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Research article

Stability and decay analysis of stress-diffusive viscoelastic rate fluids

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Abstract: This paper focused on a class of three-dimensional incompressible viscoelastic rate-type fluids with stress diffusion. To simplify the analysis, we considered a model where the elastic stress was spherical. It is worth noting that while the existence of solutions for such fluids has been studied, their stability properties remain largely unexplored. So inspired by [1], this paper employed the bootstrap argument, the Schonbek's method, and repeated use of Besov space properties to prove, for the first time, stability of the solution under some additional conditions on the initial data — but without further smallness restrictions. Our results showed that the velocity decays faster than the typical algebraic rate, while the spherical component of the elastic strain tensor exhibited global exponential decay. Finally, using the decay rates, we derived the stability result for any given globally smooth solution—namely, that a sufficiently small perturbation yielded a unique globally smooth solution which stayed close to the original reference solution. We thereby extended the analysis of [1] on inhomogeneous Navier-Stokes equations to a viscoelastic fluid with stress diffusion.

Keywords: viscoelastic fluid; Littlewood-Paley theory; decay rate; Besov space
Mathematics Subject Classification: 76A15, 35Q35, 35D30

1. Introduction

Viscoelastic fluids are ubiquitous in industrial settings (e.g., petroleum, food, and rubber industries) and in nature (e.g., animal blood, natural bitumen), characterized by their combined viscous and elastic behaviors. In modeling, rate-type fluid models incorporating a stress diffusion term are widely employed to describe complex flow behaviors such as shear banding and vorticity banding (see reviews in [2–4]), which give rise to visually striking phenomena like the rod-climbing effect. These macroscopic manifestations originate from the microscopic deformation and reorganization of embedded microstructures, such as polymer chains. To facilitate reader comprehension of such fluid, we illustrate the physical configuration below (see Figure 1).

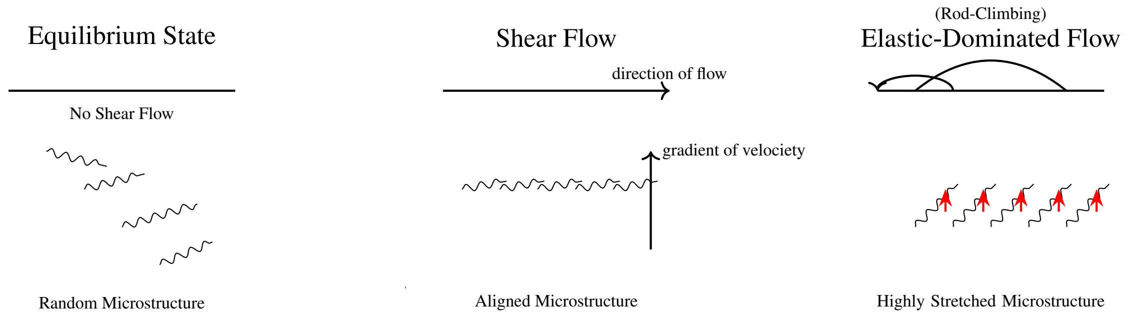

Figure 1. Physical configurations of viscoelastic fluids under different flow conditions.

For analytical simplicity, the model herein simplifies the elastic response by capturing elastic deformation effects through a spherical strain (i.e., the scalar multiplier of the identity tensor). Apparently, reducing the elastic stress to a spherical tensor weakens the model's memory effects and directional characteristics. Consequently, the findings may only apply to a highly restricted, idealized class of fluids, which limits their practical relevance. Nevertheless, the methodology and results presented in this work offer valuable insights for future studies on more comprehensive and physically realistic models. For clarity, we will study the model (3.1) and main results are presented in Section 3, preceded by a brief overview of the equation structure and existing research developments.

When the parameter $\sigma = 0$, the system (3.1) decouples into two independent physical problems: a damped transport equation governing the scalar strain field, and the classical Navier-Stokes equations describing fluid motion. The Navier-Stokes equations have been extensively studied. For instance, Abidi et al. [1] analyzed the decay and stability of solutions for the three-dimensional incompressible case with constant viscosity; He et al. [5] established the global-in-time stability of large solutions for the three-dimensional compressible case in the whole space; Gui and Zhang [6] proved the global stability of the three-dimensional Navier-Stokes system under anisotropic perturbations to the initial data of a reference solution; and Dong et al. [7] further examined the stability and exponential decay of the two-dimensional anisotropic Navier-Stokes equations with horizontal dissipation.

When the parameter $\sigma > 0$, the system displays two critical characteristics. First, a Korteweg stress term $\sigma \mathrm{div}\,(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2 \mathbb{I})$ emerges in the Cauchy stress constitutive relation within the momentum equation. It originates the system's elasticity, enabling the generation of normal stress differences responsible for purely elastic phenomena such as the rod-climbing (Weissenberg) effect and tubeless siphoning. Second, a diffusion term $\sigma \Delta b$ is incorporated into the evolution equation of the mean normal elastic stress. As an irreversible dissipative process, the stress diffusion contends with elastic relaxation during flow decay. Even under strong elastic effects, this mechanism ensures the total system energy decays continuously through viscous dissipation, ultimately restoring the system to a static equilibrium state—a key basis for mathematically proving flow decay. Simultaneously, the stress diffusion term provides essential regularization in the mathematical formulation: as a higher-order differential term, it exerts a natural smoothing effect that suppresses non-physical singularities, thereby guaranteeing the existence and smoothness of solutions under reasonable conditions. The coupling mechanism between elastic stress and microstructure in the model provides a theoretical foundation for understanding complex nonlinear phenomena such as shear banding and elastic turbulence. For the model's derivation, we refer readers to [8], which provided a rigorous derivation of the model for homogeneous compressible fluids, laying a vital theoretical groundwork for comprehending the dynamics of such complex fluids. Moreover, it demonstrated the existence of global weak solutions for this model under compressible and variable density conditions, taking into account arbitrary finite-energy initial data. The specific model studied here is obtained by imposing a divergence-free condition within the framework established in that reference.

Currently, although research on the specific model is limited, studies of other viscoelastic fluid models can provide references for this work. Ai et al. [9] established the existence of global solutions and decay estimates for a class of homogeneous incompressible rate-type viscoelastic fluids; Wang and Wen [10] investigated the global well-posedness of strong solutions near equilibrium and their temporal decay properties in Sobolev spaces for the compressible Oldroyd-B model; and Wang et al. [11] obtained precise decay estimates for the incompressible Oldroyd-B model with only fractional stress tensor diffusion in both two and three dimensions. Moatimid and Mohamed [12] investigated the nonlinear stability of two electrified viscoelastic cylindrical fluids embedded in a porous medium under an axial electric field; and they also analyze the stability of a two-layer electrified fluid system within a porous medium under a tilted electric field in [13], which systematically clarifies the effects of various physical parameters on stability under different orientations of the electric field.

Moreover, global existence results are available for related classes of diffusive rate-type viscoelastic models. Bulíček et al. [14] proved the large-data, long-time existence of weak solutions for a two-dimensional Giesekus-type viscoelastic fluid; Bulíček et al. [15] established the long-time existence of large-data weak solutions for incompressible rate-type viscoelastic fluids with stress diffusion. The problem complexity increases signicantly when considering thermal effects: Bulíček et al. [16] presented the first rigorous analysis of thermally coupled cases (assuming only spherical stress dependence), while Bulíček and Woźnicki [17] developed a large-data, long-time theory for incompressible viscoelastic heat-conducting fluids, specifically addressing the planar case.

