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Research article

Dynamics and control for a stochastic giving up smoking model

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Abstract: This paper focused on a stochastic giving-up-smoking model with harmonic mean-type incidence rate, in which the population was divided into four types. Firstly, we showed that the model has a unique global positive solution. Then, stochastic permanence of the model was discussed, which means that the population described by the model will not grow wildly or disappear. Next, sufficient conditions for the elimination of smokers (including occasional smokers, chain smokers, and quit smokers) were established. Additionally, sufficient conditions for the existence of an ergodic stationary distribution were derived, meaning that all types of smokers can be persistent. Moreover, we discussed how to control the size of the smoker population from the perspective of economics. Finally, some numerical simulations were introduced.

Keywords: stochastic smoking model; extinction and permanence; ergodic stationary distribution; smoking control
Mathematics Subject Classification: 60H10, 92D25, 92D40, 93E15

1. Introduction

Smoking is one of the main causes of health problems and continues to be one of the world's most significant health challenges (see [1]). Substantial medical research confirms that smoking is closely associated with more than twenty diseases, including lung cancer and heart disease [2, 3]. Its harm is manifested not only in individual health but also in a significant social burden. For instance, in [4], authors showed that around 55,000 deaths each year are attributable to smoking in Spain; authors in [5] pointed out that the incidence of lung cancer in smokers is ten times higher than that in non-smokers, and one out of ten smokers will die of lung cancer; furthermore, authors in [6] showed that the risk of heart attack in smokers is 70% higher than that of non-smokers. In China, tobacco control is similarly a major social issue that requires urgent resolution [7].

In recent years, mathematical models have been playing an increasingly important role in ecological fields [8], epidemiological research [9, 10], and public health policy formulation [11]. Mathematical modeling provides a powerful tool for theoretically understanding the spreading dynamics of smoking behavior and for formulating effective control strategies. Zeb, Zaman, and Momani [6] investigated a giving-up-smoking model with square root-type incidence rate, while [12] divided the population into potential smokers P, occasional smokers L, chain smokers S, and quit smokers Q, and investigated the dynamics of a giving-up-smoking model with harmonic mean type incidence rate.

However, such deterministic models typically assume constant parameters, which do not fully align with reality. In fact, the processes of disease transmission or behavioral spread in the real world are always subject to various random environmental fluctuations. These fluctuations may arise from the randomness of contact opportunities, individual behavioral differences, changes in environmental conditions, etc. Consequently, parameters in the models (such as mortality and contact rates) are not absolute constants but fluctuate randomly around some average value [13]. Neglecting this stochasticity may lead to inaccurate estimations of the system dynamics. Among the various methods for introducing stochasticity, simulating parameter perturbations via white noise is a common and effective approach [14]. Hence, many scholars have introduced randomness into models to reveal the effects of environmental noise. References [13, 15–18] discussed the dynamics of stochastic population models, while [19–23] focused on the dynamics of stochastic epidemic models. Therefore, incorporating environmental noise into deterministic models to more realistically reflect the system's dynamic behavior has become an important research direction in stochastic epidemiology and population dynamics.

Based on the above considerations, this paper aims to conduct an in-depth investigation into the transmission mechanisms and control strategies of smoking behavior by developing a stochastic smoking cessation model with harmonic mean incidence rate. The innovative contributions of this work are threefold:
(i) By introducing stochastic noise perturbations, we overcome the idealized assumption of constant parameters in deterministic models;
(ii) Through theoretical analysis and numerical simulations, we demonstrate the positive regulatory role of environmental disturbances in smoking transmission dynamics;
(iii) We discover the coupled effects of multiple factors including cigarette price, contact rate, and noise intensity on smoking transmission, thereby proposing precise tobacco control strategies.

The organization of this paper is as follows: In the next section, we offer a systematic description of the model (2.3). In Section 3, we first show that model (2.3) has a unique positive global solution. Moreover, some asymptotic properties of the solution are given. In Section 4, we establish sufficient conditions for the extinction of smokers (including occasional smokers, chain smokers, and quit smokers). In Section 5, by constructing a suitable Lyapunov function, we show that there is an ergodic stationary distribution for the solution of the model. This means that all types of smokers in model (2.3) can be persistent. In Section 6, we discuss how to control the number of occasional smokers from the perspective of economics. Numerical simulations under certain parameters are presented in Section 7. The paper ends with a conclusion.

2. Problem formulation

In this section, we provide a detailed description of the stochastic giving-up-smoking model under study.

To establish the theoretical foundation, we first introduce the classical deterministic model as the basis for our study. Reference [12] divided the population into potential smokers P, occasional smokers L, chain smokers S, and quit smokers Q, and investigated the dynamics of the following giving-up-smoking model with harmonic mean type incidence rate

$$\begin{cases} \frac{dP}{dt} = \lambda - \beta\frac{2PL}{P+L} - (d+\mu)P, \\ \frac{dL}{dt} = \beta\frac{2PL}{P+L} - (\zeta + d + \mu)L, \\ \frac{dS}{dt} = \zeta L - (\delta + d + \mu)S, \\ \frac{dQ}{dt} = \delta S - (d+\mu)Q. \end{cases} \tag{2.1}$$

Here, λ is the birth rate for potential smoker individuals; μ is the natural death rate; ζ is the rate of change from occasional smokers to chain smokers; β is the transmission coefficient; δ is quit rate of smoking; d represents the death rate for potential smokers, occasional smokers, chain smokers, and quit smokers due to smoking disease. All parameters in model (2.1) are assumed to be positive.

To further refine the model structure, Zaman [3] assumed that the mortality rates of potential smokers, occasional smokers, chain smokers, and quit smokers due to smoking diseases are d_1, d_2, d_3, and d_4, respectively. Thus, based on (2.1), one can get the following giving-up-smoking model

$$\begin{cases} \frac{dP}{dt} = \lambda - \beta\frac{2PL}{P+L} - (d_1+\mu)P, \\ \frac{dL}{dt} = \beta\frac{2PL}{P+L} - (\zeta + d_2 + \mu)L, \\ \frac{dS}{dt} = \zeta L - (\delta + d_3 + \mu)S, \\ \frac{dQ}{dt} = \delta S - (d_4+\mu)Q. \end{cases} \tag{2.2}$$

In a similar discussion as that in [12], model (2.2) has one smoking-free equilibrium point $E^0 = (\frac{\lambda}{d_1+\mu}, 0, 0, 0)$ for all parameter values. Based on dynamical systems theory, model (2.2) has one smoking-present equilibrium point $E^* = (P^*, L^*, S^*, Q^*)$ when $\mathcal{R}_0 = \frac{2\beta}{\zeta+d_2+\mu} > 1$. Here,

$$P^* = \frac{\lambda}{2\beta - \zeta}, \ \ L^* = (\mathcal{R}_0 - 1)P^*, \ \ S^* = \frac{\zeta(\mathcal{R}_0 - 1)P^*}{\delta + d_3 + \mu}, \ \ Q^* = \frac{\delta\zeta(\mathcal{R}_0 - 1)P^*}{(d_4+\mu)(\delta + d_3 + \mu)}.$$

Considering the random fluctuations of parameters in real-world environments, we assume that the death rate d_i in model (2.2) always fluctuates around some average value. In this sense,

$$-d_i \to -d_i + \sigma_i \dot{B}_i(t) \ \ (i = 1, 2, 3, 4),$$

where $B_1(t), B_2(t), B_3(t), B_4(t)$ are mutually independent Brownian motions defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. σ_i^2 is the intensity of white noise $\dot{B}_i(t)$ $(i = 1, 2, 3, 4)$. Ultimately, we establish the stochastic giving-up-smoking model with harmonic mean incidence rate as follows:

$$\begin{cases} dP(t) = \Big[\lambda - \beta\frac{2P(t)L(t)}{P(t)+L(t)} - (d_1+\mu)P(t)\Big]dt + \sigma_1 P(t)dB_1(t), \\ dL(t) = \Big[\beta\frac{2P(t)L(t)}{P(t)+L(t)} - (\zeta + d_2+\mu)L(t)\Big]dt + \sigma_2 L(t)dB_2(t), \\ dS(t) = \Big[\zeta L(t) - (\delta + d_3+\mu)S(t)\Big]dt + \sigma_3 S(t)dB_3(t), \\ dQ(t) = \Big[\delta S(t) - (d_4+\mu)Q(t)\Big]dt + \sigma_4 Q(t)dB_4(t), \end{cases} \tag{2.3}$$

with $(P(0), L(0), S(0), Q(0)) = (P_0, L_0, S_0, Q_0) \in \mathbb{R}^4_+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i > 0, i = 1, 2, 3, 4\}$. All meanings of the parameters are exact to or similar as those for model (2.2).

3. Global positive solution and asymptotic behaviors

In this section, we first show that the model has a unique positive global solution. Then, we discuss the asymptotic property of the solution. For ease, we denote

$$\begin{aligned} &X(t) = (P(t), L(t), S(t), Q(t)), \qquad X_0 = (P_0, L_0, S_0, Q_0), \\ &N(t) = P(t) + L(t) + S(t) + Q(t), \qquad \langle u(t)\rangle = \frac{1}{t}\int_0^t u(s)ds, \\ &\hat{d} = \min\{d_1, d_2, d_3, d_4\}, \ \ \check{d} = \max\{d_1, d_2, d_3, d_4\}, \ \ \sigma^2 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\}. \end{aligned}$$

3.1. Existence and uniqueness of the positive solution

In this subsection, we show that model (2.3) has a unique positive global solution with positive initial value.

Theorem 3.1. *For any given initial value $X_0 \in \mathbb{R}^4_+$, model* (2.3) *has a unique global positive solution $X(t)$ on* $[0, \infty)$*; that is, $X(t) \in \mathbb{R}^4_+$ with probability one for $t \in [0, \infty)$.*

Proof. Clearly, the coefficients of (2.3) are locally Lipschitz continuous. Thus, for any $X_0 \in \mathbb{R}^4_+$, model (2.3) has a unique maximal local solution $X(t)$ on $[0, \tau_e)$, where τ_e is the explosion time. Let $n_0 > 0$ be sufficiently large such that P_0, L_0, S_0, and Q_0 all lie within the interval $(1/n_0, n_0)$. For each integer $n \geq n_0$, define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{P(t), L(t), S(t), Q(t)\} \leq \frac{1}{n} \text{ or } \max\{P(t), L(t), S(t), Q(t)\} \geq n\right\},$$

where for empty set $\emptyset$, we set $\inf\emptyset = \infty$. It is clear that τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty}\tau_n$. Thus, τ_∞ is a stopping time and $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $X(t) \in \mathbb{R}^4_+$ a.s. for all $t \geq 0$. Now, we show that $\tau_\infty = \infty$ a.s. If this assertion is not true, then there are constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. For any $n \geq n_0$, let $\Omega_n = \{\omega \in \Omega : \tau_n(\omega) \leq T\}$. Then, for any $n \geq n_0$, we have $\mathbb{P}(\Omega_n) > \varepsilon$. Define function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ by

$$V(P, L, S, Q) = (P - 1 - \ln P) + (L - 1 - \ln L) + (S - 1 - \ln S) + (Q - 1 - \ln Q).$$

