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Research article

Exploration traveling solitary solutions to the modified Korteweg-de Vries equation with advection noise

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Abstract: In this study, we investigate the stochastic modified Korteweg-de Vries (SmKdV) equation, which is driven in the Itô sense by advection noise. We show that by solving certain deterministic counterparts of the modified Korteweg-de Vries with an extra diffusion term (for short DmKdV), and then combining the results with a solution of stochastic ordinary differential equations, the exact solution of the SmKdV equation may be discovered. We derive soliton solutions for the DmKdV problem using two distinct methods: the extended tanh function approach and the $\exp(-\psi(\eta))$-expansion method. Additionally, we study how the advection noise affects the solutions of the SmKdV equation by presenting several 3D graphs using a MATLAB program.

Keywords: stochastic solutions; simulation; mathematical model; exact solutions; analytical techniques; noise intensity
Mathematics Subject Classification: 35A20, 83C15, 60H10, 60H15, 35Q51

1. Introduction

Partial differential equations (PDEs) are incredibly important because they are used to describe various phenomena in nature, engineering, and other scientific fields. The solutions to PDEs allow us to understand and predict how systems change over time and space, which is crucial for solving real-world problems. Recently, many authors have used a number of methods to obtain the exact solutions, such as the exponential rational function method [1], the modified tanh method [2], the generalized extended function method [3], the planner dynamical scheme [4], the generalized Kudryashov technique [5, 6], the φ^6-model expansion scheme [7, 8], the modified F-expansion method [9], the Riccati equation approach [10], and the improved modified extended tanh [11]. By understanding these solutions, we harness the power of mathematics to better navigate and shape the world around us, highlighting the indispensable role of PDEs in both science and everyday life.

The modified Korteweg-de Vries (mKdV) equation is an important mathematical model used to analyze waves in various physical settings, particularly in shallow water. It is an extended version of the classical KdV equation, which originally described the propagation of solitary waves in fluid dynamics. One key feature of the mKdV equation is its ability to handle different types of wave profiles, including those that arise in real-world situations, such as in certain ocean waves or in plasma physics. Additionally, the mKdV model is employed in other areas like fluid dynamics, nonlinear optics, and even in some areas of quantum physics, illustrating its versatility and significance across different disciplines [12–15].

One interesting aspect of the mKdV equation is how it behaves with random fluctuations. In simple terms, random fluctuations refer to unexpected changes or disturbances that can occur in a system, and studying how these fluctuations impact solutions to the mKdV equation helps us understand real-world phenomena better. When we examine waves governed by the mKdV equation, we often find that they can form stable structures called solitons. These solitons are wave packets that maintain their shape while traveling at constant speeds. However, when random fluctuations are introduced into the system, they can disrupt these soliton solutions. For instance, in a water wave setup, changes in wind or additional waves can lead to unexpected variations in height or speed. Researchers study these random fluctuations to assess how robust soliton solutions are to disturbances and how the overall wave behavior is influenced by randomness. More information in this direction can be found in the following papers [16, 17].

The following SmKdV equation perturbed by advection noise is taken into consideration in this work:

$$d\mathcal{Y} + [\gamma_1 \mathcal{Y}^2 \mathcal{Y}_x + \gamma_2 \mathcal{Y}_{xxx}]dt = \varepsilon \mathcal{Y}_x d\mathcal{W}, \tag{1.1}$$

where $\mathcal{Y}$ denotes the wave's amplitude; $\gamma_1 \mathcal{Y}^2 \mathcal{Y}_x$ represents the wave's self-interaction; $\gamma_2 \mathcal{Y}_{xxx}$ is the dispersive term; γ_1 and γ_2 are non-zero real constants; $\mathcal{W}(t)$ is the standard Wiener process and, it depends only on t; ε is the intensity of noise.

The significance of the mKdV equation has led numerous authors to obtain the exact solutions for it by utilizing various methods, including the tanh method [18], the first integral method [19], the exp-function method [20], the (G'/G)-expansion [21], the tanh method [22], the bifurcation [23], the hyperbolic function approach [24], and the Sardar-subequation method [25]. Mohammed et al. [26] used the mapping method to derive the exact solutions of SmKdV Eq (1.1) perturbed by multiplicative noise in the Stratonovich sense in the form $\varepsilon \mathcal{Y} \circ \mathcal{W}_t$, while, Mohammed and Al-Askar [27] used two different methods, such as the generalizing Riccati equation mapping and Jacobi elliptic functions methods, to acquired the exact solutions of Eq (1.1) with multiplicative noise in the Itô sense in the form $\varepsilon \mathcal{Y} \mathcal{W}_t$. Furthermore, the exact solutions of the Wick-type stochastic mKdV equation were obtained by Liu [28] and Dai et al. [29] using the modified mapping approach and the exp-function method, respectively. While Yuan et al. [16] used the Darboux transformation method to find the stochastic soliton solutions of Eq (1.1) with stochastic term in the form $\mathcal{W}(t)+12\mathcal{Y}\mathcal{Y}_x \int_0^t \mathcal{W}(s)ds$.

The main motivation of this work is to obtain the exact solutions to the SmKdV Eq (1.1) with advection noise in the form $\varepsilon \mathcal{Y}_x \mathcal{W}_t$. In order to accomplish this, the SmKdV equation is divided into two equations. The stochastic ordinary differential equation (SODE) is the first equation, and the deterministic mKdV (DmKdV) equation is the second one that is derived from the SmKdV equation using Itô calculus and suitable transformation techniques. By using the $\exp(-\psi(\eta))$-expansion method and the extended tanh function method, we acquire the solutions of the DmKdV equation. After that, by combining the results that we obtained with a solution of the SODE, we obtain the solutions of the SmKdV Eq (1.1). One significant reason for studying the mKdV equation with advection noise is its ability to help scientists understand how irregularities and random influences can impact wave formations. For example, in the ocean, waves do not always travel smoothly due to changing conditions. The mKdV model can demonstrate how noise, which is essentially a collection of random forces, can cause phenomena like rogue waves exceptionally large and dangerous waves that appear unexpectedly. Therefore, we present some graphics created using MATLAB tools to demonstrate the impact of the stochastic term.

The outline of this paper is as follows: In the following section, we present a lemma for breaking down the SmKdV Eq (1.1) into a SODE and DmKdV equation with an additional diffusion term. In Section 3, the solutions for the DmKdV equation are found. In Section 4, the solutions of the SmKdV Eq (1.1) are obtained. In Section 5, we present the physical meaning and the effect of the stochastic term on the obtained solutions. Finally, the results of the article are presented.