In fact, studying the decay and stability properties of these fluids is of significant importance. Decay properties concern how the energy or disturbances within a system diminish to zero after the removal of external drivers, such as initial perturbations or applied forces. Stability, on the other hand, examines whether a fluid system will return to its equilibrium state or exhibit uncontrolled, amplified changes when subjected to minor disturbances. In practical industrial applications, understanding and utilizing these two fluid characteristics enables accurate prediction of flow dynamics, helping to prevent product defects, equipment damage, and production disruptions caused by flow instabilities. All in all, decay and stability are complementary and synergistic, collectively determining whether viscoelastic fluids can be predicted, controlled, and effectively utilized in both engineering and natural environments.

Finally, we conclude this section by reiterating the novel contributions of this work. It extends the analysis of Abidi et al. [1] on inhomogeneous Navier-Stokes equations to a viscoelastic fluid with stress diffusion, and complements recent studies on diffusive rate-type models (e.g., Bulíček et al. [8]) by focusing on decay and stability properties rather than existence theory.

2. Preliminary

Before presenting the definition of Besov spaces, we briefly review the fundamental constructs from Littlewood-Paley theory (see [18] for details).

Let $\mathcal{S}(\mathbb{R}^3)$ denote Schwarz space, and C the annulus $\{\xi \in \mathbb{R}^3 \big| \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. We consider radial functions $\varphi, \chi : \mathbb{R}^3 \to [0,1]$ satisfying

$$Supp\,\varphi \subset \{\xi \in \mathbb{R}^3 \Big| \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}, \qquad Supp\,\chi \subset \{\xi \in \mathbb{R}^3 \Big| |\xi| \leq \frac{4}{3}\},$$

$$\forall\, \xi \in \mathbb{R}^3, \quad \chi(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1.$$

The set $\tilde{C} \triangleq \mathcal{B}(0, 2/3) + C$ is an annulus satisfying $|j - j'| \geq 5 \implies 2^{j'}\tilde{C} \cap 2^j C = \emptyset$.

In the homogeneous case, the dyadic blocks $\dot{\Delta}_j$ and low-frequency cut-off operators $\dot{S}_j$ are defined by

$$\forall\, j \in \mathbb{Z},\ \dot{\Delta}_j f(x) \triangleq \varphi(2^{-j}D)f(x) = 2^{3j}\int_{\mathbb{R}^3} \mathcal{F}^{-1}\varphi(2^j y)f(x-y)dy,$$

$$\dot{S}_j f(x) \triangleq \sum_{p\leq j-1} \dot{\Delta}_p f(x) = \chi(2^{-j}D)f(x) = 2^{3j}\int_{\mathbb{R}^3} \mathcal{F}^{-1}\chi(2^j y)f(x-y)dy,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform operator. The homogeneous Littlewood-Paley decomposition thus reads

$$\forall f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3), \quad f(x) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f(x),$$

with $\mathcal{P}$ denoting the space of polynomials.

Definition 2.1. *(Homogeneous Bony's decomposition [19]) For $f, g \in \mathcal{S}'(\mathbb{R}^3)$, the Bony's decomposition is given by $fg = \dot{T}_f g + \dot{R}(f,g) = \dot{T}_f g + \dot{T}_g f + \dot{\mathcal{R}}(f,g)$, where the paraproduct and remainder terms are defined as*

$$\dot{T}_f g \triangleq \sum_{j\in Z} \dot{S}_{j-1} f \dot{\Delta}_j g, \quad \dot{R}(f,g) \triangleq \sum_{j\in Z} \dot{\Delta}_j f \dot{S}_{j+2} g, \quad \dot{\mathcal{R}}(f,g) \triangleq \sum_{j\in Z} \dot{\Delta}_j f \tilde{\dot{\Delta}}_j g, \quad \tilde{\dot{\Delta}}_j g \triangleq \sum_{|j'-j|\leq 1} \dot{\Delta}_{j'} g.$$

We now recall the definition of homogeneous Besov spaces from [20]:

Definition 2.2. *Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $f \in \mathcal{S}'(\mathbb{R}^3)$. For $1 \leq r < \infty$, we define the homogeneous space $\dot{B}^s_{p,r}$ as*

$$f \in \dot{B}^s_{p,r} \Leftrightarrow [\sum_{j\geq -1} (2^{js}\|\dot{\Delta}_j f\|_{L^p})^r]^{\frac{1}{r}} < \infty.$$

For $r = \infty$, it is defined by

$$f \in \dot{B}^s_{p,\infty} \Leftrightarrow \sup_{j\geq -1} 2^{js}\|\dot{\Delta}_j f\|_{L^p} < \infty.$$

Remark 2.1. *(1) Let k be a nonnegative integer. If* $s \in [\frac{3}{p}+k, \frac{3}{p}+k+1)$ *(or* $r=1$ *and* $s=\frac{3}{p}+k+1$*), then the homogeneous Besov space* $\dot{B}^s_{p,r}(\mathbb{R}^3)$ *consists of distributions* $f \in \mathcal{S}'(\mathbb{R}^3)$ *such that all partial derivatives* $\partial^\beta f$ *(with* $|\beta|=k$*) belong to* $\dot{B}^{s-k}_{p,r}$*.*

(2) For any positive real number s, the inclusion $\dot{B}^s_{p,r} \cap L^p = B^s_{p,r}$ *holds, with the equivalent norm*

$$\|f\|_{B^s_{p,r}} \approx \|f\|_{\dot{B}^s_{p,r}} + \|f\|_{L^p}.$$

Definition 2.3. *(Chemin-Lerner type space [21]) For* $s \in [-\infty, \frac{3}{p}]$*,* $1 \le r, \lambda, p \le \infty$*, and* $0 < t \le \infty$*. The space* $\tilde{L}^\lambda_t(\dot{B}^s_{p,r}(\mathbb{R}^3))$ *is the completion of* $C(0,t;\mathcal{S}(\mathbb{R}^3))$ *under the norm*

$$\|f\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,r})} \triangleq (\sum_{j\in Z} 2^{jrs}(\int_0^t \|\dot{\varDelta}_j f(\tau)\|^\lambda_{L^p} d\tau)^{\frac{r}{\lambda}})^{\frac{1}{r}} < \infty, \tag{2.1}$$

with the standard modification for $r=\infty$*. For* $p=r=2$*, we denote this space by* $\tilde{L}^\lambda_t(\dot{H}^s)$*. The local space* $\tilde{L}^\lambda_{\text{loc}}(\dot{H}^s)$ *is the intersection of* $\tilde{L}^\lambda_t(\dot{H}^s)$ *over all* $t>0$*; while for* $t=\infty$*, we concisely denote the space as* $\tilde{L}^\lambda(\dot{H}^s)$ *by omitting the subscript t.*

