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Research article

Analysis of solvability and representation of general solutions for anti-Hermitian constrained quaternion matrix equations

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Abstract: This paper addresses the constrained system of quaternion matrix equations incorporating anti-Hermitian properties, driven by the significance of symmetric solutions in diverse applications. Solvability conditions are determined via rank equalities and relationships derived from Moore–Penrose inverses and induced projectors. Explicit solution representations are obtained, utilizing the Moore–Penrose inverse and projections. The originality of the results is established through a novel technique based on quaternion row-column determinant theory, supported by a numerical validation. This approach retains its innovative character even when extended to complex matrix equations using conventional determinants.

Keywords: quaternion matrix; matrix equation; generalized inverse; anti-Hermitian solution; symmetric solution; noncommutative determinant; Cramer's rule; mathematical operators
Mathematics Subject Classification: 15A09, 15A15, 15A24, 15B33

1. Introduction

We consider the set of matrices with entries that belong to the quaternion skew field determined by $\mathbb{H} = \{t_0 + t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ t_0, t_1, t_2, t_3 \in \mathbb{R}\}$. For an arbitrary quaternion $t = t_0 + t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k} \in \mathbb{H}$, its conjugate quaternion is defined by $\bar{t} = t_0 - t_1\mathbf{i} - t_2\mathbf{j} - t_3\mathbf{k}$. As usual, $\mathbb{C}$ and $\mathbb{R}$ stand for the complex and real number fields, respectively. For any $A \in \mathbb{H}^{m\times n}$, the matrix $A^* \in \mathbb{H}^{n\times m}$ indicates the conjugate transpose of A. The matrix $A \in \mathbb{H}^{n\times n}$ is called Hermitian when $A^* = A$, and it is anti-Hermitian when $A^* = -A$. Sometimes, instead of referring to "anti-Hermitian", the term "skew-Hermitian" is used, especially when describing quaternion matrices. Note that for any $A \in \mathbb{H}^{m\times n}$, the half of the sum of it and its conjugate transpose $\check{A} := \frac{1}{2}(A + A^*)$ is Hermitian, and the half of their difference

$$\hat{A} := \frac{1}{2}(A - A^*) = -\hat{A}^*, \tag{1.1}$$

yields its corresponding skew-Hermitian matrix, and $A = \check{A} + \hat{A}$ holds. Both symbols $r(A)$ and $rank\,(A)$ denote the rank of A. The identity matrix of order n is denoted by I_n.

Definition 1.1. For a given matrix $A \in \mathbb{H}^{m\times n}$, the Moore–Penrose (MP-) inverse of A, denoted by $A^\dagger \in \mathbb{H}^{n\times m}$, is the unique matrix satisfying the Penrose conditions:

$$(1)\ AA^\dagger A = A,\ \ (2)\ A^\dagger AA^\dagger = A^\dagger,\ \ (3)\ (AA^\dagger)^* = AA^\dagger,\ \ (4)\ (A^\dagger A)^* = A^\dagger A.$$

Matrices satisfying the condition (2) are called generalized inverses of A.

The MP-inverse produces projectors, $L_A = I_n - A^\dagger A$ and $R_A = I_m - AA^\dagger$, that satisfy the following conditions:

$$L_A = (L_A)^* = (L_A)^2 = L_A^\dagger,\ \ R_A = (R_A)^2 = (R_A)^* = R_A^\dagger,\ \ L_{A^*} = R_A,\ \ R_{A^*} = L_A.$$

The introduction of quaternionic analysis has significantly expanded the scope of applied mathematical fields [13, 15, 40]. Quaternions are fundamental in the description of three-dimensional rotations and find widespread application in computer graphics, robotics, navigation, quantum physics, mechanics, and signal processing, as extensively documented in [1, 18]. The growing application of quaternions in various practical fields has motivated extensive research into anti-Hermitian solutions for quaternion systems of matrix equations. Furthermore, the determination of skew-Hermitian solutions for matrix equations under symmetry constraints is of critical importance in quantum mechanics [27], control theory [3], and Lie algebra [29]. Numerous problems in various engineering disciplines, including linear descriptor systems, system design, singular system control [7, 9, 38], perturbation theory [23], feedback control [39], and color image data transmission [16], necessitate the solution of Sylvester-type matrix equations. For instance, Bai [4] investigated iterative methods for solving the Sylvester equation $A_1X + XA_2 = B$. Roth [37] later provided the consistency conditions governing the solvability of its generalized form $A_1X + YA_2 = B$, while subsequent work in [5] derived its general solution structure.

In [42], Wang and He explored the general solution to the system

$$A_1X + YB_1 = C_1,\ \ A_2Z + YB_2 = C_2. \tag{1.2}$$

Lee and Vu [22] studied some solvability conditions for simultaneous solutions of (1.2). Lin and Wei [24] evaluated the condition number of (1.2). The constraint solutions to (1.2) are explored by Wang et al. in [43]. Some practical, necessary, and sufficient conditions for

$$A_1X_1 + Z_1B_1 = C_1,\ \ A_2Z_1 + X_2B_2 = C_2,$$

to have a general solution are presented by Wang and He in [17].

The generalized Sylvester matrix equation

$$AXB + CYD = E, \tag{1.3}$$

has been extensively researched. Baksalari and Kala [6] provided a comprehensive solution to the complex equation (1.3) employing the MP-inverse. This result was later extended and developed into quaternion equations by Wang [41, 44].

Liu [26] evaluated the Hermitian solution to the Sylvester-type equation

$$AXA^* + BYB^* = C \tag{1.4}$$

over $\mathbb{C}$ and expressed it in terms of generalized inverses. A nonlinear Hermitian expression was also explored in [30]. Building on the explicit solution representation using generalized inverses, the authors in [19] developed Cramer's rules for obtaining Hermitian solutions to Eq (1.4) in the quaternion skew field $\mathbb{H}$, employing row-column noncommutative determinant techniques. One can argue that a direct method for solving matrix equations can be provided by using generalized inverses. Another established strategy involves the application of various iterative methods. Using iterative approaches, [10] investigated Eq (1.4) in its findings. In [14], Hajarian developed an algorithm for determining the solution to the system

$$\begin{aligned} A_1XB_1 + C_1YD_1 &= E_1, \\ A_2ZB_2 + C_2YD_2 &= E_2. \end{aligned}$$

Alternative iterative approaches for solving coupled matrix equations have been extensively investigated, as documented in [11, 12]. Some of the latest developments in solving Sylvester-type matrix equations and quaternion matrix theory can be found in [31, 32, 35, 36]. In [33], researchers investigated the system

$$\begin{aligned} A_1UA_1^*+B_1VB_1^* = C_1,\ C_1 = -C_1^* \\ A_2WA_2^*+B_2VB_2^* = C_2,\ C_2 = -C_2^*, \end{aligned} \tag{1.5}$$

while determining the necessary and sufficient conditions governing its consistency.

Motivated by the aforementioned research and the wide-ranging applications of generalized Sylvester matrix equations in various applied fields, this paper focuses on investigating the constrained anti-Hermitian solutions to Sylvester-type matrix equations:

$$\begin{aligned} &A_3X = C_3, XB_3 = C_4,\ X^* = -X, \\ &A_4Y = C_5, YB_4 = C_6,\ Y^* = -Y, \\ &A_5Z = C_7, ZB_5 = C_8,\ Z^* = -Z, \\ &A_1XA_1^*+B_1YB_1^* = C_1,\ C_1 = -C_1^* \\ &A_2ZA_2^*+B_2YB_2^* = C_2,\ C_2 = -C_2^*, \end{aligned} \tag{1.6}$$

over the quaternion skew field $\mathbb{H}$. The principal aim of this work is to derive the complete general solution for Eq (1.6) under solvable conditions. The general solution to the equation

$$A_4X - (A_4X)^* + B_4YB_4^* + C_4ZC_4^* = D_4,\ D_4 = -D_4^*,\ Y = -Y^*,\ Z = -Z^*, \tag{1.7}$$

plays a fundamental role in deriving the main findings of this paper over $\mathbb{H}$ with anti-Hermitian properties.

The structure of this article is outlined as follows. We devote Section 2 to revisiting several definitions, fundamental properties, and lemmas that serve as the foundation for our subsequent analysis. Section 3 establishes the general solution of (1.6), including a special case. Section 4 presents an algorithm and a numerical example for the anti-Hermitian solution of (1.6). Finally, Section 5 provides a conclusion to this research.

2. Main results

This section presents fundamental lemmas, key properties, and essential mathematical tools that will underpin both the proof of our main theorem and the construction of illustrative examples.

Lemma 2.1 ([28]). *Let $K \in \mathbb{H}^{m\times n}$, $P \in \mathbb{H}^{m\times t}$, $Q \in \mathbb{H}^{l\times n}$. Then*

$$r\begin{bmatrix} K \\ Q \end{bmatrix} - r(K) = r(QL_K), \quad r\begin{bmatrix} K & P \end{bmatrix} - r(P) = r(R_P K),$$

$$r\begin{bmatrix} K & P \\ Q & 0 \end{bmatrix} - r(P) - r(Q) = r(R_P K L_Q).$$

Lemma 2.2 ([25]). *Let A_2, B_2, C_2, and D_2 be given with conformable sizes over $\mathbb{H}$. Set*

$$E = \begin{bmatrix} A_2 \\ -B_2^* \end{bmatrix}, F = \begin{bmatrix} C_2 \\ D_2^* \end{bmatrix}.$$

The system $A_2Y = C_2, YB_2 = D_2$ has the skew-Hermitian solution if and only if $R_EF = 0$ and $EF^* = -FE^*$. Under these terms, its general skew-Hermitian solution is

$$Y = E^\dagger F - (E^\dagger F)^* + E^\dagger E(E^\dagger F)^* + L_E V L_E^*,$$

where $V = -V^*$ is a free matrix over $\mathbb{H}$ with conformable size.