2. Preliminaries

The Wiener process $\{\mathcal{W}(t),\ t \geq 0\}$ is a fundamental concept in probability theory and mathematical finance [30]. It is a continuous-time stochastic process that describes the random movement of particles suspended in a fluid, which was first observed by the botanist Robert Brown in the 19th century. In mathematical terms, a Wiener process $\mathcal{W}(t)$ is characterized by several key properties. Firstly, it starts at zero, meaning ($\mathcal{W}(0) = 0$). Secondly, its paths are continuous. Moreover, the increments of the process are independent, which implies that the movement during one time interval does not affect the movement during another. Another important feature of the Wiener process is that its increments are normally distributed. Specifically, for any two time points s and t where $s < t$, the increment $\mathcal{W}(t) - \mathcal{W}(s)$ follows a normal distribution with a mean of zero and variance equal to $t - s$.

The next lemma (see for more detail [31]) demonstrates that the exact solutions of the SmKdV Eq (1.1) can be acquired by solving deterministic counterparts of the mKdV equation with an extra diffusion term and merging the result with the solution of SODE: $X_t(t) = x + \varepsilon\mathcal{W}(t)$.

Lemma 2.1. *The SmKdV Eq (1.1) has the solution* $\mathcal{Y}(t,x)= \mathcal{U}(t,X_t)$ *for* $t \in [0,T]$, *where* $\mathcal{U}$ *is the solution of the DmKdV equation:*

$$\mathcal{U}_t + \gamma_1\mathcal{U}^2\mathcal{U}_x + \gamma_2\mathcal{U}_{xxx} + \frac{\varepsilon^2}{2}\mathcal{U}_{xx} = 0, \tag{2.1}$$

where X_t *is the solution of the following SODE:*

$$dX_t = \varepsilon d\mathcal{W}, \tag{2.2}$$

with initial values

$$X_0 = x.$$

Proof. By applying the following Itô formula to the solution X_t of Eq (2.2) with the transformation $\mathcal{U}(t,X_t)$ where $\mathcal{U}(t,x)$ is the solutions of the deterministic (2.1) (see [32]):

$$d\mathcal{U}(t,X_t) = \frac{\partial\mathcal{U}}{\partial t}dt + \frac{\partial\mathcal{U}}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2\mathcal{U}}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2\mathcal{U}}{\partial x^2}(dX_t)^2 + \frac{\partial^2\mathcal{U}}{\partial t\partial x}dtdX_t.$$

Using (2.1), and (2.2), we get

$$\begin{aligned} d\mathcal{U}(t, X_t) &= [-\gamma_2 \mathcal{U}_{xxx} - \gamma_1 \mathcal{U}^2 \mathcal{U}_x - \frac{\varepsilon^2}{2} \mathcal{U}_{xx}]dt + \varepsilon \mathcal{U}_x d\mathcal{W} + \frac{\varepsilon^2}{2} \mathcal{U}_{xx} dt \\ &= [-\gamma_2 \mathcal{U}_{xxx} - \gamma_1 \mathcal{U}^2 \mathcal{U}_x]dt + \varepsilon \mathcal{U}_x d\mathcal{W}, \end{aligned}$$

where we used $(dt)^2 = dtd\mathcal{W} = 0$ and $(d\mathcal{W})^2 = dt$. Since $\mathcal{Y} = \mathcal{U}$, then we obtain Eq (1.1). □

3. Wave equation of DmKdV equation

To derive the wave equation for DmKdV Eq (2.1), we apply the next transform:

$$\mathcal{U}(t, x) = \mathcal{Z}(\eta), \ \eta = kx + \lambda t, \tag{3.1}$$

where $\mathcal{Z}$ is a deterministic function. We see that

$$\mathcal{U}_x = k\mathcal{Z}', \ \mathcal{U}_{xx} = k^2\mathcal{Z}'', \ \mathcal{U}_{xxx} = k^3\mathcal{Z}''', \ \mathcal{U}_t = \lambda\mathcal{Z}'. \tag{3.2}$$

Plugging Eqs (3.1) and (3.2) into Eq (2.1), we have

$$\lambda\mathcal{Z}' + k\gamma_1\mathcal{Z}^2\mathcal{Z}' + k^3\gamma_2\mathcal{Z}''' + \frac{\varepsilon^2}{2}k^2\mathcal{Z}'' = 0. \tag{3.3}$$

Integrating Eq (3.3) once and putting the constant of integration to be zero, we get

$$\mathcal{Z}'' + H_0\mathcal{Z}' + H_1\mathcal{Z} + H_2\mathcal{Z}^3 = 0, \tag{3.4}$$

where

$$H_0 = \frac{\varepsilon^2}{2k\gamma_2}, \ H_1 = \frac{\lambda}{k^3\gamma_2} \ \text{ and } H_2 = \frac{\gamma_1}{3k^2\gamma_2}.$$

4. Solutions of DmKdV equation

In this section, we acquire the solutions of the DmKdV Eq (2.1) by applying the $\exp(-\psi(\eta))$-expansion method and the extended tanh function method. These two methods were chosen due to their effectiveness in handling nonlinear dispersive wave equations. The $\exp(-\psi(\eta))$-expansion method is capable of producing a wide variety of solitary wave and exponential solutions, while the extended tanh function method is straightforward to apply and suitable for equations with polynomial nonlinearities and it is a generalized version of the classical tanh method. Due to the existence of the term $\mathcal{Z}'$ in Eq (3.4), we can not apply some methods such as the mapping method, the Sarder method, and the φ^6-expansion scheme.

