_2^l}{k^u}\Big)u_2^2\\
&+\frac{aa_1^u\alpha^u(c_2^2)^u}{k^l}+\frac{a_2^u\beta^u(c_1^2)^u}{k^l}
\end{aligned}\tag{3.4}$$

Let

$$a=\Big(\frac{a_2^l}{k^u}-\frac{a_1^u\alpha^uc_2^u}{k^l}-\frac{3(a_1^2)^u(\alpha^2)^u}{(k^2)^l}-3\Big)\frac{k^l}{a_1^u\alpha^u}>0$$

and

$$b=\Big(\frac{a_1^u\alpha^uc_2^u}{k^l}+\frac{3(a_2^2)^u(\beta^2)^u}{(k^2)^l}+3\Big)\frac{k^u}{a_2^l\beta^l}.$$

In view of (H_1) and (3.4), we have

$$\mathcal{L}V\le\frac{aa_1^u\alpha^u(c_2^2)^u}{k^l}+\frac{a_2^u\beta^u(c_1^2)^u}{k^l}+\frac{a(\sigma_1^2)^u}{2}+\frac{b(\sigma_2^2)^u}{2}:=H.$$

Hence,

$$\int_0^{\tau_n\wedge L}dV(u_1(t),u_2(t))\le\int_0^{\tau_n\wedge L}Hdt+\int_0^{\tau_n\wedge L}\Big[(1-\frac{a}{u_1})\sigma_1u_1dB_1(t)+(1-\frac{b}{u_2})\sigma_2u_2dB_2(t)\Big].$$

Take the expectation of the above inequality,

$$\begin{aligned}
\mathbb{E}V(u_1(\tau_n\wedge L),u_2(\tau_n\wedge L))&\le V(u_1(0),u_2(0)+\mathbb{E}\int_0^{\tau_n\wedge L}Hdt\\
&\le V(u_1(0),u_2(0)+LH.
\end{aligned}\tag{3.5}$$

For $n\ge n_1$, let $\Theta_n=\{\tau_n\le L\}$. By (3.1), $P(\Theta_n)\ge\varepsilon$. For each $\omega\in\Theta_n$, there is at least one of $u_1(\tau_n,\omega),u_2(\tau_n,\omega)$ equals either n or $\frac{1}{n}$. If $u_1(\tau_n,\omega)=n$ or $\frac{1}{n}$, then

$$V(u_1(\tau_n\wedge L),u_2(\tau_n\wedge L))\ge\Big(n-a-a\ln\frac{n}{a}\Big)\wedge\Big(\frac{1}{n}-a-a\ln\frac{1}{na}\Big).$$

If $u_2(\tau_n,\omega)=n$ or $\frac{1}{n}$, then

$$V(u_1(\tau_n\wedge L),u_2(\tau_n\wedge L))\geq\Big(n-b-b\ln\frac{n}{b}\Big)\wedge\Big(\frac{1}{n}-b-b\ln\frac{1}{nb}\Big).$$

Thus,

$$\begin{aligned}V(u_1(\tau_n\wedge L),u_2(\tau_n\wedge L))\geq&\Big(n-a-a\ln\frac{n}{a}\Big)\wedge\Big(\frac{1}{n}-a-a\ln\frac{1}{na}\Big)\\&\wedge\Big(n-b-b\ln\frac{n}{b}\Big)\wedge\Big(\frac{1}{n}-b-b\ln\frac{1}{nb}\Big).\end{aligned}\tag{3.6}$$

From Eqs (3.5) and (3.6), we have

$$\begin{aligned}V(u_1(0),u_2(0)+LH\geq&\mathbb{E}\Big[1_{\Theta_n}V(u_1(\tau_n\wedge L),u_2(\tau_n\wedge L))\Big]\\\geq&\varepsilon\Big(n-a-a\ln\frac{n}{a}\Big)\wedge\Big(\frac{1}{n}-a-a\ln\frac{1}{na}\Big)\\&\wedge\Big(n-b-b\ln\frac{n}{b}\Big)\wedge\Big(\frac{1}{n}-b-b\ln\frac{1}{nb}\Big),\end{aligned}$$

where 1_{Θ_n} is the indicator function of Θ_n. Letting $n\to\infty$ leads to the contradiction $\infty>V(u_1(0),u_2(0)+LH=\infty$. Hence, $\tau_\infty=\infty$ almost surely. System (1.3) has a unique positive solution. □

Let

$$\lambda_1(t)=\langle a_1(t)-\frac{\sigma_1^2(t)}{2}\rangle_\gamma$$

and

$$\lambda_2(t)=\langle a_2(t)+\frac{a_2(t)\beta(t)c_1^2(t)}{k(t)}-\frac{\sigma_2^2(t)}{2}\rangle_\gamma.$$

From the perspective of the development of enterprise clusters, it is crucial to study their bankruptcy and long-term survival conditions. Therefore, we present the following two results (Theorems 3.2 and 3.3).