Using Itô formula, we have, for any $t \in [0, T]$ and $n \geq n_0$,

$$\begin{aligned}\mathbb{E}V(P(t \wedge \tau_n), L(t \wedge \tau_n), S(t \wedge \tau_n), Q(t \wedge \tau_n)) =& V(P_0, L_0, S_0, Q_0) \\ &+ \mathbb{E}\int_0^{t\wedge\tau_n} \mathcal{L}V(P(s), L(s), S(s), Q(s))\,\mathrm{d}s, \end{aligned} \tag{3.1}$$

where $\mathcal{L}V : \mathbb{R}^4_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{L}V =& \Big(1 - \frac{1}{P}\Big)\Big[\lambda - \beta\frac{2PL}{P+L} - (d_1 + \mu)P\Big] + \Big(1 - \frac{1}{L}\Big)\Big[\beta\frac{2PL}{P+L} - (\zeta + d_2 + \mu)L\Big] \\ &+ \Big(1 - \frac{1}{S}\Big)\Big[\zeta L - (\delta + d_3 + \mu)S\Big] + \Big(1 - \frac{1}{Q}\Big)\Big[\delta S - (d_4 + \mu)Q\Big] + \sum_{i=1}^{4}\frac{\sigma_i^2}{2} \\ \leq& \lambda + 2\beta + 4\mu + \zeta + \delta + \sum_{i=1}^{4} d_i + \sum_{i=1}^{4}\frac{\sigma_i^2}{2} \doteq K. \end{aligned} \tag{3.2}$$

Here, $K > 0$ is a constant. Thus, from (3.1), it follows that

$$\mathbb{E}V(P(T \wedge \tau_n), L(T \wedge \tau_n), S(T \wedge \tau_n), Q(T \wedge \tau_n)) \leq V(P_0, L_0, S_0, Q_0) + KT. \tag{3.3}$$

For every $\omega \in \Omega_n$, there is at least one of $P(\tau_n, \omega)$, $L(\tau_n, \omega)$, $S(\tau_n, \omega)$, and $Q(\tau_n, \omega)$ equalling either $1/n$ or n. Hence,

$$V(P(\tau_n, \omega), L(\tau_n, \omega), S(\tau_n, \omega), Q(\tau_n, \omega)) \geq (n - 1 - \ln n) \wedge \Big(\frac{1}{n} - 1 - \ln\frac{1}{n}\Big). \tag{3.4}$$

It then follows from (3.3) and (3.4) that

$$\begin{aligned}V(P_0, L_0, S_0, Q_0) + KT \geq& \mathbb{E}\Big[I_{\Omega_n}(\omega)V(P(\tau_n, \omega), L(\tau_n, \omega), S(\tau_n, \omega), Q(\tau_n, \omega))\Big] \\ >& \varepsilon\Big[(n - 1 - \ln n) \wedge \Big(\frac{1}{n} - 1 - \ln\frac{1}{n}\Big)\Big],\end{aligned}$$

where I_{Ω_n} is the indicator function of Ω_n. Letting $n \to \infty$ leads to the contradiction

$$\infty > V(P_0, L_0, S_0, Q_0) + KT = \infty.$$

Hence, $\tau_\infty = \infty$ a.s. Thus, model (2.3) has a unique global positive solution. The proof is complete. □

3.2. Asymptotic behaviors of the solution

In this subsection, we discuss the asymptotic properties of the solution. First, we show that the sample Lyapunov exponents of the solution are non-positive.

Theorem 3.2. *For any $X_0 \in \mathbb{R}^4_+$, let $X(t)$ be a solution of model* (2.3) *with initial value X_0. Then*

$$\limsup_{t\to\infty} N(t) = \limsup_{t\to\infty}[P(t) + L(t) + S(t) + Q(t)] < \infty \ \text{ a.s.} \tag{3.5}$$

Proof. From model (2.3), it follows that

$$\begin{aligned} N(t) =& \frac{\lambda}{\mu} + \Big(N(0) - \frac{\lambda}{\mu}\Big)e^{-\mu t} + M(t) - \int_0^t e^{-\mu(t-s)}[d_1 P(s) + d_2 L(s) + d_3 S(s) + d_4 Q(s)]\mathrm{d}s \\ \leq& \frac{\lambda}{\mu} + \Big(N(0) - \frac{\lambda}{\mu}\Big)e^{-\mu t} + M(t) \\ =& N(0) + \frac{\lambda}{\mu}(1 - e^{-\mu t}) - N(0)(1 - e^{-\mu t}) + M(t) \ \text{ a.s.,} \end{aligned}$$

where

$$M(t) = e^{-\mu t}\Big[\sigma_1 \int_0^t e^{\mu s} P(s)\mathrm{d}B_1(s) + \sigma_2 \int_0^t e^{\mu s} L(s)\mathrm{d}B_2(s) + \sigma_3 \int_0^t e^{\mu s} S(s)\mathrm{d}B_3(s) + \sigma_4 \int_0^t e^{\mu s} Q(s)\mathrm{d}B_4(s)\Big].$$

It is clear that $M(t)$ is a continuous local martingale with $M(0) = 0$. Let

$$Y(t) = Y(0) + A(t) - U(t) + M(t),$$

where $Y(0) = N(0)$, $A(t) = \frac{\lambda}{\mu}(1 - e^{-\mu t})$, and $U(t) = N(0)(1 - e^{-\mu t})$. It is clear that $N(t) \leq Y(t)$ a.s. for all $t \geq 0$. Note that $A(t)$ and $U(t)$ are two continuous adapted increasing processes with $A(0) = U(0) = 0$ a.s. From [24, Theorem 1.3.9], we obtain that $\lim_{t\to\infty} Y(t) < \infty$ a.s. Thus,

$$\limsup_{t\to\infty} N(t) = \limsup_{t\to\infty} [P(t) + L(t) + S(t) + Q(t)] < \infty \ \text{ a.s.}$$

The proof is complete. □

Remark 3.1. *From Theorem 3.2 and the positivity of the solution, it follows that for any $X_0 \in \mathbb{R}^4_+$, the solution of model* (2.3) *has the properties that*

$$\limsup_{t\to\infty} \frac{\ln P(t)}{t} \leq 0, \ \ \limsup_{t\to\infty} \frac{\ln L(t)}{t} \leq 0, \ \ \limsup_{t\to\infty} \frac{\ln S(t)}{t} \leq 0, \ \ \limsup_{t\to\infty} \frac{\ln Q(t)}{t} \leq 0 \ \text{ a.s.}$$

This means that the sample Lyapunov exponents of the solution are non-positive.

Further, from Remark 3.1 and the positivity of the solution, we have the following result, which will be used in next section.

Corollary 3.1. *For any $X_0 \in \mathbb{R}^4_+$, let $X(t)$ be the solution of model* (2.3) *with initial value X_0. Then*

$$\lim_{t\to\infty} \frac{P(t)}{t} = 0, \ \ \lim_{t\to\infty} \frac{L(t)}{t} = 0, \ \ \lim_{t\to\infty} \frac{S(t)}{t} = 0, \ \ \lim_{t\to\infty} \frac{Q(t)}{t} = 0 \ \text{ a.s.}$$

Next, we discuss the stochastic permanence of model (2.3). The definition of stochastic permanence of the model is introduced as follows.

Definition 3.1 (see [23])**.** *Model* (2.3) *is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there are positive constants $\varrho = \varrho(\varepsilon)$, $\chi = \chi(\varepsilon)$, $\chi < \varrho$, such that for any $X_0 \in \mathbb{R}^4_+$, the solution $X(t)$ satisfies*

$$\liminf_{t\to\infty} \mathbb{P}\{|X(t)| \leq \varrho\} \geq 1 - \varepsilon, \quad \liminf_{t\to\infty} \mathbb{P}\{|X(t)| \geq \chi\} \geq 1 - \varepsilon.$$

The stochastic permanence implies that the population described by model (2.3) will not grow wildly or disappear.

Theorem 3.3. *For any given $X_0 \in \mathbb{R}^4_+$, model* (2.3) *is stochastically permanent.*

Proof. Define $V_1(X) = N + \frac{1}{N}$, where $X = (P, L, S, Q)$ and $N = P + L + S + Q$. From Itô formula, it follows that

$$\begin{aligned}
\mathcal{L}V_1(X) =&\lambda - \mu N - d_1 P - d_2 L - d_3 S - d_4 Q - \frac{\lambda - \mu N - d_1 P - d_2 L - d_3 S - d_4 Q}{N^2} \\
&+ \frac{\sigma_1^2 P^2 + \sigma_2^2 L^2 + \sigma_3^2 S^2 + \sigma_4^2 Q^2}{N^3} \\
\leq& -\mu N - \frac{\mu}{N} + \lambda - \frac{\lambda}{N^2} + \frac{\check{d} + 2\mu + \sigma^2}{N} \\
\leq& K_1 - \mu V_1(X),
\end{aligned} \tag{3.6}$$

where $K_1 = \frac{4\lambda^2 + (\check{d} + 2\mu + \sigma^2)^2}{4\lambda}$. Using Itô formula again, $\mathcal{L}(e^{\mu t} V_1(X)) = \mu e^{\mu t} V_1(X) + e^{\mu t} \mathcal{L}V_1(X) \leq K_1 e^{\mu t}$. Thus,

$$\mathbb{E}[e^{\mu t} V_1(X(t))] \leq V_1(X_0) + \mathbb{E} \int_0^t K_1 e^{\mu s} \mathrm{d}s = V_1(X_0) + \frac{K_1}{\mu}(e^{\mu t} - 1),$$

which implies

$$\limsup_{t\to\infty} \mathbb{E}[V_1(X(t))] \leq \limsup_{t\to\infty} \left[e^{-\mu t} V_1(X_0) + \frac{K_1}{\mu}(1 - e^{-\mu t}) \right] = \frac{K_1}{\mu}.$$

Hence,

$$\limsup_{t\to\infty} \mathbb{E}[N(t)] \leq \frac{K_1}{\mu}, \quad \limsup_{t\to\infty} \mathbb{E}\left[\frac{1}{N(t)}\right] \leq \frac{K_1}{\mu}. \tag{3.7}$$

Note that $N^2 = (P + L + S + Q)^2 \leq 4(P^2 + L^2 + S^2 + Q^2) = 4|X|^2 \leq 4(P + L + S + Q)^2 = 4N^2$. Then, together with (3.7), one yields

$$\limsup_{t\to\infty} \mathbb{E}[|X(t)|] \leq \frac{K_1}{\mu}, \quad \limsup_{t\to\infty} \mathbb{E}\left[\frac{1}{|X(t)|}\right] \leq \frac{2K_1}{\mu}. \tag{3.8}$$

For any $\varepsilon \in (0, 1)$, let $\varrho = \frac{K_1}{\mu\varepsilon}$. By Chebyshev's inequality and (3.8), we have

$$\limsup_{t\to\infty} \mathbb{P}\{|X(t)| > \varrho\} \leq \frac{\limsup_{t\to\infty} \mathbb{E}[|X(t)|]}{\varrho} = \varepsilon.$$

