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Research article

Variational analysis of Kirchhoff-type fractional p-Laplacian BVPs under simultaneous instantaneous and non-instantaneous impulses

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Abstract: This research focuses on solving Kirchhoff-type fractional p-Laplacian BVPs with hybrid impulsive effects (instantaneous and non-instantaneous types). Through the use of variational methods, the existence of solutions and multiple solutions for the aforementioned problem are established under assumptions weaker than the super-p-linear Ambrosetti-Rabinowitz type growth condition. Finally, an example demonstrates the validity of the paper's main results.

Keywords: fractional Kirchhoff problems; p-Laplacian operator; instantaneous impulses; non-instantaneous impulses; variational methods
Mathematics Subject Classification: 34A08, 34B15

1. Introduction

Impulse DEs (differential equations), as a crucial mathematical tool for describing abrupt changes in dynamical systems, demonstrate unique modeling advantages in fields such as biomedical applications (e.g., intermittent drug administration therapies), engineering control (e.g., sudden signal regulation), and ecological management (e.g., disaster interventions). Departing from conventional instantaneous impulse frameworks, Hernádez and O'Regan [1] introduced a non-instantaneous impulse theory in 2013, eliminating the idealized momentary action hypothesis. This advancement enables precise characterization of prolonged impulse dynamics, exemplified by sustained pharmacological concentration profiles and extended ecological management interventions. Recent breakthroughs [2–4] in variational methods and critical point theory have established a novel framework for studying the dynamics of non-instantaneous impulsive systems. As a representative study, Tian and Zhang [5] established existence criteria for second-order Dirichlet BVPs (boundary value problems) with mixed impulses through an innovative application of the Ekeland variational principle:

$$\begin{cases}-x''(t)=f_i(t,x(t)),\, t\in(s_i,t_{i+1}],\, i=\overline{0,m},\\ \Delta x'(t_i)=I_i(x(t_i)),\, i=\overline{1,m},\\ x'(t)=x'\left(t_i^+\right),\, t\in(t_i,s_i],\, i=\overline{1,m},\\ x'\left(s_i^+\right)=x'\left(s_i^-\right),\, i=\overline{1,m},\\ x(0)=x(T)=0.\end{cases}$$

The mixed impulsive problem studied here has a structure that alternates between continuous and impulsive intervals, as shown in Figure 1. The system evolves in two types of intervals: Continuous intervals ($[s_{i-1},t_i]$ and $[s_i,t_{i+1}]$), where it follows a differential equation, and mixed impulse intervals ($(t_i,s_i]$). At each instantaneous impulse point t_i, the derivative jumps according to $\Delta x'(t_i)=I_i(x(t_i))$, while the state $x(t)$ stays continuous. During the following mixed impulse interval $(t_i,s_i]$, the derivative remains constant at $x'(t)=x'(t_i^+)$. Finally, at the non-instantaneous impulse point s_i, the derivative becomes continuous again with $x'(s_i^+)=x'(s_i^-)$, and the system continues its continuous evolution. This repeating pattern is key to the analysis.

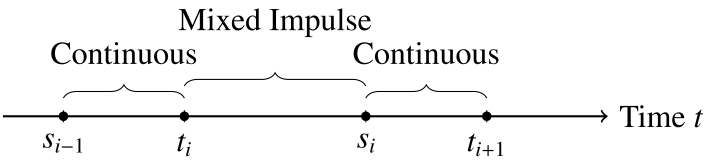

Figure 1. Schematic of mixed impulse intervals.

The demonstrated efficacy of fractional calculus in capturing non-local properties, memory effects, and complex dynamics has established FIDEs (fractional-order impulsive differential equations) as a fundamental framework for addressing interdisciplinary challenges characterized by discontinuous transitions and cumulative history dependence. Significant applications [6–9] have been developed in neural dynamics, intelligent control systems, and epidemiological modeling. As a pioneering contribution, Zhang and Liu's 2020 work [10] initially formulated the FBVPs (fractional-order boundary value problems) with mixed impulses:

$$\begin{cases}{}_tD_T^{\gamma}({}_0^CD_t^{\gamma}x(t))=f_i(t,x(t)),\, t\in(s_i,t_{i+1}],\, i=\overline{0,m},\gamma\in(1/2,1],\\ \Delta\left({}_tD_T^{\gamma-1}\left({}_0^CD_t^{\gamma}x\right)\right)(t_i)=I_i(x(t_i)),\, i=\overline{1,m},\\ {}_tD_T^{\gamma-1}\left({}_0^CD_t^{\gamma}x(t)\right)={}_tD_T^{\gamma-1}\left({}_0^CD_t^{\gamma}x\left(t_i^+\right)\right),\, t\in(t_i,s_i],\, i=\overline{1,m},\\ {}_tD_T^{\gamma-1}\left({}_0^CD_t^{\gamma}x\left(s_i^-\right)\right)={}_tD_T^{\gamma-1}\left({}_0^CD_t^{\gamma}x\left(s_i^+\right)\right),\ i=\overline{1,m},\\ x(0)=x(T)=0.\end{cases}$$

In subsequent work, Zhou et al. [11] established an extension incorporating the p-Laplacian operator:

$$\begin{cases}{}_tD_T^{\gamma}\left(\phi_p({}_0^CD_t^{\gamma}x(t))\right)+q(t)\phi_p(x(t))=f_i(t,x(t)),\, t\in(s_i,t_{i+1}],\, i=\overline{0,m},\\ \Delta\left({}_tD_T^{\gamma-1}\phi_p\left({}_0^CD_t^{\gamma}x(t_i)\right)\right)=I_i(x(t_i)),\, i=\overline{1,m},\\ {}_tD_T^{\gamma-1}\phi_p\left({}_0^CD_t^{\gamma}x(t)\right)={}_tD_T^{\gamma-1}\phi_p\left({}_0^CD_t^{\gamma}x\left(t_i^+\right)\right),\, t\in(t_i,s_i],\, i=\overline{1,m},\\ {}_tD_T^{\gamma-1}\phi_p\left({}_0^CD_t^{\gamma}x\left(s_i^-\right)\right)={}_tD_T^{\gamma-1}\phi_p\left({}_0^CD_t^{\gamma}x\left(s_i^+\right)\right),\ i=\overline{1,m},\\ x(0)=x(T)=0,\end{cases}$$

here γ lies in the interval $(\frac{1}{p}, 1]$, $\phi_p(x) = |x|^{p-2}x$, $p > 1$. As demonstrated through the least action principle, the system admitted the solution's existence. Extending the methodology, Li and collaborators [12] employed Ricceri's three-critical-point theorem to derive multiple solution existences for the problem. Subsequent studies [13,14] have significantly expanded this research direction.

Conversely, the Kirchhoff-type equation serves as a generalization of the classical D'Alembert wave equation. It was originally derived by Kirchhoff [15] in 1883 during his investigation into the free vibrations of elastic strings. An essential characteristic of Kirchhoff-type models is the incorporation of a nonlocal term $\int_0^L \left|\frac{\partial x}{\partial t}\right|^2 dt$, which fundamentally alters the nature of the problem from pointwise-defined to be nonlocal in scope. In later literature, mathematical problems incorporating nonlocal terms of this form has become conventionally termed Kirchhoff problems. With broad applications spanning diverse disciplines, the Kirchhoff equation is utilized in non-Newtonian mechanics, astrophysics, elastic electrodynamics, and population dynamics. The past few years have seen remarkable developments in Kirchhoff equation research [16–18]. For illustration, we cite Liang's 2014 work [19] that employed variational principles to determine positive solution existence for Kirchhoff-type equations of the form:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla x|^2\right)\Delta x = f(t,x),\ in\,\Omega, \\ x=0,\ on\,\partial\Omega. \end{cases}$$