Theorem 3.2. Assume that assumption (H_1) holds. If $\lambda_i^u<0$, $i=1,2$. Then, for any initial value $(u_1(0),u_2(0))\in\mathbb{R}_+^2$, the solution $(u_1(t),u_2(t))$ of system (1.3) has the following asymptotic property: $\lim_{t\to\infty}u_i(t)=0$ almost surely, where i=1,2.

Proof. Using the Itôs formula to the first equation of system (1.3), we have

$$\begin{aligned}d\ln u_1(t)=&\Big[a_1(t)(1-\frac{u_1(t)}{k(t)}-\frac{\alpha(t)(u_2(t)-c_2(t))^2}{k(t)})-\frac{\sigma_1^2(t)}{2}\Big]dt+\sigma_1(t)dB_1(t)\\\leq&\Big(a_1(t)-\frac{\sigma_1^2(t)}{2}\Big)dt+\sigma_1(t)dB_1(t).\end{aligned}\tag{3.7}$$

Integrating from 0 to t and dividing by t on both sides of (3.7) yield

$$\frac{1}{t}\Big(\ln u_1(t)-\ln u_1(0)\Big)\leq\frac{1}{t}\int_0^t\Big(a_1(s)-\frac{\sigma_1^2(s)}{2}\Big)ds+\frac{1}{t}\int_0^t\sigma_1(s)dB_1(s).\tag{3.8}$$

It follows by the strong law of large numbers for local martingales (see [16]) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_1(s)dB_1(s)=0\text{ almost surely.}\tag{3.9}$$

By assumption (H_1), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big(a_1(s)-\frac{\sigma_1^2(s)}{2}\Big)ds=\frac{1}{\gamma}\int_0^\gamma\Big(a_1(s)-\frac{\sigma_1^2(s)}{2}\Big)ds. \tag{3.10}$$

Thus,

$$\limsup_{t\to\infty}\frac{\ln u_1(t)}{t}\le\lambda_1^u<0$$

which implies $\lim_{t\to\infty}u_1(t)=0$ almost surely.

Similar to the above proof, using Itôs formula to the second equation of system (1.3), we have

$$\begin{aligned} d\ln u_2(t)&=\Big[a_2(t)(1-\frac{u_2(t)}{k(t)}+\frac{\beta(t)(u_1(t)-c_1(t))^2}{k(t)})-\frac{\sigma_2^2(t)}{2}\Big]dt+\sigma_2(t)dB_2(t)\\ &\le\Big(a_2(t)+\frac{a_2(t)\beta(t)(u_1(t)-c_1(t))^2}{k(t)}-\frac{\sigma_2^2(t)}{2}\Big)dt+\sigma_2(t)dB_2(t). \end{aligned} \tag{3.11}$$

Integrating from 0 to t and dividing by t on both sides of (3.11) yield

$$\frac{1}{t}\Big(\ln u_2(t)-\ln u_2(0)\Big)\le\frac{1}{t}\int_0^t\Big(a_2(s)+\frac{a_2(s)\beta(s)(u_1(s)-c_1(s))^2}{k(s)}-\frac{\sigma_2^2(s)}{2}\Big)ds+\frac{1}{t}\int_0^t\sigma_2(s)dB_2(s). \tag{3.12}$$

We also have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_2(s)dB_2(s)=0 \text{ amost surely.} \tag{3.13}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big(a_2(s)+\frac{a_2(s)\beta(s)(u_1(s)-c_1(s))^2}{k(s)}-\frac{\sigma_2^2(s)}{2}\Big)ds=\frac{1}{\gamma}\int_0^\gamma\Big(a_2(s)+\frac{a_2(s)\beta(s)c_1^2(s)}{k(s)}-\frac{\sigma_2^2(s)}{2}\Big)ds. \tag{3.14}$$

It follows by Eqs (3.12) and (3.14) that

$$\limsup_{t\to\infty}\frac{\ln u_2(t)}{t}\le\lambda_2^u<0$$

which implies $\lim_{t\to\infty}u_2(t)=0$ almost surely. □

Consider the following Logistic equation with periodic coefficients:

$$d\phi(t)=\phi(t)\Big[a_1(t)-\frac{a_1(t)\phi(t)}{k(t)}\Big]dt+\sigma_1(t)\phi(t)dB_1(t)$$

with initial value $\phi(0)=u_1(0)>0$. Using Lemma 2.1 in [17], we have

$$\limsup_{t\to\infty}\langle\phi(t)\rangle_\gamma=\frac{k^u\lambda_1^u}{a_1^l}, \tag{3.15}$$

provided that $\lambda_1^l>0$.