Lemma 2.1. *(Bernstein's inequality [22]) Let* C *be an annulus and* $\mathcal{B}$ *be a ball of* $\mathbb{R}^3$*. There exists a constant* $C>0$ *such that for any* $\lambda \in \mathbb{R}^+$*,* $k \in \mathbb{N}$*, and* $1 \le a_1 \le a_2 \le \infty$*, the following hold*

$$\begin{aligned} &Supp\, \hat{f} \subset \lambda\mathcal{B} \Rightarrow \sup_{|\theta|=k} \|\partial^\theta f\|_{L^{a_2}} \le C^{k+1}\lambda^{k+3(\frac{1}{a_1}-\frac{1}{a_2})}\|f\|_{L^{a_1}},\\ &Supp\, \hat{f} \subset \lambda C \Rightarrow C^{k+1}\lambda^k\|f\|_{L^{a_1}} \le \sup_{|\theta|=k}\|\partial^\theta f\|_{L^{a_1}} \le C^{-k-1}\lambda^k\|f\|_{L^{a_1}}. \end{aligned} \tag{2.2}$$

Lemma 2.2. *(Interpolation inequality [21]) For* $0 \le \gamma \le 1$*, it holds that*

$$\|f\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,r})} \le \|f\|^\gamma_{\tilde{L}^{\lambda_1}_t(\dot{B}^{s_1}_{p,r})}\|f\|^{1-\gamma}_{\tilde{L}^{\lambda_2}_t(\dot{B}^{s_2}_{p,r})},$$

where the exponents satisfy

$$\frac{1}{\lambda} = \frac{\gamma}{\lambda_1} + \frac{1-\gamma}{\lambda_2} \quad and \quad s = \gamma s_1 + (1-\gamma)s_2.$$

Furthermore, the Chemin-Lerner type spaces relate to classical Besov spaces via the Minkowski's inequality

$$\begin{aligned} &if \quad \lambda \le r, \quad \|f\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,r})} \le \|f\|_{L^\lambda_t(\dot{B}^s_{p,r})};\\ &if \quad r \le \lambda, \quad \|f\|_{L^\lambda_t(\dot{B}^s_{p,r})} \le \|f\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,r})}. \end{aligned}$$

Lemma 2.3 ([18])**.** *Let* $s>0$ *and* $f \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$*. If* $F:\mathbb{R}\to\mathbb{R}$ *is a smooth function with* $F(0)=0$*, then*

$$\|F(f)\|_{H^s} \le C(1+\|f\|_{L^\infty})^{\lfloor s\rfloor+1}\|f\|_{H^s}. \tag{2.3}$$

Lemma 2.4. *(Aubin-Lions lemma [23]) Assume that* $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$*, where* X, Z *are reflexive spaces, and X is dense in Z. Let* $W \triangleq \{f \in L^{p_0}(0,T;X),\ f_t \in L^{p_1}(0,T;X),\ 1 < p_0, p_1 \le \infty\}$*, then there holds that* $W \hookrightarrow\hookrightarrow L^{p_0}(0,T;Y)$*.*

Lemma 2.5 ([1]). *Let* $1 \le r \le \infty$, $f \in \dot B^s_{2,r}(\mathbb{R}^3)$, *and* $\mathbf{u} \in \dot B^{\frac52}_{2,1}(\mathbb{R}^3)$ *with* $\operatorname{div}\mathbf{u} = 0$. *Thus,* *(1) For* $s \in (-\frac52, \frac52)$ *(or* $r = 1$ *with* $s = \frac52$*),* $\|[\dot\Delta_j, \mathbf{u}\cdot\nabla]f\|_{L^2} \le C2^{-sj}\|\mathbf{u}\|_{\dot B^{\frac52}_{2,1}}\|f\|_{\dot B^s_{2,r}}$; *(2) For* $s \in (-\frac52, \infty)$ *with* $\nabla f \in L^\infty(\mathbb{R}^3)$, $\mathbf{u} \in \dot B^s_{2,r}(\mathbb{R}^3)$, $\|[\dot\Delta_j, \mathbf{u}\cdot\nabla]f\|_{L^2} \le C2^{-sj}(\|f\|_{\dot B^s_{2,r}}\|\mathbf{u}\|_{\dot B^{\frac52}_{2,1}} + \|\nabla f\|_{L^\infty}\|\mathbf{u}\|_{\dot B^s_{2,r}})$; *(3) For* $s \in (-1, \infty)$, $\|[\dot\Delta_j, \mathbf{u}\cdot\nabla]u\|_{L^2} \le C2^{-sj}\|\mathbf{u}\|_{\dot B^s_{2,r}}\|\nabla\mathbf{u}\|_{L^\infty}$.

We shall prove the following commutator estimates, which will be used in the next sections.

Lemma 2.6. *Let*
$$\nabla\bar b \in L^\infty_t(\dot B^{\frac32}_{2,1}(\mathbb{R}^3)) \cap L^1_t(\dot B^{\frac52}_{2,1}(\mathbb{R}^3)).$$
Then, there holds the following:
$$\|[\dot\Delta_j, w''(\bar b)]\nabla\bar b\|_{L^1(T_1,t;L^2)} \le C2^{-\frac52 j}\|\bar b\|_{L^1(T_1,t;\dot B^{\frac72}_{2,1})}\Big(\|\bar b\|_{L^\infty(T_1,t;\dot B^{\frac32}_{2,1})} + \|\bar b\|^3_{L^\infty(T_1,t;\dot B^{\frac32}_{2,1})}\Big).$$

Proof. Denote $w_1''(\bar b) \triangleq w''(\bar b) - w''(0)$, and it's obvious that $[\dot\Delta_j, w''(\bar b)]\nabla\bar b = [\dot\Delta_j, w_1''(\bar b)]\nabla\bar b$. Therefore, our focus will be on calculating $\|[\dot\Delta_j, w_1''(\bar b)]\nabla\bar b\|_{L^1(T_1,t;L^2)}$.

By Definition 2.1, the following identities hold:
$$\dot\Delta_j(w_1''(\bar b)\nabla\bar b) = \dot\Delta_j(\dot T_{w_1''(\bar b)}\nabla\bar b) + \dot\Delta_j(\dot T_{\nabla\bar b}w_1''(\bar b)) + \dot\Delta_j\dot{\mathcal{R}}(w_1''(\bar b), \nabla\bar b),$$
$$w_1''(\bar b)\dot\Delta_j\nabla\bar b = \dot T_{w_1''(\bar b)}\dot\Delta_j\nabla\bar b + \dot R(w_1''(\bar b), \dot\Delta_j\nabla\bar b).$$

Consequently, we have
$$[\dot\Delta_j, w_1''(\bar b)]\nabla\bar b = [\dot\Delta_j, \dot T_{w_1''(\bar b)}]\nabla\bar b + \dot\Delta_j(\dot T_{\nabla\bar b}w_1''(\bar b)) + \dot\Delta_j\dot{\mathcal{R}}(w_1''(\bar b), \nabla\bar b) - \dot R(w_1''(\bar b), \dot\Delta_j\nabla\bar b). \tag{2.4}$$