Later work by Fiscella and Valdinoci [20] incorporated the nonlocal characteristics of tension, originating from the nonlocal metric of the string's fractional-dimensional length. Fiscella and Valdinoci generalized the conventional Kirchhoff equation to its fractional counterpart, establishing existence results for non-negative solutions to certain Kirchhoff-type problems containing nonlocal operators. Nevertheless, studies concerning Kirchhoff-type FIDEs have shown notable deficiencies in the last ten years. The theoretical framework remains incomplete when nonlocal effects are coupled with fractional-order operators and impulsive disturbances. The constructive theory demands thorough investigation, especially regarding Kirchhoff-type FBVPs incorporating p-Laplacian operators coupled with mixed impulses. Against this background, the application of variational approaches emerges as a pivotal turning point. As a representative example, Wang and Tian's 2023 work [21] solved Kirchhoff-type FBVPs with (p,q)-Laplacian and instantaneous impulses using critical point theory:

$$\begin{cases} M_\gamma\left(\|x\|_\gamma^p\right)\, {}_tD_T^\gamma\left(\mu(t)\,\phi_p\left({}_0^CD_t^\gamma x(t)\right)\right) + \nu(t)\,\phi_p(x(t)) \\ = F_x(t,x(t),y(t)) + \lambda G_x(t,x(t),y(t)),\ t\neq t_j, a.e.\, t\in[0,T], \\ M_\beta\left(\|y\|_\beta^q\right)\, {}_tD_T^\beta\left(\xi(t)\,\phi_q\left({}_0^CD_t^\beta y(t)\right)\right) + \eta(t)\,\phi_q(y(t)) \\ = F_y(t,x(t),y(t)) + \lambda G_y(t,x(t),y(t)),\ t\neq t'_i, a.e.\, t\in[0,T], \\ \Delta\left(M_\gamma\left(\left\|x\left(t_j\right)\right\|_\gamma^p\right)\right) {}_tD_T^{\gamma-1}\left(\mu\left(t_j\right)\phi_p\left({}_0^CD_t^\gamma x\left(t_j\right)\right)\right) = H_j\left(x\left(t_j\right)\right), j=\overline{1,m}, \\ \Delta\left(M_\beta\left(\|y(t_i)\|_\beta^q\right)\right) {}_tD_T^{\beta-1}\left(\xi(t'_i)\,\phi_q\left({}_0^CD_t^\beta y(t'_i)\right)\right) = J_i(y(t'_i)), i=\overline{1,n}, \\ x(0)=x(T)=y(0)=y(T)=0, \end{cases}$$

here the exponents γ and β range in $\left(\frac{1}{p},1\right]$ and $\left(\frac{1}{q},1\right]$. Nevertheless, existing research has been limited to cases with purely instantaneous impulses. For theoretical completeness, Yao and Zhang's 2025 work [22] shifted focus to more complex mixed impulsive systems. Through variational techniques, they examined multiple solutions for fractional p-Kirchhoff BVPs with mixed-type impulses, requiring the nonlinear term to fulfill an Ambrosetti-Rabinowitz condition beyond p-th order growth.

In summary, although the existing literature provides an important theoretical foundation for Kirchhoff-type fractional p-Laplacian problems with mixed impulses, their key conclusions heavily rely on the classical Ambrosetti-Rabinowitz-type super-p-linear growth condition. This restrictive condition considerably limits the choice of nonlinear terms, thereby constraining the model's applicability in describing more complex nonlinear phenomena. In contrast, the core innovation and significant advantage of our study lie in systematically breaking through the limitations of this traditional framework. Specifically, first, we propose a class of growth assumptions that are considerably weaker than the classical Ambrosetti-Rabinowitz condition. This enables our theoretical framework to accommodate a broader range of more complex nonlinear functions, significantly expanding the potential physical applications of the model. Second, compared to the results obtained in [22] under specific strong growth conditions, our work not only proves the existence and multiplicity of weak solutions under weaker assumptions but also establishes a more universal existence theory by incorporating nonlinear terms that combine concave-convex features and weakened superlinear conditions. Finally, the variational framework and technical approaches developed in this study open new avenues for analyzing impulsive problems that do not satisfy the traditional Ambrosetti-Rabinowitz condition, thereby enriching the research toolkit in this field. Therefore, this work is not merely a simple extension of existing research but represents a substantial breakthrough in fundamental theoretical limitations, establishing a more general and comprehensive theoretical framework for the study of such problems.

To achieve a breakthrough in fundamental theoretical limitations, this paper aims to establish a more general theoretical framework, with the core objective of systematically weakening the traditional growth restrictions on nonlinear terms. Specifically, we will investigate a class of Kirchhoff-type fractional p-Laplacian boundary value problems that incorporate both instantaneous and non-instantaneous impulses:

$$\begin{cases} M\left(\|u\|^p\right) {}_tD_T^{\alpha}\left(h(t)\,\phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right) + a(t)\,\phi_p(u(t)) = \lambda f_i(t,u(t)),\ t\in(s_i,t_{i+1}], i=\overline{0,n}, \\ \Delta\left(M\left(\|u(t_i)\|^p\right) {}_tD_T^{\alpha-1}\left(h(t_i)\,\phi_p\left({}_0^CD_t^{\alpha}u(t_i)\right)\right)\right) = \mu I_i(u(t_i)),\ i=\overline{1,n}, \\ M\left(\|u\|^p\right) {}_tD_T^{\alpha-1}\left(h(t)\,\phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right) \\ = M\left(\left\|u(t_i^+)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(t_i^+)\,\phi_p\left({}_0^CD_t^{\alpha}u(t_i^+)\right)\right),\ t\in(t_i,s_i], i=\overline{1,n}, \\ M\left(\left\|u(s_i^-)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(s_i^-)\,\phi_p\left({}_0^CD_t^{\alpha}u(s_i^-)\right)\right) \\ = M\left(\left\|u(s_i^+)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(s_i^+)\,\phi_p\left({}_0^CD_t^{\alpha}u(s_i^+)\right)\right),\ i=\overline{1,n}, \\ u(0)=u(T)=0, \end{cases} \tag{1.1}$$

where the exponent p ranges in $(1,+\infty)$ while α takes values in $(\frac{1}{p},1]$. $\phi_p(u)=|u|^{p-2}u\,(u\neq 0)$, $\phi_p(0)=0$, $\lambda,\mu>0$. The operators ${}_0^CD_t^{\alpha}$ (left Caputo type) and ${}_tD_T^{\alpha}$ (right Riemann-Liouville type) constitute the α-th order fractional differential operators in their respective formulations. The function h belongs to $L^{\infty}([0,T],\mathbb{R}^+)$, and its essential infimum $h_0=\operatorname{ess\,inf}_{t\in[0,T]}h(t)$ is strictly positive. The function a belongs to $C([0,T],\mathbb{R}^+)$ with the existence of positive constants a_0 and a^0 satisfying the uniform bounds $0<a_0\le a(t)\le a^0$ throughout the interval. Each I_i belongs to $C^1(\mathbb{R},\mathbb{R})$, with at least one index $i\in\{1,2,\cdots,n\}$ satisfying $I_i(u(t_i))\neq 0$. $f_i\in C^1((s_i,t_{i+1}]\times\mathbb{R},\mathbb{R})$, $0=s_0<t_1<s_1<t_2<\cdots<s_n<t_{n+1}=T$. $M\in C([0,+\infty),\mathbb{R})$ admits strictly positive lower and upper bounds $M_0=\inf_{s\ge 0}M(s)$ and

$M^0 = \sup_{s\geq 0} M(s)$ fulfilling $M_0 \leq M(s) \leq M^0$ globally. Specifically, $M(x) = (a + bx^p)^{p-1}$, where the coefficients a, b are positive real numbers. The precise definition of the norm $\|u\|$ appears in subsequent Eq (2.1). Instantaneous impulses produce discontinuous state jumps at discrete points t_i, whereas non-instantaneous impulses induce persistent state variations over finite intervals $(t_i, s_i]$. Beyond this,

$$\begin{aligned}
&\Delta\left(M\left(\|u(t_i)\|^p\right)\right) {}_tD_T^{\alpha-1}\left(h(t_i)\phi_p\left({}_0^CD_t^\alpha u(t_i)\right)\right)\\
=&M\left(\left\|u(t_i^+)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(t_i^+)\phi_p\left({}_0^CD_t^\alpha u(t_i^+)\right)\right) - M\left(\left\|u(t_i^-)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(t_i^-)\phi_p\left({}_0^CD_t^\alpha u(t_i^-)\right)\right),\\
&M\left(\left\|u(t_i^\pm)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(t_i^\pm)\phi_p\left({}_0^CD_t^\alpha u(t_i^\pm)\right)\right) = \lim_{t\to t_i^\pm} M\left(\|u\|^p\right) {}_tD_T^{\alpha-1}\left(h(t)\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right),\\
&M\left(\left\|u(s_i^\pm)\right\|^p\right) {}_tD_T^{\alpha-1}\left(h(s_i^\pm)\phi_p\left({}_0^CD_t^\alpha u(s_i^\pm)\right)\right) = \lim_{t\to s_i^\pm} M\left(\|u\|^p\right) {}_tD_T^{\alpha-1}\left(h(t)\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right).
\end{aligned}$$

Based on variational methods and critical point theory, this paper investigates the existence and multiplicity of weak solutions to the boundary value problem (1.1). By constructing a variational framework, we first prove the existence of solutions to problem (1.1) under weakened super-p-linear growth conditions using the mountain pass theorem. Subsequently, by decomposing the nonlinear term into components satisfying super-p-linear and sub-p-linear growth conditions, we establish the multiplicity of solutions to problem (1.1) via genus theory.

The innovations and advantages of this study are mainly reflected in the following aspects: First, we propose a class of growth assumptions that are significantly weaker than the classical super-p-linear Ambrosetti-Rabinowitz condition; second, by introducing nonlinear terms that incorporate both concave-convex components and weakened super-p growth conditions, we establish a more universal theory of solution multiplicity. Compared with the classical Ambrosetti-Rabinowitz case treated in reference [22], this work not only substantially relaxes the growth restrictions on nonlinear terms and develops a more general theory for solution existence and multiplicity but also opens new avenues for analyzing impulsive problems that do not satisfy the traditional Ambrosetti-Rabinowitz condition.