Theorem 3.3. Assume that (H_1) holds. If $\lambda_1^l>0$ and $\lambda_3^u+\frac{a_2^u\beta^u(c_1^u)^2}{k^l}+\frac{\beta^u a_2^u(k^u\lambda_1^u)^2}{k^l(a_1^l)^2}<0$, where $\lambda_3(t)=\langle a_2(t)-\frac{\sigma_2^2(t)}{2}\rangle_\gamma$. Then, the solution $(u_1(t),u_2(t))$ of system (1.3) has the following asymptotic properties:

$$\limsup_{t\to\infty}\langle u_1(t)\rangle_\gamma=\frac{k^u\lambda_1^u}{a_1^l} \text{ almost surely} \tag{3.16}$$

and

$$\lim_{t\to\infty} u_2(t) = 0 \text{ almost surely.} \tag{3.17}$$

Proof. From the first equation of system (1.3), we have

$$du_1(t) \le a_1(t)u_1(t)[1 - \frac{u_1(t)}{k(t)}]dt + \sigma_1(t)u_1(t)dB_1(t)$$

together with (3.15) that

$$\lim_{t\to\infty} \sup\langle u_1(t)\rangle_\gamma = \frac{k^u\lambda_1^u}{a_1^l} \text{ almost surely.} \tag{3.18}$$

Hence, (3.16) holds. Using the Itôs formula to the second equation of system (1.3), we have

$$\begin{aligned} d\ln u_2(t) &= \Big[a_2(t)[1 - \frac{u_2(t)}{k(t)} + \frac{\beta(t)(u_1(t) - c_1(t))^2}{k(t)}] - \frac{\sigma_2^2(t)}{2}\Big]dt + \sigma_2(t)dB_2(t) \\ &\le \Big(a_2(t) - \frac{\sigma_2^2(t)}{2} + \frac{a_2^u\beta^u(c_1^u)^2}{k^l} + \frac{a_2^u\beta^u}{k^l}u_1^2(t)\Big)dt + \sigma_2(t)dB_2(t). \end{aligned} \tag{3.19}$$

Integrating from 0 to t and dividing by t on both sides of (3.19), and using (3.18) and $\lambda_3(t) = \langle a_2(t) - \frac{\sigma_2^2(t)}{2}\rangle_\gamma$, we have

$$\begin{aligned} \frac{1}{t}\Big(\ln u_2(t) - \ln u_2(0)\Big) &\le \lambda_3^u + \frac{a_2^u\beta^u(c_1^u)^2}{k^l} + \frac{1}{t}\int_0^t \frac{a_2^u\beta^u}{k^l}u_1^2(s)ds + \frac{1}{t}\int_0^t \sigma_2(s)dB_2(s) \\ &\le \lambda_3^u + \frac{a_2^u\beta^u(c_1^u)^2}{k^l} + \frac{\beta^u a_2^u(k^u\lambda_1^u)^2}{k^l(a_1^l)^2} + \frac{1}{t}\int_0^t \sigma_2(s)dB_2(s) \end{aligned}$$

and

$$\lim_{t\to\infty}\sup \frac{\ln u_2(t)}{t} \le \lambda_3^u + \frac{a_2^u\beta^u(c_1^u)^2}{k^l} + \frac{\beta^u a_2^u(k^u\lambda_1^u)^2}{k^l(a_1^l)^2} < 0$$

which implies $\lim_{t\to\infty} u_2(t) = 0$ almost surely. Hence, (3.17) holds. □

Now, we consider the existence of a positive γ−periodic solution of system (1.3) by constructing a suitable Lyapunov function and the theory belonging to Khasminskii [15], and the following theorem can be presented.

Theorem 3.4. Suppose that assumption (H$_1$) holds. Then, for any initial value $(u_1(0), u_2(0)) \in \mathbb{R}_+^2$, system (1.3) has a γ−periodic solution, provided that

$$\delta_1 = h_1(t) - \frac{\alpha^u(c_2^u)^2}{k^l} > 0,$$

where $h_1(t) = \langle 1 - \frac{1}{2}\sigma_1^2(t) + 1 - \frac{1}{2}\sigma_2^2(t)\rangle_\gamma$.