The second term on the righthand side of (2.4) is rewritten as
$$\dot\Delta_j(\dot T_{\nabla\bar b}w_1''(\bar b)) = \sum_{|j-j'|\le4}\dot\Delta_j(\dot S_{j-1}\nabla\bar b\dot\Delta_{j'}w_1''(\bar b)),$$

then there holds
$$\begin{aligned}\|\dot\Delta_j(\dot T_{\nabla\bar b}w_1''(\bar b))\|_{L^1(T_1,t;L^2)} &\le C\sum_{|j-j'|\le4}\|\dot S_{j-1}\nabla\bar b\|_{L^1(T_1,t;L^\infty)}\|\dot\Delta_{j'}w_1''(\bar b)\|_{L^\infty(T_1,t;L^2)}\\ &\le C2^{-\frac52 j}\|\bar b\|_{L^1(T_1,t;\dot B^{\frac72}_{2,1})}\|w_1''(\bar b)\|_{L^\infty(T_1,t;\dot B^{\frac32}_{2,1})},\end{aligned}$$

where Lemma 2.1 is used, so that
$$\|\dot S_{j-1}\nabla\bar b\|_{L^1(T_1,t;L^\infty)} \le C\sum_{k\le j-2}2^k\|\dot\Delta_k\bar b\|_{L^1(T_1,t;L^\infty)} \le C\sum_{k\le j-2}2^k2^{\frac32 k}\|\dot\Delta_k\bar b\|_{L^1(T_1,t;L^2)} \le C2^{-j}\|\bar b\|_{L^1(T_1,t;\dot B^{\frac72}_{2,1})}.$$

For the third term in (2.4), by Lemma 2.1, we get
$$\begin{aligned}\|\dot\Delta_j\dot{\mathcal{R}}(w_1''(\bar b), \nabla\bar b)\|_{L^1(T_1,t;L^2)} &\le C\sum_{j'\ge j-3}2^{-\frac52 j'}\|w_1''(\bar b)\|_{L^\infty(T_1,t;\dot B^{\frac32}_{2,1})}\|\bar b\|_{L^1(T_1,t;\dot B^{\frac72}_{2,1})}\\ &\le C2^{-\frac52 j}\|w_1''(\bar b)\|_{L^\infty(T_1,t;\dot B^{\frac32}_{2,1})}\|\bar b\|_{L^1(T_1,t;\dot B^{\frac72}_{2,1})}.\end{aligned}$$

The definition of $\dot{R}(w_1''(\overline{b}),\dot{\Delta}_j\nabla\overline{b})$ and Lemma 2.1 give that

$$\begin{aligned}
\|\dot{R}(w_1''(\overline{b}),\dot{\Delta}_j\nabla\overline{b})\|_{L^1(T_1,t;L^2)} &\le C\sum_{j'\ge j-2}2^{j}\|\dot{\Delta}_j\overline{b}\|_{L^1(T_1,t;L^\infty)}\|\dot{\Delta}_{j'}w_1''(\overline{b})\|_{L^\infty(T_1,t;L^2)}\\
&\le C2^{\frac{5}{2}j}\|\dot{\Delta}_j\overline{b}\|_{L^1(T_1,t;L^2)}\sum_{j'\ge j-2}\|\dot{\Delta}_{j'}w_1''(\overline{b})\|_{L^\infty(T_1,t;L^2)}\\
&\le C2^{-\frac{5}{2}j}\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}\|w_1''(\overline{b})\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}.
\end{aligned}$$

We now estimate the first term $[\dot{\Delta}_j,T_{w_1''(\overline{b})}]\nabla\overline{b}$. By the definition of $\dot{\Delta}_j$, we have

$$\begin{aligned}
\dot{\Delta}_j,\dot{T}_{w_1''(\overline{b})}t\nabla\overline{b} &= \sum_{|j-j'|\le 4}[\dot{\Delta}_j,\dot{S}_{j'-1}w_1''(\overline{b})]\nabla\dot{\Delta}_{j'}\overline{b}\\
&=\sum_{|j-j'|\le 4}2^{3j}\int_{\mathbb{R}^3}\mathcal{F}^{-1}\varphi(2^j(x-y))(\dot{S}_{j'-1}w_1''(\overline{b}(x))-\dot{S}_{j'-1}w_1''(\overline{b}(y)))\nabla\dot{\Delta}_{j'}\overline{b}(y)\,dy\\
&=\sum_{|j'-j|\le 4}2^{4j}\int_{\mathbb{R}^3}\Big[\int_0^1 y\cdot\nabla\dot{S}_{j'-1}w_1''(\overline{b}(x-\tau y))\,d\tau\Big]\cdot\nabla\mathcal{F}^{-1}\varphi(2^jy)\dot{\Delta}_j\overline{b}(x-y)\,dy\\
&\quad+\sum_{|j'-j|\le 4}2^{3j}\int_{\mathbb{R}^3}\mathcal{F}^{-1}\varphi(2^j(x-y))\nabla\dot{S}_{j'-1}w_1''(\overline{b}(y))\dot{\Delta}_j\overline{b}(y)\,dy,
\end{aligned}$$

which combined with the Minkowski inequality yields that

$$\begin{aligned}
\|[\dot{\Delta}_j,\dot{T}_{w_1''(\overline{b})}]\nabla\overline{b}\|_{L^1(T_1,t;L^2)} &\le C\sum_{|j'-j|\le 4}\|\dot{\Delta}_j\overline{b}\|_{L^1(T_1,t;L^2)}\|\nabla\dot{S}_{j'-1}w_1''(\overline{b})\|_{L^\infty(T_1,t;L^\infty)}\\
&\le C2^{-\frac{5}{2}j}\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}\|w_1''(\overline{b})\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}.
\end{aligned}$$

Combining the above estimates, we directly obtain

$$\begin{aligned}
\|[\dot{\Delta}_j,w_1''(\overline{b})]\nabla\overline{b}\|_{L^1(T_1,t;L^2)} &\le C2^{-\frac{5}{2}j}\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}\|w_1''(\overline{b})\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\\
&\le C2^{-\frac{5}{2}j}\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}\Big(\|\overline{b}\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}+\|\overline{b}\|^3_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\Big),
\end{aligned}\tag{2.5}$$

where we used Lemma 2.3 so that

$$\|w_1''(\overline{b})\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\le C(1+\|\overline{b}\|_{L^\infty(T_1,t;L^\infty)})^2\|\overline{b}\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\le C(1+\|\overline{b}\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})})^2\|\overline{b}\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}.$$

□

Remark 2.2. *Compared to the previous lemma, the commutator here involves only the interaction between the multiplication operator $w''(b)$ and the function ∇b. Its essence is to quantify the commutation error between the "pointwise multiplication" operation and the "localized filtering" process. In sharp contrast, the commutator $[\dot{\Delta}_j,\mathbf{u}\cdot\nabla]$ treated in Lemma 2.5 is associated with the first-order differential operator $\mathbf{u}\cdot\nabla$, and therefore measures the commutation error between the "differentiation" operation and the "localized filtering" process.*

Lemma 2.7 ([1]). *Let* $\mathbf{u}$ *be a divergence-free vector field with* $\nabla\mathbf{u} \in L^1_T(\dot{B}^{\frac{3}{2}}_{2,1})$. *For* $-\frac{5}{2} < s \le \frac{5}{2}$, *if* $f_0 \in \dot{B}^s_{2,1}$ *and* $g \in L^1_T(\dot{B}^s_{2,1})$, *the transport equation*

$$\begin{cases}\partial_t f + \mathbf{u}\cdot\nabla f = g, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+,\\ f(0,x) = f_0(x),\end{cases}$$

admits a unique solution $f \in C(0,T;\dot{B}^s_{2,1})$. *For* $0 \le t \le T$,

$$\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,1})} \le \|f_0\|_{\dot{B}^s_{2,1}} + C\int_0^t \|f(\tau)\|_{\dot{B}^s_{2,1}}\|\mathbf{u}(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,d\tau + C\|g\|_{L^1_t(\dot{B}^s_{2,1})}. \tag{2.6}$$

By Grönwall's inequality:

$$\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,1})} \le \exp\{C\|\mathbf{u}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})}\}(\|f_0\|_{\dot{B}^s_{2,1}} + C\|g\|_{L^1_t(\dot{B}^s_{2,1})}). \tag{2.7}$$

This lemma will serve as a fundamental tool in our subsequent derivation of energy estimates for the density equation within the Besov space framework.