This implies

$$\liminf_{t\to\infty} \mathbb{P}\{|X(t)| \leq \varrho\} \geq 1 - \varepsilon.$$

Similarly, let $\chi = \frac{\mu\varepsilon}{2K_1}$. Then, from Chebyshev's inequality and (3.8), it follows that

$$\limsup_{t\to\infty} \mathbb{P}\{|X(t)| < \chi\} \leq \limsup_{t\to\infty} \chi \mathbb{E}\left[\frac{1}{|X(t)|}\right] = \varepsilon,$$

which means

$$\liminf_{t\to\infty} \mathbb{P}\{|X(t)| \geq \chi\} \geq 1-\varepsilon.$$

Let $\varepsilon \in (0,1)$ be sufficiently small such that $\chi < \varrho$. Then, from Definition 3.1, model (2.3) is stochastically permanent. □

Remark 3.2. *From the proof of Theorem 3.3, for any $X_0 \in \mathbb{R}^4_+$, the solution of model* (2.3) *satisfies*

$$\mathbb{E}[N(t)] \leq e^{-\mu t}N_0 + \frac{K_1}{\mu}(1-e^{-\mu t}) = \Big[N_0 - \frac{K_1}{\mu}\Big]e^{-\mu t} + \frac{K_1}{\mu}.$$

Thus, we have $\mathbb{E}[N(t)] \leq \max\{N_0, \frac{K_1}{\mu}\} \doteq \bar{K}_1$.
This, together with the positivity of the solution, yields

$$\max\Big\{\mathbb{E}[P(t)],\ \mathbb{E}[L(t)],\ \mathbb{E}[S(t)],\ \mathbb{E}[Q(t)]\Big\} \leq \bar{K}_1.$$

This means that the mathematical expectation of the solution of model (2.3) *is bounded.*

To conclude this subsection, we show that the solution of model (2.3) is p-th ($p > 1$) moment bounded.

Theorem 3.4. *Assume that $p > 1$ and $\bar{\mu} = (\mu + \hat{d}) - \frac{p-1}{2}\sigma^2 > 0$. Let $X(t)$ be the solution of model* (2.3) *with initial value $X_0 \in \mathbb{R}^4_+$. Then, for any $k \in (0, p\bar{\mu})$,*

$$\limsup_{t\to\infty} \mathbb{E}[N^p(t)] \leq \frac{\lambda}{k}\Big[\frac{\lambda(p-1)}{p\bar{\mu}-k}\Big]^{p-1}.$$

Proof. For any $k \in (0, p\bar{\mu})$, define $V_2(t,X) = e^{kt}N^p$, where $X = (P, L, S, Q)$ and $N = P + L + S + Q$. From Itô formula, we obtain

$$\mathbb{E}[V_2(t, X(t))] = V_2(0, X_0) + \mathbb{E}\int_0^t \mathcal{L}(V_2(s, X(s)))\mathrm{d}s, \tag{3.9}$$

where

$$\begin{aligned}\mathcal{L}V_2(t,X) =& ke^{kt}N^p + pe^{kt}N^{p-2}\Big[\lambda N - \mu N^2 - (d_1P + d_2L + d_3S + d_4Q)N \\ &+ \frac{p-1}{2}(\sigma_1^2P^2 + \sigma_2^2L^2 + \sigma_3^2S^2 + \sigma_4^2Q^2)\Big] \\ \leq& ke^{kt}N^p + pe^{kt}N^{p-2}\Big[\lambda N - \Big(\mu + \hat{d} - \frac{p-1}{2}\sigma^2\Big)N^2\Big] \\ =& pe^{kt}N^{p-2}\Big[-\Big(\bar{\mu} - \frac{k}{p}\Big)N^2 + \lambda N\Big].\end{aligned}$$

It is clear that function $f(x) = x^{p-2}[-(\bar{\mu} - \frac{k}{p})x^2 + \lambda x]$ reaches it's maximum value at $x = \frac{\lambda(p-1)}{p\bar{\mu}-k} > 0$ and $f_{\max} = \frac{\lambda}{p}[\frac{\lambda(p-1)}{p\bar{\mu}-k}]^{p-1} \doteq H$. Then, together with (3.9), one yields

$$\mathbb{E}[V_2(t, X(t))] \leq V_2(0, X_0) + \mathbb{E}\int_0^t pHe^{ks}\mathrm{d}s - N_0 + \frac{pH}{k}(e^{kt} - 1).$$

Thus, we have $\mathbb{E}[N^p(t)] \leq N_0e^{-kt} + \frac{pH}{k}(1 - e^{-kt})$, which implies that the conclusion holds. □

Remark 3.3. *From the proof of Theorem 3.4, for any $p > 1$ and $k \in (0, p\bar{\mu})$*

$$\mathbb{E}[N^p(t)] \leq N_0 e^{-kt} + \frac{pH}{k}(1 - e^{-kt}).$$

Thus, $\mathbb{E}[N^p(t)] \leq \max\{\frac{\lambda}{k}[\frac{\lambda(p-1)}{p\bar{\mu}-k}]^{p-1},\ N_0^p\}$. *This, together with the positivity of the solution, yields that there is a constant $M = M(p) > 0$ such that the solution of* (2.3) *with initial value $X_0 \in \mathbb{R}_+^4$ satisfies*

$$\max\Big\{\mathbb{E}[P^p(t)],\ \mathbb{E}[L^p(t)],\ \mathbb{E}[S^p(t)],\ \mathbb{E}[Q^p(t)]\Big\} \leq M.$$

This means that the solution of the model is p-th ($p > 1$) *moment bounded.*

4. Extinction of smokers

In this section, we provide the sufficient conditions for the extinction of occasional smokers (including occasional smokers, chain smokers, and quit smokers) in model (2.3). To prove our results, we first give the following result.

Lemma 4.1. *Let $X(t)$ be the solution of model* (2.3) *with initial value $X_0 \in \mathbb{R}_+^4$. If $p > 2$ and $\bar{\mu} > 0$, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P(s)\mathrm{d}B_1(s) = 0,\quad \lim_{t\to\infty} \frac{1}{t}\int_0^t L(s)\mathrm{d}B_2(s) = 0,\quad \text{a.s.}$$
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\mathrm{d}B_3(s) = 0,\quad \lim_{t\to\infty} \frac{1}{t}\int_0^t Q(s)\mathrm{d}B_4(s) = 0,\quad \text{a.s.}$$

Proof. Denote $X_1(t) = \int_0^t P(s)\mathrm{d}B_1(s)$. From Burkholder-Davis-Gundy inequality (see [24, Theorem 1.7.3]) and Hölder inequality, we can claim that for $p > 0$ and $t \geq 0$,

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|X_1(s)|^p\Big] \leq C_p\mathbb{E}\Big[\int_0^t P^2(s)\mathrm{d}s\Big]^{\frac{p}{2}} \leq C_p t^{\frac{p}{2}-1}\Big[\int_0^t \mathbb{E}(P^p(s))\mathrm{d}s\Big]. \tag{4.1}$$

Here, $C_p > 0$ (depending only on p) is a constant. From (4.1) and Remark 3.3, it follows that for $t \geq 0$,

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|X_1(s)|^p\Big] \leq C_p t^{\frac{p}{2}-1}\Big[\int_0^t \mathbb{E}(P^p(s))\mathrm{d}s\Big] \leq C_p M t^{\frac{p}{2}}.$$

Thus, for any positive integer n, we have

$$\mathbb{E}\Big[\sup_{n\leq t\leq n+1}|X_1(t)|^p\Big] \leq \mathbb{E}\Big[\sup_{0\leq t\leq n+1}|X_1(t)|^p\Big] \leq C_p M(n+1)^{\frac{p}{2}}.$$

Let $\varepsilon > 0$ be arbitrary. By Chebyshev's inequality, we have

$$\mathbb{P}\Big\{\sup_{n\leq t\leq n+1}|X_1(t)|^p > n^{1+\varepsilon+\frac{p}{2}}\Big\} \leq \frac{1}{n^{1+\varepsilon+\frac{p}{2}}}\mathbb{E}\Big[\sup_{n\leq t\leq n+1}|X_1(t)|^p\Big] \leq \frac{C_p M(n+1)^{\frac{p}{2}}}{n^{1+\varepsilon+\frac{p}{2}}}. \tag{4.2}$$

Since $\sum_{n=0}^{\infty}\frac{C_pM(n+1)^{\frac{p}{2}}}{n^{1+\varepsilon+\frac{p}{2}}}<\infty$ for $\varepsilon>0$, the Borel-Cantelli lemma (see [24, Lemma 1.2.1]) shows that for almost all $\omega\in\Omega$, there exists a positive integer $n_0=n_0(\omega)$ such that for any $n\geq n_0$,

$$\sup_{n\leq t\leq n+1}|X_1(t)|^p\leq n^{1+\varepsilon+\frac{p}{2}}.$$

That is,

$$\frac{\ln|X_1(t)|^p}{\ln t}\leq\frac{(1+\varepsilon+\frac{p}{2})\ln n}{\ln n}=1+\varepsilon+\frac{p}{2}.$$

Hence, $\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln t}\leq\frac{1+\varepsilon+\frac{p}{2}}{p}$ a.s. Let $\varepsilon\downarrow 0$, we have

$$\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln t}\leq\frac{1}{p}+\frac{1}{2}\ \text{ a.s.}$$

This implies that for any $0<\xi<\frac{1}{2}-\frac{1}{p}$ $(p>2)$, there is $T=T(\omega)>0$ such that $|X_1(t)|\leq t^{\frac{1}{p}+\frac{1}{2}+\xi}$ for $t\geq T$. Thus, from $\frac{1}{p}+\frac{1}{2}+\xi<1$, we have

$$\limsup_{t\to\infty}\frac{|X_1(t)|}{t}\leq\limsup_{t\to\infty}\frac{t^{\frac{1}{p}+\frac{1}{2}+\xi}}{t}=0.$$

This, together with $\liminf_{t\to\infty}\frac{|X_1(t)|}{t}\geq 0$, yields $\lim_{t\to\infty}\frac{|X_1(t)|}{t}=0$ a.s. Thus,

$$\lim_{t\to\infty}\frac{X_1(t)}{t}=\lim_{t\to\infty}\frac{1}{t}\int_0^t P(s)\mathrm{d}B_1(s)=0\ \text{ a.s.}$$

In a similar discussion as above, we can get the required assertion. The proof is complete. □

Theorem 4.1. *Let* $\mathcal{R}_s\doteq\frac{2\beta}{\zeta+d_2+\mu+\frac{\sigma_2^2}{2}}$. *For any* $X_0\in\mathbb{R}_+^4$, *let* $X(t)$ *be the solution of model* (2.3) *with initial value* X_0. *Then,*

$$\limsup_{t\to\infty}\frac{\ln[L(t)]}{t}\leq\Big(\zeta+d_2+\mu+\frac{\sigma_2^2}{2}\Big)(\mathcal{R}_s-1)\ \text{ a.s.}$$

Further, if $\mathcal{R}_s<1$, *then*

$$\lim_{t\to\infty}L(t)=0,\ \ \lim_{t\to\infty}S(t)=0,\ \ \lim_{t\to\infty}Q(t)=0\ \text{ a.s.}$$