Lemma 2.3 ([34]). *Let A_4, B_4, C_4, and $D_4 = -D_4^*$, be coefficient matrices in (1.7) over $\mathbb{H}$ with agreeable sizes. Assume that*

$$A = R_{A_4}B_4, \ B = R_{A_4}C_4, \ C = R_{A_4}D_4R_{A_4}, \ M = R_AB, \ S = BL_M.$$

The following conditions are equivalent:

(1) The system (1.7) has a solution (X, Y, Z), where Y and Z are anti-Hermitian matrices.
(2) $R_MR_AC = 0$ and $R_ACR_B = 0$.
(3)

$$r\begin{bmatrix} D_4 & A_4 & B_4 & C_4 \\ A_4^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_4 & B_4 & C_4 \end{bmatrix} + r(A_4),$$

$$r\begin{bmatrix} D_4 & A_4 & B_4 \\ A_4^* & 0 & 0 \\ C_4^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_4 & B_4 \end{bmatrix} + r\begin{bmatrix} A_4 & C_4 \end{bmatrix}.$$

Under these conditions, X, $Y^* = -Y$, and $Z^* = -Z$ are given below

$$X = A_4^\dagger[D_4 - B_4YB_4^* - C_4ZC_4^*] - \frac{1}{2}A_4^\dagger[D_4 - B_4YB_4^* - C_4ZC_4^*](A_4^\dagger)^*A_4^* - U_2^*(A_4^\dagger)^*A_4^* -$$

$$
\begin{aligned}
&-A_4^{\dagger}U_2A_4^{*}+L_{A_4}U_1,\\
Y=&-Y^{*}=A^{\dagger}C(A^{\dagger})^{*}-\frac{1}{2}(A^{\dagger}BM^{\dagger}C[I+(B^{\dagger})^{*}S^{*}](A^{\dagger})^{*}-A^{\dagger}[I+SB^{\dagger}]C(M^{\dagger})^{*}B^{*}(A^{\dagger})^{*})-\\
&-A^{\dagger}SU_6S^{*}(A^{\dagger})^{*}+L_AU_4-U_4^{*}L_A,\\
Z=&-Z^{*}=\frac{1}{2}M^{\dagger}C(B^{\dagger})^{*}[I+(S^{\dagger}S)^{*}]-\frac{1}{2}(I+S^{\dagger}S)B^{\dagger}C(M^{\dagger})^{*}+L_MU_6L_M+\\
&+U_5L_B-L_BU_5^{*}+L_ML_SU_3-(L_ML_SU_3)^{*},
\end{aligned}
$$

where $U_1,\cdots,U_5$*, and* $U_6=-U_6^{*}$ *are free matrices with acceptable sizes.*

Lemma 2.3 expresses the system solution (1.7) in terms of the MP-inverses and inducted projectors. While this representation offers a method for finding a solution, it necessitates a procedure for computing the MP-inverse, which possesses properties distinct from those of an ordinary inverse. The ordinary matrix inverse has a well-known determinantal (D)-representation using the cofactor matrix. A similar representation is desirable for generalized inverses. But, defining the determinant of a quaternion matrix itself presents significant difficulties (see e.g., [2,8] for details). Recent progress in addressing this problem has been made possible by the theory of column-row determinants developed in [21].

For $A=(a_{ij})\in\mathbb{H}^{n\times n}$, we produce n row ($\mathfrak{R}$-)determinants and n column ($\mathfrak{C}$-)determinants similar to usual, but stating a certain order of factors in each term.

- *The i-th* $\mathfrak{R}$*-determinant* of A, for a row index $i\in I_n=\{1,\ldots,n\}$, is given by

  $$\mathrm{rdet}_iA:=\sum_{\sigma\in S_n}(-1)^{n-r}\,(a_{i\,i_{k_1}}a_{i_{k_1}i_{k_1+1}}\ldots a_{i_{k_1+l_1}\,i})\ldots(a_{i_{k_r}i_{k_r+1}}\ldots a_{i_{k_r+l_r}i_{k_r}}),$$

  whereat S_n is the symmetric group on I_n. The permutation σ is a product of mutually disjoint cycles ordered from the left to right by the rules

  $$\begin{gathered}\sigma=(i\,i_{k_1}i_{k_1+1}\ldots i_{k_1+l_1})\,(i_{k_2}i_{k_2+1}\ldots i_{k_2+l_2})\ldots(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r}),\\ i_{k_t}<i_{k_t+s},\ \ i_{k_2}<i_{k_3}<\cdots<i_{k_r},\ \ \forall\ t=2,\ldots,r,\quad s=1,\ldots,l_t.\end{gathered}$$

- For an arbitrary column index $j\in I_n$, *the j-th* $\mathfrak{C}$*-determinant* of A is defined as follows

  $$\mathrm{cdet}_jA=\sum_{\tau\in S_n}(-1)^{n-r}(a_{j_{k_r}j_{k_r+l_r}}\cdots a_{j_{k_r+1}j_{k_r}})\cdots(a_{j\,j_{k_1+l_1}}\cdots a_{j_{k_1+1}j_{k_1}}a_{j_{k_1}j}),$$

  while a permutation τ is ordered from right to left in the following way:

  $$\begin{gathered}\tau=(j_{k_r+l_r}\cdots j_{k_r+1}j_{k_r})\cdots(j_{k_2+l_2}\cdots j_{k_2+1}j_{k_2})\ (j_{k_1+l_1}\cdots j_{k_1+1}j_{k_1}j),\\ j_{k_t}<j_{k_t+s},\quad j_{k_2}<j_{k_3}<\cdots<j_{k_r}.\end{gathered}$$

In general, all $\mathfrak{R}$- and $\mathfrak{C}$-determinants are different. However, for any Hermitian matrix $A\in\mathbb{H}^{n\times n}$, the following equalities ensure the existence of a unique determinant

$$\mathrm{rdet}_1A=\cdots=\mathrm{rdet}_nA=\mathrm{cdet}_1A=\cdots=\mathrm{cdet}_nA=:\det A\in\mathbb{R}.$$

For more details on quaternion column-row determinants, see [21].

The following notations are used. Let $A_{i.}(b)$ and $A_{.j}(c)$ stand for matrices obtained by replacing the i-th row and j-th column of A with the row vector $b \in \mathbb{H}^{1\times n}$ and the column vector $c \in \mathbb{H}^{m}$, respectively. Let $\alpha := \{\alpha_1, \ldots, \alpha_r\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_r\} \subseteq \{1, \ldots, n\}$. Then, the notation A^{α}_{β} stands for a submatrix of $A \in \mathbb{H}^{m\times n}$, with rows and columns indexed by α and β, respectively. Furthermore, A^{α}_{α} and $|A|^{\alpha}_{\alpha}$ represent principal submatrices and principal minors of A when A is Hermitian. Denote

$$
\begin{aligned}
&I_{r,m} := \{\alpha : 1 \leq \alpha_1 < \cdots < \alpha_r \leq m\}, \quad J_{r,n} := \{\beta : 1 \leq \beta_1 < \cdots < \beta_r \leq n\},\\
&I_{r,m}\{j\} := \{\alpha \in I_{r,m} : j \in \alpha\}, \quad J_{r,n}\{i\} := \{\beta \in J_{r,n} : i \in \beta\}.
\end{aligned}
$$

Lemma 2.4 ([21]). *Suppose that $A \in \mathbb{H}^{n\times m}$ and rank $(A) = r$. Then for any $s \leq r$, we have*

$$
\sum_{\alpha \in I_{s,m}} |A^*A|^{\alpha}_{\alpha} = \sum_{\beta \in J_{s,n}} |AA^*|^{\beta}_{\beta} \in \mathbb{R}.
$$

Now, we provide a method for $\mathcal{D}$-representing the quaternion MP-inverse.

Lemma 2.5. *[20, Theorem 4.5] Let $A \in \mathbb{H}^{m\times n}$ with rank $(A) = r$. Then, the MP-inverse $A^{\dagger} = (a^{\dagger}_{ij}) \in \mathbb{H}^{n\times m}$ has the following two $\mathcal{D}$-representations:*

$$
a^{\dagger}_{ij} = \frac{1}{\delta} \sum_{\beta \in J_{r,n}\{i\}} \operatorname{cdet}_i \left((A^*A)_{.i} \left(a^*_{.j} \right) \right)^{\beta}_{\beta} = \frac{1}{\delta} \sum_{\alpha \in I_{r,m}\{j\}} \operatorname{rdet}_j \left((AA^*)_{j.} (a^*_{i.}) \right)^{\alpha}_{\alpha},
$$

where

$$
\delta = \sum_{\beta \in J_{r,n}} |A^*A|^{\beta}_{\beta} = \sum_{\alpha \in I_{r,m}} |AA^*|^{\alpha}_{\alpha},
$$

and $a^*_{.j}$ and $a^*_{i.}$ are the j-th column and the i-th row of A^*, respectively.

Lemma 2.5 presents novel $\mathcal{D}$-representations of the MP-inverse for any complex matrix by substituting ordinary determinants for row-column noncommutative determinants.

Note that another direct method of calculating the MP-inverse is based on the use of Quaternion Singular Value Decomposition (QSVD), which can be computed in several ways by converting the quaternion matrix into a complex matrix (using symplectic decomposition) and computing the SVD in complex arithmetic, or using specialized algorithms for quaternion matrices, such as those based on quaternion Householder transformations and quaternion QR decomposition. But, in both cases, it works reliably for small- to medium-sized matrices.

3. General solution and solvability condition of (1.6)

This section presents the main theorem of the paper.