Lemma 2.8 ([1]). *Let* $\mathbf{u}$ *be a divergence-free vector field with* $\mathbf{u} \in L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})$, $g \in \tilde{L}^1_T(\dot{B}^s_{2,r})$, *and* $a \in L^\infty_T(\dot{H}^2) \cap L^\infty_T(\dot{H}^{s+\frac{3}{2}})$ *with* $1 + a \ge c > 0$. *For* $-\frac{3}{2} < s < 1$, $r = 1$, *or* 2, *if* $f_0 \in \dot{B}^s_{2,r}$ *and* $(f,P) \in L^1_T(\dot{B}^{s+\frac{3}{2}}_{2,1}) \cap L^1_T(\dot{H}^1)$ *solves*

$$\begin{cases}\partial_t f + \mathbf{u}\cdot\nabla f - (1+a)(\Delta f - \nabla P) = g, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+,\\ \operatorname{div} f = 0,\\ f(0,x) = f_0(x),\end{cases} \tag{2.8}$$

then we get that for all $0 \le t \le T$,

$$\begin{aligned}\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,r})} + \|f\|_{\tilde{L}^1_t(\dot{B}^{s+2}_{2,r})} + \|\nabla P\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})} \le \|f_0\|_{\dot{B}^s_{2,r}} + C\int_0^t \|f(\tau)\|_{\dot{B}^s_{2,1}}\|\mathbf{u}(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,d\tau\\ +C\|g\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})} + C\|a\|_{L^\infty_t(\dot{H}^{s+\frac{3}{2}})}\|\nabla P\|_{L^1_t(L^2)} + C\|a\|_{L^\infty_t(\dot{H}^2)}\|f\|_{L^1_t(\dot{B}^{s+\frac{3}{2}}_{2,r})}.\end{aligned} \tag{2.9}$$

Consequently, we arrive at the following result:

$$\begin{aligned}&\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,r})} + \|f\|_{\tilde{L}^1_t(\dot{B}^{s+2}_{2,r})} + \|\nabla P\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})} \le C\exp\{C\|\mathbf{u}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})}\}\\ &\times\Big(\|f_0\|_{\dot{B}^s_{2,r}} + \|g\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})} + \|a\|_{L^\infty_t(\dot{H}^{s+\frac{3}{2}})}\|\nabla P\|_{L^1_t(L^2)} + \|a\|_{L^\infty_t(\dot{H}^2)}\|f\|_{L^1_t(\dot{B}^{s+\frac{3}{2}}_{2,r})}\Big).\end{aligned} \tag{2.10}$$

This lemma will serve as a fundamental tool in our subsequent derivation of energy estimates for the momentum equation within the Besov space framework.

Remark 2.3. *If* $-1 < s < 1$ *and* $u = f$ *in Eq* (2.8), *there holds that*

$$\begin{aligned}&\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,r})} + \|f\|_{\tilde{L}^1_t(\dot{B}^{s+2}_{2,r})} + \|\nabla P\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})} \le C\exp\{C\|\nabla f\|_{L^1_t(L^\infty)}\}\\ &\times\Big(\|f_0\|_{\dot{B}^s_{2,r}} + \|g\|_{L^1_t(\dot{B}^s_{2,r})} + \|a\|_{L^\infty_t(\dot{H}^{s+\frac{3}{2}})}\|\nabla P\|_{L^1_t(L^2)} + \|a\|_{L^\infty_t(\dot{H}^2)}\|f\|_{L^1_t(\dot{B}^{s+\frac{3}{2}}_{2,r})}\Big).\end{aligned} \tag{2.11}$$

Corollary 2.1. *Let* $\mathbf{u}$ *be a divergence-free vector field with* $\mathbf{u} \in L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})$. *For* $-\frac{5}{2} < s < \frac{5}{2}$, $r = 1, 2$, *(or* $r = 1$ *with* $s = \frac{5}{2}$*), if* $f_0 \in \dot{B}^{s+1}_{2,r}$, $g \in \tilde{L}^1_T(\dot{B}^{s+1}_{2,1})$, *and* $f \in L^1_T(\dot{B}^{s+1}_{2,r})$ *solves*

$$\begin{cases} \partial_t f + \mathbf{u} \cdot \nabla f - \Delta f = g, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \\ f(0,x) = f_0(x), \end{cases} \tag{2.12}$$

then for $0 \le t \le T$,

$$\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,r})} + \|f\|_{\tilde{L}^1_t(\dot{B}^{s+2}_{2,r})} \le \|f_0\|_{\dot{B}^s_{2,r}} + C\int_0^t \|f(\tau)\|_{\dot{B}^s_{2,r}} \|\mathbf{u}(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \, d\tau + C\|g\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})}, \tag{2.13}$$

which implies that

$$\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,r})} + \|f\|_{\tilde{L}^1_t(\dot{B}^{s+2}_{2,r})} \le \exp\{C\|\mathbf{u}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})}\}(\|f_0\|_{\dot{B}^s_{2,r}} + C\|g\|_{\tilde{L}^1_t(\dot{B}^s_{2,r})}). \tag{2.14}$$

This corollary will serve as a fundamental tool in our subsequent derivation of energy estimates for the elasticity equation within the Besov space framework.

Proof. This follows directly from Lemma 2.8 with $a = 0$ and $P = 0$. □

When addressing the regularity issues of nonlinear diffusion equations, Besov spaces and Chemin–Lerner type spaces offer significant advantages over the traditional Sobolev space framework. By leveraging the frequency decomposition based on Littlewood–Paley theory, Besov spaces enable a more refined characterization of function smoothness and achieve sharp estimates for nonlinear terms via Bony's paraproduct decomposition, thereby facilitating the treatment of large initial data problems. Meanwhile, Chemin–Lerner spaces optimize the order of defining space–time norms, ensuring perfect compatibility with heat semigroup estimates, and providing an indispensable analytical framework for establishing local or global well-posedness in critical spaces.

3. Problem formulation and main results

Prior to introducing our model, we first provide a complete list of all mathematical notations used throughout this paper for the reader's reference (see Table 1).