Proof. Applying Itô formula to $\ln[L(t)]$ leads to

$$\frac{\ln[L(t)]}{t}\leq\Big[2\beta-\Big(\zeta+d_2+\mu+\frac{\sigma_2^2}{2}\Big)\Big]+\frac{\sigma_2B_2(t)}{t}+\frac{\ln(L_0)}{t}.\tag{4.3}$$

From the strong law of large numbers (see [24, Theorem 1.4.2]), $\lim_{t\to\infty}\frac{\sigma_2B_2(t)}{t}=0$ a.s. Hence,

$$\limsup_{t\to\infty}\frac{\ln[L(t)]}{t}\leq 2\beta-\Big(\zeta+d_2+\mu+\frac{\sigma_2^2}{2}\Big)=\Big(\zeta+d_2+\mu+\frac{\sigma_2^2}{2}\Big)(\mathcal{R}_s-1)\ \text{a.s.}$$

Further, if $\mathcal{R}_s < 1$, then $\limsup_{t\to\infty} \frac{\ln[L(t)]}{t} < 0$ a.s. Thus,

$$\lim_{t\to\infty} L(t) = 0 \text{ a.s.}$$

Consider the following stochastic differential equation $\mathrm{d}x(t) = -(\delta + d_3 + \mu)x(t)\mathrm{d}t + \sigma_3 x(t)\mathrm{d}B_3(t)$, with $x(0) = S_0$. It is clear that the solution of the above equation satisfies

$$\lim_{t\to\infty} x(t) = 0 \text{ a.s.}$$

From $\lim_{t\to\infty} L(t) = 0$ a.s., for sufficiently small $\varepsilon > 0$, there is a constant $T > 0$ and a set $\Omega_\varepsilon \subset \Omega$ such that $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$ and $\zeta L(t) < \varepsilon$ for $t \geq T$ and $\omega \in \Omega_\varepsilon$. Thus, from (2.3), it follows that

$$\mathrm{d}S(t) \leq [\varepsilon - (\delta + d_3 + \mu)S(t)]\mathrm{d}t + \sigma_3 S(t)\mathrm{d}B_3(t). \tag{4.4}$$

$$\mathrm{d}S(t) \geq -(\delta + d_3 + \mu)S(t)\mathrm{d}t + \sigma_3 S(t)\mathrm{d}B_3(t). \tag{4.5}$$

Note that ε is arbitrary. Letting $\varepsilon \to 0$, it then follows from (4.4) and (4.5) that $\mathrm{d}S(t) = \mathrm{d}x(t)$ a.s. Thus, $S(t)$ has the same positivity with $x(t)$, that is,

$$\lim_{t\to\infty} S(t) = 0 \text{ a.s.}$$

Similarly, we also have

$$\lim_{t\to\infty} Q(t) = 0 \text{ a.s.}$$

The proof is complete. □

Theorem 4.2. *For any $X_0 \in \mathbb{R}^4_+$, let $X(t)$ be the solution of model* (2.3) *with initial value X_0. Assume that for some $p > 2$, $\bar{\mu} = (\mu + \hat{d}) - \frac{p-1}{2}\sigma^2 > 0$. If $\mathcal{R}_s < 1$, then*

$$\lim_{t\to\infty} \langle P(t) \rangle = \frac{\lambda}{d_1 + \mu} \text{ a.s.}$$

Meanwhile, the distribution of $P(t)$ converges weakly to the measure that has the density

$$\pi(x) = C\sigma_1^{-2} x^{-2-\frac{2(d_1+\mu)}{\sigma_1^2}} \exp\left\{-\frac{2\lambda}{\sigma_1^2 x}\right\}, \quad x > 0, \tag{4.6}$$

where C is a constant satisfying $\int_0^\infty \pi(x)\mathrm{d}x = 1$.

Proof. Note that $\mathcal{R}_s < 1$. From Theorem 4.1, $\lim_{t\to\infty} L(t) = 0$, $\lim_{t\to\infty} S(t) = 0$, and $\lim_{t\to\infty} Q(t) = 0$ a.s. Using L'Hôpital's rule, we have

$$\lim_{t\to\infty} \langle L(t) \rangle = 0, \quad \lim_{t\to\infty} \langle S(t) \rangle = 0, \quad \lim_{t\to\infty} \langle Q(t) \rangle = 0 \text{ a.s.} \tag{4.7}$$

Moreover, it follows from (2.3) that

$$\frac{N(t) - N(0)}{t} = \lambda - (d_1 + \mu)\langle P(t) \rangle - (d_2 + \mu)\langle L(t) \rangle - (d_3 + \mu)\langle S(t) \rangle$$

$$-(d_4+\mu)\langle Q(t)\rangle + \frac{1}{t}\int_0^t \sigma_1 P(s)\mathrm{d}B_1(s) + \frac{1}{t}\int_0^t \sigma_2 L(s)\mathrm{d}B_2(s) + \frac{1}{t}\int_0^t \sigma_3 S(s)\mathrm{d}B_3(s) + \frac{1}{t}\int_0^t \sigma_4 Q(s)\mathrm{d}B_4(s).$$

Then, together with Corollary 3.1, Lemma 4.1, and (4.7), one yields

$$\lim_{t\to\infty}\langle P(t)\rangle = \frac{\lambda}{d_1+\mu} \text{ a.s.}$$

Now, we show the statement (4.6). Consider the stochastic equation described by

$$\mathrm{d}x(t) = [\lambda - (d_1+\mu)x(t)]\,\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t), \tag{4.8}$$

with $x(0) = P_0 > 0$. From Theorem 1.16 in Kutoyants (see [25]) (or the condition of existence for invariant density (see [26]), system (4.8) has the ergodic property, and the invariant density is given by

$$\pi(x) = C\sigma_1^{-2}x^{-2-\frac{2(d_1+\mu)}{\sigma_1^2}}\exp\left\{-\frac{2\lambda}{\sigma_1^2 x}\right\},\quad x>0,$$

where C is a constant satisfying $\int_0^\infty \pi(x)\mathrm{d}x = 1$. From Theorem 4.1, if $\mathcal{R}_s < 1$, then $\lim_{t\to\infty} L(t) = 0$ a.s. This, together with the positivity of the solution, yields

$$0 \le \lim_{t\to\infty}\beta\frac{2P(t)L(t)}{P(t)+L(t)} \le \lim_{t\to\infty} 2\beta L(t) = 0 \text{ a.s.}$$

Thus, for sufficiently small $\varepsilon > 0$, there is a constant $T > 0$ and a set $\Omega_\varepsilon \subset \Omega$ such that $\mathbb{P}(\Omega_\varepsilon) > 1-\varepsilon$ and $0 \le \beta\frac{2P(t)L(t)}{P(t)+L(t)} \le \varepsilon$ for $t \ge T$ and $\omega \in \Omega_\varepsilon$. Hence,

$$\mathrm{d}P(t) \le [\lambda - (d_1+\mu)P(t)]\,\mathrm{d}t + \sigma_1 P(t)\mathrm{d}B_1(t), \tag{4.9}$$

$$\mathrm{d}P(t) \ge [\lambda - \varepsilon - (d_1+\mu)P(t)]\,\mathrm{d}t + \sigma_1 P(t)\mathrm{d}B_1(t). \tag{4.10}$$

Letting $\varepsilon \to 0$, it then follows from (4.9) and (4.10) that $\mathrm{d}P(t) = \mathrm{d}x(t)$ a.s. Thus, the Markov process $P(t)$ has the same invariant density with $x(t)$, that is, the statement (4.6) holds. The proof is complete. □

5. Stationary distribution and ergodicity

In this section, we show that the model has an ergodic stationary distribution, which means that all types of smokers in the model can persist. Let $X(t)$ be a homogeneous Markov process in E_d (denotes d-dimensional Euclidean space), described by the following equation

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t),\ X(0) = X_0. \tag{5.1}$$

The diffusion matrix of process $X(t)$ is defined as $J(X) = g(X)g^{\mathbb{T}}(X) = (a_{ij}(X))$.

Lemma 5.1 (see [22])**.** *Assume that there is a bounded domain $D \subset E_d$ with regular boundary Γ and*

(A1) *there is a constant $M > 0$ such that $\sum_{i,j=1}^{d} a_{ij}(X)\xi_i\xi_j \geq M|\xi|^2$, $X \in D$, $\xi \in \mathbb{R}^d$;*

(A2) *there is a nonnegative C^2-function V such that there is a constant $C > 0$, such that $LV \leq -C$ for any $X \in E_d \backslash D$.*

Then, the Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$. Moreover, if $f(\cdot)$ is a function integrable with respect to the measure μ, then

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))\mathrm{d}t = \int_{E_d} f(x)\mu(\mathrm{d}x)\right\} = 1.$$

Now, we show that the model has a stationary distribution.

Theorem 5.1. *For any $X_0 \in \mathbb{R}_+^4$, let $X(t)$ be the solution of model* (2.3) *with initial value X_0. If $\mathcal{R}^s = \frac{2\beta(\bar{d}+\mu)}{(d_1+\mu+\frac{\sigma_1^2}{2})(\zeta+d_2+\mu+\frac{\sigma_2^2}{2})} > 1$, then model* (2.3) *has a stationary distribution $\mu(\cdot)$, and the solution $X(t)$ has the ergodic property. Here, $\bar{d} = \min\{d_1, d_2\}$.*

Proof. From Itô formula and model (2.3), it follows that

$$\begin{aligned}\mathcal{L}(P + L + S + Q) &= \lambda - (d_1 + \mu)P - (d_2 + \mu)L - (d_3 + \mu)S - (d_4 + \mu)Q \\ &= \lambda - (\bar{d} + \mu)(P + L) - (d_1 - \bar{d})P - (d_2 - \bar{d})L - (d_3 + \mu)S - (d_4 + \mu)Q, \\ \mathcal{L}(-\ln P) &= -\frac{\lambda}{P} + \frac{2\beta L}{P+L} + \Big(d_1 + \mu + \frac{\sigma_1^2}{2}\Big), \\ \mathcal{L}(-\ln L) &= -\frac{2\beta P}{P+L} + \Big(\zeta + d_2 + \mu + \frac{\sigma_2^2}{2}\Big),\end{aligned}$$

where $\bar{d} = d_1 \wedge d_2$. Define the function

$$V_1(P, L, S, Q) = (P + L + S + Q) - k_1 \ln P - k_2 \ln L,$$

where k_1 and k_2 are positive constants to be determined later. Using Itô formula,

$$\begin{aligned}\mathcal{L}V_1 =& -(\bar{d} + \mu)(P + L) - \frac{k_1\lambda}{P} - \frac{k_2 2\beta P}{P+L} + k_1\Big(d_1 + \mu + \frac{\sigma_1^2}{2}\Big) + \lambda - (d_1 - \bar{d})P \\ &+ k_2\Big(\zeta + d_2 + \mu + \frac{\sigma_2^2}{2}\Big) + \frac{k_1 2\beta L}{P+L} - (d_2 - \bar{d})L - (d_3 + \mu)S - (d_4 + \mu)Q \\ \leq& -3\Big[2k_1k_2\lambda\beta(\bar{d} + \mu)\Big]^{\frac{1}{3}} + \lambda + k_1\Big(d_1 + \mu + \frac{\sigma_1^2}{2}\Big) \\ &+ k_2\Big(\zeta + d_2 + \mu + \frac{\sigma_2^2}{2}\Big) + \frac{k_1 2\beta L}{P+L}.\end{aligned}$$