Theorem 3.1. *Let $A_i \in \mathbb{H}^{m\times n}$, $B_i \in \mathbb{H}^{m\times q}$ for all $i = 1, \ldots, 5$, $C_i \in \mathbb{H}^{m\times k}$ for all $i = 3, \ldots, 8$, and $C_i = -C^*_i \in \mathbb{H}^{m\times m}$ for $i = 1, 2$. Assign the following*

$$
A_6 = \begin{bmatrix} A_3 \\ -B^*_3 \end{bmatrix}, C_9 = \begin{bmatrix} C_3 \\ C^*_4 \end{bmatrix}, A_7 = \begin{bmatrix} A_4 \\ -B^*_4 \end{bmatrix},
$$

$$
\begin{aligned}
&C_{10} = \begin{bmatrix} C_5 \\ C_6^* \end{bmatrix}, A_8 = \begin{bmatrix} A_5 \\ -B_5^* \end{bmatrix}, C_{11} = \begin{bmatrix} C_7 \\ C_8^* \end{bmatrix},\\
&A_9 = A_1 L_{A_6}, B_9 = B_1 L_{A_7}, C_{12} = C_1 - A_1 X_{01} A_1^* - B_1 Y_{01} B_1^*,\\
&X_{01} = A_6^\dagger C_9 - (A_6^\dagger C_9)^* + A_6^\dagger A_6 (A_6^\dagger C_9)^*,\ Y_{01} = A_7^\dagger C_{10} - (A_7^\dagger C_{10})^* + A_7^\dagger A_7 (A_7^\dagger C_{10})^*,\\
&M_1 = R_{A_9} B_9,\ S_1 = B_9 L_{M_1},\ A_{10} = A_2 L_{A_8},\ B_{10} = B_2 L_{A_7},\\
&C_{13} = C_2 - A_2 Z_{01} A_2^* - B_2 Y_{01} B_2^*,\ Z_{01} = A_8^\dagger C_{11} - (A_8 C_{11})^* + A_8^\dagger A_8 (A_8^\dagger C_{11})^*,\\
&M_2 = R_{A_{10}} B_{10},\ S_2 = B_{10} L_{M_2},\ E_1 = L_{M_1}, F_1 = L_{M_2},\\
&D_1 = \begin{bmatrix} L_{M_1} L_{S_1} & -L_{B_9} & -L_{M_2} L_{S_2} & L_{B_{10}} \end{bmatrix},\\
&G_1 = V_{02} - V_{01},\ V_{02} = \frac{1}{2} M_2^\dagger C_{13} (B_{10}^\dagger)^* (I_m + S_2^\dagger S_2) - \frac{1}{2} (I_m + S_2^\dagger S_2) B_{10}^\dagger C_{13} (M_2^\dagger)^*,\\
&V_{01} = \frac{1}{2} M_1^\dagger C_{12} (B_9^\dagger)^* (I_m + S_1^\dagger S_1) - \frac{1}{2} (I_m + S_1^\dagger S_1) B_9^\dagger C_{12} (M_1^\dagger)^*,\\
&D_2 = R_{D_1} E_1, E_2 = R_{D_1} F_1, G_2 = R_{D_1} G_1 R_{D_1}, M_3 = R_{D_2} E_2, S_3 = E_2 L_{M_3}.
\end{aligned} \tag{3.1}
$$

The following statements are equivalent:

(1) System (1.5) has at least one solution.

(2) The following relations hold true:

$$
\begin{aligned}
&R_{A_6} C_9 = 0, A_6 C_9^* = -C_9 A_6^*,\\
&R_{A_7} C_{10} = 0, A_7 C_{10}^* = -C_{10} A_7^*,\\
&R_{A_8} C_{11} = 0, A_8 C_{11}^* = -C_{11} A_8^*,\\
&R_{A_9} C_{12} R_{B_9} = 0,\ \ R_{M_1} R_{A_9} C_{12} = 0,\\
&R_{A_{10}} C_{13} R_{B_{10}} = 0,\ \ R_{M_2} R_{A_{10}} C_{13} = 0,\\
&R_{D_2} G_2 R_{D_2} = 0,\ \ R_{M_3} R_{D_2} G_2 = 0.
\end{aligned} \tag{3.2}
$$

(3) The following equalities hold for the ranks:

$$
\begin{aligned}
&r\left(\begin{bmatrix} C_9 \\ A_6 \end{bmatrix}\right) = r(A_6), A_6 C_9^* = -C_9 A_6^*,\\
&r\left(\begin{bmatrix} C_{10} \\ A_7 \end{bmatrix}\right) = r(A_7), A_7 C_{10}^* = -C_{10} A_7^*,\\
&r\left(\begin{bmatrix} C_{11} \\ A_8 \end{bmatrix}\right) = r(A_8), A_8 C_{11}^* = -C_{11} A_8^*,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&r\left(\begin{bmatrix} C_1 & A_1 & B_1C_{10}^* \\ B_1^* & 0 & A_7^* \\ C_9A_1^* & A_6 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 \\ A_6 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 \\ A_7 \end{bmatrix}\right),\\
&r\left(\begin{bmatrix} C_1 & B_1 & A_1 \\ C_{10}B_1^* & A_7 & 0 \\ C_9A_1^* & 0 & A_6 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 & B_1 \\ 0 & A_7 \\ A_6 & 0 \end{bmatrix}\right),\\
&r\left(\begin{bmatrix} C_2 & A_2 & B_2C_{10}^* \\ B_2^* & 0 & A_7^* \\ C_{11}A_2^* & A_8 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 \\ A_8 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_2 \\ A_7 \end{bmatrix}\right),\\
&r\left(\begin{bmatrix} C_2 & B_2 & A_2 \\ C_{10}B_2^* & A_7 & 0 \\ C_{11}A_2^* & 0 & A_8 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 & B_2 \\ A_8 & 0 \\ 0 & A_7 \end{bmatrix}\right),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&r\left(\begin{bmatrix}
0 & 0 & B_2^* & 0 & B_1^* & 0 & 0 & 0 & A_7^* & 0 & 0 & 0 \\
0 & 0 & 0 & B_1^* & B_1^* & 0 & 0 & 0 & 0 & A_7^* & 0 & 0 \\
0 & 0 & 0 & 0 & B_1^* & B_2^* & 0 & 0 & 0 & 0 & A_7^* & 0 \\
0 & 0 & B_2^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_7^* \\
B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)\\
&= r\left(\begin{bmatrix}
B_1 & -B_1 & -B_1 & 0 & 0 \\
B_2 & 0 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 & 0 \\
0 & 0 & B_1 & A_1 & 0 \\
0 & 0 & 0 & 0 & A_2 \\
A_7 & 0 & 0 & 0 & 0 \\
0 & A_7 & 0 & 0 & 0 \\
0 & 0 & A_7 & 0 & 0 \\
0 & 0 & 0 & A_6 & 0 \\
0 & 0 & 0 & 0 & A_8
\end{bmatrix}\right) + r\left(\begin{bmatrix}
B_1 & B_1 & 0 & 0 \\
B_2 & 0 & 0 & 0 \\
0 & B_2 & A_2 & 0 \\
0 & 0 & 0 & A_1 \\
A_7 & 0 & 0 & 0 \\
0 & A_7 & 0 & 0 \\
0 & 0 & A_8 & 0 \\
0 & 0 & 0 & A_6
\end{bmatrix}\right),
\end{aligned}
\tag{3.5}
$$

$$r\left(\begin{bmatrix} 0 & 0 & B_1^* & B_2^* & 0 & 0 & 0 & A_7^* & 0 \\ 0 & 0 & B_1^* & 0 & B_2^* & 0 & 0 & 0 & A_7^* \\ -B_2 & B_2 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_6 & 0 & 0 \end{bmatrix}\right)$$
$$= r\left(\begin{bmatrix} B_2 & 0 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & 0 & A_1 \\ B_1 & B_1 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 \\ A_7 & 0 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 & 0 \\ 0 & 0 & A_7 & 0 & 0 \\ 0 & 0 & 0 & A_8 & 0 \\ 0 & 0 & 0 & 0 & A_6 \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & B_1 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ B_2 & 0 & 0 & A_2 \\ A_7 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 \\ 0 & 0 & A_6 & 0 \\ 0 & 0 & 0 & A_8 \end{bmatrix}\right). \tag{3.6}$$

With these assumptions, the general solution to (1.5) takes the form:

$$\begin{aligned} X &= A_6^\dagger C_9 - (A_6^\dagger C_9)^* + A_6^\dagger A_6 (A_6^\dagger C_9)^* + L_{A_6} U L_{A_6}^*, \\ Y &= A_7^\dagger C_{10} - (A_7^\dagger C_{10})^* + A_7^\dagger A_7 (A_7^\dagger C_{10})^* + L_{A_7} V L_{A_7}^*, \\ Z &= A_8^\dagger C_{11} - (A_8^\dagger C_{11})^* + A_8^\dagger A_8 (A_8^\dagger C_{11})^* + L_{A_8} W L_{A_8}^*, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} U =& A_9^\dagger C_{12} (A_9^\dagger)^* - \frac{1}{2} A_9^\dagger B_9 M_1^\dagger C_{12} [I_m + (B_9^\dagger)^* S_1^*] (A_9^\dagger)^* \\ &+ \frac{1}{2} A_1^\dagger [I_m + S_1 B_9^\dagger] C_{12} (M_1^\dagger)^* B_9^* (A_9^\dagger)^* - A_9^\dagger S_1 U_1 S_1^* (A_9^\dagger)^* + L_{A_9} U_2 - U_2^* L_{A_9}, \\ W =& A_{10}^\dagger C_{13} (A_{10}^\dagger)^* - \frac{1}{2} A_{10}^\dagger B_{10} M_2^\dagger C_{13} [I_m + (B_{10}^\dagger)^* S_2^*] (A_{10}^\dagger)^* \\ &+ \frac{1}{2} A_{10}^\dagger [I_m + S_2 B_{10}^\dagger] C_{13} M_2^* B_{10}^* (A_{10}^\dagger)^* - A_{10}^\dagger S_2 W_1 S_2^* (A_{10}^\dagger)^* + L_{A_{10}} W_2 - W_2^* L_{A_{10}}, \end{aligned} \tag{3.8}$$