The organizational framework of this manuscript is presented below: The second section assembles the requisite definitions and technical lemmas that underpin our later theoretical arguments. The variational approach developed in Section 3 yields existence and multiplicity results for solutions to FBVPs (1.1). The paper concludes with Section 4, which offers a comprehensive summary and a perspective on future research trajectories.

2. Preliminaries

Definition 2.1. *([23]) The left Caputo derivative ${}_0^CD_t^\alpha u(t)$ and the right Riemann-Liouville derivative ${}_tD_T^\alpha u(t)$ of order $\alpha > 0$ ($n = \lceil\alpha\rceil$) for the function $u(t)$ are defined respectively as*

$$ {}_0^CD_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, $$

$$ {}_tD_T^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^T (s-t)^{n-\alpha-1}u(s)ds. $$

Definition 2.2. *([23]) Let $\alpha \in (0,1]$, $p \in (1,+\infty)$. We define the fractional-order function space*

$$ E_0^{\alpha,p} = \left\{u : [0,T] \to \mathbb{R} \,|\, u, {}_0^CD_t^\alpha u(t) \in L^p\left((0,T),\mathbb{R}\right), u(0) = u(T) = 0\right\}, $$

where the associated norm is given by

$$\|u\|_{E^{\alpha,p}} = \left(\|u\|_{L^p}^p + \left\|{}_0^C D_t^\alpha u\right\|_{L^p}^p\right)^{\frac{1}{p}},$$

where $\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}$ *is the norm of* $L^p\left((0,T),\mathbb{R}^N\right)$*. Additionally, we define* $E_0^{\alpha,p}$ *to be the completion of* $C_0^\infty\left((0,T),\mathbb{R}^N\right)$ *with respect to* $\|\cdot\|_{E^{\alpha,p}}$*. Since* $h(t) \in L^\infty([0,T],\mathbb{R}^+)$*,* $h_0 = \mathrm{essinf}_{t\in[0,T]} h(t) > 0$*,* $a(t) \in C([0,T],\mathbb{R}^+)$*, and* $0 < a_1 \leq a(t) \leq a_2$*, so* $\|u\|_{E^{\alpha,p}}$ *is equivalent to*

$$\|u\|_{\alpha,p} = \left(\int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt + \int_0^T a(t)|u(t)|^p dt\right)^{\frac{1}{p}}.$$

Lemma 2.1. *([23]) Let* $\alpha \in (0,1]$*,* $p \in (1,+\infty)$*. Then* $E_0^{\alpha,p}$ *enjoys both reflexivity and separability as a Banach space.*

Lemma 2.2. *([24]) Let* $q_1, q_2 \geq 1$*,* $\beta > 0$*, and either* $\frac{1}{q_1} + \frac{1}{q_2} \leq 1 + \beta$ *or* $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \beta$*,* $q_1 \neq 1$*,* $q_2 \neq 1$*. Then for any* $x \in L^{q_1}\left([0,T],\mathbb{R}^N\right)$*,* $y \in L^{q_2}\left([0,T],\mathbb{R}^N\right)$*, the following equality holds:*

$$\int_0^T \left[{}_0D_t^{-\beta} x(t)\right] y(t)\, dt = \int_0^T \left[{}_tD_T^{-\beta} y(t)\right] x(t)\, dt.$$

Lemma 2.3. *([23]) Let* $\alpha \in (0,1]$*,* $1 < p < \infty$*. Then*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left\|{}_0^C D_t^\alpha u\right\|_{L^p},\ \forall u \in E_0^{\alpha,p}.$$

If $\alpha > \frac{1}{p}$*,* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}\left\|{}_0^C D_t^\alpha u\right\|_{L^p},$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ *is the norm of* $C([0,T],\mathbb{R})$*.*

Lemma 2.4. *([23]) Let* $\frac{1}{p} < \alpha \leq 1$*,* $1 < p < \infty$*. If* $u_k \rightharpoonup u$ *in* $E_0^{\alpha,p}$*, then* $u_k \to u$ *in* $C([0,T],\mathbb{R}^N)$ *with* $\|u_k - u\|_\infty \to 0$ *as* $k \to \infty$*.*

In the space $E_0^{\alpha,p}$, define a new norm

$$\|u\| = \left(\int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt + \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)|u(t)|^p dt\right)^{\frac{1}{p}}. \tag{2.1}$$

Lemma 2.5. *([22]) The two norms* $\|u\|_{\alpha,p}$ *and* $\|u\|$ *are equivalent for* $\forall u \in E_0^{\alpha,p}$*, in the sense that there are* ϱ_1 *and* $\varrho_2 > 0$ *for which*

$$\varrho_1\|u\|_{\alpha,p} \leq \|u\| \leq \varrho_2\|u\|_{\alpha,p}.$$

Lemma 2.6. *([22]) Assuming* $\alpha \in (\frac{1}{p},+\infty)$*,* $\frac{1}{p} + \frac{1}{q} = 1$*, and* $1 < p < \infty$*, the infinity norm of u satisfies the estimate* $\|u\|_\infty \leq K\|u\|$*, in which the constant* K *is defined as*

$$K := \frac{T^{\alpha-\frac{1}{p}} h_0^{-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}.$$

Lemma 2.7. *([25]) Suppose E is a reflexive Banach space over the real field. Additionally, let* Θ *be bounded and weakly order-closed, with* φ *being order-weakly lower semicontinuous over* Θ. *Then, for the function* $\varphi : \Theta \subseteq E \to [-\infty, +\infty]$, *and under the condition that* $\Theta \neq \phi$, *the minimization problem* $\min_{u\in\Theta}\varphi(u) = \varepsilon$ *admits a solution.*

Definition 2.3. *([26]) (Palais-Smale condition). Let E be a real Banach space and* $\varphi \in C^1(E, \mathbb{R})$. *The functional* φ *is said to satisfy the Palais-Smale condition (PS-condition), if any sequence* $\{u_n\} \subset E$ *such that* $\{\varphi(u_n)\}$ *is bounded and* $\varphi'(u_n) \to 0$ *in the dual space* E^* *as* $n \to \infty$, *admits a convergent subsequence in E.*

Lemma 2.8. *([26]) Assume* $\varphi \in C^1(E, \mathbb{R})$ *fulfills the PS-condition with* $\varphi(0) = 0$. *Suppose additionally that* φ *verifies the following two conditions:*

(i) We can find $\rho, \sigma > 0$ *with the following property: if any element* $u_0 \in E$ *has its norm equal to* ρ, *then the functional value* $\varphi(u_0)$ *is bounded below by* σ;

(ii) The functional φ *attains values below* σ *at some point* $u_1 \in E$ *beyond the norm threshold* ρ.
Consequently, φ *admits a critical value* ε *satisfying* $\varepsilon \geq \sigma$. *Additionally,* ε *can be expressed as* $\varepsilon = \inf_{k\in Y}\max_{s\in[0,1]}\varphi(k(s))$, *where*

$$Y = \left\{k \in C^1([0,1], E) : k(0) = u_0, k(1) = u_1\right\}.$$

Definition 2.4. *([26]) (Krasnoselskii Genus). Let E be a real Banach space and U be a closed symmetric subset of* $E \setminus \{0\}$. *The Krasnoselskii genus (or genus) of U is defined as the smallest integer n for which there exists an odd continuous mapping* $\Psi : U \to \mathbb{R}^n \setminus \{0\}$. *If no such integer exists, we define* $\gamma(U) = +\infty$. *By convention,* $\gamma(\emptyset) = 0$.

Lemma 2.9. *([26]) Given a* C^1 *even functional* φ *on the Banach space E, which satisfies the Palais-Smale compactness property. For* $\forall n \in \mathbb{N}$, $z \in \mathbb{R}$, *set* $\Sigma = \{U \subset E - \{0\} : U \subset E$ *be closed and* 0*-symmetric*$\}$, $\Sigma_n = \{U \in \Sigma : \gamma(U) \geq n\}$, $K_z = \{u \in E : \varphi(u) = z, \varphi'(u) = 0\}$, $z_n = \inf\limits_{U\in\Sigma_n}\sup\limits_{u\in U}\varphi(u)$.
Consequently,

(i) Given a nonempty class Σ_n *and a real number* z_n, *the value* z_n *constitutes a critical value for the functional* φ.

(ii) Suppose for some natural number l, the sequence $z_n = z_{n+1} = z_{n+2} = \cdots = z_{n+l} = z \in \mathbb{R}$ *stabilizes at a real value* $z \neq \varphi(0)$. *Then the Krasnoselskii genus of* K_z *satisfies* $\gamma(K_z) \geq l + 1$.