4.1. *The* $\exp(-\psi(\eta))$*-expansion method*

Consider the $\exp(-\psi(\eta))$-expansion method that is stated in [33, 34]. Assuming the solutions of Eq (3.4) as

$$\mathcal{Z}(\eta) = \sum_{\kappa=0}^{m} \ell_\kappa e^{-\kappa\psi(\eta)}, \ \text{such that } \ell_N \neq 0, \tag{4.1}$$

where the constants $\ell_0, \ell_1 \ldots, \ell_N$ are to be computed later, and $\psi = \psi(\eta)$ satisfies:

$$\psi' = b + a\exp(\psi) + \exp(-\psi), \tag{4.2}$$

where a and b are constants. Utilizing the homogeneous balance rule, we equate the highest power of $\mathcal{Z}^3$ with the highest power of $\mathcal{Z}''$ in (3.4) as:

$$3m = m + 2 \quad \Rightarrow \quad m = 1.$$

Hence, Eq (4.1) with $m = 1$ becomes

$$\mathcal{Z}(\eta) = \ell_0 + \ell_1[\exp(-\psi(\eta))]. \tag{4.3}$$

Substituting Eqs (4.2) and (4.3) into Eq (3.4), we attain

$$(2\ell_1 + H_2\ell_1^3)e^{-3\psi} + (3b\ell_1 - k\ell_1 + 3H_2\ell_0\ell_1^2)e^{-2\psi} + (2a\ell_1 + b^2\ell_1 - bH_0\ell_1$$

$$+\ell_1 H_1 + 3H_2\ell_0^2\ell_1)e^{-\psi} + (ab\ell_1 - aH_0\ell_1 + H_1\ell_0 + H_2\ell_0^3) = 0.$$

Putting the coefficients of $\exp(-\kappa\psi)$ equal to zero, for $\kappa = 3,\ 2,\ 1$, and 0, we obtain

$$2\ell_1 + H_2\ell_1^3 = 0,$$

$$3b\ell_1 - k\ell_1 + 3H_2\ell_0\ell_1^2 = 0,$$

$$2a\ell_1 + b^2\ell_1 - bH_0\ell_1 + \ell_1 H_1 + 3H_2\ell_0^2\ell_1 = 0,$$

and

$$ab\ell_1 - aH_0\ell_1 + H_1\ell_0 + H_2\ell_0^3 = 0. \tag{4.4}$$

Solving these equations, we get

$$\ell_0 = \pm\sqrt{\frac{-H_1}{H_2}}, \quad \ell_1 = \pm\sqrt{\frac{-2}{H_2}},\ a = \frac{3H_1}{2},\ b = \frac{1}{3}H_0 \pm \sqrt{2H_1}. \tag{4.5}$$

Substituting the values of ℓ_0 and ℓ_1 into Eq (4.3), we have

$$\mathcal{Z}(\eta) = \pm\left(\sqrt{\frac{-H_1}{H_2}} - \sqrt{\frac{-2}{H_2}}\exp(-\psi(\eta))\right). \tag{4.6}$$

Equation (4.2) has many solutions based on the values of a and b as follows:
Case I: If $\Omega = b^2 - 4a > 0$ and $a \neq 0$, then Eq (4.2) has the solutions:

$$\psi(\eta) = \ln\left(\frac{\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2}(\eta + \mathcal{E})\right) + b}{-2a}\right), \tag{4.7}$$

and

$$\psi(\eta) = \ln\left(\frac{\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2}(\eta + \mathcal{E})\right) + b}{-2a}\right), \tag{4.8}$$

where $\mathcal{E}$ is the integration constant.

Now, the solutions of DmKdV Eq (2.1), by substituting Eqs (4.7) and (4.8) into Eq (4.6) and utilizing Eq (3.1), are

$$\mathcal{U}(t,x) = \pm\Bigg(\sqrt{\frac{-H_1}{H_2}} + \sqrt{\frac{-2}{H_2}}\frac{2a}{\sqrt{\Omega}\tanh\Big(\frac{\sqrt{\Omega}}{2}(kx+\lambda t+\mathcal{E})\Big)+b}\Bigg), \tag{4.9}$$

and

$$\mathcal{U}(t,x) = \pm\Bigg(\sqrt{\frac{-H_1}{H_2}} + \sqrt{\frac{-2}{H_2}}\frac{2a}{\sqrt{\Omega}\coth\Big(\frac{\sqrt{\Omega}}{2}(kx+\lambda t+\mathcal{E})\Big)+b}\Bigg). \tag{4.10}$$

Case II: If $\Omega < 0$ and $a \neq 0$, then Eq (4.2) has the solutions:

$$\psi(\eta) = \ln\Bigg(\frac{\sqrt{-\Omega}\tan\Big(\frac{\sqrt{-\Omega}}{2}(\eta+\mathcal{E})\Big)-b}{2a}\Bigg), \tag{4.11}$$

and

$$\psi(\eta) = \ln\Bigg(\frac{-\sqrt{-\Omega}\cot\Big(\frac{\sqrt{-\Omega}}{2}(\eta+\mathcal{E})\Big)-b}{2a}\Bigg). \tag{4.12}$$

Thus, the solutions of DmKdV Eq (2.1), by plugging Eqs (4.11) and (4.12) into (4.6) and utilizing Eq (3.1), are

$$\mathcal{U}(t,x) = \pm\Bigg(\sqrt{\frac{-H_1}{H_2}} - \sqrt{\frac{-2}{H_2}}\frac{2a}{\sqrt{-\Omega}\tan\Big(\frac{\sqrt{-\Omega}}{2}(\eta+\mathcal{E})\Big)-b}\Bigg), \tag{4.13}$$

and

$$\mathcal{U}(t,x) = \pm\Bigg(\sqrt{\frac{-H_1}{H_2}} + \sqrt{\frac{-2}{H_2}}\frac{-2a}{\sqrt{-\Omega}\cot\Big(\frac{\sqrt{-\Omega}}{2}(\eta+\mathcal{E})\Big)+b}\Bigg). \tag{4.14}$$

Case III: If $a = 0$ (i.e $H_1 = 0$) and $b \neq 0$, then the Eq (4.2) has the solution:

$$\psi(\eta) = -\ln\Big(\frac{-b}{1-\exp(b\eta+b\mathcal{E})}\Big). \tag{4.15}$$

Hence, the solution of DmKdV Eq (2.1), by plugging Eq (4.15) into Eq (4.6) and utilizing Eq (3.1), is

$$\mathcal{U}(t,x) = \pm\sqrt{\frac{-2}{H_2}}\Big(\frac{-b}{1-\exp(bkx+b\mathcal{E})}\Big). \tag{4.16}$$

Case IV: If $\Omega = 0$, $a \neq 0$ and $b \neq 0$, hence the solutions of Eq (4.2) is

$$\psi(\eta) = \ln\Big(-\frac{4+2b(\eta+\mathcal{E})}{b^2(\eta+\mathcal{E}))}\Big). \tag{4.17}$$