and the matrix V can be found in two different ways

$$\begin{aligned} V =& \frac{1}{2} M_1^\dagger C_{12} (B_9^\dagger)^* (I_m + S_1^\dagger S_1) - \frac{1}{2} (I_m + S_1^\dagger S_1) B_9^\dagger C_{12} (M_1^\dagger)^* + L_{M_1} U_1 L_{M_1} \\ &+ L_{M_1} L_{S_1} V_1 - V_1^* L_{S_1} L_{M_1} + V_2 L_{B_9} - L_{B_9} V_2^*, \\ V =& \frac{1}{2} M_2^\dagger C_{13} (B_{10}^\dagger)^* (I_m + S_2^\dagger S_2) - \frac{1}{2} (I_m + S_2^\dagger S_2) B_{10}^\dagger C_{13} (M_2^\dagger)^* + L_{M_2} W_1 L_{M_2} \\ &+ L_{M_2} L_{S_2} V_3 - V_3^* L_{S_2} L_{M_2} + V_4 L_{B_{10}} - L_{B_{10}} V_4^*. \end{aligned} \tag{3.9}$$

Here, the matrices U_1, and W_1 are determined by

$$\begin{aligned} U_1 =& D_2^{\dagger}G_2(D_2^{\dagger})^* - \frac{1}{2}D_2^{\dagger}E_2M_3^{\dagger}G_2(I_m + (E_2^{\dagger})^*S_3^*)(D_2^{\dagger})^* \\ &+\frac{1}{2}D_2^{\dagger}(I_m + S_3E_2^{\dagger})G_2(M_3^{\dagger})^*E_2^*(D_2^{\dagger})^* - D_2^{\dagger}S_3U_3(D_2^{\dagger}S_3)^* + L_{D_2}U_4 - U_4^*L_{D_2}, \\ T_2 =& - W_1 = \frac{1}{2}M_3^{\dagger}G_2(E_2^{\dagger})^*(I_m + S_3^{\dagger}S_3)H - \frac{1}{2}(I_m + S_3^{\dagger}S_3)E_2^{\dagger}G_2^*(M_3^{\dagger})^* + L_{M_3}U_3L_{M_3} \\ &+L_{M_3}L_{S_3}T_3 - T_3^*L_{S_3}L_{M_3} + T_4L_{E_2} - L_{E_2}T_4^*, \end{aligned} \tag{3.10}$$

and the matrices V_1, V_2, V_3, V_4 are $m \times m$-blocks of the matrix

$$T_1^* = \begin{bmatrix} V_1^* & V_2 & V_3^* & V_4 \end{bmatrix} \in \mathbb{H}^{m\times 4m},$$

where

$$\begin{aligned} T_1 =& D_1^{\dagger}(G_1 - E_1U_1E_1^* - F_1T_2F_1^*) - \frac{1}{2}D_1^{\dagger}(G_1 - E_1U_1E_1^* - F_1T_2F_1^*)(D_1^{\dagger})^*D_1^* \\ &-D_1^{\dagger}T_4D_1^* - T_4^*(D_1^{\dagger})^*D_1^* + L_{D_1}T_6. \end{aligned} \tag{3.11}$$

While U_2, W_2, T_3, T_4, and $T_6 = -T_6^*$ are any matrices of admissible sizes over $\mathbb{H}$.

Proof. We write the equations in (1.6) as follows:

$$\begin{aligned} &A_3X = C_3, XB_3 = C_4,\ X^* = -X, \\ &A_4Y = C_5, YB_4 = C_6,\ Y^* = -Y, \\ &A_1XA_1^* + B_1YB_1^* = C_1,\ C_1 = -C_1^* \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &A_4Y = C_5, YB_4 = C_6,\ Y^* = -Y, \\ &A_5Z = C_7, ZB_5 = C_8,\ Z^* = -Z, \\ &A_2ZA_2^* + B_2YB_2^* = C_2,\ C_2 = -C_2^*. \end{aligned} \tag{3.13}$$

By Lemma 2.2, the general solution to $A_3X = C_3, XB_3 = C_4,\ X^* = -X$, and $A_4Y = C_5, YB_4 = C_6,\ Y^* = -Y$ is given by

$$X = A_6^{\dagger}C_9 - (A_6^{\dagger}C_9)^* + A_6^{\dagger}A_6(A_6^{\dagger}C_9)^* + L_{A_6}UL_{A_6}^*, \tag{3.14}$$

$$Y = A_7^{\dagger}C_{10} - (A_7^{\dagger}C_{10})^* + A_7^{\dagger}A_7(A_7^{\dagger}C_{10})^* + L_{A_7}VL_{A_7}^* \tag{3.15}$$

respectively. Using (3.14) and (3.15) in the last equation of (3.12) and making some calculations, we obtain

$$A_9UA_9^* + B_9VB_9^* = C_{12}. \tag{3.16}$$

A solution to Eq (3.16) exists precisely when

$$R_{A_9}C_{12}R_{B_9} = 0, \quad R_{M_1}R_{A_9}C_{12} = 0.$$

Then, the general solution of (3.16) takes the form:

$$U = A_9^{\dagger}C_{12}(A_9^{\dagger})^* - \frac{1}{2}A_9^{\dagger}B_9M_1^{\dagger}C_{12}[I_m + (B_9^{\dagger})^*S_1^*](A_9^{\dagger})^*$$

$$
\begin{aligned}
&-\frac{1}{2}A_9^{\dagger}[I_m + S_1B_9^{\dagger}]C_{12}(M_1^{\dagger})^*B_9^*(A_9^{\dagger})^* - A_9^{\dagger}S_1U_1S_1^*(A_9^{\dagger})^* + L_{A_9}U_2 - U_2^*L_{A_9},\\
V =\ &\frac{1}{2}M_1^{\dagger}C_{12}(B_9^{\dagger})^*(I_m + S_1^{\dagger}S_1) + \frac{1}{2}(I_m + S_1^{\dagger}S_1)B_9^{\dagger}C_{12}(M_1^{\dagger})^*\\
&+ L_{M_1}U_1L_{M_1} + L_{M_1}L_{S_1}V_1 - V_1^*L_{S_1}L_{M_1} + V_2L_{B_9} - L_{B_9}V_2^*.
\end{aligned} \tag{3.17}
$$

Applying Lemma 2.2 to the system $A_5Z = C_7, ZB_5 = C_8,\ Y^* = -Y$ gives

$$
Z = A_8^{\dagger}C_{11} - (A_8^{\dagger}C_{11})^* + A_8^{\dagger}A_8(A_8^{\dagger}C_{11})^* + L_{A_8}WL_{A_8}^*. \tag{3.18}
$$

Using (3.15) and (3.18) in the last equation of (3.13) and making some calculations, we gain

$$
A_{10}WA_{10}^* + B_{10}VB_{10}^* = C_{13}. \tag{3.19}
$$

Equation (3.19) is solvable if and only if

$$
R_{A_{10}}C_{13}R_{B_{10}} = 0, \quad R_{M_2}R_{A_{10}}C_{13} = 0.
$$

Combining Lemma 2.3 with Eq (1.1), we deduce that the general solution to (3.19) has the representation:

$$
\begin{aligned}
W =\ & A_{10}^{\dagger}C_{13}(A_{10}^{\dagger})^* - \frac{1}{2}A_{10}^{\dagger}B_{10}M_2^{\dagger}C_{13}[I_m + (B_{10}^{\dagger})^*S_2^*](A_{10}^{\dagger})^*\\
&+\frac{1}{2}A_{10}^{\dagger}[I_m + S_2B_{10}^{\dagger}]C_{13}(M_2^*B_2^*(A_{10}^{\dagger})^* - A_{10}^{\dagger}S_2W_1S_2^*(A_{10}^{\dagger})^* + L_{A_{10}}W_2 - W_2^*L_{A_{10}},\\
V =\ & \frac{1}{2}M_2^{\dagger}C_{13}(B_{10}^{\dagger})^*(I_m + S_2^{\dagger}S_2) - \frac{1}{2}(I_m + S_2^{\dagger}S_2)B_{10}^{\dagger}C_{13}(M_2^{\dagger})^*\\
&+ L_{M_2}W_1L_{M_2} + L_{M_2}L_{S_2}V_3 - V_3^*L_{S_2}L_{M_2} + V_4L_{B_{10}} - L_{B_{10}}V_4^*,
\end{aligned} \tag{3.20}
$$

where W_2 is a free matrix of adequate shapes over $\mathbb{H}$. The matrices $U_1, W_1, V_1, V_2, V_3, V_4$ are determined as follows.

Denote $T_1^* = \begin{bmatrix} V_1^* & V_2 & V_3^* & V_4 \end{bmatrix}$, $E_1 = L_{M_1}$, $F_1 = L_{M_2}$, $T_2 = -W_1$, $G_1 = V_{02} - V_{01}$. Equating (3.17) and (3.20), we obtain

$$
D_1T_1 - (D_1T_1)^* + E_1U_1E_1{}^* + F_1T_2F_1^* = G_1. \tag{3.21}
$$

Lemma 2.3 establishes that Eq (3.21) admits a solution precisely when the equalities in (3.2) are satisfied, with the general solution then taking the form (3.7)–(3.11).