Remark 2.1. *([26, Remark 7.3]) implies that when the critical set* K_z *belongs to* Σ *and has genus greater than* 1, K_z *must contain an infinite number of distinct elements.*

Definition 2.5. *The function* $u \in E_0^{\alpha,p}$ *is a weak solution of (1.1) if u satisfies the following equation:*

$$\begin{aligned}&M(\|u\|^p)\left(\int_0^T h(t)\,\phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)\,\phi_p(u(t))\,v(t)\,dt\right)\\&= \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} f_i(t, u(t))\,v(t)\,dt - \mu\sum_{i=1}^{n} I_i(u(t_i))\,v(t_i), \forall v \in E_0^{\alpha,p}.\end{aligned} \tag{2.2}$$

Consider the energy functional $\varphi_\lambda : E_0^{\alpha,p} \to \mathbb{R}$ *given by*

$$\varphi_\lambda(u) = \frac{1}{p}\mathcal{M}(\|u\|^p) - \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t, u(t))\,dt + \mu\sum_{i=1}^{n} J_i(u(t_i)), \tag{2.3}$$

with $\mathcal{M}(u) = \int_0^u M(s)\,ds$, $F_i(t,u(t)) = \int_0^u f_i(t,s)\,ds$, *and* $J_i(u) = \int_0^u I_i(s)\,ds$. *Due to the continuity of* M, f_i, *and* I_i, *it follows that* $\varphi_\lambda \in C^1\left(E_0^{\alpha,p}, \mathbb{R}\right)$, *and*

$$\begin{aligned}\langle \varphi_\lambda'(u), v\rangle = \; & M(\|u\|^p)\left(\int_0^T h(t)\,\phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)\,\phi_p(u(t))\,v(t)\,dt\right) \\ & - \lambda \sum_{i=0}^n \int_{s_i}^{t_{i+1}} f_i(t,u(t))\,v(t)\,dt + \mu \sum_{i=1}^n I_i(u(t_i))\,v(t_i), \;\; \forall v \in E_0^{\alpha,p}.\end{aligned} \tag{2.4}$$

In this framework, weak solutions to (1.1) bijectively correspond to critical points of φ_λ.

3. Main results

This paper first presents the assumptions necessary for its main theorems and lemmas.

(H_1) For $p > 1$, we can find growth exponents $0 \le l_i < p-1$ and $A_i > 0, B_i > 0 \;\; (i = 1, 2, \cdots, n)$ satisfying the subcritical growth condition:

$$|I_i(u)| \le A_i |u|^{l_i} + B_i, \;\; u \in \mathbb{R}.$$

(H_2) $\limsup\limits_{|u|\to 0} \frac{F_i(t,u)}{|u|^p} = 0$, $(i = 0, 1, \cdots, n)$, holds almost everywhere uniformly for $t \in (s_i, t_{i+1}]$.

(H_3) There exists a superlinear exponent $\sigma > p$ such that the asymptotic inequality

$$\limsup_{|u|\to\infty} \frac{\sigma F_i(t,u) - f_i(t,u)\,u}{|u|^p} \le 0, \;\; (i = 0, 1, \cdots, n),$$

holds almost everywhere uniformly for $t \in (s_i, t_{i+1}]$.

(H_4) There are positivity subsets $\Omega_i \subset (s_i, t_{i+1}]$, $meas(\Omega_i) > 0$ where the nonlinear potential satisfies

$$\liminf_{|u|\to\infty} \frac{F_i(t,u)}{|u|^p} > 0, \;\; (i = 0, 1, \cdots, n),$$

uniformly for almost every $t \in \Omega_i$.

Remark 3.1. *(On conditions* (H_3) *and* (H_4)*). The requirement of "uniform convergence for almost every t" is sufficient for the following reasons:*

1) Measurability: For a fixed t, the limit in u, when uniform for almost every t, results in a limit function $f(t)$ *that is measurable in t. This is because the limit is ultimately taken over a sequence of measurable functions (in t), and uniformity outside a set of measure zero preserves measurability.*

2) Integrability control: Combined with the growth assumptions, this uniform convergence condition provides a uniformly integrable bound for sequences like $\frac{F_i(t,u_n)}{|u_n|^p}$. *This is crucial for applying convergence theorems (e.g., the Lebesgue dominated convergence theorem), allowing the interchange of limits and integrals, which is essential for establishing the continuity of the energy functional in our variational framework.*

Thus, "uniform convergence for almost every t" is a condition that is weaker than uniform convergence everywhere, yet sufficiently strong to ensure mathematical rigor in our analysis.

Lemma 3.1. *Given the validity of conditions* (H_1) *and* (H_3)*, the energy functional* φ_λ *fulfills the PS-condition.*

Proof. Consider a sequence $\{u_m\}_{m\in\mathbb{N}}$ in $E_0^{\alpha,p}$ with $\{\varphi_\lambda(u_m)\}_{m\in\mathbb{N}}$ bounded and $\lim_{m\to\infty}\varphi'_\lambda(u_m) = 0$. Consequently, we can find $N > 0$ satisfying the uniform bounds:

$$|\varphi_\lambda(u_m)| \le N, \quad \left\|\varphi'_\lambda(u_m)\right\|_* \le N, \ m \in \mathbb{N}, \tag{3.1}$$

with $\|\cdot\|_*$ being the dual space norm of $E_0^{\alpha,p}$.

We proceed by contradiction to prove the boundedness of $\{u_m\}$. Suppose $\{u_m\}$ is unbounded, then $\|u_m\| \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|}$, then $\|v_m\| = 1$. According to Lemma 2.4, assuming $v_m \rightharpoonup v_0$ in $E_0^{\alpha,p}$, then $v_m \to v_0$ in $C^1([0,T],\mathbb{R})$, as $m \to \infty$. Combining Lemma 2.4, Remark 2.1, and condition (H_1), it follows from (3.1) that there is a fixed constant $N_1 > 0$ satisfying

$$\begin{aligned}
\left(\frac{\sigma}{p}-1\right)M_0\|u_m\|^p &\le \frac{\sigma}{p}\mathcal{M}\left(\|u_m\|^p\right) - M\left(\|u_m\|^p\right)\|u_m\|^p \\
&= \sigma\varphi_\lambda(u_m) - \varphi'_\lambda(u_m)u_m - \sigma\mu\sum_{i=1}^{n} J_i(u_m(t_i)) + \mu\sum_{i=1}^{n} I_i(u_m(t_i))u_m(t_i) \\
&\quad + \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left(\sigma F_i(t,u_m(t)) - f_i(t,u_m(t))u_m(t)\right)dt \\
&\le N_1(1+\|u_m\|) + \sigma\mu\sum_{i=1}^{n}\left(\frac{A_iK^{l_i+1}\|u_m\|^{l_i+1}}{l_i+1} + B_iK\|u_m\|\right) \\
&\quad + \mu\sum_{i=1}^{n}\left(A_iK^{l_i+1}\|u_m\|^{l_i+1} + B_iK\|u_m\|\right) \\
&\quad + \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left(\sigma F_i(t,u_m(t)) - f_i(t,u_m(t))u_m(t)\right)dt.
\end{aligned}$$

From $\|u_m\| \to \infty$, it follows that

$$\begin{aligned}
\left(\frac{\sigma}{p}-1\right)M_0\|v_m\|^p \le &\frac{N_1(1+\|u_m\|)}{\|u_m\|^p} + \sigma\mu\sum_{i=1}^{n}\left(\frac{A_iK^{l_i+1}\|u_m\|^{l_i+1}}{(l_i+1)\|u_m\|^p} + \frac{B_iK\|u_m\|}{\|u_m\|^p}\right) \\
&+ \mu\sum_{i=1}^{n}\left(\frac{A_iK^{l_i+1}\|u_m\|^{l_i+1}}{\|u_m\|^p} + \frac{B_iK\|u_m\|}{\|u_m\|^p}\right) \\
&+ \lambda\frac{\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left(\sigma F_i(t,u_m(t)) - f_i(t,u_m(t))u_m(t)\right)dt}{\|u_m\|^p}.
\end{aligned} \tag{3.2}$$

From (H_3), it follows that there exists $\Omega_{i0} \subset (s_i, t_{i+1}]$, $meas(\Omega_{i0}) = 0$, such that the asymptotic inequality

$$\limsup_{|u|\to\infty}\frac{\sigma F_i(t,u) - f_i(t,u)u}{|u|^p} \le 0$$

holds uniformly for $t \in (s_i, t_{i+1}] \setminus \Omega_{i0}$. We assert that

$$\limsup_{m\to\infty} \frac{\sigma F_i\,(t, u_m\,(t)) - f_i\,(t, u_m\,(t))\,u_m\,(t)}{\|u_m\|^p} \le 0, \;\; t \in (s_i, t_{i+1}] \setminus \Omega_{i0}. \tag{3.3}$$

If it does not exist, then there exists $t_0 \in (s_i, t_{i+1}] \setminus \Omega_{i0}$ and a subsequence (relabeled as $\{u_m\}$) satisfying

$$\limsup_{m\to\infty} \frac{\sigma F_i\,(t_0, u_m\,(t_0)) - f_i\,(t_0, u_m\,(t_0))\,u_m\,(t_0)}{\|u_m\|^p} > 0. \tag{3.4}$$