Proof. According to Lemma 2.2, for obtaining a γ−periodic solution of system (1.3), we only need to verify the conditions (i) and (ii) of Lemma 2.2. Construct a function $V(t, u_1, u_2) \in C^2(\mathbb{R}_+^3, \mathbb{R})$ as follows:

$$\begin{aligned} V(t, u_1, u_2) &= M\Big(- \ln u_1 - \ln u_2 + w_1(t)\Big) + e^{w_2(t)}\frac{1}{2}(u_1 + u_2)^2 + e^{w_2(t)}(u_1 + u_2) \\ &= V_1(t, u_1, u_2) + V_2(t, u_1, u_2), \end{aligned}$$

where $M > 0$ will be determined later,

$$w_1'(t) = 1 - \frac{1}{2}\sigma_1^2(t) + 1 - \frac{1}{2}\sigma_2^2(t) - h_1(t) - \frac{\alpha(t)u_2^2(t)}{k(t)},$$

$$w_2'(t) = -\frac{2a_2u_2\beta(t)(u_1 - c_1(t))^2}{k(t)} + h_2(t),$$

$$h_1(t) = \langle 1 - \frac{1}{2}\sigma_1^2(t) + 1 - \frac{1}{2}\sigma_2^2(t)\rangle_\gamma, h_2(t) = \langle 2a_1(t) + 4a_2(t) + \sigma_2^2(t)\rangle_\gamma.$$

For showing the condition (i) of Lemma 2.2 holds, we only need to prove

$$\inf_{(t,u_1,u_2)\in\mathbb{R}_+\times(\mathbb{R}_+^2\setminus U_m)} V(t, u_1, u_2) \to \infty \text{ as } m \to \infty,$$

where $U_m = (\frac{1}{m}, m) \times (\frac{1}{m}, m)$. Since all the coefficients of the quadratic term in $V(t, u_1, u_2)$ are positive, the condition (i) of Lemma 2.2 holds.

Next, we show that the condition (ii) of Lemma 2.2 holds. Using It$\hat{o}$'s formula, we have

$$\begin{aligned} \mathcal{L}V_1(t, u_1, u_2) &\leq -M\Big(1 - \frac{1}{2}\sigma_1^2(t) - \frac{\alpha(t)c_2^2(t)}{k(t)} - \frac{\alpha(t)u_2^2(t)}{k(t)}\Big) + \frac{Mu_1(t)}{k(t)} \\ &\quad - M\Big(1 - \frac{1}{2}\sigma_2^2(t)\Big) + \frac{Mu_2(t)}{k(t)} + Mw_1'(t) \\ &\leq M\Big(-\delta_1 + \frac{1}{k^l}u_1(t) + \frac{1}{k^l}u_2(t)\Big), \end{aligned} \tag{3.20}$$

where $\delta_1 > 0$, and

$$\begin{aligned} \mathcal{L}V_2(t, u_1, u_2) &\leq w_2'(t)e^{w_2(t)}\frac{1}{2}(u_1 + u_2)^2 + e^{w_2(t)}(u_1 + u_2) \\ &\quad \times \Big(2a_1(t)u_1 - \frac{a_1u_1^2}{k(t)} + 2a_2(t)u_2 - \frac{a_2u_2^2}{k(t)} + \frac{2a_2u_2\beta(t)(u_1 - c_1(t))^2}{k(t)} + w_2'(t)\Big) \\ &\quad + \frac{1}{2}e^{w_2(t)}\Big(\sigma_1^2(t)u_1^2 + \sigma_2^2(t)u_2^2\Big) \\ &\leq e^{w_2(t)}\Big[h_2(t)u_1 + h_2(t)u_2 + \Big(\frac{w_2'(t) + 4a_1(t) + 2a_2(t)}{2} + \frac{1}{2}\sigma_1^2(t)\Big)u_1^2 \\ &\quad + \Big(\frac{w_2'(t) + 2a_1(t) + 4a_2(t)}{2} + \frac{1}{2}\sigma_2^2(t)\Big)u_2^2 - \frac{a_1(t)}{k(t)}u_1^3 - \frac{a_2(t)}{k(t)}u_2^3\Big] \\ &\leq e^{|w_2^u|}|h_2^u|u_1 + e^{|w_2^u|}|h_2^u|u_2 + \frac{1}{2}e^{|w_2^u|}\Big(|w_2'(t)|^u + 4a_1^u + 2a_2^u + (\sigma_1^u)^2\Big)u_1^2 \\ &\quad + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3. \end{aligned} \tag{3.21}$$

From Eqs (3.20) and (3.21), we have

$$\begin{aligned} \mathcal{L}V(t, u_1, u_2) &\leq -M\delta_1 + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_1 + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_2 \\ &\quad + qu_1^2 + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3, \end{aligned}$$

where $q = \frac{1}{2}e^{|w_2^u|}\Big(|w_2'(t)|^u + 4a_1^u + 2a_2^u + (\sigma_1^u)^2\Big)$. Define the following bounded open set:

$$\mathcal{U}_\varepsilon = \Big\{(u_1, u_2) : \varepsilon < u_1 < \frac{1}{\varepsilon},\ \varepsilon < u_2 < \frac{1}{\varepsilon}\Big\}.$$