The primary model studied in this paper is given by the following equations:

$$\begin{cases} \varrho_t + \mathbf{v} \cdot \nabla \varrho = 0, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \\ \varrho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) - \Delta \mathbf{v} + \nabla P + \sigma \mathrm{div}\,(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2 \mathbb{I}) = 0, \\ b_t + \mathbf{v} \cdot \nabla b + \frac{1}{\nu}(w'(b) - \sigma \Delta b) = 0, \\ \mathrm{div}\, \mathbf{v} = 0, \\ (\varrho, \mathbf{v}, b)(x,t)|_{t=0} = (\varrho_0, \mathbf{v}_0, b_0)(x). \end{cases} \tag{3.1}$$

The unknowns $\varrho = \varrho(x,t)$, $\mathbf{v} = (v^1(x,t), v^2(x,t), v^3(x,t))$, and $b = b(x,t)$ denote the density, fluid velocity field, and spherical component of the elastic strain tensor, respectively. The scalar pressure function is denoted by $P = P(x,t)$. Additionally, $w(\cdot)$ is assumed to be a smooth strictly convex function of b, with its second and third derivatives denoted $w''(b)$ and $w'''(b)$ being bounded, specifically, $m_0^{-1} \leq w''(b) \leq m_0$ and $|w'''(b)| \leq m_1$ for constants $m_0 > 0$ and $m_1 \geq 0$. The coefficients ν and σ are positive constants. Since their exact values do not play an essential role in our subsequent analysis, we set $\nu = \sigma = 1$ for simplicity. This is achieved through a standard nondimensionalization procedure, which simplifies the governing equations without loss of physical generality.

Table 1. Notation.

Symbol	**Description**
C	Generic positive constant independent of t
$x \lesssim y$	$x \leq Cy$ for some $C > 0$
$C(I; X)$	Continuous functions from I to Banach space X
$C_b(I; X)$	Bounded continuous functions in $C(I; X)$
$L^q(\mathbb{R}^3)$	Lebesgue spare of p-integrable functions on $\mathbb{R}^3$
L^q	Shorthand for $L^q(\mathbb{R}^3)$ (unless otherwise specified)
$L_t^p(X)$	Shorthand for $L^p(0, t; X)$
$\mathcal{S}(\mathbb{R}^3)$	Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^3$
$\mathcal{S}'(\mathbb{R}^3)$	The space of tempered distributions, dual of $\mathcal{S}(\mathbb{R}^3)$
$\mathcal{P}$	The space of polynomials on $\mathbb{R}^3$
$H^s(\mathbb{R}^3)$	Sobolev space of order s on $\mathbb{R}^3$
$[P, Q]$	Commutator: $PQ - QP$
$\hat{\mathbf{v}}, \mathcal{F}\mathbf{v}$	Fourier transform of $\mathbf{v}$
$\check{\mathbf{v}}, \mathcal{F}^{-1}\mathbf{v}$	Inverse Fourier transform operator of $\mathbf{v}$

This system of equations describes a class of incompressible yet density-inhomogeneous complex fluids, whose physical essence is highly relevant to systems such as nematic liquid crystals and other viscoelastic fluids. Here, the density ϱ acts as a flow-transported quantity that marks the fluid's inherent inhomogeneity. Central to the system's physics is the introduction of an elastic strain b and its strong coupling with the flow: the evolution equation for b, which combines advection, relaxation (driven by the function $w(b)$), and diffusion (representing the system's tendency to homogenize its internal structure), collectively models the dynamics of microscopic configurations in this class of fluids, such as the orientation of liquid crystal molecules or the conformation of polymer chains. Changes in this microstructure directly influence the macroscopic momentum balance through a mechanism effected by the stress tensor $\sigma \mathrm{div}\,(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2 \mathbb{I})$ in the momentum equation; this nonlinear term, originating from spatial gradients of the internal variable, is the fundamental source of non-Newtonian behaviors, such as normal stress differences, and leads to characteristic phenomena like the Weissenberg effect (rod-climbing). Simultaneously, the viscous dissipation term $-\Delta \mathbf{v}$ provides the primary mechanism for momentum diffusion, smoothing the velocity field and dissipating kinetic energy into heat, while the elastic diffusion term $-\sigma \Delta b$ in the b-equation governs the relaxation of the internal microstructure. Consequently, the model fully captures several hallmark features of viscoelastic or complex fluids: upon cessation of flow, the internal variable b relaxes toward the minima of its potential via the relaxation term, causing the associated elastic stress to decay—a manifestation of stress relaxation. In summary, this system of equations provides a rigorous mathematical framework for analyzing the dynamics of a class of incompressible complex fluids with orientational elasticity (notably nematic liquid crystals), clearly elucidating their fundamental viscoelastic physical nature. Meanwhile, to facilitate the reader's understanding of the coupling relationships between the fluid velocity, stress diffusion, and elastic strain, we present a schematic diagram as Figure 2.

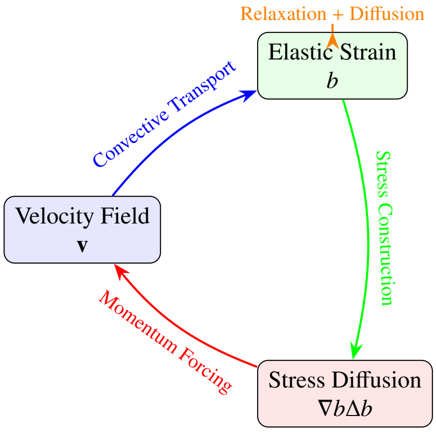

Figure 2. Coupling relationships between velocity, stress diffusion, and elastic strain in viscoelastic fluids.

In what follows, we shall investigate the large-time decay and stability of any given smooth solution to (3.1) with a constant viscosity coefficient. Let $a = \frac{1}{\varrho} - 1$, and base on the identity $\operatorname{div}(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2\mathbb{I}) = \nabla b\Delta b$, then model (3.1) can be rewritten as

$$\begin{cases} a_t + \mathbf{v}\cdot\nabla a = 0, \qquad (x,t)\in\mathbb{R}^3\times\mathbb{R}^+, \\ \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + (1+a)(\nabla P - \Delta\mathbf{v}) + (1+a)\nabla b\Delta b = 0, \\ b_t + \mathbf{v}\cdot\nabla b + w'(b) - \Delta b = 0, \\ \operatorname{div}\mathbf{v} = 0, \\ (a,\mathbf{v},b)(x,t)|_{t=0} = (a_0,\mathbf{v}_0,b_0)(x). \end{cases} \tag{3.2}$$

Our first result establishes the global stability for the solutions to (3.2) with initial density ϱ_0 near a positive constant. This extends the inhomogeneous Navier-Stokes system's stability result in [1].