(2) $\Leftrightarrow$ (3) : Lemma 2.3 implies the following rank equalities.

$$
\begin{aligned}
&r(R_{A_6}C_9) = 0 \Leftrightarrow r\left(\begin{bmatrix} C_9 & A_6 \end{bmatrix}\right) = r(A_6),\ A_6C_9^* = -C_9A_6^*,\\
&r(R_{A_7}C_{10}) = 0 \Leftrightarrow r\left(\begin{bmatrix} C_{10} & A_7 \end{bmatrix}\right) = r(A_7),\ A_7C_{10}^* = -C_{10}A_7^*,\\
&r(R_{A_8}C_{11}) = 0 \Leftrightarrow r\left(\begin{bmatrix} C_{11} & A_8 \end{bmatrix}\right) = r(A_8),\ A_8C_{11}^* = -C_{11}A_8^*,\\
&r(R_{A_9}C_{12}B_9) = 0 \Leftrightarrow r\left(\begin{bmatrix} C_{12} & A_9 \\ B_9^* & 0 \end{bmatrix}\right) = r(A_9) + r(B_9)
\end{aligned}
$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_1 - A_1X_{01}A_1^* - B_1Y_{01}B_1^* & A_1L_{A_6} \\ L_{A_7}B_1^* & 0 \end{bmatrix}\right) = r(A_1L_{A_6}) + r(L_{A_7}B_1^*),$$

$$r\left(\begin{bmatrix} C_1 - A_1X_{01}A_1^* - B_1Y_{01}B_1^* & A_1 & 0 \\ B_1^* & 0 & A_7^* \\ 0 & A_6 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 \\ A_6 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 \\ A_7 \end{bmatrix}\right),$$

$$r\left(\begin{bmatrix} C_1 & A_1 & B_1C_{10}^* \\ B_1^* & 0 & A_7^* \\ C_9A_1^* & A_6 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 \\ A_6 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 \\ A_7 \end{bmatrix}\right),$$

$$r(R_{M_1}R_{A_9}C_{12}) = 0 \Leftrightarrow r\left(\begin{bmatrix} R_{A_9}C_{12} & M_1 \end{bmatrix}\right) = r(M_1) \Leftrightarrow r\left(\begin{bmatrix} R_{A_9}C_{12} & R_{A_9}B_9 \end{bmatrix}\right) = r(R_{A_9}B_9),$$

$$r\left(\begin{bmatrix} C_{12} & B_9 & A_9 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_9 & B_9 \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_1 - A_1X_{01}A_1^* - B_1Y_{01}B_1* & B_1L_{A_7} & A_1L_{A_6} \end{bmatrix}\right) = r\left(\begin{bmatrix} B_1L_{A_7} & A_1L_{A_6} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_1 - A_1X_{01}A_1^* - B_1Y_{01}B_1* & B_1 & A_1 \\ 0 & A_7 & 0 \\ 0 & 0 & A_6 \end{bmatrix}\right) = r\left(\begin{bmatrix} B_1 & A_1 \\ A_7 & 0 \\ 0 & A_6 \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_1 & B_1 & A_1 \\ C_{10} & A_7 & 0 \\ C_9A_1^* & 0 & A_6 \end{bmatrix}\right) = r\left(\begin{bmatrix} B_1 & A_1 \\ A_7 & 0 \\ 0 & A_6 \end{bmatrix}\right),$$

$$r(R_{A_{10}}C_{13}B_{10}) = 0 \Leftrightarrow r\left(\begin{bmatrix} C_{13} & A_{10} \\ B_{10}^* & 0 \end{bmatrix}\right) = r(A_{10}) + r(B_{10})$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_2 - A_2Z_{01}A_2^* - B_2Y_{01}B_2^* & A_2L_{A_8} \\ L_{A_7}B_2^* & 0 \end{bmatrix}\right) = r(A_2L_{A_8}) + r(L_{A_7}B_2^*)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_2 - A_2Z_{01}A_2^* - B_2Y_{01}B_2^* & A_2 & 0 \\ B_2^* & 0 & A_7^* \\ 0 & A_8 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 \\ A_8 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_2 \\ A_7 \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_2 & A_2 & B_2C_{10}^* \\ B_2^* & 0 & A_7^* \\ C_{11}A_1^* & A_8 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 \\ A_8 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_2 \\ A_7 \end{bmatrix}\right),$$

$$r(R_{M_2}R_{A10}C_{13}) = 0 \Leftrightarrow r\left(\begin{bmatrix} R_{A_{10}}C_{13} & M_2 \end{bmatrix}\right) = r(M_1)$$

$$\Leftrightarrow r\left(\begin{bmatrix} R_{A_{10}}C_{13} & R_{A_{10}}B_{10} \end{bmatrix}\right) = r(R_{A_{10}}B_{10})$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_{13} & B_{10} & A_{10} \end{bmatrix}\right) = r\left(\begin{bmatrix} A_{10} & B_{10} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_2 - A_2Z_{01}A_2^* - B_2Y_{01}B_2^* & B_2L_{A_7} & A_2L_{A_8} \end{bmatrix}\right) = r\left(\begin{bmatrix} B_2L_{A_7} & A_2L_{A_8} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_2 - A_2Z_{01}A_2^* - B_2Y_{01}B_2^* & B_2 & A_2 \\ 0 & A_7 & 0 \\ 0 & 0 & A_8 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 & B_2 \\ A_8 & 0 \\ 0 & A_7 \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} C_2 & B_2 & A_2 \\ C_{10}B_2^* & A_7 & 0 \\ C_{11}A_2^* & 0 & A_8 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_2 & B_2 \\ A_8 & 0 \\ 0 & A_7 \end{bmatrix}\right),$$

$$R_{D_2}G_2R_{D_2}^* = 0 \Leftrightarrow r\left(\begin{bmatrix} G_2 & D_2 \\ D_2^* & 0 \end{bmatrix}\right) = r(D_2) + r(E_2),$$

$$r\left(\begin{bmatrix} R_{D_1}G_1R_{D_1}^* & R_{D_1}E_1 \\ F_1^*R_{D_1} & 0 \end{bmatrix}\right) = r(R_{D_1}E_1) + r(R_{D_1}F_1)$$

$$\Leftrightarrow r\left(\begin{bmatrix} G_1 & E_1 & D_1 \\ F_1^* & 0 & 0 \\ D_1^* & 0 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} D_1 & E_1 \end{bmatrix}\right) + r\left(\begin{bmatrix} D_1 & F_1 \end{bmatrix}\right),$$

$$r\left(\begin{bmatrix} V_{02} - V_{01} & L_{M_1} & L_{M_1}L_{S_1} & -L_{B_9} & -L_{M_2}L_{S_2} & L_{B_{10}} \\ L_{M_2} & 0 & 0 & 0 & 0 & 0 \\ L_{S_1}L_{M_1} & 0 & 0 & 0 & 0 & 0 \\ -L_{B_9} & 0 & 0 & 0 & 0 & 0 \\ L_{S_2}L_{M_2} & 0 & 0 & 0 & 0 & 0 \\ L_{B_{10}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} L_{M_1} & L_{M_1}L_{S_1} & -L_{B_9} & -L_{M_2}L_{S_2} & L_{B_{10}} \end{bmatrix}\right) + r\left(\begin{bmatrix} L_{M_2} & L_{S_1}L_{M_1} & -L_{B_9} & L_{S_2}L_{M_2} & L_{B_{10}} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} V_{02} - V_{01} & L_{M_1} & -L_{B_9} & -L_{M_2}L_{S_2} & L_{B_{10}} \\ L_{M_2} & 0 & 0 & 0 & 0 \\ L_{S_1}L_{M_1} & 0 & 0 & 0 & 0 \\ -L_{B_9} & 0 & 0 & 0 & 0 \\ L_{B_{10}} & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} -I & -L_{M_2} & I & I \\ B_9 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & B_{10} & 0 \\ 0 & 0 & 0 & M_1 \end{bmatrix}\right) + r\left(\begin{bmatrix} L_{M_1} & -I & I & I \\ S_1 & 0 & 0 & 0 \\ 0 & B_9 & 0 & 0 \\ 0 & 0 & B_{10} & 0 \\ 0 & 0 & 0 & M_2 \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} V_{01} - V_{02} & I & -I & -L_{M_2} & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & M_2^* & 0 & 0 & 0 \\ L_{M_1} & 0 & 0 & 0 & 0 & 0 & S_1^* & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & B_2^* & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{10} \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} -I & -L_{M_2} & I & I \\ B_9 & 0 & 0 & 0 \\ 0 & B_1L_{M_{10}} & 0 & 0 \\ 0 & 0 & B_{10} & 0 \\ 0 & 0 & 0 & R_{A_9}B_9 \end{bmatrix}\right) + r\left(\begin{bmatrix} L_{M_1} & -I & I & I \\ B_9L_{M_1} & 0 & 0 & 0 \\ 0 & B_9 & 0 & 0 \\ 0 & 0 & B_{10} & 0 \\ 0 & 0 & 0 & R_{A_{10}}B_{10} \end{bmatrix}\right)$$

$$
\Leftrightarrow r\left(\begin{bmatrix} 0 & 0 & B_{10}^{*} & 0 & B_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_9^{*} & B_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_9 & B_{10}^{*} & 0 & 0 \\ 0 & 0 & B_{10}^{*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{10}^{*} & 0 & 0 & 0 & 0 & 0 \\ B_{10} & 0 & -C_{13} & 0 & 0 & 0 & A_{10} & 0 \\ 0 & -B_9 & 0 & -C_{12}^{*} & 0 & 0 & 0 & A_9 \end{bmatrix}\right)
$$