Let

$$\mathcal{U}_\varepsilon^1 = \Big\{(u_1, u_2) : 0 < u_1 \le \varepsilon,\ u_2 \in \mathbb{R}_+\Big\},\ \mathcal{U}_\varepsilon^2 = \Big\{(u_1, u_2) : 0 < u_2 \le \varepsilon,\ u_1 \in \mathbb{R}_+\Big\},$$

$$\mathcal{U}_\varepsilon^3 = \Big\{(u_1, u_2) : u_1 \ge \frac{1}{\varepsilon},\ u_2 \in \mathbb{R}_+\Big\},\ \mathcal{U}_\varepsilon^4 = \Big\{(u_1, u_2) : u_2 \ge \frac{1}{\varepsilon},\ u_1 \in \mathbb{R}_+\Big\}.$$

Obviously, $\mathcal{U}_\varepsilon^c = \mathcal{U}_\varepsilon^1 \cup \mathcal{U}_\varepsilon^2 \cup \mathcal{U}_\varepsilon^3 \cup \mathcal{U}_\varepsilon^4$. In the following, we show that $\mathcal{L}V(t, u_1, u_2) \le -1$ on $\mathbb{R}_+ \times \mathcal{U}_\varepsilon^c$. For $u_i < \frac{k^l\delta_1}{2},\ i = 1, 2$, let

$$M_{1i} = \frac{2k^l}{k^l\delta_1 - 2u_i} \max_{(u_1,u_2)\in\mathbb{R}_+^2} \Big\{2, qu_1^2 + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big\},\ i = 1, 2. \tag{3.22}$$

For $u_i \ge \frac{k^l\delta_1}{2},\ i = 1, 2$, let

$$M_{21} = \frac{2}{\delta_1} \max_{(u_1,u_2)\in\mathbb{R}_+^2} \Big\{2, \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big\}, \tag{3.23}$$

$$M_{22} = \frac{2}{\delta_1} \max_{(u_1,u_2)\in\mathbb{R}_+^2} \Big\{2, qu_1^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3\Big\}. \tag{3.24}$$

Define

$$M = \max\Big\{M_{i,j},\quad i, j = 1, 2\Big\}. \tag{3.25}$$

Choose a sufficient small ε satisfying

$$0 < \varepsilon \le \frac{k^l\delta_1}{4}. \tag{3.26}$$

Case 1. If $(t, u_1, u_2) \in \mathbb{R}_+ \times \mathcal{U}_\varepsilon^1$, we have

$$\begin{aligned}\mathcal{L}V(t, u_1, u_2) \le{}& -\frac{M\delta_1}{4} + \Big[-\frac{M\delta_1}{4} + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)\varepsilon\Big] \\ &+ \Big[-\frac{M\delta_1}{2} + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_2 + qu_1^2 + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big].\end{aligned}$$

If $u_2 < \frac{k^l\delta_1}{2}$, in view of (3.22), (3.25), and (3.26), we have

$$\mathcal{L}V(t, u_1, u_2) \le -\frac{M\delta_1}{4} \le -1.$$

If $u_2 \ge \frac{k^l\delta_1}{2}$, in view of (3.23)–(3.26), we have

$$\begin{aligned}\mathcal{L}V(t, u_1, u_2) &\le -\frac{M\delta_1}{4} + \Big[-\frac{M\delta_1}{4} + (\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|)\varepsilon\Big] \\ &\quad + \Big(-\frac{M\delta_1}{2} + qu_1^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3\Big) \\ &\le -\frac{M\delta_1}{4} \le -1.\end{aligned}$$

Case 2. If $(t, u_1, u_2) \in \mathbb{R}_+ \times \mathcal{U}_\varepsilon^2$, we have

$$\begin{aligned}\mathcal{L}V(t,u_1,u_2) \le & -\frac{M\delta_1}{4} + \Big[-\frac{M\delta_1}{4} + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)\varepsilon\Big] \\ & + \Big[-\frac{M\delta_1}{2} + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_1 + qu_1^2 + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big].\end{aligned}$$

If $u_1 < \frac{k^l\delta_1}{2}$, is sufficiently small, in view of (3.22), (3.25), and (3.26), we have

$$\mathcal{L}V(t,u_1,u_2) \le -\frac{M\delta_1}{4} \le -1.$$

If $u_1 \ge \frac{k^l\delta_1}{2}$, in view of (3.23)–(3.26), we have

$$\begin{aligned}\mathcal{L}V(t,u_1,u_2) & \le -\frac{M\delta_1}{4} + \Big[-\frac{M\delta_1}{4} + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)\varepsilon\Big] \\ & \quad + \Big(-\frac{M\delta_1}{2} + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big) \\ & \le -\frac{M\delta_1}{4} \le -1.\end{aligned}$$