Let $k_1(d_1 + \mu + \frac{\sigma_1^2}{2}) = k_2(\zeta + d_2 + \mu + \frac{\sigma_2^2}{2}) = \lambda$, then $k_1 = \frac{\lambda}{d_1+\mu+\frac{\sigma_1^2}{2}}$ and $k_2 = \frac{\lambda}{\zeta+d_2+\mu+\frac{\sigma_2^2}{2}}$. As a consequence

$$\mathcal{L}V_1 \leq -3\left[\left(\frac{2\lambda^3\beta(\bar{d} + \mu)}{(d_1 + \mu + \frac{\sigma_1^2}{2})(\zeta + d_2 + \mu + \frac{\sigma_2^2}{2})}\right)^{\frac{1}{3}} - \lambda\right] + \frac{k_1 2\beta L}{P+L}$$

$$= -3\lambda\Big[(\mathcal{R}^s)^{\frac{1}{3}} - 1\Big] + \frac{k_1 2\beta L}{P+L}.$$

Further, define

$$V_2(P,L,S,Q) = MV_1(P,L,S,Q) - \ln P - \ln S - \ln Q + (P+L+S+Q),$$

where a positive constant M satisfies

$$-M\bar{\lambda} + \lambda + 2\beta + d_1 + d_3 + d_4 + \delta + 3\mu + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2} \le -2, \tag{5.2}$$

and $\bar{\lambda} = 3\lambda[(\mathcal{R}^s)^{\frac{1}{3}} - 1] > 0$. It is easy to see that

$$\liminf_{k\to\infty,(P,L,S,Q)\in\mathbb{R}_+^4\setminus U_k} V_2(P,L,S,Q) = +\infty,$$

where $U_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$. From the continuity of $V_2(P,L,S,Q)$, we know that $V_2(P,L,S,Q)$ has a minimum point $(\bar{P}_0, \bar{L}_0, \bar{S}_0, \bar{Q}_0)$ in the interior of $\mathbb{R}_+^4$. Then, we define a nonnegative C^2-function V_3: $\mathbb{R}_+^2 \to \mathbb{R}$ as follows:

$$V_3(P,L,S,Q) = V_2(P,L,S,Q) - V_2(\bar{P}_0, \bar{L}_0, \bar{S}_0, \bar{Q}_0).$$

From Itô formula, we have

$$\begin{aligned}
\mathcal{L}V_3 =& M\mathcal{L}V_1 + \mathcal{L}(-\ln P) + \mathcal{L}(-\ln S) + \mathcal{L}(-\ln Q) + \mathcal{L}(P+L+S+Q) \\
\le& -M\bar{\lambda} + \frac{k_1 M 2\beta L}{P+L} - \frac{\lambda}{P} - \frac{\zeta L}{S} - \frac{\delta S}{Q} - (d_1+\mu)P - (d_2+\mu)L - (d_3+\mu)S \\
& - (d_4+\mu)Q + \lambda + 2\beta + d_1 + d_3 + d_4 + \delta + 3\mu + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2}{2}.
\end{aligned}$$

Now, define the bounded closed set

$$D = \left\{(P,L,S,Q) \in \mathbb{R}_+^4 : \epsilon \le P \le \frac{1}{\epsilon}, \epsilon^2 \le L \le \frac{1}{\epsilon^2}, \epsilon^3 \le S \le \frac{1}{\epsilon^3}, \epsilon^4 \le Q \le \frac{1}{\epsilon^4}\right\},$$

where $0 < \epsilon < 1$ sufficiently small. Let $K_2 = k_1 M 2\beta + \lambda + 2\beta + d_1 + d_3 + d_4 + \delta + 3\mu + \frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}$. In the set $\mathbb{R}_+^4 \setminus D$, we can choose ϵ sufficiently small such that

$$-\frac{\lambda}{\epsilon} + K_2 \le -1, \tag{5.3}$$

$$2k_1 M\beta\epsilon \le 1, \tag{5.4}$$

$$-\frac{\zeta}{\epsilon} + K_2 \le -1, \tag{5.5}$$

$$-\frac{\delta}{\epsilon} + K_2 \le -1, \tag{5.6}$$

$$-\frac{d_1+\mu}{\epsilon} + K_2 \le -1, \tag{5.7}$$

$$-\frac{d_2+\mu}{\epsilon^2}+K_2\le -1, \tag{5.8}$$

$$-\frac{d_3+\mu}{\epsilon^3}+K_2\le -1, \tag{5.9}$$

$$-\frac{d_4+\mu}{\epsilon^4}+K_2\le -1, \tag{5.10}$$

For convenience, we divide $\mathbb{R}_+^4 \setminus D$ into the following eight domains

$$\begin{aligned}
D_1 &= \left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ 0<P<\epsilon\right\},\\
D_2 &= \left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ 0<L<\epsilon^2,\ P\ge\epsilon\right\},\\
D_3 &= \left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ 0<S<\epsilon^3,\ P\ge\epsilon,\ L\ge\epsilon^2\right\},\\
D_4 &= \left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ 0<Q<\epsilon^4,\ P\ge\epsilon,\ L\ge\epsilon^2,\ S\ge\epsilon^3\right\},\\
D_5 &= \left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ P>\frac{1}{\epsilon}\right\},\ D_6=\left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ L>\frac{1}{\epsilon^2}\right\},\\
D_7 &= \left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ S>\frac{1}{\epsilon^3}\right\},D_8=\left\{(P,L,S,Q)\in\mathbb{R}_+^4:\ Q>\frac{1}{\epsilon^4}\right\}.
\end{aligned}$$

Next, we show that $LV_3(P,L,S,Q)\le -1$ on $\mathbb{R}_+^4\setminus D$, which is equivalent to verifying it on the above eight domains.

Case 1. If $(P,L,S,Q)\in D_1$, then

$$\begin{aligned}
\mathcal{L}V_3 \le& k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-\frac{\lambda}{P}\\
=&K_2-\frac{\lambda}{P}\le K_2-\frac{\lambda}{\epsilon}.
\end{aligned}$$

Then, together with (5.3), one yields $\mathcal{L}V_3\le -1$ for any $(P,L,S,Q)\in D_1$.

Case 2. If $(P,L,S,Q)\in D_2$, then

$$\begin{aligned}
\mathcal{L}V_3 \le& -M\bar{\lambda}+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}+\frac{k_1M2\beta L}{P+L}\\
\le& -M\bar{\lambda}+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}+\frac{k_1M2\beta\epsilon^2}{\epsilon}.
\end{aligned}$$

Thus, from (5.2) and (5.4), $\mathcal{L}V_3\le -2+k_1M2\beta\epsilon\le -1$ for any $(P,L,S,Q)\in D_2$.

Case 3. If $(P,L,S,Q)\in D_3$, then

$$\begin{aligned}
\mathcal{L}V_3 \le& k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-\frac{\zeta L}{S}\\
=&K_2-\frac{\zeta L}{S}\le K_2-\frac{\zeta\epsilon^2}{\epsilon^3}=K_2-\frac{\zeta}{\epsilon}.
\end{aligned}$$

Then, together with (5.5), one yields $\mathcal{L}V_3\le -1$ for any $(P,L,S,Q)\in D_3$.

Case 4. If $(P,L,S,Q)\in D_4$, then

$$\mathcal{L}V_3 \le k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-\frac{\delta S}{Q}$$

$$=K_2-\frac{\delta S}{Q}\leq K_2-\frac{\delta\epsilon^3}{\epsilon^4}=K_2-\frac{\delta}{\epsilon}.$$

Thus, from (5.6), we have $\mathcal{L}V_3 \leq -1$ for any $(P, L, S, Q) \in D_4$.

Case 5. If $(P, L, S, Q) \in D_5$, then

$$\begin{aligned}\mathcal{L}V_3 \leq& k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-(d_1+\mu)P\\=&K_2-(d_1+\mu)P\leq K_2-\frac{d_1+\mu}{\epsilon}.\end{aligned}$$

Then, together with (5.7), one yields $\mathcal{L}V_3 \leq -1$ for any $(P, L, S, Q) \in D_5$.

Case 6. If $(P, L, S, Q) \in D_6$, then

$$\begin{aligned}\mathcal{L}V_3 \leq& k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-(d_2+\mu)L\\=&K_2-(d_2+\mu)L\leq K_2-\frac{d_2+\mu}{\epsilon^2}.\end{aligned}$$

Thus, from (5.8), it follows that $\mathcal{L}V_3 \leq -1$ for any $(P, L, S, Q) \in D_6$.

Case 7. If $(P, L, S, Q) \in D_7$, then

$$\begin{aligned}\mathcal{L}V_3 \leq& k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-(d_3+\mu)S\\=&K_2-(d_3+\mu)S\leq K_2-\frac{d_3+\mu}{\epsilon^3}.\end{aligned}$$

Then, together with (5.9), one yields $\mathcal{L}V_3 \leq -1$ for any $(P, L, S, Q) \in D_7$.

Case 8. If $(P, L, S, Q) \in D_8$, then

$$\begin{aligned}\mathcal{L}V_3 \leq& k_1M2\beta+\lambda+2\beta+d_1+d_3+d_4+\delta+3\mu+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}-(d_4+\mu)Q\\=&K_2-(d_4+\mu)Q\leq K_2-\frac{d_4+\mu}{\epsilon^4}.\end{aligned}$$

Thus, from (5.10), we have $\mathcal{L}V_3 \leq -1$ for any $(P, L, S, Q) \in D_8$. Hence, for a sufficiently small $\epsilon > 0$, one has $\mathcal{L}V_3(P, L, S, Q) \leq -1$ for all $(P, L, S, Q) \in \mathbb{R}^4_+ \setminus D$. This means that (A2) in Lemma 5.1 is satisfied.

The diffusion matrix of stochastic model (2.3) is $A = (a_{ij})_{4\times4} = \mathrm{diag}(\sigma_1^2P^2, \sigma_2^2L^2, \sigma_3^2S^2, \sigma_4^2Q^2)$. Then, for any $(P, L, S, Q) \in D$ and $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$, we have

$$\sum_{i,j=1}^{4}a_{ij}(P,L,S,Q)\xi_i\xi_j=\sigma_1^2P^2\xi_1^2+\sigma_2^2L^2\xi_2^2+\sigma_3^2S^2\xi_3^2+\sigma_4^2Q^2\xi_4^2\geq K_3|\xi|^2,$$

where $K_3 = \min\{\epsilon^2\sigma_1^2, \epsilon^4\sigma_2^2, \epsilon^6\sigma_3^2, \epsilon^8\sigma_4^2\}$. Thus, condition (A1) of Lemma 5.1 holds. According to Lemma 5.1, the model (2.3) has a stationary distribution $\mu(\cdot)$, and the solution has the ergodic property.
□

6. Control of occasional smokers

Here, we discuss how to control the number of occasional smokers. As in [7], we assume that the effective contact rate is $\beta = \theta\rho$, where the positive constant θ is the contact rate, and ρ is the probability of becoming a smoker due to contact with a smoker. Moreover, as in [7], we suppose

$$\rho(r) = \frac{1}{1 + kr},$$

where k is a positive constant, and r is the price of cigarettes. Hence, the effective contact rate is $\beta = \theta\rho = \frac{\theta}{1+kr}$.