$$
= r\left(\begin{bmatrix} B_9 & -B_9 & B_9 & 0 & 0 \\ B_{10} & 0 & 0 & 0 & 0 \\ 0 & B_{10} & 0 & 0 & 0 \\ 0 & 0 & B_9 & A_9 & 0 \\ B_{10} & 0 & 0 & 0 & A_{10} \end{bmatrix}\right) + r\left(\begin{bmatrix} B_9 & B_9 & 0 & 0 \\ B_{10} & 0 & 0 & 0 \\ 0 & B_{10} & A_{10} & 0 \\ 0 & 0 & 0 & A_9 \end{bmatrix}\right),
$$

$$
r\left(\begin{bmatrix} 0 & 0 & B_2^{*} & 0 & B_1^{*} & 0 & 0 & 0 & A_7^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1^{*} & B_1^{*} & 0 & 0 & 0 & 0 & A_7^{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_1^{*} & B_2^{*} & 0 & 0 & 0 & 0 & A_7^{*} & 0 \\ 0 & 0 & B_2^{*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_7^{*} \\ B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)
$$

$$
= r\left(\begin{bmatrix} B_1 & -B_1 & -B_1 & 0 & 0 \\ B_2 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & B_1 & A_1 & 0 \\ 0 & 0 & 0 & 0 & A_2 \\ A_7 & 0 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 & 0 \\ 0 & 0 & A_7 & 0 & 0 \\ 0 & 0 & 0 & A_6 & 0 \\ 0 & 0 & 0 & 0 & A_8 \end{bmatrix}\right) + r\left(\begin{bmatrix} B_1 & B_1 & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & B_2 & A_2 & 0 \\ 0 & 0 & 0 & A_1 \\ A_7 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 \\ 0 & 0 & A_8 & 0 \\ 0 & 0 & 0 & A_6 \end{bmatrix}\right),
$$

$$
r(R_{M_3}R_{D_2}G_2] = 0 \Leftrightarrow r\left(\begin{bmatrix} R_{D_2}G_2 & M_3 \end{bmatrix}\right) = r(M_3) \Leftrightarrow r\left(\begin{bmatrix} R_{D_2}G_2 & R_{D_2}E_2 \end{bmatrix}\right) = r(M_3)
$$

$$
\Leftrightarrow r\left(\begin{bmatrix} G_2 & E_2 & D_2 \end{bmatrix}\right) = r\left(\begin{bmatrix} D_2 & E_2 \end{bmatrix}\right)
$$

$$
\Leftrightarrow r\left(\begin{bmatrix} R_{D_1}G_1R_{D_1}^{*} & R_{D_1}F_1 & R_{D_1}E_1 \end{bmatrix}\right) = r\left(\begin{bmatrix} F_1 & D_1 & E_1 \end{bmatrix}\right) + r(D_1),
$$

$$
r\left(\begin{bmatrix} V_{02} - V_{01} & L_{M_2} & L_{M_1} & L_{M_1}L_{S_1} & -L_{B_4} & -L_{M_2}L_{S_2} & L_{B_{10}} \\ L_{S_1}L_{M_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -L_{B_9} & 0 & 0 & 0 & 0 & 0 & 0 \\ -L_{S_2}L_{M_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{B_{10}} & 0 & & 0 & 0 & 0 & 0 \end{bmatrix}\right)
$$

$$
= r\left(\begin{bmatrix} L_{M_2} & L_{M_1} & L_{M_1}L_{S_1} & -L_{B_9} & -L_{M_2}L_{S_2} & L_{B_{10}} \end{bmatrix}\right) + r\left(\begin{bmatrix} L_{M_1}L_{S_1} & -L_{B_9} & -L_{M_2}L_{S_2} & L_{B_{10}} \end{bmatrix}\right)
$$

$$\Leftrightarrow r\left(\begin{bmatrix} V_{02}-V_{01} & I_m & I_m & -I_m & -L_{M_2} & I_m & 0 & 0 & 0 & 0 \\ L_{M_1} & 0 & 0 & 0 & 0 & 0 & S_1^* & 0 & 0 & 0 \\ -I_m & 0 & 0 & 0 & 0 & 0 & 0 & B_9^* & 0 & 0 \\ L_{M_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_2^* & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{10}^* \\ 0 & M_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{10} & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} L_{M_2} & L_{M_1} & -L_{B_9} & L_{B_{10}} \end{bmatrix}\right) + r\left(\begin{bmatrix} L_{M_1}L_{S_1} & -L_{B_9} & -L_{M_2}L_{S_2} & L_{B_{10}} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} V_{02}-V_{01} & I_m & I_m & -I_m & -L_{M_2} & I_m & 0 & 0 & 0 & 0 \\ L_{M_1} & 0 & 0 & 0 & 0 & 0 & L_{M_1}B_9^* & 0 & 0 & 0 \\ -I_m & 0 & 0 & 0 & 0 & 0 & 0 & B_9^* & 0 & 0 \\ L_{M_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_{M_2}B_{10}^* & 0 \\ I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{10}^* \\ 0 & R_{A_{10}}B_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{A_9}B_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{10}L_{M_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{10} & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} I_m & I_m & -I_m & I_m \\ M_2 & 0 & 0 & 0 \\ 0 & M_1 & 0 & 0 \\ 0 & 0 & B_9 & 0 \\ 0 & 0 & 0 & B_{10} \end{bmatrix}\right) + r\left(\begin{bmatrix} L_{M_1} & -I_m & -L_{M_2} & I_m \\ S_1 & 0 & 0 & 0 \\ 0 & B_9 & 0 & 0 \\ 0 & 0 & S_2 & 0 \\ 0 & 0 & 0 & B_{10} \end{bmatrix}\right).$$

Expanding the above equations and using

$$A_9U_{01}A_9^* + B_9V_{01}B_9^* = C_{12},$$
$$A_{10}W_{01}A_{10}^* + B_{10}V_{02}B_{10}^* = C_{13},$$

and doing some simplifications in it, we have

$$r\left(\begin{bmatrix} 0 & 0 & -B_9^* & B_{10}^* & 0 & 0 & 0 \\ 0 & 0 & -B_9^* & 0 & B_{10}^* & 0 & 0 \\ -B_{10} & B_{10} & 0 & 0 & 0 & A_{10} & 0 \\ B_9 & 0 & 0 & 0 & 0 & 0 & A_9 \\ 0 & B_9 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} B_{10} & 0 & 0 & A_{10} & 0 \\ 0 & B_9 & 0 & 0 & A_9 \\ B_9 & B_9 & B_9 & 0 & 0 \\ 0 & 0 & B_{10} & 0 & 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & B_9 & 0 & 0 \\ B_9 & 0 & 0 & 0 \\ 0 & B_{10} & 0 & 0 \\ 0 & 0 & A_9 & 0 \\ B_{10} & 0 & 0 & A_{10} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} 0 & 0 & -L_{A_7}B_1^* & L_{A_7}B_2^* & 0 & 0 & 0 \\ 0 & 0 & -L_{A_7}B_1^* & 0 & L_{A_7}B_2^* & 0 & 0 \\ -B_{10} & B_{10} & 0 & 0 & 0 & A_2L_{A_8} & 0 \\ L_{A_7}B_1^* & 0 & 0 & 0 & 0 & 0 & A_1L_{A_6} \\ 0 & L_{A_7}B_1^* & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$=r\left(\begin{bmatrix} B_2L_{A_7} & 0 & 0 & A_2L_{A_8} & 0 \\ 0 & B_1L_{A_7} & 0 & 0 & A_9 \\ B_1L_{A_7} & B_1L_{A_7} & B_1L_{A_7} & 0 & 0 \\ 0 & 0 & B_2L_{A_7} & 0 & 0 \end{bmatrix}\right)+r\left(\begin{bmatrix} 0 & B_1L_{A_7} & 0 & 0 \\ B_1L_{A_7} & 0 & 0 & 0 \\ 0 & B_2L_{A_7} & 0 & 0 \\ 0 & 0 & A_1LA_6 & 0 \\ B_2L_{A_7} & 0 & 0 & A_2L_{A_8} \end{bmatrix}\right)$$

$$\Leftrightarrow r\left(\begin{bmatrix} 0 & 0 & B_1^* & B_2^* & 0 & 0 & 0 & A_7^* & 0 \\ 0 & 0 & B_1^* & 0 & B_2^* & 0 & 0 & 0 & A_7^* \\ -B_2 & B_2 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_6 & 0 & 0 \end{bmatrix}\right)$$

$$=r\left(\begin{bmatrix} B_2 & 0 & 0 & A_2 & 0 \\ 0 & B_1 & 0 & 0 & A_1 \\ B_1 & B_1 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 \\ A_7 & 0 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 & 0 \\ 0 & 0 & A_7 & 0 & 0 \\ 0 & 0 & 0 & A_8 & 0 \\ 0 & 0 & 0 & 0 & A_6 \end{bmatrix}\right)+r\left(\begin{bmatrix} 0 & B_1 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ B_2 & 0 & 0 & A_2 \\ A_7 & 0 & 0 & 0 \\ 0 & A_7 & 0 & 0 \\ 0 & 0 & A_6 & 0 \\ 0 & 0 & 0 & A_8 \end{bmatrix}\right).$$

□

Now we discuss a particular case of our system.

If A_i, B_i, $(i = 3, 4, 5)$, and C_j, $(j = 3, \ldots, 8)$, are all equal to zero in Theorem 3.1, then the following consequence holds.