Assuming the boundedness of $\{u_m\,(t_0)\}$, we can find $N_2 > 0$ satisfying $|u_m\,(t_0)| \le N_2$, $\forall m \in \mathbb{N}$. Since f_i is continuous on Ω_{i0}, it follows that

$$\frac{\sigma F_i\,(t_0, u_m\,(t_0)) - f_i\,(t_0, u_m\,(t_0))\,u_m\,(t_0)}{\|u_m\|^p} \le \frac{(\sigma + 1)\,N_2}{\|u_m\|^p} \to 0, \; m \to \infty,$$

which contradicts (3.4). Therefore, $\{u_m\,(t_0)\}$ admits a subsequence exhibiting divergence: $|u_m\,(t_0)| \to \infty$, $m \to \infty$. Hence,

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{\sigma F_i\,(t_0, u_m\,(t_0)) - f_i\,(t_0, u_m\,(t_0))\,u_m\,(t_0)}{\|u_m\|^p} \\
&= \limsup_{m\to\infty} \frac{\sigma F_i\,(t_0, u_m\,(t_0)) - f_i\,(t_0, u_m\,(t_0))\,u_m\,(t_0)}{|u_m\,(t_0)|^p} \cdot |v_m\,(t_0)|^p \\
&= \limsup_{m\to\infty} \frac{\sigma F_i\,(t_0, u_m\,(t_0)) - f_i\,(t_0, u_m\,(t_0))\,u_m\,(t_0)}{|u_m\,(t_0)|^p} \cdot \lim_{m\to\infty} |v_m\,(t_0)|^p \\
&\le 0.
\end{aligned}$$

This contradicts (3.4). Hence, (3.3) holds.

Through Eqs (3.2) and (3.3), it follows that $\limsup_{m\to\infty} \left(\frac{\sigma}{p} - 1\right) M_0 \|v_m\|^p \le 0$. The condition $\sigma > p$ implies the convergence $\|v_m\|^p \to 0$ as $m \to \infty$, which contradicts $\|v_m\| = 1$. As a result, $\{u_m\}$ is energy-bounded in $E_0^{\alpha,p}$.

The following proves that $u_m \to u$ in $E_0^{\alpha,p}$. Indeed, by reflexivity of $E_0^{\alpha,p}$, we may extract a subsequence (still denoted u_m) converging weakly: $u_m \rightharpoonup u$ in $E_0^{\alpha,p}$. Consequently, $u_m \to u$ in

$C^1([0,T],\mathbb{R})$. Equation (2.4) shows that

$$\begin{aligned}
&\left\langle \varphi'_\lambda(u_m) - \varphi'_\lambda(u), u_m - u \right\rangle \\
= \ & M(\|u_m\|^p) \int_0^T h(t)\left(\phi_p\left({}_0^C D_t^\alpha u_m(t)\right) - \phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)\left({}_0^C D_t^\alpha (u_m(t) - u(t))\right) dt \\
&+ (M(\|u_m\|^p) - M(\|u\|^p)) \int_0^T h(t)\phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha (u_m(t) - u(t))\right) dt \\
&+ M(\|u_m\|^p) \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)\left(\phi_p(u_m(t)) - \phi_p(u(t))\right)(u_m(t) - u(t))\, dt \\
&+ (M(\|u_m\|^p) - M(\|u\|^p)) \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)\phi_p(u(t))(u_m(t) - u(t))\, dt \\
&- \lambda \sum_{i=0}^n \int_{s_i}^{t_{i+1}} (f_i(t, u_m(t)) - f_i(t, u(t)))(u_m(t) - u(t))\, dt \\
&+ \mu \sum_{i=1}^n (I_i(u_m(t_i)) - I_i(u(t_i)))(u_m(t_i) - u(t_i)).
\end{aligned}$$

From Lemma 2.4 and the boundedness of $M(\|u_m\|^p) - M(\|u\|^p)$, it follows that

$$(M(\|u_m\|^p) - M(\|u\|^p)) \int_0^T h(t)\phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha (u_m(t) - u(t))\right) dt \to 0, m \to \infty,$$

$$(M(\|u_m\|^p) - M(\|u\|^p)) \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)\phi_p(u(t))(u_m(t) - u(t))\, dt \to 0, m \to \infty,$$

$$\left\langle \varphi'_\lambda(u_m) - \varphi'_\lambda(u), u_m - u \right\rangle \to 0,\ m \to \infty,$$

$$(I_i(u_m(t)) - I_i(u(t)))(u_m(t) - u(t)) \to 0,\ m \to \infty,$$

$$\int_{s_i}^{t_{i+1}} (f_i(t, u_m(t)) - f_i(t, u(t)))(u_m(t) - u(t))\, dt \to 0,\ m \to \infty.$$

According to [27, Eq (2.2)], we can find $c_1, c_2 > 0$ satisfying

$$\begin{aligned}
&M(\|u_m\|^p) \int_0^T h(t)\left(\phi_p\left({}_0^C D_t^\alpha u_m(t)\right) - \phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)\left({}_0^C D_t^\alpha (u_m(t) - u(t))\right) dt \\
&+ M(\|u_m\|^p) \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)\left(\phi_p(u_m(t)) - \phi_p(u(t))\right)(u_m(t) - u(t))\, dt \\
\geq &\begin{cases} lc_1 M(\|u_m\|^p)\left(\int_0^T h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^p dt + \sum_{i=0}^n \int_{s_i}^{t_{i+1}} a(t)|u_m(t) - u(t)|^p dt\right), p \geq 2, \\ c_2 M(\|u_m\|^p)\left(\int_0^T \frac{h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_m(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{2-p}} dt + \sum_{i=0}^n \int_{s_i}^{t_{i+1}} \frac{a(t)|u_m(t) - u(t)|^2}{(|u_m(t)| + |u(t)|)^{2-p}} dt\right), 1 < p < 2. \end{cases}
\end{aligned}$$

In summary, when $p \geq 2$, $\|u_m - u\| \to 0, m \to +\infty$.

In the case $1 < p < 2$, the Hölder's inequality implies

$$\begin{aligned}&\int_0^T h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^p dt\\ &\leq 2^{\frac{(p-1)(2-p)}{2}}\left(\int_0^T \frac{h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_m(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{2-p}}dt\right)^{\frac{p}{2}}(\|u_m\| + \|u\|)^{\frac{p(2-p)}{2}},\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)|u_m(t) - u(t)|^p dt\\ &\leq 2^{\frac{(p-1)(2-p)}{2}}\left(\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} \frac{a(t)|u_m(t) - u(t)|^2}{(|u_m(t)| + |u(t)|)^{2-p}}dt\right)^{\frac{p}{2}}(\|u_m\| + \|u\|)^{\frac{p(2-p)}{2}}.\end{aligned}$$

Consequently,

$$\begin{aligned}&M(\|u_m\|^p)\left[\int_0^T h(t)\left(\phi_p\left({}_0^C D_t^\alpha u_m(t)\right) - \phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)\left({}_0^C D_t^\alpha (u_m(t) - u(t))\right)dt\right.\\ &\left.+\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)\left(\phi_p(u_m(t)) - \phi_p(u(t))\right)(u_m(t) - u(t))\,dt\right]\\ &\geq \frac{c_2 M(\|u_m\|^p)}{2^{\frac{(p-1)(2-p)}{2}} \times (\|u_m\| + \|u\|)^{2-p}} \times \left[\left(\int_0^T h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{2}{p}}\right.\\ &\left.+\left(\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)|u_m(t) - u(t)|^p dt\right)^{\frac{2}{p}}\right]\\ &\geq \frac{c_2 M(\|u_m\|^p)}{2^{\frac{(p-1)(2-p)}{2}} \times \max\left\{2^{\frac{2}{p}-1}, 1\right\}} \times \frac{\|u_m - u\|^2}{(\|u_m\| + \|u\|)^{2-p}}.\end{aligned}$$

To summarize, $\|u_m - u\| \to 0, m \to +\infty$. That is, $u_m \to u$ in $E_0^{\alpha,p}$. □

Theorem 3.1. *If conditions* (H_1)–(H_4) *hold and the constant* $\Delta = \frac{M_0\delta^p}{pK^p} - \mu\sum_{i=1}^{n}\left(\frac{A_i}{l_i+1}\delta^{l_i+1} + B_i\delta\right) > 0$*, then for each* $\lambda \in \left(0, \frac{\Delta}{kT\delta^p}\right)$*, problem (1.1) possesses at least two distinct weak solutions.*

Proof. Denote by B_r the open ball of radius r centered at the origin in the space $E_0^{\alpha,p}$. Here, ∂B_r stands for the boundary of B_r, and $\overline{B}_r$ for its closure. Standard arguments demonstrate the weak closedness and boundedness of $\overline{B}_{\frac{\delta}{K}}$. Through meticulous verification, we demonstrate that $\varphi_\lambda(u)$ is w.l.s.c. in $E_0^{\alpha,p}$. According to Lemma 2.7, $\varphi_\lambda(u)$ attains a local minimizer u_0 within $\overline{B}_{\frac{\delta}{K}}$, that is, $\varphi_\lambda(u_0) \leq \varphi_\lambda(0) = 0$.