Let

$$\begin{aligned}C_1 = \max_{(u_1,u_2)\in\mathbb{R}_+^2} & \Big\{\Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_1 + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_2 \\ & + qu_1^2 + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - \frac{1}{2}e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big\}\end{aligned}$$

and

$$\begin{aligned}C_2 = \max_{(u_1,u_2)\in\mathbb{R}_+^2} & \Big\{\Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_1 + \Big(\frac{M}{k^l} + e^{|w_2^u|}|h_2^u|\Big)u_2 \\ & + qu_1^2 + \frac{1}{2}e^{|w_2^u|}(|h_2^u| + |w_2'|^u)u_2^2 - e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 - \frac{1}{2}e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\Big\}\end{aligned}$$

such that

$$-M\delta_1 + C_1 + 1 \le \frac{1}{2}e^{|w_2^l|}\frac{a_1^l}{\varepsilon^3 k^u}, \tag{3.27}$$

and

$$-M\delta_1 + C_2 + 1 \le \frac{1}{2}e^{|w_2^l|}\frac{a_2^l}{\varepsilon^3 k^u}. \tag{3.28}$$

Case 3. If $(t, u_1, u_2) \in \mathbb{R}_+ \times \mathcal{U}_\varepsilon^3$, in view of (3.27), we have

$$\begin{aligned}\mathcal{L}V(t,u_1,u_2) & \le -M\delta_1 + C_1 - \frac{1}{2}e^{|w_2^l|}\frac{a_1^l}{k^u}u_1^3 \\ & \le -M\delta_1 + C_1 - \frac{1}{2}e^{|w_2^l|}\frac{a_1^l}{\varepsilon^3 k^u} \\ & \le -1.\end{aligned}$$

Case 4. If $(t, u_1, u_2) \in \mathbb{R}_+ \times \mathcal{U}_\varepsilon^4$, in view of (3.28), we have

$$\begin{aligned}\mathcal{L}V(t, u_1, u_2) &\leq -M\delta_1 + C_2 - \frac{1}{2}e^{|w_2^l|}\frac{a_2^l}{k^u}u_2^3\\ &\leq -M\delta_1 + C_2 - \frac{1}{2}e^{|w_2^l|}\frac{a_2^l}{\varepsilon^3 k^u}\\ &\leq -1.\end{aligned}$$

Therefore, we obtain that $\mathcal{L}V_1(t, u_1, u_2) \leq -1$ for all $(t, u_1, u_2) \in \mathbb{R}_+ \times \mathcal{U}_\varepsilon^c$, i.e., the condition (ii) of Lemma 2.2 is satisfied. Therefore, system (1.3) has a $\gamma-$periodic solution. □

4. Examples

In this section, we give two numerical examples for verifying our main results. To this end, based on the method in [18], system (1.3) can be discretized to the following form at $t = (k+1)\Delta t,\ k = 0, 1, \cdots$:

$$\begin{aligned}u_1^{k+1} &= u_1^k + a_1(k\Delta t)u_1^k[1 - \frac{u_1(k\Delta t)}{k(k\Delta t)} - \frac{\alpha(k\Delta t)(u_2^k - c_2(k\Delta t))^2}{k(k\Delta t)}]\Delta t\\ &\quad + \sigma_1(k\Delta t)u_1^k(\Delta t)^{\frac{1}{2}}\xi_k + \frac{\sigma_1^2(k\Delta t)}{2}(u_1^k)^2(\xi_k^2\Delta t - \Delta t)\\ u_2^{k+1} &= u_2^k + a_2(k\Delta t)u_2^k[1 - \frac{u_2^k}{k(k\Delta t)} + \frac{\beta(k\Delta t)(u_1^k - c_1(k\Delta t))^2}{k(k\Delta t)}]\Delta t\\ &\quad + \sigma_2(k\Delta t)u_2^k(\Delta t)^{\frac{1}{2}}\eta_k + \frac{\sigma_1^2(k\Delta t)}{2}(u_2^k)^2(\eta_k^2\Delta t - \Delta t),\end{aligned}$$

where ξ_k and η_k are the $N(0, 1)$-distribution Gaussian stochastic variables. In the system (1.3), take the periodic coefficients as follows:

$$a_1(t) = 0.6 + 0.01\sin\frac{\pi t}{20},\ a_2(t) = 0.5 + 0.01\cos\frac{\pi t}{20},\ k(t) = 20 + 0.01\sin\frac{\pi t}{20},$$

$$\alpha(t) = 0.65 + 0.001\sin\frac{\pi t}{20},\ \beta(t) = 0.7 + 0.001\cos\frac{\pi t}{20},\ c_1(t) = c_2(t) = 0.5 + 0.0001\sin\frac{\pi t}{20}.$$