6.1. *Effects of r and θ on occasional smokers*

(I) From the expressions of $\mathcal{R}_s$, we have $\mathcal{R}_s = \frac{2\theta}{(1+kr)(\zeta+d_2+\mu+\frac{\sigma_2^2}{2})}$. It is clear that $\frac{\partial \mathcal{R}_s}{\partial r} < 0$ and $\frac{\partial \mathcal{R}_s}{\partial \theta} > 0$. Note that $L(t)$ will be eliminated as long as $\mathcal{R}_s < 1$. This, together with the decreasing property of $\mathcal{R}_s$ with respect to r, yields that if

$$r > \frac{2\theta - (\zeta + d_2 + \mu + 0.5\sigma_2^2)}{k(\zeta + d_2 + \mu + 0.5\sigma_2^2)} \doteq r_1,$$

then $L(t)$ will be eliminated gradually. Similarly, if

$$\theta < \frac{(1 + kr)(\zeta + d_2 + \mu + 0.5\sigma_2^2)}{2} \doteq \theta_1,$$

then occasional smokers $L(t)$ will be eliminated gradually.

(II) From the expressions of $\mathcal{R}^s$, we have $\mathcal{R}^s = \frac{2\theta(\bar{d}+\mu)}{(1+kr)(d_1+\mu+\frac{\sigma_1^2}{2})(\zeta+d_2+\mu+\frac{\sigma_2^2}{2})}$. Note that the model has a stationary distribution as long as $\mathcal{R}^s > 1$. This means that when $\mathcal{R}^s > 1$, occasional smokers can persist. It is clear that $\frac{\partial \mathcal{R}^s}{\partial r} < 0$ and $\frac{\partial \mathcal{R}^s}{\partial \theta} > 0$. Thus, from the decreasing property of $\mathcal{R}^s$ with respect to r, if

$$r < \frac{2\theta(\bar{d} + \mu) - (d_1 + \mu + 0.5\sigma_1^2)(\zeta + d_2 + \mu + 0.5\sigma_2^2)}{k(d_1 + \mu + 0.5\sigma_1^2)(\zeta + d_2 + \mu + 0.5\sigma_2^2)} \doteq r_2,$$

then occasional smokers will be persistent. Similarly, if

$$\theta > \frac{(1 + kr)(d_1 + \mu + 0.5\sigma_1^2)(\zeta + d_2 + \mu + 0.5\sigma_2^2)}{2(\bar{d} + \mu)} \doteq \theta_2,$$

then occasional smokers $L(t)$ can persist.

It is clear that $r_2 < r_1$ and $\theta_2 > \theta_1$. From the above analysis, in order to reduce the number of smokers, the government can adopt certain policies to increase the price of cigarettes r and reduce the contact rate θ. To reduce the contact rate θ, the government can design smoking areas, which diminishes the contact chances between smokers and potential smokers. If $\mathcal{R}^s > 1$, then occasional smokers $L(t)$ can be persist. However, the above analysis does not point out how the price of cigarettes r and the contact rate θ affect the number of occasional smokers in the presence of occasional smokers. We will discuss this problem through numerical simulation in the next section.

6.2. Effects of σ_i^2 on occasional smokers

Here, we discuss how the noise intensity σ_i^2 affects the dynamic behavior of the model. Note that $\mathcal{R}_s$ only depends on the noise intensity σ_2^2 and decreases monotonically with respect to σ_2^2. Hence, if

$$\sigma_2^2 > 2\left[\frac{2}{1+kr} - \zeta - d_2 - \mu\right],$$

the occasional smokers will be gradually eliminated. Moreover, the elimination of occasional smokers will lead to the gradual elimination of both chain smokers and quit smokers. From Theorem 5.1, when $\mathcal{R}^s > 1$, the model has a stationary distribution. This means that occasional smokers $L(t)$ in the model will be persistent. Note that $\mathcal{R}^s$ depends not only on noise intensity σ_1^2 but also on noise intensity σ_2^2. Further, according to the expression of $\mathcal{R}^s$, when the noise intensities σ_1^2 and σ_2^2 are both small, occasional smokers $L(t)$ can persist. However, in the presence of $L(t)$, we need to answer the following two questions.

(i) How does noise intensity σ_1^2 and noise intensity σ_2^2 affect the number of occasional smokers?

(ii) Which noise has a greater impact on the number of occasional smokers?

In the next section, we will solve the above problems by numerical simulation.

7. Numerical simulations

In this section, we use the Milstein method (see [27]) to give some numerical simulations.

7.1. Verify the theoretical results

Here, we make numerical simulations to substantiate our results (Theorems 4.1, 4.2, and 5.1). Numerical experiments of (2.3) are made by using $(P_0, L_0, S_0, Q_0) = (4, 6, 3, 5)$ and $\lambda = 0.2$, $\mu = 0.0211$, $d_1 = 0.0019$, $d_2 = 0.0019$, $d_3 = 0.0020$, $d_4 = 0.0021$, $\zeta = 0.0021$, $\delta = 0.0041$.

(i) Take $d_1 = d_2 = d_3 = d_4 = d$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = 0$, then we can get the deterministic model (2.1). Take $\beta = 0.03$, then $\mathcal{R}_0 = 1.3636 > 1$. Thus, model (2.2) admits a unique smoking-present equilibrium $E_1^* = (5.1282, 1.8648, 0.6109, 1.0797)$ (see Figure 1).

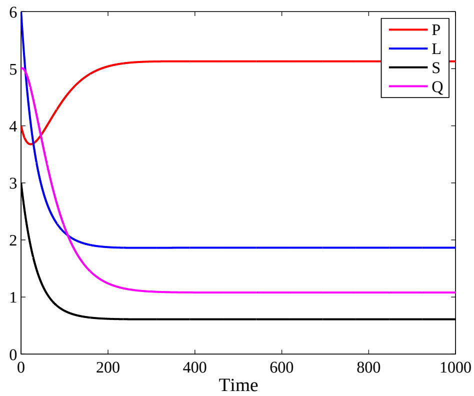
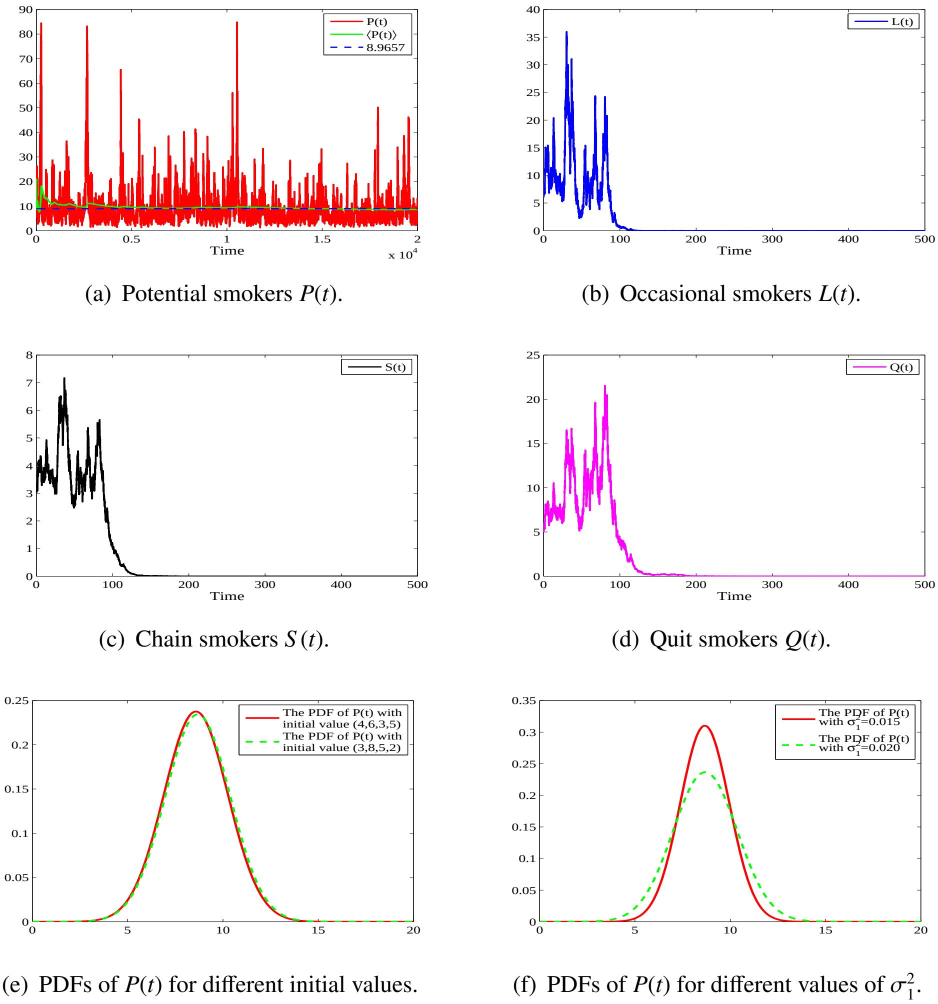


Figure 1. Trajectories of deterministic model (2.2) with $\beta = 0.03$. (Color figure online).

(ii) Let $\beta = 0.03$, $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.04$, $\sigma_3^2 = 0.01$, and $\sigma_4^2 = 0.02$. Then, $\mathcal{R}_s = 0.9375 < 1$. Thus, by Theorem 4.1, $\lim_{t\to\infty} L(t) = 0$, $\lim_{t\to\infty} S(t) = 0$, and $\lim_{t\to\infty} Q(t) = 0$ (see Figure 2(b)–2(d)). Take $p = 2.02$, then $(\mu + \hat{d}) - \frac{p-1}{2}\sigma^2 = 0.0026 > 0$. Thus, from Theorem 4.2, $\lim_{t\to\infty}\langle P(t)\rangle = 8.9657$ (see Figure 2(a)), and potential smokers have the probability density, which depends on the noise intensity σ_1^2 (see Figure 2(e) and 2(f)).



(a) Potential smokers $P(t)$.

(b) Occasional smokers $L(t)$.

(c) Chain smokers $S(t)$.

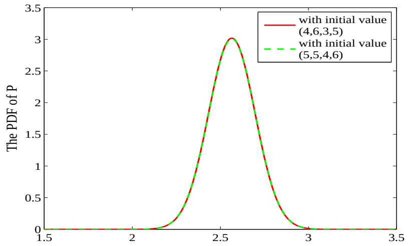
(d) Quit smokers $Q(t)$.

(e) PDFs of $P(t)$ for different initial values.

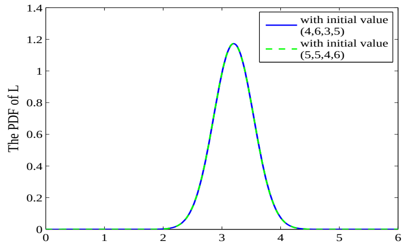
(f) PDFs of $P(t)$ for different values of σ_1^2.