Therefore, the solution of DmKdV Eq (2.1), by plugging Eq (4.17) into Eq (4.6) and utilizing Eq (3.1), is

$$\mathcal{U}(t,x) = \pm\Big(\sqrt{\frac{-H_1}{H_2}} - \sqrt{\frac{-2}{H_2}}\frac{b^2(kx+\lambda t+\mathcal{E})}{4+2b(kx+\lambda t+\mathcal{E}))}\Big). \tag{4.18}$$

4.2. Extended tanh function method

Now, we use the extended tanh function method [35] to get the solutions of DmKdV Eq (2.1). Supposing the solutions of Eq (3.4), with $N = 1$, are as follows:

$$\mathcal{Z}(\eta) = \alpha_0 + \alpha_1 \mathcal{F}, \tag{4.19}$$

where $\mathcal{F}$ solves

$$\mathcal{F}' = \mathcal{F}^2 + \varpi. \tag{4.20}$$

Substituting Eq (4.19) into Eq (3.4), we have

$$(2\alpha_1 + H_2\alpha_1^3)\mathcal{F}^3 + (\alpha_1 H_0 + 3\alpha_0\alpha_1^2 H_2)\mathcal{F}^2 + (2\varpi\alpha_1$$

$$+\alpha_1 H_1 + 3\alpha_0^2\alpha_1 H_2)\mathcal{F} + (\varpi\alpha_1 H_0 + \alpha_0 H_1 + \alpha_0^3 H_2) = 0.$$

Putting the coefficients of $\mathcal{F}^J$ equal to zero, for $J = 3,\ 2,\ 1$, and 0, we get

$$2\alpha_1 + H_2\alpha_1^3 = 0,$$

$$\alpha_1 H_0 + 3\alpha_0\alpha_1^2 H_2 = 0,$$

$$2\varpi\alpha_1 + \alpha_1 H_1 + 3\alpha_0^2\alpha_1 H_2 = 0,$$

and

$$\varpi\alpha_1 H_0 + \alpha_0 H_1 + \alpha_0^3 H_2 = 0.$$

By solving these equations, we have

$$\alpha_0 = \pm\frac{H_0}{3\sqrt{-2H_2}},\ \alpha_1 = \mp\sqrt{\frac{-2}{H_2}},\ H_1 = \frac{2}{9}H_0^2 \text{ and } \varpi = -\frac{1}{36}H_0^2.$$

Since $\varpi < 0$, hence the solutions of Eq (4.20) are

$$\mathcal{F}(\eta) = -\sqrt{-\varpi}\tanh\left(\sqrt{-\varpi}\eta\right), \tag{4.21}$$

$$\mathcal{F}(\eta) = -\sqrt{-\varpi}\coth\left(\sqrt{-\varpi}\eta\right), \tag{4.22}$$

$$\mathcal{F}(\eta) = -\sqrt{-\varpi}\Big(\coth(\sqrt{-4\varpi}\eta) \pm \operatorname{csch}(\sqrt{-4\varpi}\eta)\Big), \tag{4.23}$$

and

$$\mathcal{F}(\eta) = \frac{-1}{2}\sqrt{-\varpi}\Big(\tanh(\frac{1}{2}\sqrt{-\varpi}\eta) + \coth(\frac{1}{2}\sqrt{-\varpi}\eta)\Big). \tag{4.24}$$

Now, the solutions of DmKdV Eq (2.1), by plugging Eqs (4.21)–(4.24) into (4.19) and using Eq (3.1), are

$$\mathcal{U}(t,x) = \pm\frac{H_0}{3\sqrt{-2H_2}}\Big(1 + \tanh\Big(\frac{H_0}{6}(kx + \lambda t)\Big)\Big), \tag{4.25}$$

$$\mathcal{U}(t,x) = \pm\frac{H_0}{3\sqrt{-2H_2}}\Big(1 + \coth\Big(\frac{H_0}{6}(kx + \lambda t)\Big)\Big), \tag{4.26}$$

$$\mathcal{U}(t,x) = \pm\frac{H_0}{3\sqrt{-2H_2}}\Big(1 + \coth\Big(\frac{H_0}{3}(kx + \lambda t)\Big) \pm \operatorname{csch}\Big(\frac{H_0}{3}(kx + \lambda t)\Big)\Big), \tag{4.27}$$

and

$$\mathcal{U}(t,x) = \pm\frac{H_0}{6\sqrt{-2H_2}}\Big(2 + \tanh\Big(\frac{H_0}{12}(kx + \lambda t)\Big) + \coth\Big(\frac{H_0}{12}(kx + \lambda t)\Big)\Big). \tag{4.28}$$

5. Exact solutions of SmKdV equation

To acquire the exact solutions of the SmKdV Eq (1.1), let us integrate Eq (2.2) from 0 to t:

$$X_t = x + \varepsilon \mathcal{W}(t). \tag{5.1}$$

Now, using Lemma 2.1 and the previous section to obtain the solutions of the SmKdV Eq (1.1) as follows:

5.1. *The* exp$(-\psi(\eta))$*-expansion method*

Case I: If $\Omega = b^2 - 4a > 0$ and $a \neq 0$, then the solutions of the SmKdV Eq (1.1) are

$$\mathcal{Y}(t,x) = \pm\Big(\sqrt{\frac{-H_1}{H_2}} + \sqrt{\frac{-2}{H_2}}\frac{2a}{\sqrt{\Omega}\tanh\Big(\frac{\sqrt{\Omega}}{2}(kx + k\varepsilon\mathcal{W}(t) + \lambda t + \mathcal{E})\Big) + b}\Big), \tag{5.2}$$

and

$$\mathcal{Y}(t,x) = \pm\Big(\sqrt{\frac{-H_1}{H_2}} + \sqrt{\frac{-2}{H_2}}\frac{2a}{\sqrt{\Omega}\coth\Big(\frac{\sqrt{\Omega}}{2}(kx + \varepsilon k\mathcal{W}(t) + \lambda t + \mathcal{E})\Big) + b}\Big). \tag{5.3}$$

Case II: If $\Omega < 0$ and $a \neq 0$, then the solutions of the SmKdV Eq (1.1) are

$$\mathcal{Y}(t,x) = \pm\Big(\sqrt{\frac{-H_1}{H_2}} - \sqrt{\frac{-2}{H_2}}\frac{2a}{\sqrt{-\Omega}\tan\Big(\frac{\sqrt{-\Omega}}{2}(kx + \varepsilon k\mathcal{W}(t) + \mathcal{E})\Big) - b}\Big), \tag{5.4}$$

and

$$\mathcal{Y}(t,x) = \pm\Big(\sqrt{\frac{-H_1}{H_2}} + \sqrt{\frac{-2}{H_2}}\frac{-2a}{\sqrt{-\Omega}\cot\Big(\frac{\sqrt{-\Omega}}{2}(kx + \varepsilon k\mathcal{W}(t) + \mathcal{E})\Big) + b}\Big). \tag{5.5}$$