Corollary 3.2. *Let* $A_1 \in \mathbb{C}^{m\times n}$, $A_2 \in \mathbb{C}^{m\times q}$, $B_i \in \mathbb{C}^{m\times k}$, *and* $C_i = -C_i^* \in \mathbb{C}^{m\times m}$ *for* $i = 1, 2$. *Assign*

$$M_1 = R_{A_1}B_1,\ S_1 = B_1L_{M_1},\ M_2 = R_{A_2}B_2,\ S_2 = B_2L_{M_2},\ A_4 = R_{A_3}L_{M_1},\ B_4 = R_{A_3}L_{M_2},$$
$$A_3 = \begin{bmatrix} L_{B_2}^* & -L_{B_1} & L_{M_1}L_{S_1} & -L_{M_2}L_{S_2} \end{bmatrix},\ M_3 = R_{A_4}B_4,\ S_3 = B_4L_{M_3}$$
$$C_3 = V_{02} - V_{01},\ V_{02} = \frac{1}{2}M_2^{\dagger}C_2(B_2^{\dagger})^*(I_m + S_2^{\dagger}S_2) - \frac{1}{2}(I_m + S_2^{\dagger}S_2)B_2^{\dagger}C_2(M_2^{\dagger})^*,$$
$$V_{01} = \frac{1}{2}M_1^{\dagger}C_1(B_1^{\dagger})^*(I_m + S_1^{\dagger}S_1) - \frac{1}{2}(I_m + S_1^{\dagger}S_1)B_1^{\dagger}C_1(M_1^{\dagger})^*,\ C_4 = R_{A_3}C_3R_{A_3}.$$

Then the following statements are equivalent:

(1) The consistency of system (1.5) holds.

(2) The system yields these equalities:

$$R_{A_1}C_1R_{B_1}=0,\ \ R_{M_1}R_{A_1}C_1=0,$$
$$R_{A_2}C_2R_{B_2}=0,\ \ R_{M_2}R_{A_2}C_2=0,$$
$$R_{A_4}C_4R_{B_4}=0,\ \ R_{M_3}R_{A_4}C_4=0.$$

(3) The ranks obey the equalities:

$$r\left(\begin{bmatrix} C_1 & A_1\\ B_1^* & 0\end{bmatrix}\right)=r(A_1)+r(B_1),\quad r\left(\begin{bmatrix} C_1 & B_1 & A_1\end{bmatrix}\right)=r\left(\begin{bmatrix} A_1 & B_1\end{bmatrix}\right),$$
$$r\left(\begin{bmatrix} C_2 & A_2\\ B_2^* & 0\end{bmatrix}\right)=r(A_2)+r(B_2),\quad r\left(\begin{bmatrix} C_2 & B_2 & A_2\end{bmatrix}\right)=r\left(\begin{bmatrix} A_2 & B_2\end{bmatrix}\right),$$
$$r\left(\begin{bmatrix} 0 & 0 & 0 & B_2^* & B_1 & 0 & 0\\ 0 & 0 & 0 & -B_2^* & 0 & B_1^* & 0\\ B_1 & 0 & 0 & 0 & 0 & C_1 & A_1\\ 0 & B_2 & 0 & -C_2 & 0 & 0 & 0\\ -B_2 & -B_2 & B_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & A_2^* & 0 & 0 & 0\end{bmatrix}\right)$$
$$=r\left(\begin{bmatrix} -B_1 & 0 & -B_1 & A_1\\ B_2 & B_2 & 0 & 0\\ 0 & B_1 & 0 & 0\\ 0 & 0 & B_2 & 0\end{bmatrix}\right)+r\left(\begin{bmatrix} B_2 & 0 & 0 & A_2\\ -B_2 & B_2 & -B_2 & 0\\ 0 & B_1 & 0 & 0\\ 0 & 0 & B_1 & 0\end{bmatrix}\right),$$
$$r\left(\begin{bmatrix} 0 & 0 & -B_1^* & B_2^* & 0 & 0 & 0 & 0\\ 0 & 0 & -B_1^* & 0 & B_1^* & 0 & 0 & 0\\ 0 & 0 & -B_1^* & 0 & 0 & B_2^* & 0 & 0\\ -B_1^* & -B_1^* & 0 & 0 & -C_1 & 0 & A_1 & 0\\ B_2 & 0 & 0 & 0 & 0 & C_2 & A_2 & 0\\ 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\right)$$
$$=r\left(\begin{bmatrix} -B_1 & 0 & -B_1 & A_1\\ B_2 & B_2 & 0 & 0\\ 0 & B_1 & 0 & 0\\ 0 & 0 & B_2 & 0\end{bmatrix}\right)+\left(\begin{bmatrix} B_2 & 0\\ B_1 & 0\\ 0 & B_2\end{bmatrix}\right)+r(B_1).$$

With these assumptions, the general solution of (1.5) takes the form:

$$\begin{aligned}X=&A_1^\dagger C_1(A_1^\dagger)^*-\frac{1}{2}A_1^\dagger B_1M_1^\dagger C_1[I_m+(B_1^\dagger)^*S_1^*](A_1^\dagger)^*\\&+\frac{1}{2}A_1^\dagger[I_m+S_1B_1^\dagger]C_1(M_1^\dagger)^*B_1^*(A_1^\dagger)^*-A_1^\dagger S_1U_1S_1^*(A_1^\dagger)^*+L_{A_1}V_1-V_1^*L_{A_1},\\Z=&A_2^\dagger C_2(A_2^\dagger)^*-\frac{1}{2}A_2^\dagger B_2M_2^\dagger C_2[I_m+(B_2^\dagger)^*S_2^*](A_2^\dagger)^*\\&+\frac{1}{2}A_2^\dagger[I_m+S_2B_2^\dagger]C_2(M_2^\dagger)^*B_2^*(A_2^\dagger)^*-A_2^\dagger S_2U_4S_2^*(A_2^\dagger)^*+L_{A_2}V_2-V_2^*L_{A_2},\end{aligned}$$

$$Y = \frac{1}{2}M_1^{\dagger}C_1(B_1^{\dagger})^*(I_m + S_1^{\dagger}S_1) - \frac{1}{2}(I_m + S_1^{\dagger}S_1)B_1^{\dagger}C_1(M_1^{\dagger})^* + L_{M_1}U_1L_{M_1} + L_{M_1}L_{S_1}U_2 - U_2^*L_{S_1}L_{M_1} + U_3L_{B_1} - L_{B_1}U_3^*,$$

or

$$Y = \frac{1}{2}M_2^{\dagger}C_2(B_2^{\dagger})^*(I_m + S_2^{\dagger}S_2) - \frac{1}{2}(I_m + S_2^{\dagger}S_2)B_2^{\dagger}C_2(M_2^{\dagger})^* + L_{M_2}U_4L_{M_2} + L_{M_2}L_{S_2}U_5 - U_5^*L_{S_2}L_{M_2} + U_6L_{B_2} - L_{B_2}U_6^*,$$

with

$$\begin{aligned} U_6^* &= [\ I_k \quad 0 \quad 0 \quad 0\ ]Z, \\ U_3^* &= [\ 0 \quad I_k \quad 0 \quad 0\ ]Z, \\ U_2 &= [\ 0 \quad 0 \quad I_k \quad 0\ ]Z, \\ U_5 &= [\ 0 \quad 0 \quad 0 \quad I_k\ ]Z, \end{aligned}$$

where

$$\begin{aligned} Z =& A_3^{\dagger}(C_3 - L_{M_1}U_1L_{M_1} - L_{M_2}U_4L_{M_2}) - \frac{1}{2}A_3^{\dagger}(C_3 - L_{M_1}U_1L_{M_1} - L_{M_2}U_4L_{M_2})A_3A_3^{\dagger} \\ & - A_3^{\dagger}U_7A_3^* - U_7^*A_3A_3^{\dagger} + L_{A_3}U_8, \\ U_1 =& A_4^{\dagger}C_4(A_4^{\dagger})^* - \frac{1}{2}A_4^{\dagger}B_4M_3^{\dagger}C_4(I_m + (B_4^{\dagger})^*S_3^*)(A_4^{\dagger})^* \\ & + \frac{1}{2}A_4^{\dagger}(I_m + S_3B_4^{\dagger})C_4(M_3^{\dagger})^*B_4^*(A_4^{\dagger})^* - A_4^{\dagger}S_3U_9(A_4^{\dagger}S_3)^* + L_{A_4}U_{10} - U_{10}^*L_{A_4}, \\ U_4 =& \frac{1}{2}M_3^{\dagger}C_4(B_4^{\dagger})^*(I_m + S_3^{\dagger}S_3) - \frac{1}{2}(I_m + S_3^{\dagger}S_3)B_4^{\dagger}C_4(M_3^{\dagger})^* + L_{M_3}U_{11}L_{M_3} \\ & + L_{M_3}L_{S_3}U_{12} - U_{12}^*L_{S_3}L_{M_3} + U_{13}L_{B_4} - L_{B_4}U_{13}^*. \end{aligned}$$

Here $V_1,\ V_2,\ U_7, \ldots, U_{13},\ U_9 = -U_9^*, U_{11} = -U_{11}^*$ *are any matrices of admissible sizes over* $\mathbb{H}$.

4. An algorithm and an example

We derive an explicit solution algorithm for system (1.5) based on the foundations established in Theorem 3.1. This algorithm implements the $\mathcal{D}$-representations of the MP-inverse to construct general solutions, thus realizing theoretical principles in the final computation of concrete examples.

Algorithm 1 (H).
1) Input the matrices A_i, B_i, $(i = 1, \ldots, 5)$, and C_j, $(j = 1, \ldots, 8)$. Ensure they have conformable sizes over $\mathbb{H}$, and some of them are skew-Hermitian.
2) Compute the requisite matrices as prescribed in (3.1).
3) Evaluate the consistency of the system using either the matrix equations stated in (3.2) or rank conditions (3.3)–(3.6). If they do not hold, return "inconsistent".
4) Given satisfied consistency conditions:
   (a) Evaluate T_1, T_2, U_1, U, W, and V from (3.8)–(3.11).

(b) Derive the solution X, Y, Z through (3.7).