Example 4.1. Let $\sigma_1 = 1.25 + 0.05\sin\frac{\pi t}{20}$ and $\sigma_2 = 1.16 + 0.02\sin\frac{\pi t}{20}$. By simple computation, we have $\lambda_1^u \approx -0.181 < 0$ and $\lambda_2^u \approx -0.143 < 0$. It follows by Theorem 3.2 that the solution $(u_1(t), u_2(t))$ of system (1.3) has the asymptotic property: $\lim_{t\to\infty} u_1(t) = 0$ and $\lim_{t\to\infty} u_2(t) = 0$ almost surely. Figure 1(a) and (b) verify the above results. Furthermore, let $\sigma_1 = 0.82 + 0.03\sin\frac{\pi t}{20}$ and $\sigma_2 = 6.15 + 0.01\sin\frac{\pi t}{20}$. We have

$$\lambda_1^l \approx 0.263 > 0,\ \ \lambda_3^u + \frac{a_2^u\beta^u(c_1^u)^2}{k^l} + \frac{\beta^u a_2^u(k^u\lambda_1^u)^2}{k^l(a_1^l)^2} \approx -14.5 < 0.$$

Then, all conditions of Theorem 3.3 hold. Therefore, the solution $(u_1(t), u_2(t))$ of system (1.3) has the following asymptotic properties:

$$\lim_{t\to\infty}\sup\langle u_1(t)\rangle_\gamma = \frac{k^u\lambda_1^u}{a_1^l} \approx 8.67 \text{ almost surely}$$

and

$$\lim_{t\to\infty} u_2(t) = 0 \text{ almost surely.}$$

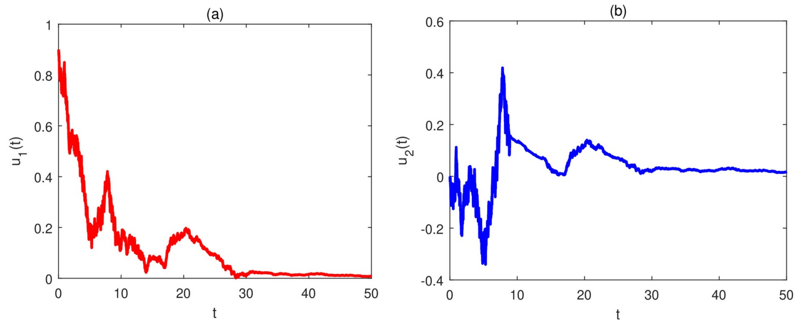

Figure 1. The stochastic dynamics behaviors of system (1.3) with $\sigma_1 = 1.25 + 0.05\sin\frac{\pi t}{20}$ and $\sigma_2 = 1.16 + 0.02\sin\frac{\pi t}{20}$.

Figures 2(a) and (b) verify the results of Theorem 3.3.

From Figure 1, when the external environment of an enterprise cluster is harsh (the densities of white noises are very high), it will go bankrupt. From Figure 2, the density of white noise has a direct impact on the survival of enterprises, and enterprises can overcome difficulties by implementing various strategies.

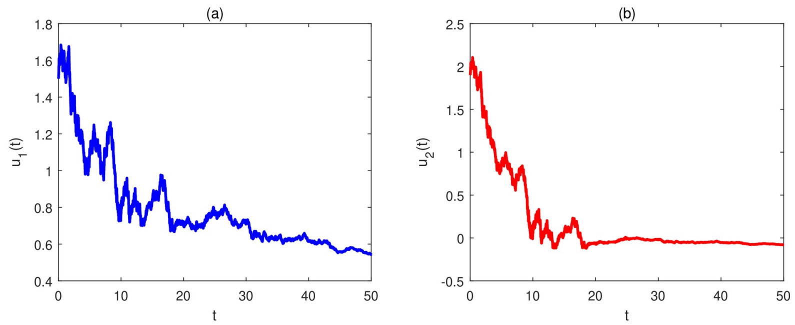

Figure 2. The stochastic dynamics behaviors of system (1.3) with $\sigma_1 = 0.82 + 0.03\sin\frac{\pi t}{20}$ and $\sigma_2 = 6.15 + 0.01\sin\frac{\pi t}{20}$.

Now, we verify the theoretical results of Theorem 3.4 and obtain that system (1.3) has a non-trivial stochastically positive periodic solution.