Case III: If $b \neq 0$ and $a = 0$, then the solution of the SmKdV Eq (1.1) is

$$\mathcal{Y}(t,x) = \pm\sqrt{\frac{-2}{H_2}}\Big(\frac{b}{\exp(b(kx + \varepsilon k\mathcal{W}(t) + \mathcal{E})) - 1}\Big). \tag{5.6}$$

Case IV: If $\Omega = 0$, $a \neq 0$, and $b \neq 0$, then the solution of the SmKdV Eq (1.1) is

$$\mathcal{Y}(t,x) = \pm\Big(\sqrt{\frac{-H_1}{H_2}} - \sqrt{\frac{-2}{H_2}}\frac{b^2(kx + \varepsilon k\mathcal{W}(t) + \lambda t + \mathcal{E})}{4 + 2b(kx + \varepsilon k\mathcal{W}(t) + \lambda t + \mathcal{E}))}\Big). \tag{5.7}$$

5.2. *Extended tanh function method*

Using Eqs (4.25)–(4.28), the solutions of the SmKdV Eq (1.1) are

$$\mathcal{Y}(t,x) = \pm\frac{H_0}{3\sqrt{-2H_2}}\Big(1 + \tanh\Big(\frac{H_0}{6}(kx + \varepsilon k\mathcal{W}(t) + \lambda t)\Big)\Big), \tag{5.8}$$

$$\mathcal{Y}(t,x) = \pm\frac{H_0}{3\sqrt{-2H_2}}\Big(1+\coth\Big(\frac{H_0}{6}(kx+\varepsilon k\mathcal{W}(t)+\lambda t)\Big)\Big), \tag{5.9}$$

$$\begin{aligned}\mathcal{Y}(t,x) \;&=\; \pm\frac{H_0}{3\sqrt{-2H_2}}\Big(1+\coth\Big(\frac{H_0}{3}(kx+\varepsilon k\mathcal{W}(t)+\lambda t)\Big)\\ &\quad \pm\text{csch}\Big(\frac{H_0}{3}(kx+\varepsilon k\mathcal{W}(t)+\lambda t)\Big)\Big),\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\mathcal{Y}(t,x) \;&=\; \pm\frac{H_0}{6\sqrt{-2H_2}}\Big(2+\tanh\Big(\frac{H_0}{12}(kx+k\varepsilon\mathcal{W}(t)+\lambda t)\Big)\\ &\quad +\coth\Big(\frac{H_0}{12}(kx+k\varepsilon\mathcal{W}(t)+\lambda t)\Big)\Big).\end{aligned} \tag{5.11}$$

Remark 5.1. *The solutions (5.8)–(5.11) that are obtained by the extended tanh function method, depend on the noise intensity ε. If we let $\varepsilon = 0$, then all these solutions equal zero. While the solutions (5.2)–(5.7) that are obtained by the* $\exp(-\psi(\eta))$*-expansion method do not depend on ε. So, the solutions (5.2)–(5.7) are more acceptable than the solutions (5.8)–(5.11).*

6. Physical meaning and noise impacts

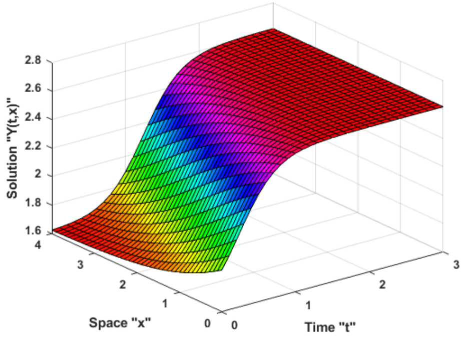
Physical meaning: In this paper, we obtained the stochastic solutions for SmKdV Eq (1.1). The solution $\mathcal{Y}(t,x)$ reported in Eq (5.2) represents a stochastic traveling solitary wave (soliton-like), whose position fluctuates randomly over time due to the presence of a Wiener process term $\mathcal{W}(t)$. The wave maintains its shape but exhibits random motion, typical of noise-driven solitons in nonlinear dispersive media. The solution $\mathcal{Y}(t,x)$ reported in Eq (5.3) represents a stochastic traveling singular wave. Its profile includes a sharp front or potential singularity near the wave center (due to the behavior of coth near zero), with the wave decaying to constant values at spatial infinity. The solution $\mathcal{Y}(t,x)$ reported in Eq (5.4) represents a periodic singular traveling wave with sharp oscillations and blow-up behavior at regular spatial intervals. It is not localized but periodically structured, and its center moves stochastically due to the presence of the Wiener process. The solution $\mathcal{Y}(t,x)$ reported in Eq (5.5) represents a periodic singular traveling wave that repeats at regular spatial intervals with sharp spikes or blow-up points, due to the cotangent structure. The stochastic term $\varepsilon\mathcal{W}(t)$ causes the wave to randomly shift in space over time, without changing its shape. This behavior is typical of nonlinear periodic wave trains under the influence of random noise, seen in systems like nonlinear optics, plasma waves, and turbulent fluid interfaces. The solution $\mathcal{Y}(t,x)$ reported in Eq (5.6) represents a stochastic singular traveling wave, exhibiting a sharp blow-up (singularity) at a moving location determined by the stochastic term $\varepsilon\mathcal{W}(t)$. The solution $\mathcal{Y}(t,x)$ reported in Eq (5.7) describes a nonlinear traveling wave with a potentially singular or sharply peaked structure. Its form is governed by a rational function that approaches a finite constant at infinity but may exhibit a blow-up at a specific point. The solution $\mathcal{Y}(t,x)$ stated in Eq (5.8) represents a smooth, monotonic traveling wave front. It models a transition between two stable states (from 0 to a constant value $\pm\frac{H_0}{3\sqrt{-2H_2}}$) and moves through space with deterministic drift (due to λ) and stochastic fluctuation (due to the Wiener process $\mathcal{W}(t)$). The solution $\mathcal{Y}(t,x)$ stated in Eq (5.9) represents a singular traveling wave front, where the wave profile sharply transitions from

0 to $\pm\frac{H_0}{3\sqrt{-2H_2}}$, but with a vertical asymptote (blow-up) at the center- due to the behavior of the coth function. This indicates a discontinuity or infinite gradient at that point. The wave moves forward with deterministic speed λ, and its position is randomly perturbed by noise (via $\mathcal{W}(t)$). The solution $\mathcal{Y}(t, x)$ stated in Eq (5.10) describes a stochastic singular wave front, sharply transition from one state to another. The $coth + csch$ combination produces a wave with a sharp rise (or dip) and a singular peak at its center. As time progresses, the wave moves deterministically (via λ) and wanders randomly (due to the Wiener process $\mathcal{W}(t)$).