By hypothesis (H_2), when $k > 0$, $i = 0, 1, \cdots, n$, one can find $\delta \in (0, k)$ ensuring that for a.e. $t \in (s_i, t_{i+1}]$, $u \in \mathbb{R}$ satisfying $|u| \leq \delta$, the following holds:

$$|F_i(t, u)| \leq k|u|^p. \tag{3.5}$$

Assume $\|u\| \leq \frac{\delta}{K}$. By Lemma 2.6, we have $\|u\|_\infty \leq \delta$. Then for $u \in \partial B_r, (r \leq \frac{\delta}{K})$, combining Lemma 2.6, (H_1), and (3.5), we obtain

$$\begin{aligned}\varphi_\lambda(u) &\geq \frac{1}{p}\mathcal{M}(\|u\|^p) - \lambda kTK^p\|u\|^p - \mu\sum_{i=1}^{n}\left(\frac{A_iK^{l_i+1}}{l_i+1}\|u\|^{l_i+1} + B_iK\|u\|\right)\\ &\geq \left(\frac{M_0}{p} - \lambda kTK^p\right)r^p - \mu\sum_{i=1}^{n}\left(\frac{A_iK^{l_i+1}}{l_i+1}r^{l_i+1} + B_iKr\right).\end{aligned}$$

Then, for all $u \in \partial B_{\frac{\delta}{K}}$, we have

$$\varphi_\lambda(u) \geq \left(\frac{M_0}{p} - \lambda kTK^p\right)\frac{\delta^p}{K^p} - \mu\sum_{i=1}^{n}\left(\frac{A_i}{l_i+1}\delta^{l_i+1} + B_i\delta\right) = N_\lambda.$$

Given that $\lambda \in \left(0, \frac{\Delta}{kT\delta^p}\right)$, it follows that $\varphi_\lambda(u) = N_\lambda > 0 \geq \varphi_\lambda(u_0)$ holds for all $u \in \partial B_{\frac{\delta}{K}}$. Therefore, $\inf_{u\in\partial B_{\frac{\delta}{K}}}\varphi_\lambda(u) > \varphi_\lambda(u_0)$.

For implementation purposes, each segment $i = 0, 1, \cdots, n$ under constraints (H_3)–(H_4) with continuous f_i generates design parameters $\rho > \frac{(\sigma-p)M^0K^p}{p^2\lambda}$, $N_3 > 0$ and operational domain Ω_i (positive measure) satisfying

$$|F_i(t,u)| \geq \frac{p\rho}{\sigma-p}|u|^p - N_3,\ t \in \Omega_i, u \in \mathbb{R}. \tag{3.6}$$

Select $u_0(t) \in E_0^{\alpha,p}$ with $\|u_0\| \leq K$ and $\int_{\Omega_i}|u_0(t)|^p dt = 1$. For $\xi > 0$, it follows from (3.6) and (H_1) that,

$$\begin{aligned}\varphi_\lambda(\xi u_0) &\leq \frac{1}{p}\mathcal{M}(\|\xi u_0\|^p) - \lambda\sum_{i=0}^{n}\int_{\Omega_i}F_i(t,\xi u_0(t))\,dt + \mu\sum_{i=1}^{n}\int_0^{\xi u_0(t_i)}I_i(s)\,ds\\ &\leq \frac{M^0\xi^p}{p}\|u_0\|^p - \lambda(n+1)\frac{p\rho\xi^p}{\sigma-p} + \lambda\sum_{i=0}^{n}N_3 meas(\Omega_i)\\ &\quad + \mu\sum_{i=1}^{n}\left(\frac{A_iK^{l_i+1}\xi^{l_i+1}}{l_i+1}\|u_0\|^{l_i+1} + B_i\xi K\|u_0\|\right)\\ &\leq \left(\frac{M^0K^p}{p} - \lambda(n+1)\frac{p\rho}{\sigma-p}\right)\xi^p + \lambda\sum_{i=0}^{n}N_3 meas(\Omega_i)\\ &\quad + \mu\sum_{i=1}^{n}\left(\frac{A_iK^{2(l_i+1)}\xi^{l_i+1}}{l_i+1} + B_i\xi K^2\right).\end{aligned}$$

Noting that $\rho > \frac{(\sigma-p)M^0K^p}{p^2\lambda(n+1)}$, it follows that $\varphi_\lambda(\xi u_0) \to -\infty$, $\xi \to \infty$. A rigorous analysis demonstrates the existence of $u_1 > 0$ with $\|u_1\| > \frac{\delta}{K}$ that satisfies $\inf_{u\in\partial B_{\frac{\delta}{K}}}\varphi_\lambda(u) > \varphi_\lambda(u_1)$. According to Lemmas 2.8 and 3.1, it is straightforward to see that there exists $u_2 \in E_0^{\alpha,p}$ satisfying $\varphi'_\lambda(u_2) = 0$ and $\varphi_\lambda(u_2) > \max\{\varphi_\lambda(u_0), \varphi_\lambda(u_1)\}$. In conclusion, u_0 and u_2 are two distinct weak solutions to problem (1.1). □

In what follows, we examine the scenario in problem (1.1) with the decomposition $f_i(t,u) = f_{i1}(t,u) + f_{i2}(t,u)$. In which $f_{i1}(t,u)$ exhibits superlinear growth as $|u| \to \infty$, whereas $f_{i2}(t,u)$ shows sublinear behavior at infinity. We formally define the integral functions $F_{i1}(t,u) = \int_0^u f_{i1}(t,s)\,ds$

and $F_{i2}(t,u) = \int_0^u f_{i2}(t,s)\,ds$ for subsequent analysis. The rigorous mathematical assumptions are enumerated below:

$(H_2{}')$ Given $i = 0, 1, \cdots, n$, one can find $\sigma > p$ for which $\lim\limits_{|u|\to\infty} \frac{F_{i1}(t,u)}{|u|^{\sigma}} = \infty$, uniformly in $t \in (s_i, t_{i+1}]$;

$(H_3{}')$ For each $i = 0, 1, \cdots, n$, one can find positive numbers $k_0 > 0$, $D > 0$ for which $f_{i1}(t,u)u - \sigma F_{i1}(t,u) \geq -k_0 u^p$ is satisfied whenever $t \in (s_i, t_{i+1}]$, $|u| \geq D$;

$(H_4{}')$ Given $i = 0, 1, \cdots, n$, one can find $1 < \tau < p$ and $\mathcal{K}_1 \in L^1((s_i, t_{i+1}], \mathbb{R}^+)$ for which $|f_{i2}(t,u)| \leq \mathcal{K}_1(t)|u|^{\tau-1}$ is satisfied whenever $t \in (s_i, t_{i+1}]$, $u \in \mathbb{R}$;

(H_5) Given $i = 0, 1, \cdots, n$, one can find $\mathcal{K}_2 \in C^1((s_i, t_{i+1}], \mathbb{R}^+)$ for which $F_{i2}(t,u) \geq \mathcal{K}_2(t)|u|^{\tau}$ is satisfied whenever $t \in (s_i, t_{i+1}]$, $u \in \mathbb{R}$.

Lemma 3.2. *The validity of conditions* (H_1), $(H_2{}')$–$(H_4{}')$ *implies that* φ_λ *fulfills the PS-condition.*

Proof. Our first step establishes the boundedness of $\{u_m\}_{m\in\mathbb{N}} \subset E_0^{\alpha,p}$. We verify this by contradiction. Assume that $\|u_m\| \to \infty, m \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|}$, then $\|v_m\| = 1$. According to Lemma 2.4, if $v_m \rightharpoonup v_0$ in $E_0^{\alpha,p}$, then $v_m \to v_0$ in $C^1([0,T], \mathbb{R})$ as $m \to \infty$. By virtue of hypothesis $(H_4{}')$, we obtain the growth conditions:

$$|f_{i2}(t,u)u| \leq \mathcal{K}_1(t)|u|^{\tau},\ |F_{i2}(t,u)| \leq \frac{1}{\tau}\mathcal{K}_1(t)|u|^{\tau}. \tag{3.7}$$

The subsequent analysis divides into two mutually exclusive cases: the trivial case $v_0 \equiv 0$ and the nontrivial case $v_0 \neq 0$.