Theorem 3.1. *Let* $(\bar{a},\bar{\mathbf{v}},\bar{b})$ *be a given global solution of the system* (3.2) *with initial data*

$$\begin{aligned} &(\bar{a}_0,\bar{\mathbf{v}}_0,\bar{b}_0)\in B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)\times B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)\times B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3), \quad \operatorname{div}\bar{\mathbf{v}}_0 = 0, \\ &m\le 1+\bar{a}_0\le M, \quad \text{for some } m, M>0. \end{aligned} \tag{3.3}$$

Suppose the solution possesses the following regularity:

$$\begin{aligned} &\bar{a}\in C(0,\infty;B^{\frac{5}{2}}_{2,1}), \quad \bar{\mathbf{v}}\in C(0,\infty;B^{\frac{3}{2}}_{2,1})\cap L^1_{\mathrm{loc}}(0,\infty;\dot{B}^{\frac{7}{2}}_{2,1}), \\ &\bar{b}\in C(0,\infty;B^{\frac{5}{2}}_{2,1})\cap L^1_{\mathrm{loc}}(0,\infty;\dot{B}^{\frac{9}{2}}_{2,1}). \end{aligned}$$

Then, there exist positive constants c_1, c_2, c_3*, and a sufficiently large* $T_1 \triangleq T_1(\bar{a}_0,\bar{\mathbf{v}}_0,\bar{b}_0)$ *such that if*

$$\|\bar{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\exp\{c_2\int_0^{T_1}\|\nabla\bar{\mathbf{v}}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}dt\}\le c_1, \tag{3.4}$$

then for any initial perturbation $(\tilde{a}_0, \tilde{\mathbf{v}}_0, \tilde{b}_0)$ *satisfying*

$$\|\tilde{a}_0\|_{B^{\frac{3}{2}}_{2,1}} + \|\tilde{\mathbf{v}}_0\|_{B^{\frac{1}{2}}_{2,1}} + \|\tilde{b}_0\|_{B^{\frac{3}{2}}_{2,1}} \leq c_3, \tag{3.5}$$

the perturbed initial data $(a_0, \mathbf{v}_0, b_0) = (\overline{a}_0 + \tilde{a}_0, \overline{\mathbf{v}}_0 + \tilde{\mathbf{v}}_0, \overline{b}_0 + \tilde{b}_0)$ *also admits a unique global solution* $(a, \mathbf{v}, b)$ *to system* (3.2)*, which satisfies the regularity:*

$$\begin{aligned} &a \in C_b(0,\infty; B^{\frac{5}{2}}_{2,1}), \quad \mathbf{v} \in C_b(0,\infty; B^{\frac{3}{2}}_{2,1}) \cap L^1(0,\infty; \dot{B}^{\frac{7}{2}}_{2,1}), \\ &b \in C_b(0,\infty; B^{\frac{5}{2}}_{2,1}) \cap L^1(0,\infty; \dot{B}^{\frac{9}{2}}_{2,1}). \end{aligned} \tag{3.6}$$

Next, to study the global stability of general smooth solutions to (3.2), we first require global-in-time estimation frameworks (e.g., (3.6)) for the reference system. Inspired by [1], we note that two key challenges necessitate a revised approach: first, the intrinsic hyperbolic nature of the continuity equation in (3.2); second, the analytical complexity in quantifying the pressure term in the momentum equation. We therefore focus on analyzing the large-time decay behavior of these reference solutions, and from which the desired result follows.

Theorem 3.2. *Let* $(a, \mathbf{v}, b)$ *be a global solution to* (3.2) *with initial data* $(a_0, \mathbf{v}_0, b_0)$*, satisfying*

$$\begin{aligned} &a \in C(0,\infty; B^{\frac{5}{2}}_{2,1}), \quad \mathbf{v} \in C(0,\infty; B^{\frac{3}{2}}_{2,1}) \cap L^1_{\mathrm{loc}}(0,\infty; \dot{B}^{\frac{7}{2}}_{2,1}), \\ &b \in C(0,\infty; B^{\frac{5}{2}}_{2,1}) \cap L^1_{\mathrm{loc}}(0,\infty; \dot{B}^{\frac{9}{2}}_{2,1}), \end{aligned}$$

where $\overline{a}_0$ *satisfies* (3.3)*,* $a_0 \in B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$*,* $b_0 \in B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$*, and* $\mathbf{v}_0 \in B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ $(1 < q < \frac{6}{5})$ *with* $\operatorname{div} \mathbf{v}_0 = 0$*. Define* $\delta(q) \triangleq \frac{3}{4}(\frac{2}{q} - 1)$*. Then, there exists a time* $t_2 > 0$ *such that for all* $t > t_2$*,*

$$\begin{aligned} &\|\mathbf{v}(t)\|_{L^2} \leq C_1(1+t)^{-\delta(q)}, \quad \|\nabla \mathbf{v}(t)\|_{L^2} \leq C_1(1+t)^{-\frac{1+2\delta(q)}{2}}, \quad \|\nabla b(t)\|_{L^2} \leq C_1 e^{-m_0^{-1} t}, \\ &\|\nabla^2 b(t)\|_{L^2} \leq C_1 e^{-\frac{1}{2} m_0^{-1} t}, \quad \int_{t_2}^{\infty} e^{m_0^{-1}\tau}(\|\nabla b_\tau\|_{L^2}^2 + \|\nabla^3 b\|_{L^2}^2) d\tau \leq C_1, \\ &\int_{t_2}^{\infty} (1+\tau)^l (\|\mathbf{v}_\tau\|_{L^2}^2 + \|\Delta \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) d\tau \leq C_1 \quad (0 < l < 1 + 2\delta(q)), \end{aligned} \tag{3.7}$$

and

$$\int_{t_2}^{\infty} (\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty} + \|\nabla^2 b\|_{L^\infty} + \|\mathbf{v}\|_{L^\infty} + \|\nabla \mathbf{v}\|_{L^\infty})(\tau) d\tau \leq C_1, \tag{3.8}$$

where C_1 *depends only on* m_0*,* m_1 *(from* $w(b)$*),* m*,* M *(given in* (3.3)*),* $\|a_0\|_{L^2}$*,* $\|b_0\|_{H^2}$*,* $\|\mathbf{v}_0\|_{L^q}$*, and* $\|\mathbf{v}_0\|_{H^1}$*.*

Remark 3.1. *The decay rates we obtained—algebraic decay for the velocity field* $\mathbf{v}$ *and exponential decay for the elastic variable* b *— accurately characterize two distinct yet coexisting energy dissipation mechanisms in viscoelastic fluids: viscous dissipation and elastic relaxation. The algebraic decay of the velocity field* $\mathbf{v}$ *is a typical feature of momentum diffusion in classical Newtonian fluids. This process is inherently nonlocal and gradual, with its decay rate depending on the spatial distribution scale of the initial disturbance, reflecting the persistence and gradual dissipation of macroscopic inertial effects in the fluid. In contrast, the exponential decay of the elastic variable* b *reflects the* *rapid relaxation of memory effects of microstructural memory effects following flow disturbances. Governed by an intrinsic relaxation timescale in the system, once initiated, the stored elastic energy is irreversibly converted into thermal energy in a determinate and rapid manner, yielding strong exponential decay. The coexistence of exponential and algebraic decay directly manifests the dual characteristics of "viscosity" and "elasticity" in the long-term dynamic behavior of viscoelastic fluids. Moreover, the exponential decay of* ∇b *signifies the rapid release of internal elastic stress generated by microstructural inhomogeneities. This explains why viscoelastic fluids, such as polymer solutions, can quickly "relax" and exhibit a rapid decline in their resistance to deformation once agitation ceases. In other words, driven by the principle of increasing entropy, the system evolves rapidly and irreversibly toward a state where* $\nabla b \to 0$*, that is, toward a spatially uniform and more disordered equilibrium state.*

By leveraging Theorem 3.2 and the proof technique developed in [1], we establish the global stability result for the general smooth solutions of (3.2) as follows.