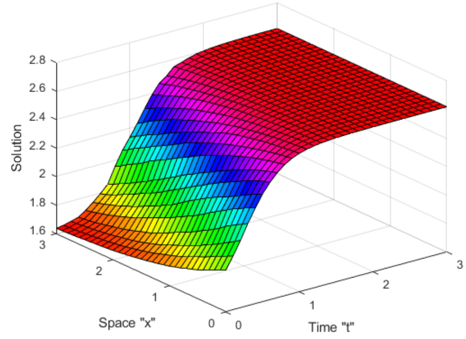
Effect of noise: Let us simulate some solutions such as the solutions stated in Eqs (5.2), (5.6)–(5.8), to demonstrate how advection noise affects them as follows:

7. Conclusions

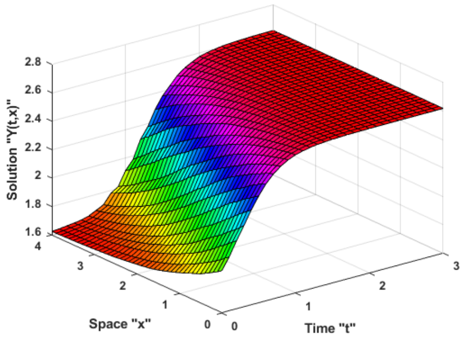
The stochastic modified KdV equation (SmKdV) Eq (1.1) driven by multiplicative noise was examined in this study. We broke down the SmKdV equation into a SODE and a deterministic mKdV (DmKdV) equation by utilizing transformation techniques and Itô calculus. We acquired the solutions of the DmKdV equation by using two various methods, such as the $\exp(-\psi(\eta))$-expansion method and the extended tanh function method. After that, we obtained the solutions of the SmKdV Eq (1.1) by combining the results that we acquired with the solution of a SODE. Moreover, by simulating specific solutions, we evaluated the impact of advective noise on the obtained solutions. We concluded from Figures 1–4 that the stochastic term $\varepsilon\mathcal{W}(t)$ introduces randomness in the wave's position and timing, causing early-time jitter or fluctuations, especially at higher noise levels. However, the wave shape and final value remain stable, showing that the system is robust to noise in the long run but sensitive to it during propagation.



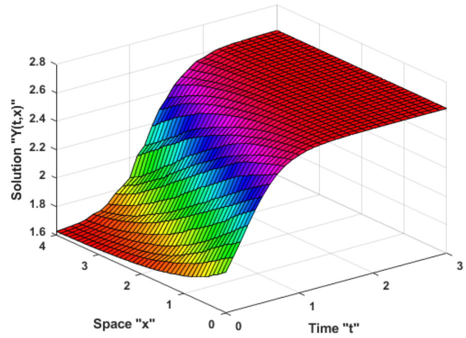
(a) $\varepsilon = 0$



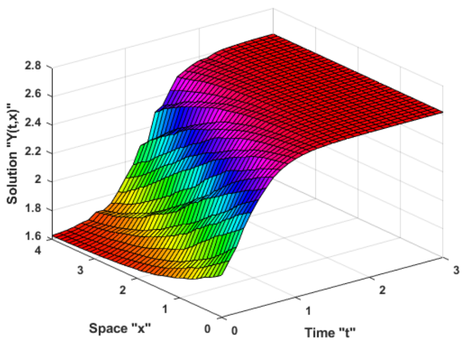
(b) $\varepsilon = 0.3$



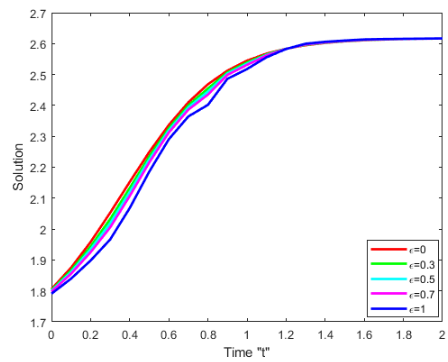
(c) $\varepsilon = 0.5$



(d) $\varepsilon = 0.7$

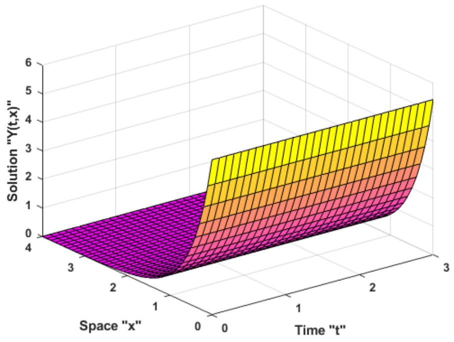


(e) $\varepsilon = 1$

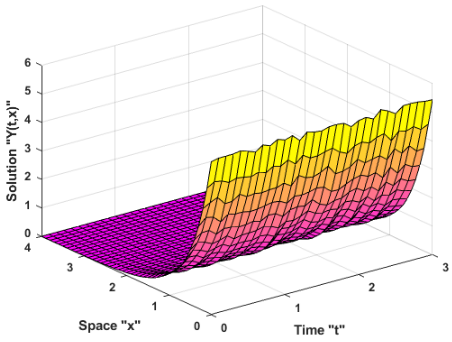


(f) $\varepsilon = 0,\ 0.3,\ 0.5,\ 0.7,\ 1$

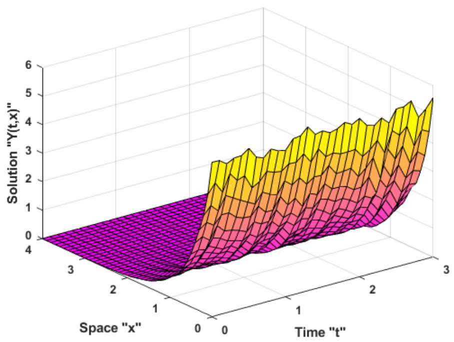
Figure 1. (a–e) show 3D-profile for the amplitude solution $\mathcal{Y}(t, x)$ reported in Eq (5.2) with $\gamma_1 = 6$, $\gamma_2 = -1$, $k = a = 1$, $b = \sqrt{5}$ $\lambda = -2$, $C = 0$, x, $t \in [0, 3]$ and with various ε, (f) shows 2D-profile for this solution with various ε. With these parameters, we get $H_0 = \frac{-\varepsilon^2}{2}$; $H_1 = 2$; $H_2 = -2$ and $\Omega = 1$. The stochastic term $\varepsilon \mathcal{W}(t)$ causes visible spatial fluctuations in the wave front. With larger ε, the wave front becomes less smooth in time and space. It appears slightly randomly shifted.