Case 1: $v_0 \equiv 0$. In view of assumption $(H_3{}')$ and the continuity of f_i, we can find a constant $k_1 > 0$ for which the following inequality holds:

$$f_{i1}(t,u)u - \sigma F_{i1}(t,u) \geq -k_0 u^p - k_1, \quad t \in (s_i, t_{i+1}],\ u \in \mathbb{R}. \tag{3.8}$$

Therefore, from (3.1), (3.7), and (3.8), we obtain

$$\begin{aligned}
o(1) = \frac{\sigma N + N\|u_m\|}{\|u_m\|^p} &\geq \frac{\sigma\varphi_\lambda(u_m) - \varphi'_\lambda(u_m)u_m}{\|u_m\|^p} \\
&\geq \left(\frac{\sigma}{p} - 1\right)M_0 + \frac{\mu}{\|u_m\|^p}\sum_{i=1}^{n}\left(\sigma\int_0^{u_m(t_i)} I_i(s)\,ds - I_i(u_m(t_i))u_m(t_i)\right) \\
&\quad + \frac{\lambda}{\|u_m\|^p}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}(f_i(t,u_m(t))u_m(t) - \sigma F_i(t,u_m(t)))\,dt \\
&\geq \left(\frac{\sigma}{p} - 1\right)M_0 - \frac{\mu}{\|u_m\|^p}\sum_{i=1}^{n}\left(\left(\frac{A_i\sigma}{l_i+1} + A_i\right)\|u_m\|_\infty^{l_i+1} + (\sigma+1)B_i\|u_m\|_\infty\right) \\
&\quad - \frac{\lambda}{\|u_m\|^p}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left(k_0|u_m|^p + k_1 + \left(1 + \frac{\sigma}{\tau}\right)\mathcal{K}_1(t)|u_m|^{\tau}\right)dt \\
&\geq \left(\frac{\sigma}{p} - 1\right)M_0 - \frac{\mu}{\|u_m\|^p}\sum_{i=1}^{n}\left(\left(\frac{A_i\sigma}{l_i+1} + A_i\right)K^{l_i+1}\|u_m\|^{l_i+1} + (\sigma+1)B_iK\|u_m\|\right) \\
&\quad - \frac{\lambda T}{\|u_m\|^p}(k_0K^p\|u_m\|^p + k_1) - \frac{\lambda}{\|u_m\|^p}\left(1 + \frac{\sigma}{\tau}\right)\|\mathcal{K}_1\|_{L^1}K^{\tau}\|u_m\|^{\tau} \\
&\geq \left(\frac{\sigma}{p} - 1\right)M_0 - \lambda T k_0 K^p,\ m \to \infty.
\end{aligned}$$

This indicates that $\{u_m\}$ is bounded in $E_0^{\alpha,p}$.

Case 2: $v_0 \neq 0$. Let $\Omega'_i = \{t \in (s_i, t_{i+1}] : |v_0(t)| > 0\}$, $i = 0, 1, \cdots, n$, then $meas(\Omega'_i) > 0$. Observe that $\|u_m\| \to \infty$, $(m \to \infty)$ together with the identity $|u_m(t)| = |v_m(t)| \cdot \|u_m\|$ necessarily leads to $|u_m(t)| \to \infty, (m \to \infty)$ when $t \in \Omega'_i$. We therefore deduce from the combination of (3.1) and (3.7) that

$$\begin{aligned}
&\sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} F_{i1}(t, u_m(t))\, dt \\
&= -\frac{1}{\lambda}\varphi_\lambda(u_m) + \frac{1}{p\lambda}\mathcal{M}(\|u_m\|^p) + \frac{\mu}{\lambda}\sum_{i=1}^{n}\int_0^{u_m(t_i)} I_i(s)\, ds - \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_{i2}(t, u(t))\, dt \\
&\leq \frac{N}{\lambda} + \frac{M^0}{p\lambda}\|u_m\|^p + \frac{\mu}{\lambda}\sum_{i=1}^{n}\left(\frac{A_i}{l_i+1}\|u_m\|_\infty^{l_i+1} + B_i\|u_m\|_\infty\right) + \frac{1}{\tau}\int_0^T \mathcal{K}_1(t)\, |u_m(t)|^\tau dt \\
&\leq \frac{N}{\lambda} + \frac{M^0}{p\lambda}\|u_m\|^p + \frac{\mu}{\lambda}\sum_{i=1}^{n}\left(\frac{A_i K^{l_i+1}}{l_i+1}\|u_m\|^{l_i+1} + B_i K\, \|u_m\|\right) + \frac{K^\tau}{\tau}\|\mathcal{K}_1\|_{L^1}\|u_m\|^\tau.
\end{aligned}$$

Observing the parameter constraints $\sigma > p$, where $1 \leq l_i + 1 < p$ and simultaneously $1 < \tau < p$, we consequently derive

$$\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} \frac{F_{i1}(t, u_m(t))}{\|u_m\|^\sigma} dt \leq o(1), \quad m \to \infty. \tag{3.9}$$

Nevertheless, through an application of Fatou's lemma combined with hypothesis $(H_2{}')$, one obtains

$$\begin{aligned}
\lim_{m\to\infty}\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} \frac{F_{i1}(t, u_m(t))}{\|u_m\|^\sigma} dt &\geq \lim_{m\to\infty}\sum_{i=0}^{n}\int_{\Omega'_i} \frac{F_{i1}(t, u_m(t))}{\|u_m\|^\sigma} dt \\
&= \lim_{m\to\infty}\sum_{i=0}^{n}\int_{\Omega'_i} \frac{F_{i1}(t, u_m(t))}{|u_m(t)|^\sigma} |v_m(t)|^\sigma dt \\
&= \infty.
\end{aligned}$$

This contradicts (3.9). In summary, the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in $E_0^{\alpha,p}$. Subsequently, employing an analogous approach to that used in proving Lemma 3.1, we establish the convergence $\|u_m - u\| \to 0$ in the space $E_0^{\alpha,p}$ when $m \to \infty$. □

Theorem 3.2. *Under hypotheses* (H_1), $(H_2{}')$–$(H_4{}')$, *and* (H_5), *if the functions* $I_i(u)(i = 1, 2, \cdots, n)$ *and* $f_i(t, u)(i = 0, 1, \cdots, n)$ *are odd with respect to u, then (1.1) possesses an infinite number of weak solutions.*

Proof. Clearly, φ_λ is an even function and $\varphi_\lambda(0) = 0$. Let $\{e_m\}_{m=1}^{\infty}$ be an orthonormal basis of $E_0^{\alpha,p}$, i.e., $\left\|e_q\right\| = 1$, $\left\langle e_q, e_{q'}\right\rangle = 0$, $1 \leq q \neq q'$. For each natural number m, let E_m be the linear span of the vectors $e_1, e_2, \cdots, e_m$, and let S_m denote the set of all unit vectors in E_m. So, for every vector $u \in E_m$, we can find real numbers $\theta_1, \theta_2, \cdots, \theta_m$ satisfying

$$u(t) = \sum_{j=1}^{m}\theta_j e_j(t),\ t \in [0, T]. \tag{3.10}$$

In other words,

$$\begin{aligned}\|u\|^{p} &= \int_{0}^{T} h(t)\left|{}_{0}^{C}D_{t}^{\alpha}u(t)\right|^{p}dt + \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)\,|u(t)|^{p}dt \\ &= \sum_{j=0}^{m}\theta_j^{p}\left(\int_{0}^{T} h(t)\left|{}_{0}^{C}D_{t}^{\alpha}e_j(t)\right|^{p}dt + +\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)\left|e_j(t)\right|^{p}dt\right) \\ &= \sum_{j=0}^{m}\theta_j^{p}\left\|e_j\right\|^{p} = \sum_{j=0}^{m}\theta_j^{p}.\end{aligned} \tag{3.11}$$

Conversely, hypothesis (H_5) guarantees that for each open bounded interval Π_i contained in $(s_i, t_{i+1}]$ (where i ranges from 0 to n), we may select a uniform lower bound $\mathcal{K}_3 > 0$ ensuring

$$F_{i2}(t, u(t)) \geq \mathcal{K}_2(t)\,|u(t)|^{\tau} \geq \mathcal{K}_3|u(t)|^{\tau} \tag{3.12}$$

holds throughout $\Pi_i \times \mathbb{R}$. Regarding the open sets Π_i previously introduced, the growth condition $(H_2{}')$ requires positive coefficients $\mathcal{K}_4, \mathcal{K}_5$ with the property that

$$F_{i1}(t, u(t)) \geq \mathcal{K}_4|u|^{\sigma} - \mathcal{K}_5 \tag{3.13}$$

holds uniformly for all $t \in \Pi_i$ and $u \in \mathbb{R}$. Therefore, for $\forall u \in S_m$, it follows from (3.11)–(3.13) that

$$\begin{aligned}\varphi_{\lambda}(\eta u) &= \frac{1}{p}\mathcal{M}\|\eta u\|^{p} - \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t, \eta u(t))\,dt + \mu\sum_{i=1}^{n}\int_{0}^{\eta u(t_i)} I_i(s)\,ds \\ &\leq \frac{M^{0}}{p}\|\eta u\|^{p} - \lambda\sum_{i=0}^{n}\int_{\Pi_i} F_i(t, \eta u(t))\,dt + \mu\sum_{i=1}^{n}\int_{0}^{\eta u(t_i)} I_i(s)\,ds \\ &\leq \frac{M^{0}\eta^{p}}{p}\|u\|^{p} - \lambda\eta^{\sigma}\mathcal{K}_4\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\theta_j e_j(t)\right|^{\sigma}dt + \lambda\mathcal{K}_5 T \\ &\quad - \lambda\eta^{\tau}\mathcal{K}_3\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\theta_j e_j(t)\right|^{\tau}dt + \mu\sum_{i=1}^{n}\left(\frac{A_i K^{l_i+1}\eta^{l_i+1}}{l_i+1}\|u\|^{l_i+1} + B_i K\eta\,\|u\|\right) \\ &= \frac{M^{0}\eta^{p}}{p} - \lambda\eta^{\sigma}\mathcal{K}_4\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\theta_j e_j(t)\right|^{\tau}dt + \lambda\mathcal{K}_5 T \\ &\quad - \lambda\eta^{\tau}\mathcal{K}_3\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\theta_j e_j(t)\right|^{\tau}dt + \mu\sum_{i=1}^{n}\left(\frac{A_i K^{l_i+1}\eta^{l_i+1}}{l_i+1} + B_i K\eta\right).\end{aligned}$$