Example 4.2. Let $\sigma_1 = 0.05 + 0.02\sin\frac{\pi t}{20}$ and $\sigma_2 = 0.03 - 0.02\sin\frac{\pi t}{20}$. We obtain that

$$\delta_1 = h_1(t) - \frac{\alpha^u (c_2^u)^2}{k^l} \approx 1.956 > 0.$$

Thus, all assumptions of Theorem 3.4 are satisfied. Based on Theorem 3.4, system (1.3) has a non-trivial stochastic positive periodic solution. Figure 3(a) and (b) verify the results of Theorem 3.4. The simulation results show that given some reasonable periodic coefficients, system (1.3) has a non-trivial stochastic positive periodic solution.

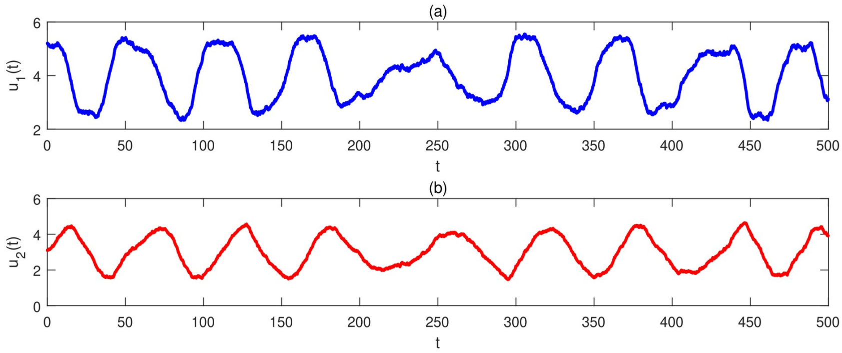

Figure 3. Stochastic positive periodic solution of system (1.3) with $\sigma_1 = 0.05 + 0.02\sin\frac{\pi t}{20}$ and $\sigma_2 = 0.03 - 0.02\sin\frac{\pi t}{20}$.

5. Discussions and conclusions

Enterprise clusters are always in a complex competitive environment and are influenced by various uncontrollable factors. The study of the dynamic mechanism of system (1.3) can help us formulate reasonable strategies and promote the development of enterprises. The periodic coefficients and white noise densities of system (1.3) are directly related to the asymptotic behavior and the existence of positive periodic solutions. Specifically, studying the existence of stochastic periodic solutions for enterprise clusters has significant practical value in promoting the development of enterprises and society. In this paper, we introduced a stochastic non-autonomous enterprise cluster model, and investigated the asymptotic behavior and the existence of stochastically positive periodic solutions of system (1.3). By using the methods belonging to [19], we first studied the existence and uniqueness of nontrivial global positive solutions to system (1.3), and then obtained the sufficient conditions for the asymptotic behavior by the stochastic analysis technique and It$\hat{o}$'s formula. For showing the existence of stochastically positive periodic solutions, we used the Lyapunov function method, It$\hat{o}$'s formula, and the theory of Khasminskii. Two numerical examples verified our main results and further demonstrated that the dynamics and existence of positive periodic solutions are intimately associated with the periodic coefficients and densities of white noise. The theoretical and numerical results in this paper show:

• If densities of white noise are sufficiently large, the conditions of Theorem 3.2 hold, and both the competitive enterprises u_1 and u_2 (see Fig. 1) will go bankrupt. This requires managers of enterprises to increase their efficiency through internal reforms when facing severe external environments, in order to achieve the survival and development of the enterprise. On the other hand, the government should also reduce various factors that are unfavorable to the development of enterprises from a macro perspective, strengthen positive factors, and support the development of enterprises.

• From Theorem 3.3, we find that competing enterprises do not always coexist (enterprise u_1 is continuously surviving, and u_2 is bankrupt). Fig. 2 shows the status. The main reason for the above situation is the periodic coefficients and densities of white noise. It is very important to increase the intrinsic growth rate of enterprise u_2 and reduce the density of white noise in order to avoid its bankruptcy.

• Periodic phenomena and behaviors are widely present in the development of nature and human society. The existence of a stochastic positive periodic solution for system (1.3) can be understood as the long-term coexistence of two competing enterprises. Fig. 3 shows the existence of a stochastic positive periodic solution. We can ensure the existence of positive periodic solutions by controlling the parameters (including intrinsic growth rates of enterprises, the carrying capacity, the competitive coefficients of enterprises, etc.) of the system.

It would also be interesting to study other important issues. In the real world, system (1.3) is often affected by time delay, so studying the dynamic mechanism of stochastic enterprise cluster systems with variable time delay will receive increasing attention. At the same time, we should also pay attention to the impact of pulse effects and regime switching on the dynamic behavior of system (1.3). We leave these problems for future research.

Author contributions

Xiuguo Lian: Methodology, Formal analysis, Investigation, Writing−original draft; Xiwang Cheng: Investigation, Methodology, Writing−original draft; Famei Zheng: Formal analysis, Software.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest in this paper.

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