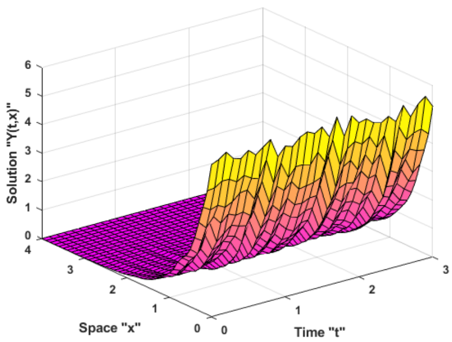
(a) $\varepsilon = 0$



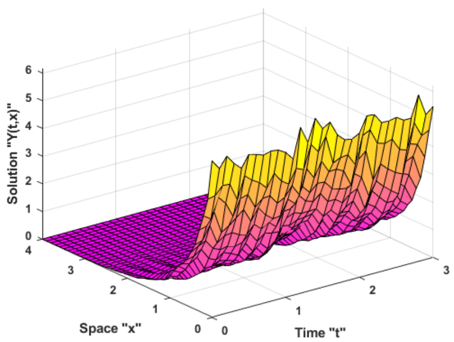
(b) $\varepsilon = 0.3$



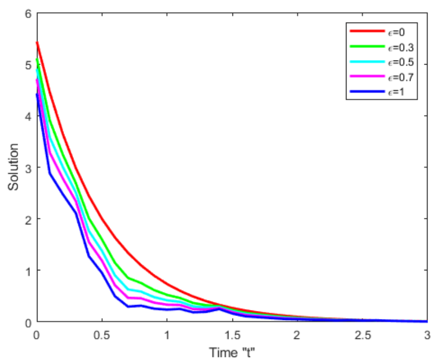
(c) $\varepsilon = 0.5$



(d) $\varepsilon = 0.7$

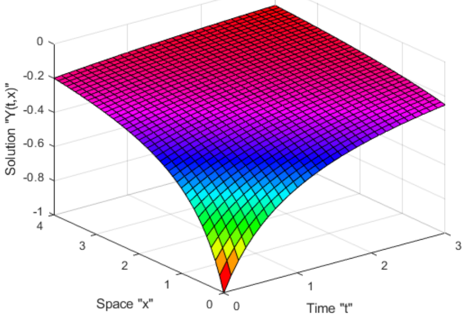


(e) $\varepsilon = 1$

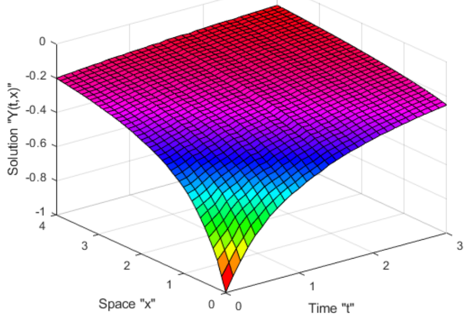


(f) $\varepsilon = 0,\ 0.3,\ 0.5,\ 0.7,\ 1$

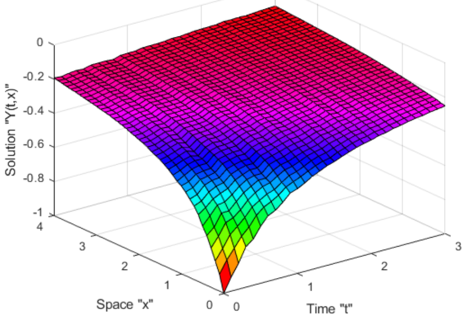
Figure 2. (a–e) show 3D-profile for the amplitude solution $\mathcal{Y}(t, x)$ reported in Eq (5.6) with $\gamma_1 = 6,\ \gamma_2 = -1,\ k = b = 1,\ C = 0,\ x,\ t \in [0, 3]$, and with various ε , (f) shows 2D-profile for this solutions with various ε. With these parameters, we get $H_2 = -2$. In the first plot, at $\varepsilon = 0$, the surface appears smooth and evolves in a predictable manner over time. In contrast, the other plots illustrate the impact of randomness: the surfaces become increasingly rough, and there are fluctuations in amplitude.