Furthermore, one may readily establish the positivity: $\sum_{i=0}^{n}\int_{\Pi_i}\left|\sum_{j=1}^{m}\theta_j e_j(t)\right|^{\tau}dt > 0$. Noting the parameter constraints $1 < \tau < p$, $\sigma > p$, and $1 \leq l_i + 1 < p$ (for all $i = 1, \cdots, n$), we can find positive numbers ξ, ω guaranteeing

$$\varphi_{\lambda}(\omega u) < -\xi,\ u \in S_m. \tag{3.14}$$

Let $S_m^\omega = \{\omega u : u \in S_m\}$, $\Theta = \left\{(\omega_1, \omega_2, \cdots, \omega_m) \in \mathbb{R}^m : \sum_{j=1}^{m} \theta_j^p < \omega^p\right\}$. From (3.14), it follows that $\varphi_\lambda(u) < -\xi,\ u \in S_m^\omega$. Combined with the even function $\varphi_\lambda \in C^1\left(E_0^{\alpha,p}, \mathbb{R}\right)$, we obtain $S_m^\omega \subset \varphi_\lambda^{-\xi} \in \Sigma$.

As a direct consequence of (3.10) and (3.11), we deduce an odd C^1-smooth boundary mapping $\Psi : \partial\Theta \to S_m^\omega$ with homomorphic properties. From the genus theory, the energy level set satisfies:

$$\gamma\left(\varphi_\lambda^{-\xi}\right) \geq \gamma\left(S_m^\omega\right) = m. \tag{3.15}$$

Therefore, $\varphi_\lambda^{-\xi} \in \Sigma_m$, which implies $\Sigma_m \neq \phi$. Let $z_m = \inf_{U\in\Sigma_m}\sup_{u\in U}\varphi_\lambda(u)$. The minimax sequence satisfies $-\infty \leq z_m < -\xi < 0$, as guaranteed by (3.15) and the lower semicontinuity of φ_λ in $E_0^{\alpha,p}$. In other words, $\forall m \in \mathbb{N}$, $z_m \in \mathbb{R}^-$. Consequently, an application of Lemma 2.9 combined with Remark 2.2 shows that φ_λ admits an infinite sequence of nontrivial critical points. In other words, (1.1) possesses an infinite number of nontrivial solutions in the weak sense. □

Example 3.1. *Let* $\alpha = 0.6, h(t) = T = 1, n = 1, p = 3, \lambda = \frac{1}{4}, \mu = 2$. *We now study an FBVPs:*

$$\begin{cases} M\left(\|u\|^3\right)\left({}_tD_1^{0.6}\left(\phi_3\left({}_0^CD_t^{0.6}u(t)\right)\right)\right) + a(t)\phi_3(u(t)) \\ = \dfrac{u^5(1+\sin t)}{4} + \dfrac{u^{\frac{1}{3}}(2+\cos t)}{4}, t \in (s_i, t_{i+1}], i = 0, 1, \\ \Delta\left(M\left(\|u(t_1)\|^3\right)\right){}_tD_1^{-0.4}\left(\phi_3({}_0^CD_t^{0.6}u)\right)(t_1) = 2I_1(u(t_1)), \\ M\left(\|u\|^3\right){}_tD_1^{-0.4}\left(\phi_3({}_0^CD_t^{0.6}u)\right)(t) = M\left(\left\|u(t_1^+)\right\|^3\right){}_tD_1^{-0.4}\left(\phi_3({}_0^CD_t^{0.6}u)\right)(t_1^+), t \in (t_1, s_1], \\ M\left(\left\|u(s_1^-)\right\|^3\right){}_tD_1^{-0.4}\left(\phi_3({}_0^CD_t^{0.6}u)\right)(s_1^-) = M\left(\left\|u(s_1^+)\right\|^3\right){}_tD_1^{-0.4}\left(\phi_3({}_0^CD_t^{0.6}u)\right)(s_1^+), \\ u(0) = u(1) = 0, \end{cases} \tag{3.16}$$

where $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$. *Let us select* $M(u) = 5 + \frac{u}{1+u}, u \in \mathbb{R}^+, a(t) = \ln\left(1 + t^2\right)$, *where* $t \in [0, 1]$. *Then, we have* $a_0 = 0 \leq a(t) \leq \ln 2 = a^0$. *Choose* $I_1(u) = \sin u$, *and there exist* $A_1 = B_1 = 2, l_1 = \frac{1}{2}$, *such that condition* (H_1) *holds. Integration gives the potential terms:* $F_{i1} = \frac{u^6(1+\sin t)}{6}, F_{i2} = \frac{3u^{\frac{4}{3}}(2+\cos t)}{4}$. *Through calculation, when we set* $k_0 = 2, D = 3, \sigma = 4$, *it can be verified that both conditions* (H_2') *and* (H_3') *are fulfilled. With the choices* $\tau = \frac{4}{3}, \mathcal{K}_1(t) = \frac{t}{3} + 5, \mathcal{K}_2(t) = \frac{1+t}{4}$, *hypotheses* (H_4') *and* (H_5) *hold. Trivially, both* $f_i(t, u)$ *and* $I_1(u)$ *possess odd parity in the u-variable. Whence, the full satisfaction of Theorem 3.2's assumptions yields the existence of an unbounded sequence of nontrivial weak solutions to (3.16).*

4. Conclusions

This study has established the existence and multiplicity of weak solutions for a class of Kirchhoff-type fractional p-Laplacian boundary value problems incorporating mixed instantaneous and non-instantaneous impulses. The main contributions are twofold. First, by employing the mountain pass theorem, we have proven the existence of at least two nontrivial weak solutions under growth assumptions that are strictly weaker than the classical Ambrosetti-Rabinowitz super-p-linear condition (Theorem 3.1). Second, utilizing the genus theory in critical point theory, we have demonstrated the existence of multiple weak solutions when the nonlinear term exhibits a combined superlinear and sublinear growth structure (Theorem 3.2).

The novelty of this study is primarily reflected in the following aspects: First, a systematic weakened variational framework suitable for mixed impulse problems has been constructed, proposing growth assumptions that are more lenient than the classical Ambrosetti-Rabinowitz condition. Second, an analytical approach has been developed to simultaneously handle concave-convex nonlinearities and weakened super-p-growth conditions, establishing a more universal theory for the existence and multiplicity of solutions. Third, it breaks through the dependence of traditional variational methods on strong growth conditions, opening up new avenues for studying impulse problems under non-standard growth conditions. Compared with the existing literature [22], this paper not only significantly relaxes the constraints on the growth conditions of nonlinear terms but also achieves substantial breakthroughs in the theoretical framework and research methodology.

From an application perspective, the theoretical model established in this work shows promising potential for describing complex dynamical systems characterized by instantaneous mutations and memory-dependent after-effects. Typical application scenarios include simulating the mechanical response and recovery process of composite materials under impact loads, modeling the pharmacokinetic behavior of rapid drug injection coupled with sustained release, and analyzing self-recovering circuit systems subject to transient disturbances. These potential connections provide viable pathways for translating the theoretical findings into engineering practice.

Looking ahead, several promising research directions emerge from this work. On the theoretical front, the present framework could be extended to problems involving variable exponents or more general fractional operators. Investigating critical growth conditions also represents a meaningful direction, though this would likely require developing new analytical tools. From an applied perspective, we recommend focusing on two key tasks: First, developing effective numerical computation methods (such as finite element methods) to visually demonstrate the dynamic characteristics of weak solutions and validate theoretical findings; second, incorporating stochastic disturbances by establishing random impulse models to more accurately describe uncertainties in practical systems. These extended investigations would effectively compensate for the limitations of purely theoretical analysis and facilitate the translation of research outcomes into practical applications.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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