To validate the efficacy and practical utility of Algorithm 1, we provide the following numerical demonstration. The matrices defined below are employed to find a solution to the equations in (1.5).

$$A_1 = \begin{bmatrix} 3+3\mathbf{i} & 3\mathbf{j}+3\mathbf{k} & 3\mathbf{i}-3 \\ 3\mathbf{j}-3\mathbf{k} & 3\mathbf{i}-3 & -3\mathbf{j}-3\mathbf{k} \\ -3\mathbf{i}+3 & 3\mathbf{j}-3\mathbf{k} & 3\mathbf{i}+3 \end{bmatrix}, A_2 = \begin{bmatrix} 3\mathbf{i} & 3\mathbf{j} & 3\mathbf{k} \\ -3 & 3\mathbf{k} & -3\mathbf{j} \\ -3\mathbf{j} & 3\mathbf{i} & 3 \end{bmatrix}, A_3 = \begin{bmatrix} -1+\mathbf{i} & 1+\mathbf{i} & -1-\mathbf{i} \\ -1-\mathbf{i} & -1+\mathbf{i} & 1-\mathbf{i} \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -1+\mathbf{j} & 1+\mathbf{j} \\ -1-\mathbf{j} & -1+\mathbf{j} \end{bmatrix}, A_5 = \begin{bmatrix} 1+\mathbf{k} & -1+\mathbf{k} & 1-\mathbf{k} \\ 2\mathbf{k} & -2 & 2 \\ -1+\mathbf{k} & -1-\mathbf{k} & 1+\mathbf{k} \end{bmatrix}, B_1 = \begin{bmatrix} 1 & \mathbf{j} \\ -\mathbf{k} & \mathbf{i} \\ \mathbf{i} & \mathbf{k} \end{bmatrix}, B_2 = \begin{bmatrix} \mathbf{i} & \mathbf{k} \\ -\mathbf{k} & \mathbf{i} \\ \mathbf{j} & -1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -1-\mathbf{i} & -2\mathbf{i} & 1-\mathbf{i} \\ 1-\mathbf{i} & 2 & 1+\mathbf{i} \\ -1+\mathbf{i} & -2 & -1-\mathbf{i} \end{bmatrix}, B_4 = \begin{bmatrix} 1-\mathbf{j} & -1-\mathbf{j} \\ 1+\mathbf{j} & 1-\mathbf{j} \end{bmatrix}, B_5 = \begin{bmatrix} -\mathbf{i}+\mathbf{j} & -2\mathbf{i} & -\mathbf{i}-\mathbf{j} \\ \mathbf{i}+\mathbf{j} & 2\mathbf{j} & -\mathbf{i}+\mathbf{j} \\ -\mathbf{i}-\mathbf{j} & -2\mathbf{j} & \mathbf{i}-\mathbf{j} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} \mathbf{i} & -1+\mathbf{i} & 1-\mathbf{j} \\ 1+\mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1-\mathbf{j} & \mathbf{k} & \mathbf{k} \end{bmatrix}, C_2 = \begin{bmatrix} \mathbf{k} & -1+\mathbf{j} & 1-\mathbf{k} \\ 1+\mathbf{j} & \mathbf{i} & \mathbf{k} \\ -1-\mathbf{k} & \mathbf{k} & \mathbf{j} \end{bmatrix}, C_4 = \begin{bmatrix} -1+\mathbf{i} & -2 & -1-\mathbf{i} \\ -1-\mathbf{i} & -2\mathbf{i} & 1-\mathbf{i} \\ 1+\mathbf{i} & 2\mathbf{i} & -1+\mathbf{i} \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1+\mathbf{i} & 1-\mathbf{i} & -1+\mathbf{i} \\ -1+\mathbf{i} & 1+\mathbf{i} & -1-\mathbf{i} \end{bmatrix}, C_5 = \begin{bmatrix} 1+\mathbf{j} & 1-\mathbf{j} \\ -1+\mathbf{j} & 1+\mathbf{j} \end{bmatrix}, C_6 = \begin{bmatrix} -1-\mathbf{j} & -1+\mathbf{j} \\ 1-\mathbf{j} & -1-\mathbf{j} \end{bmatrix},$$

$$C_7 = \begin{bmatrix} 1-\mathbf{k} & 1+\mathbf{k} & -1-\mathbf{k} \\ 2 & 2\mathbf{k} & -2\mathbf{k} \\ 1+\mathbf{k} & -1+\mathbf{k} & 1-\mathbf{k} \end{bmatrix}, C_8 = \begin{bmatrix} \mathbf{i}+\mathbf{j} & 2\mathbf{j} & \mathbf{i}-\mathbf{j} \\ \mathbf{i}-\mathbf{j} & 2\mathbf{i} & \mathbf{i}+\mathbf{j} \\ -\mathbf{i}+\mathbf{j} & -2\mathbf{i} & -\mathbf{i}-\mathbf{j} \end{bmatrix}.$$

Using these matrices, we implement the procedure established in Theorem 3.1 to compute the solution of system (1.5).

- [Step 1] Using Lemma 2.5, calculate the MP-inverses of the given matrices. As an illustration, we obtain:

$$A_6^{\dagger} = \frac{1}{36} \begin{bmatrix} -1-\mathbf{i} & -1+\mathbf{i} & 1+\mathbf{i} & 2\mathbf{i} & -1+\mathbf{i} \\ 1-\mathbf{i} & -1-\mathbf{i} & -1+\mathbf{i} & -2 & -1-\mathbf{i} \\ -1+\mathbf{i} & 1+\mathbf{i} & 1-\mathbf{i} & 2 & 1+\mathbf{i} \end{bmatrix}.$$

  In particular, after computing the MP-inverses and considering the specific structure of our system, we find that the matrices S_i, $(i = 1,2,3)$, D_2, E_2, T_1, T_2, and U_1 turn out to be zero matrices. We also consider zero matrices as arbitrary matrices U_2, W_2, T_3, T_4, and $T_6 = -T_6^*$.
- [Step 2] Verify the given matrices by checking their compliance with the representation in (3.2) and the rank conditions (3.3)–(3.6) to ensure system consistency.
- [Step 3] The next step is to calculate the matrices U, W, and V. These matrices are used to construct the general solution to our system. Their calculation relies on values computed earlier and is integral to finalizing the solution according to our algorithm.
- [Step 5] With all necessary matrices computed, we are now in a position to present the solution (3.7) to the system defined in (1.5).

$$X = \frac{1}{3456} \begin{bmatrix} -1731\mathbf{i}-430\mathbf{j}+\mathbf{k} & 6+2\mathbf{j}+860\mathbf{k} & 1731+\mathbf{j}+430\mathbf{k} \\ -6+2\mathbf{j}+860\mathbf{k} & -3468\mathbf{i}+1720\mathbf{j}-4\mathbf{k} & -6\mathbf{i}+860\mathbf{j}-2\mathbf{k} \\ -1731+\mathbf{j}+430\mathbf{k} & -6\mathbf{i}+860\mathbf{j}-2\mathbf{k} & -1731\mathbf{i}+430\mathbf{j}-\mathbf{k} \end{bmatrix},$$

$$Y = \frac{1}{96}\begin{bmatrix} 4\mathbf{i} - 47\mathbf{j} + 9\mathbf{k} & -49 - 9\mathbf{i} + 4\mathbf{k} \\ 49 - 9\mathbf{i} + 4\mathbf{k} & -4\mathbf{i} - 47\mathbf{j} - 9\mathbf{k} \end{bmatrix},$$

$$Z = \frac{1}{6912}\begin{bmatrix} -45\mathbf{i} + 198\mathbf{j} - 2543\mathbf{k} & 1994 - 15\mathbf{i} - 36\mathbf{j} - 27\mathbf{k} & -2375 + 183\mathbf{i} + 9\mathbf{j} - 27\mathbf{k} \\ -1994 - 15\mathbf{i} - 36\mathbf{j} - 27\mathbf{k} & 45\mathbf{i} + 174\mathbf{j} - 2561\mathbf{k} & 27 + 9\mathbf{i} + 189\mathbf{j} + 2357\mathbf{k} \\ 2375 + 183\mathbf{i} + 9\mathbf{j} - 27\mathbf{k} & -27 + 9\mathbf{i} + 189\mathbf{j} + 2357\mathbf{k} & 18\mathbf{i} + 6\mathbf{j} - 2180\mathbf{k} \end{bmatrix}.$$

Note that we provide exact numerical values, rather than approximations, because we utilize the direct method for solving the quaternion matrix system (1.5).

5. Conclusions

This work investigates a constrained system of anti-Hermitian Sylvester matrix equations. We establish necessary and sufficient conditions for the existence of solutions, presented in two equivalent forms:

(i) algebraic characterization: Via relations involving the Moore–Penrose (MP-) inverse and induced projectors of the coefficient matrices;
(ii) rank-based criteria: Expressed through rank conditions on the system's coefficients.

Additionally, we derive an explicit anti-Hermitian solution constructed from the MP-inverse and its associated projectors. The main theorem not only proves solvability but also yields a computational algorithm, which we demonstrate via a numerical example. This example employs innovative techniques for $\mathcal{D}$-representations of the MP-inverse, leveraging the theory of row-column noncommutative determinants. All computations are implemented in Maple 2024 using the Clifford package, showcasing the method's practical applicability.

Author contributions

A.R.: Conceptualization, Software, Writing review and editing; I.K.: Conceptualization, Investigation, Writing original draft preparation, Writing review and editing, Supervision, Project administration; K.A.K.: Conceptualization, Methodology, Validation, Formal analysis, Writing original draft preparation; T.A.: Methodology, Software, Validation, Investigation, Visualization; S.E.: Validation, Formal analysis, Visualization.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

All data generated or analyzed during this study are included in the paper.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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