Figure 2. (a)–(d) Trajectories of stochastic model (2.3) with $\beta = 0.03$, $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.04$, $\sigma_3^2 = 0.01$, and $\sigma_4^2 = 0.02$. (e) PDFs of $P(t)$ for different initial values. (f) PDFs of $P(t)$ for different values of σ_1^2.

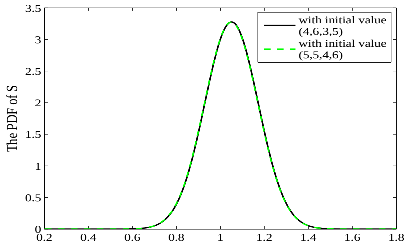
(iii) Take $\beta = 0.05$, $\sigma_1^2 = \sigma_2^2 = \sigma_4^2 = 0.002$, $\sigma_3^2 = 0.001$. Then, $\mathcal{R}^s = 2.1296 > 1$. From Theorem 5.1, model (2.3) has a stationary distribution (see Figure 3). Note that the ergodic stationary distribution reflects the weak stability and persistence of the model to some certain extent. Moreover, from Figure 4, all kinds of smokers in model (2.3) can persist.



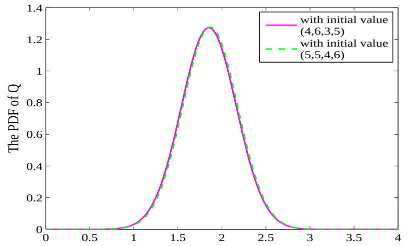
(a) PDF of potential smokers $P(t)$.



(b) PDF of occasional smokers $L(t)$.

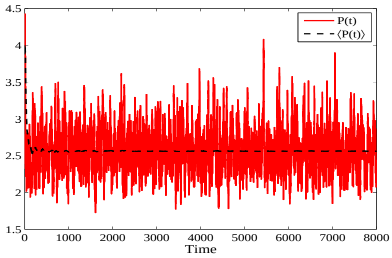


(c) PDF of chain smokers $S(t)$.

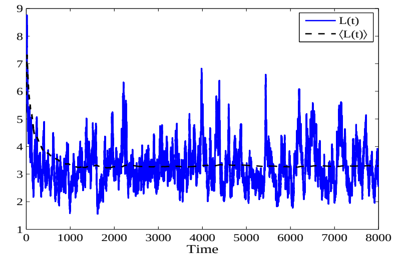


(d) PDF of quit smokers $Q(t)$.

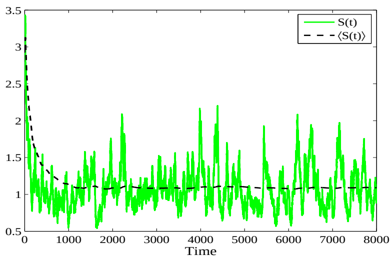
Figure 3. The density functions of $P(t)$, $L(t)$, $S(t)$, and $Q(t)$ in model (2.3) at time $t = 800000$ with different initial value based on 5 stochastic simulations.



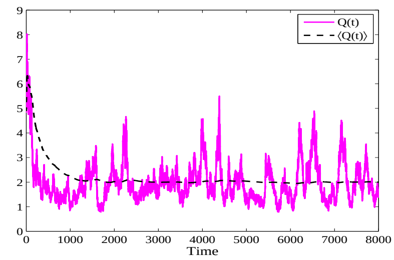
(a) Potential smokers $P(t)$.



(b) Occasional smokers $L(t)$.



(c) Chain smokers $S(t)$.



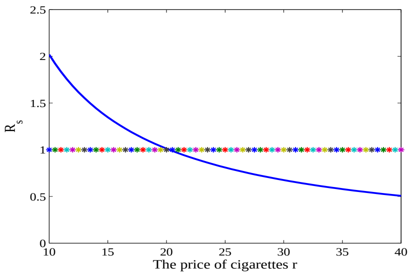
(d) Quit smokers $Q(t)$.

Figure 4. Trajectories of model (2.3) with $\beta = 0.05$, $\sigma_1^2 = 0.002$, $\sigma_2^2 = 0.002$, $\sigma_3^2 = 0.001$, and $\sigma_4^2 = 0.002$.

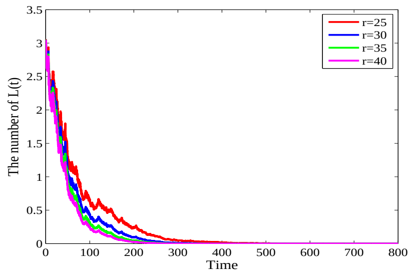
7.2. The effects of r, θ, and σ_i^2 on occasional smokers

In this subsection, we use numerical simulation to explain the effects of cigarette price r, contact rate θ, and noise intensities σ_i^2 $(i = 1, 2)$, on the number of occasional smokers in model (2.3). Here, we choose $\lambda = 0.21$, $d_1 = 0.0019$, $d_2 = 0.0019$, $d_3 = 0.0020$, $d_4 = 0.0021$, $\mu = 0.0211$, $\zeta = 0.0021$, $\delta = 0.0041$, $\sigma_3^2 = \sigma_4^2 = 0.02$, and $(P_0, L_0, S_0, Q_0) = (4, 3, 3, 1)$.

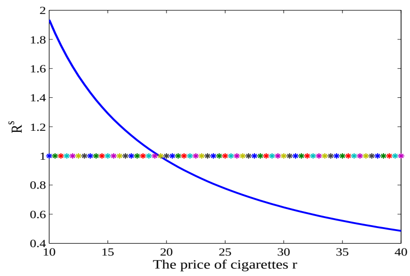
(i) **To find out the effect of** r, we take $k = 18$, $\sigma_1^2 = \sigma_2^2 = 0.002$ and $\theta = 8.2192$. When $r \in [10, 40]$, $\mathcal{R}_s$ and $\mathcal{R}^s$ are displayed in Figure 5(a) and 5(c), respectively. By a simple calculation, if $r > 20.2388$, then $\mathcal{R}_s < 1$. That is, occasional smokers will be eliminated gradually (see Figure 5(b)). If $r < 19.3932$, then $\mathcal{R}^s > 1$. Figure 5(d) shows the mean change of occasional smokers when $r < 19.3932$. It can be seen from Figure 5(b) that when $r > 20.2388$, increased r results in the elimination of $L(t)$. From Figure 5(d), we can see that when $r < 19.3932$, $\langle L(t) \rangle$ will decrease with the increase of r. This seems to be reasonable due to the fact that the number of smokers will decrease as the price of tobacco increases. However, numerical simulations show that it takes a long time $(200 - 300)$ for the smokers to disappear by raising the price of cigarettes.



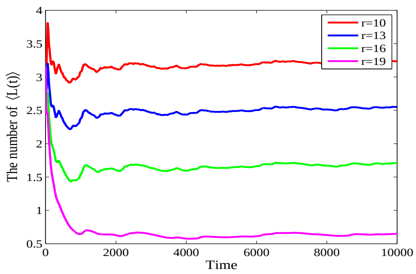
(a) $\mathcal{R}_s$ changes with price r.



(b) Occasional smokers $L(t)$.



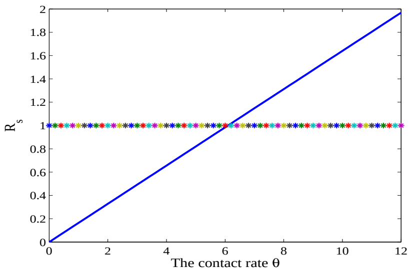
(c) $\mathcal{R}^s$ changes with price r.



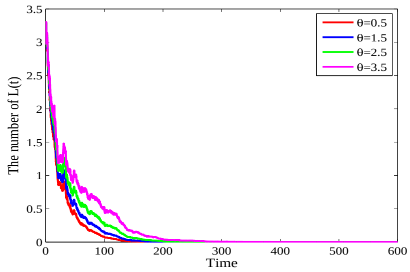
(d) Occasional smokers $\langle L(t) \rangle$.

Figure 5. (a) Variation of basic reproduction number $\mathcal{R}_s$ with r; (b) number of occasional smokers when $r > 20.2388$; (c) variation of basic reproduction number $\mathcal{R}^s$ with r; (d) mean number of occasional smokers when $r < 19.3932$.

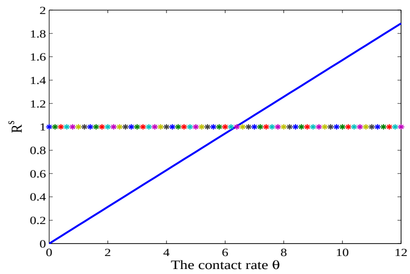
(ii) **To find out the effect of** θ, we choose $k = 18$, $\sigma_1^2 = \sigma_2^2 = 0.02$, and $r = 15$. When $\theta \in [0, 12]$, the variations of $\mathcal{R}_s$ are displayed in Figure 6(a). Obviously, if $\theta < 6.0975$, then $\mathcal{R}_s < 1$. Figure 6(b) shows the change of $L(t)$ with time when $\theta < 6.0975$. As can be seen from Figure 6(b), when $\theta < 6.0975$, $L(t)$ will be eliminated gradually. When $\theta \in [0, 12]$, the variations of $\mathcal{R}^s$ are displayed in Figure 6(c). Clearly, if $\theta > 6.3626$, then $\mathcal{R}^s > 1$. Figure 6(d) depicts the change of $\langle L(t) \rangle$ when $\theta > 6.3626$. Moreover, it can be seen from Figure 6(d) that when $\theta > 6.3626$, $\langle L(t) \rangle$ will increase with the increase of θ. Further, from Figure 6(b), when $\theta < 6.0975$, with a decrease of θ, the faster the elimination of occasional smokers is. This seems reasonable because θ represents the contact rate between potential smokers and occasional smokers. However, numerical simulations show that it takes a long time (200 − 300) for the smokers to disappear by lowering the contact rate.



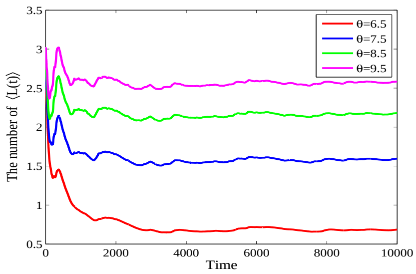
(a) $\mathcal{R}_s$ changes with θ.



(b) Occasional smokers $L(t)$.



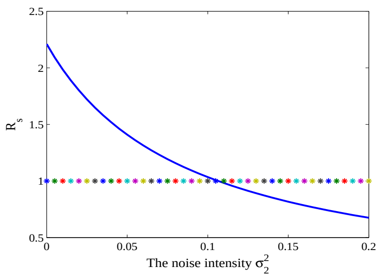
(c) $\mathcal{R}^s$ changes with θ.



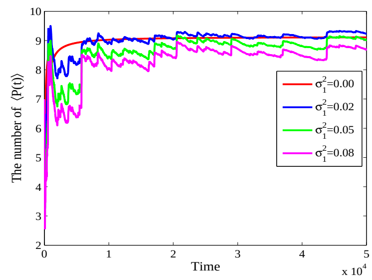
(d) Occasional smokers $\langle L(t)\rangle$.