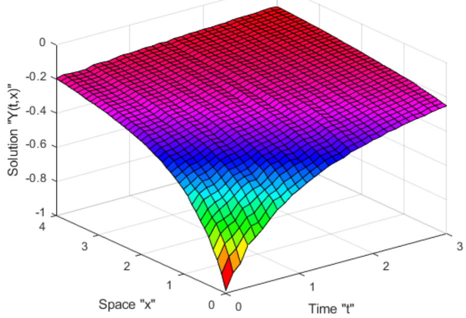
(a) $\varepsilon = 0$



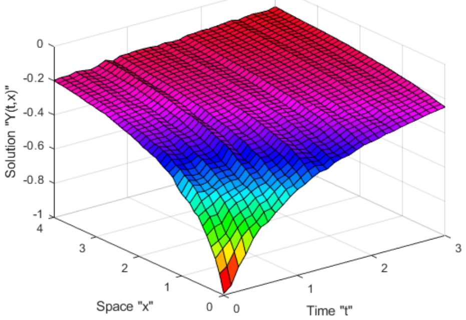
(b) $\varepsilon = 0.3$



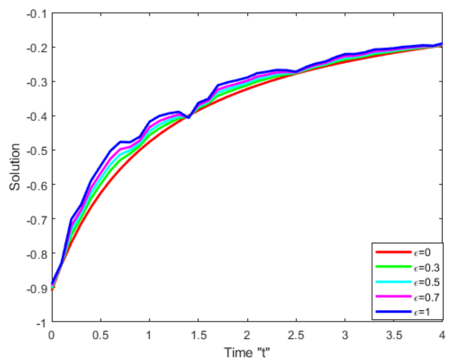
(c) $\varepsilon = 0.5$



(d) $\varepsilon = 0.7$

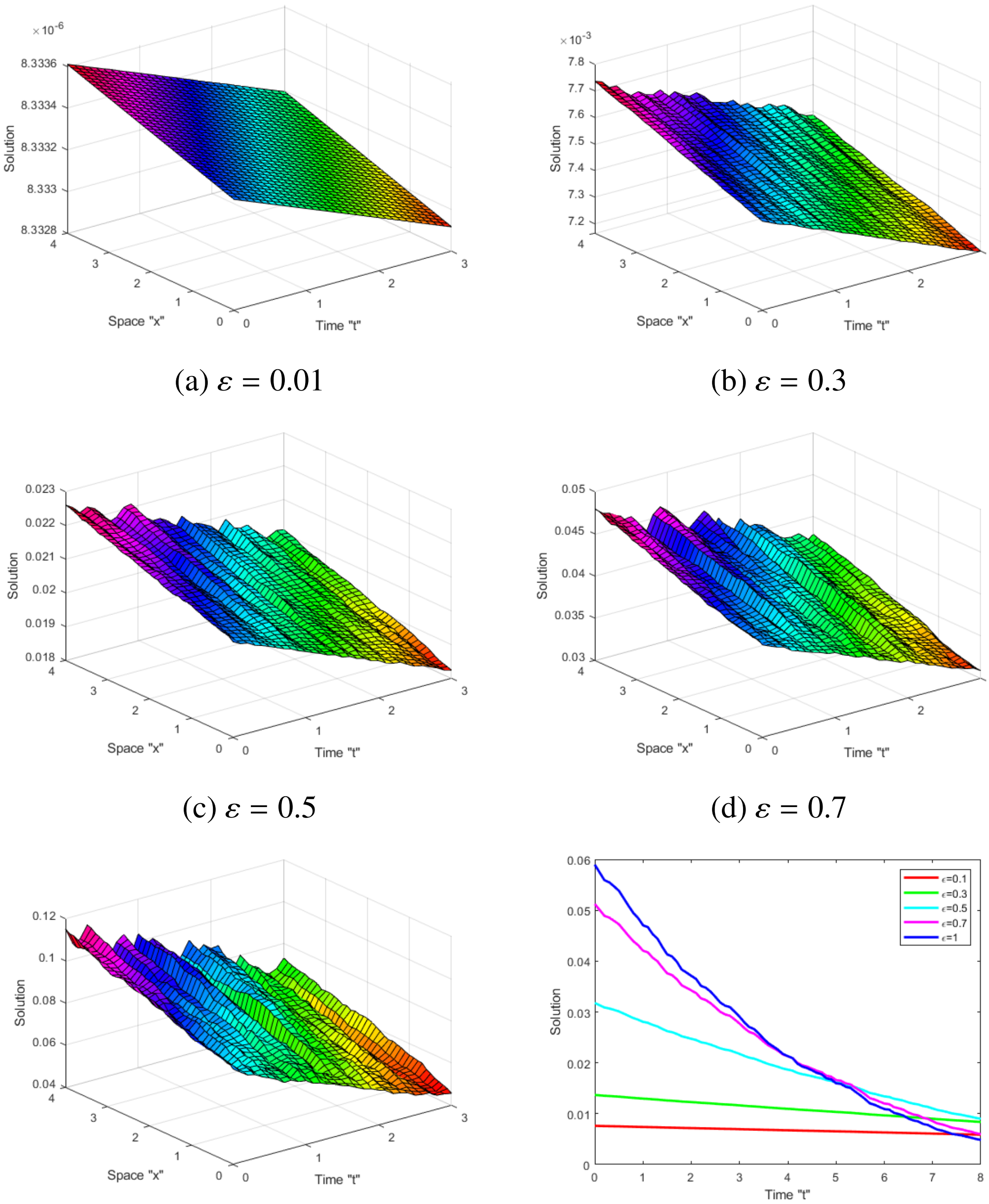


(e) $\varepsilon = 1$



(f) $\varepsilon = 0,\ 0.3,\ 0.5,\ 0.7,\ 1$

Figure 3. (a–e) show 3D-profile for the amplitude solution $\mathcal{Y}(t, x)$ reported in Eq (5.7) with $\gamma_1 = 6,\ \gamma_2 = -1,\ b = 2,\ C = 0,\ \lambda = k = a = 1,\ x,\ t \in [0, 3]$, and with various ε, (f) shows 2D-profile for this solutions with various ε. With these parameters, we get $H_0 = \frac{\varepsilon^2}{2};\ H_1 = 1;\ H_2 = -2$ and $\Omega = 0$. In the first plot, at $\varepsilon = 0$, the surface appears smooth, and there are no local oscillations or irregularities. In other plots and for higher ε, the solution becomes rougher and less predictable, though the global (expected) trend remains the same.



(a) $\varepsilon = 0.01$ (b) $\varepsilon = 0.3$

(c) $\varepsilon = 0.5$ (d) $\varepsilon = 0.7$

(e) $\varepsilon = 1$ (f) $\varepsilon = 0,\ 0.3,\ 0.5,\ 0.7,\ 1$

Figure 4. (a–e) show 3D-profile for the solution $\mathcal{Y}(t, x)$ reported in Eq (5.8) with $\gamma_1 = 6$, $\gamma_2 = -1$, $k = 1$, $\lambda = -2$, $t \in [0, 3]$, $x \in [0, 4]$, and with various ε, (f) shows 2D-profile for this solutions with various ε. With these parameters, we get $H_0 = \frac{-\varepsilon^2}{2}$; $H_1 = 2$; $H_2 = -2$. We conclude that when the stochastic term increases, the solutions deviate more from the deterministic profile.

Author contributions

Sofian T. Obeidat: methodology, validation, formal analysis, funding acquisition, writing original draft preparation, writing review and editing; Doaa Rizk: methodology, funding acquisition, validation, formal analysis, writing original draft preparation, writing review and editing; Wael W. Mohammed: methodology, software, validation, formal analysis, writing original draft preparation, writing review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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