Figure 6. (a) Variation of $\mathcal{R}_s$ with θ; (b) number of occasional smokers when $\theta < 6.0975$; (c) variation of $\mathcal{R}^s$ with θ; (d) the mean number of occasional smokers when $\theta > 6.3626$.

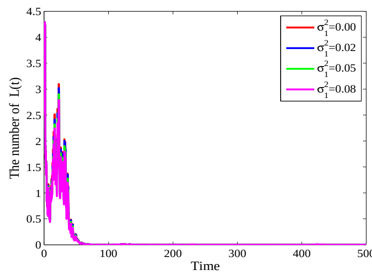
(iii) Now, we discuss the effect of noise intensities σ_1^2 and σ_2^2. Note that $\mathcal{R}_s$ depends only upon the noise intensity σ_2^2. Here, we choose $k = 12$, $r = 16$, and $\theta = 8.2192$. When $\sigma_2^2 \in [0, 0.2]$, the corresponding variations of $\mathcal{R}_s$ are displayed in Figure 7(a). Obviously, if $\sigma_2^2 > 0.1065$, then $\mathcal{R}_s < 1$. Thus, $\lim_{t\to\infty}\langle P(t)\rangle = 8.9657$ and $\lim_{t\to\infty} L(t) = 0$ a.s. Figure 7(b) and 7(c) show, respectively, the mean change of potential smokers and the number of occasional smokers under different noise intensities σ_1^2.



(a) $\mathcal{R}_s$ changes with σ_2^2.



(b) Potential smokers $\langle P(t)\rangle$.

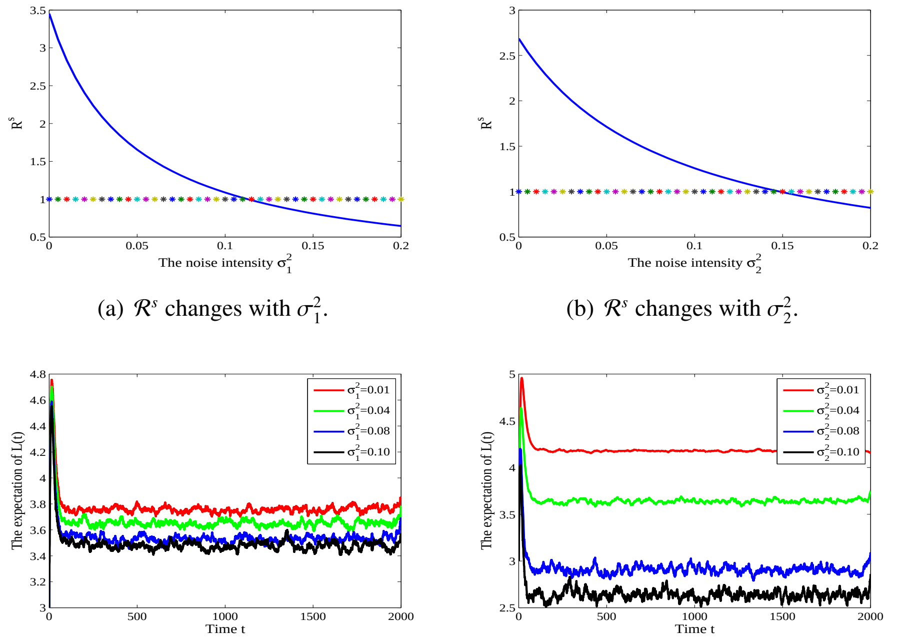


(c) Occasional smokers $L(t)$.

Figure 7. (a) Variation of basic reproduction number $\mathcal{R}^S$ with θ; (b) mean number of occasional smokers when $\theta > 6.0975$; (c) number of occasional smokers when $\theta < 6.0975$.

Note that $\mathcal{R}^s$ depends not only on σ_1^2 but also on σ_2^2. Obviously, $\mathcal{R}^s(\sigma_1^2, \sigma_2^2)$ is monotonically decreasing with respect to σ_1^2 and σ_2^2. Here, we choose $k = 18$, $r = 10$, and $\theta = 20$. When $\sigma_2^2 = 0.04$,

Figure 8(a) shows the variations of $\mathcal{R}^s$ with $\sigma_1^2 \in [0, 0.16]$. When $\sigma_1^2 = 0.04$, Figure 8(b) shows the variations of $\mathcal{R}^s$ with $\sigma_2^2 \in [0, 0.16]$. By a simple computation, we have $\mathcal{R}^s(0.10, 0.04) = 1.0879 > 1$ and $\mathcal{R}^s(0.04, 0.10) = 1.2575 > 1$. When $\sigma_2^2 = 0.04$, Figure 8(c) shows the expectation of $L(t)$ with different noise intensities σ_1^2 $(0.01, 0.04, 0.08, 0.10)$ based on 80,000 stochastic simulations. When $\sigma_1^2 = 0.04$, Figure 8(d) shows the expectation of $L(t)$ with different noise intensities σ_2^2 $(0.01, 0.04, 0.08, 0.10)$ based on 80,000 stochastic simulations. It can be seen from the numerical simulations that when occasional smokers are present and one of the noise intensities σ_1^2 and σ_2^2 is fixed, the mathematical expectation of the occasional smokers decreases as the other noise intensity increases. This is concretely reflected in the following observations: As shown in Figure 8(c), with σ_2^2 held constant, the expected number of occasional smokers, $\mathbb{E}[L(t)]$, decreases with increasing σ_1^2. Conversely, Figure 8(d) shows that for a fixed σ_1^2, increasing σ_2^2 results in a more substantial reduction in $\mathbb{E}[L(t)]$.



(a) $\mathcal{R}^s$ changes with σ_1^2.

(b) $\mathcal{R}^s$ changes with σ_2^2.

(c) The expectation of $L(t)$ with different σ_1^2.

(d) The expectation of $L(t)$ with different σ_2^2.

Figure 8. (a) Variation of basic reproduction number $\mathcal{R}^s$ with θ; (b) mean number of occasional smokers when $\theta > 6.0975$; (c) the number of occasional smokers when $\theta < 6.0975$.

8. Discussions and conclusions

This paper conducts a rigorous dynamic analysis of a stochastic giving-up-smoking model with harmonic mean type incidence rate, whose theoretical results form a mathematical system with tightly-knit internal logic. The system begins with Theorem 3.1 (the existence and uniqueness of a global positive solution for the model), which ensures the meaningfulness of subsequent theoretical analysis. Building upon this, Theorem 3.2, Remark 3.1, and Corollary 3.1 reveal the asymptotic boundedness of the solution: a fundamental property that provides crucial support for proving the stochastic permanence of the system (Theorem 3.3). The stochastic permanence (Theorem 3.3) itself naturally leads to the boundedness of the first-order moment of the solution (Remark 3.2), which, together with the boundedness of the p-th moment established under stronger conditions (Theorem 3.4 and Remark 3.3), collectively forms the core tool for analyzing the long-term behavior of the system. Utilizing this tool, we prove Lemma 4.1, which serves as a bridge connecting the stochastic model to deterministic limits. Based on this, the core of the theoretical system is established: Theorem 4.1 provides the threshold condition for the extinction of smokers, while Theorem 4.2 fully characterizes the system state post-extinction on this basis. In contrast, Theorem 5.1 proves that under another threshold condition, smokers will persist, and a stationary distribution exists. Therefore, Theorems 4.1 and 5.1 collectively reveal the phase transition behavior of the system governed by a pair of stochastic threshold parameters ($\mathcal{R}_s$ and $\mathcal{R}^s$). This series of rigorous derivations, from basic properties to threshold theorems, lays a solid mathematical foundation for the control analysis from an economic perspective (Section 6).

More importantly, we discuss how to control the number of smokers from the perspective of economics. Finally, we introduce some numerical simulations to support the theoretical results obtained and analyze the effects of the price of cigarettes, the contact rate, and noise intensity on the size of occasional smokers.

From the expressions of $\mathcal{R}_s$ and $\mathcal{R}^s$, under the assumptions of Figure 1, we have $\mathcal{R}_s = \mathcal{R}^s = \mathcal{R}_0$. For model (2.1), by Theorems 4.1 and 4.2, if $\mathcal{R}_0 < 1$, then smokers (including occasional smokers, chain smokers, and quit smokers) in model (2.1) will be eliminated gradually; from Theorem 5.1, if $\mathcal{R}_0 > 1$, then all types of smokers in model (2.1) can persist. This means that $\mathcal{R}_0$ is the threshold of model (2.1), which is consistent with the result of [12]. Moreover, from the expressions of $\mathcal{R}_s$ and $\mathcal{R}_0$, it is easy to see that $\mathcal{R}_s < \mathcal{R}_0$. This means that if smokers in model (2.1) disappear, smokers in model (2.3) must disappear. Thus, we can assert that noise is beneficial to the control the number of smoking individuals.

To sum up, if $r > r_1$ (or $\theta < \theta_1$), smokers (including occasional smokers, chain smokers, and quit smokers) will gradually disappear; if $r < r_2$ (or $\theta > \theta_2$), all types of smokers will persist. It can be seen from Figure 5(d) that in the presence of occasional smokers, the mean size of occasional smokers decreases as the cigarettes price increases. From Figure 6(d), in the presence of occasional smokers, the mean size of occasional smokers decreases as the contact rate increases. Moreover, regardless of the intensity of the other three noises, as long as the intensity of noise $\dot{B}_2(t)$ satisfies $\sigma_2^2 > 2(\frac{2}{1+kr} - \zeta - d_2 - \mu)$, smokers will gradually disappear (see Figure 2). This means that greater intensity of noise $\dot{B}_2(t)$ can result in smokers extinction. When the intensities of noises $\dot{B}_1(t)$ and $\dot{B}_2(t)$ are both weak, all types of smokers in model (2.3) will always exist. However, from the numerical simulations, when occasional smokers are present and one of the noise intensities σ_1^2 and σ_2^2 is fixed, the mathematical expectation of occasional smokers decreases as the other noise intensity increases (see Figure 8(c) and 8(d)). Moreover, by comparing Figure 8(c) and 8(d), it can be seen that noise intensity σ_2^2 has a greater influence on the mathematical expectation of occasional smokers than noise intensity σ_1^2 in the presence of occasional smokers.

To effectively control the number of smokers, this study proposes the following comprehensive intervention strategies: First, implement price regulation mechanisms, such as introducing floating tax rates to increase cigarette prices. Second, optimize spatial management by randomly adjusting the functional layout of public spaces and the locations of smoking areas to systematically reduce both the contact rate and contact certainty between potential smokers and occasional smokers. Finally, enhance noise intervention by adopting non-periodic, dynamic publicity strategies alongside ongoing public health campaigns on the harms of smoking, with a focus on introducing stochastic disturbances targeting the critical transition stage from occasional to established smoking. Numerical simulations indicate that relying solely on price increases or contact reduction requires a considerable amount of time (200-300 time units) to achieve noticeable effects. Therefore, the adoption of an integrated approach combining the above strategies is recommended.

Author contributions

Xin Yi: Responsible for the review and editing of the manuscript; Rong Liu: Responsible for preparation of the original draft of the manuscript. Yanmei Wang: Responsible for the numerical simulation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest in this paper.

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