Theorem 3.3. *Assume that* $\overline{a}_0 \in B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$ *satisfies* (3.3)*,* $\overline{b}_0 \in B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$*, and* $\overline{\mathbf{v}}_0 \in B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *for some* $1 < q < \frac{6}{5}$ *with* $\operatorname{div} \overline{\mathbf{v}}_0 = 0$*. Let* $(\overline{a}, \overline{\mathbf{v}}, \overline{b})$ *be a global solution of system* (3.2) *with initial data* $(\overline{a}_0, \overline{\mathbf{v}}_0, \overline{b}_0)$ *satisfying the following regularity conditions:*

$$\overline{a} \in C(0,\infty; B^{\frac{5}{2}}_{2,1}), \quad \overline{\mathbf{v}} \in C(0,\infty; B^{\frac{3}{2}}_{2,1}) \cap L^1_{\mathrm{loc}}(0,\infty; \dot{B}^{\frac{7}{2}}_{2,1}), \quad \overline{b} \in C(0,\infty; B^{\frac{5}{2}}_{2,1}) \cap L^1_{\mathrm{loc}}(0,\infty; \dot{B}^{\frac{9}{2}}_{2,1}).$$

Then, there exists a positive constant c_4 *such that for any*

$$(\tilde{a}_0, \tilde{\mathbf{v}}_0, \tilde{b}_0) \in B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3) \times (B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)) \times B^{\frac{5}{2}}_{2,1}(\mathbb{R}^3),$$
$$\text{with} \quad G_0 \triangleq \|\tilde{a}_0\|_{B^{\frac{3}{2}}_{2,1}} + \|\tilde{\mathbf{v}}_0\|_{H^1} + \|\tilde{\mathbf{v}}_0\|_{L^p} + \|\tilde{b}_0\|_{H^2} \le c_4,$$

the system (3.2) *with initial data* $(a_0, \mathbf{v}_0, b_0) = (\overline{a}_0 + \tilde{a}_0, \overline{\mathbf{v}}_0 + \tilde{\mathbf{v}}_0, \overline{b}_0 + \tilde{b}_0)$ *admits a unique global smooth solution* $(a, \mathbf{v}, b)$ *satisfying the regularity*

$$\begin{aligned} &a \in C_b(0,\infty; B^{\frac{5}{2}}_{2,1}), \quad \mathbf{v} \in C_b(0,\infty; B^{\frac{3}{2}}_{2,1} \cap L^q) \cap L^1_{\mathrm{loc}}(0,\infty; \dot{B}^{\frac{7}{2}}_{2,1}), \\ &b \in C_b(0,\infty; B^{\frac{5}{2}}_{2,1}) \cap L^1_{\mathrm{loc}}(0,\infty; \dot{B}^{\frac{9}{2}}_{2,1}). \end{aligned} \tag{3.9}$$

Moreover, for any $s \in [\frac{1}{2}, \frac{3}{2}]$*, the solution* $(a, \mathbf{v}, b)$ *satisfies the stability estimates*

$$\begin{aligned} &\|a - \overline{a}\|_{\tilde{L}^\infty(0,\infty; B^{s+1}_{2,1})} + \|\mathbf{v} - \overline{\mathbf{v}}\|_{\tilde{L}^\infty(0,\infty; L^p)} + \|\mathbf{v} - \overline{\mathbf{v}}\|_{\tilde{L}^\infty(0,\infty; B^{s}_{2,1})} \\ &+ \|\mathbf{v} - \overline{\mathbf{v}}\|_{L^1(0,\infty; \dot{B}^{s+2}_{2,1})} + \|b - \overline{b}\|_{\tilde{L}^\infty(0,\infty; B^{s+1}_{2,1})} + \|b - \overline{b}\|_{L^1(0,\infty; \dot{B}^{s+3}_{2,1})} \le C G_0^{\frac{3-2s}{2}}. \end{aligned} \tag{3.10}$$

In summary, Theorem 3.1 establishes the stability of a given solution to the model. Theorem 3.3 then generalizes this result to the case of arbitrary solutions. Owing to the challenges posed by the hyperbolic nature of the continuity equation in (3.2) and the estimation of the pressure term in the momentum equation, we therefore first study the large-time decay of the reference solutions in Theorem 3.2. The rest of this paper is structured as follows. In Section 4, we establish the global stability of a specific solution to system (3.2), as formulated in Theorem 3.1. Section 5 focuses on analyzing the temporal decay rate of global solutions, which corresponds to Theorem 3.2. Section 6 is dedicated to proving Theorem 3.3, where the stability results are generalized to arbitrary smooth solutions of (3.2). Finally, in Section 7, we not only provide a synthesis of the principal findings but also critically examine the limitations inherent in our model, while simultaneously delineating prospective avenues for future research and presenting the strategic roadmap for our subsequent investigative work.

4. Proof of Theorem 3.1

This section is devoted to proving the global stability of a given solution to system (3.2) under the assumption of bounded density, as stated in Theorem 3.1. We begin by establishing the uniform boundedness of the reference solution via energy estimates in Besov spaces and a bootstrap argument. Following this, the system for the perturbations, derived from the difference between the perturbed and reference equations, is shown to be bounded using the same technique. The global regularity of the solution to the perturbed system is thereby established. The detailed proof proceeds as follows.

First, to establish the global well-posedness of the reference solution $(\overline{a}, \overline{\mathbf{v}}, \overline{b})$ to system (3.2) with initial data $(\overline{a}_0, \overline{\mathbf{v}}_0, \overline{b}_0)$, we derive uniform-in-time estimates for $(\overline{a}, \overline{\mathbf{v}}, \overline{b})$. We define the density variable $\overline{\varrho} \triangleq \frac{1}{1+\overline{a}}$, and perform classical energy analysis on (3.2).

A direct consequence of the continuity Eq $(3.1)_1$ is the invariance property: for any $1 \le p \le \infty$,

$$\|\overline{\varrho} - 1\|_{L^p} = \|\overline{\varrho}_0 - 1\|_{L^p}. \tag{4.1}$$

Next, multiplying the momentum Eq $(3.1)_2$ by the velocity field $\overline{\mathbf{v}}$ and integrating over $\mathbb{R}^3$ yields:

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\overline{\varrho}}\,\overline{\mathbf{v}}(t)\|_{L^2}^2 + \|\nabla\overline{\mathbf{v}}\|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla\overline{b}\Delta\overline{b}\cdot\overline{\mathbf{v}}dx = 0. \tag{4.2}$$

Taking the L^2 inner product of $(3.1)_3$ with $w'(\overline{b})$ and $-\Delta\overline{b}$, respectively, and integrating by parts, we derive the following equations:

$$\frac{d}{dt}\|w(\overline{b})\|_{L^1} + \|w'(\overline{b})\|_{L^2}^2 + \int_{\mathbb{R}^3} w''(\overline{b})|\nabla\overline{b}|^2 dx = 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla\overline{b}(t)\|_{L^2}^2 + \|\Delta\overline{b}\|_{L^2}^2 - \int_{\mathbb{R}^3} \overline{\mathbf{v}}\cdot\nabla\overline{b}\Delta\overline{b}dx + \int_{\mathbb{R}^3} w''(\overline{b})|\nabla\overline{b}|^2 dx = 0. \tag{4.3}$$

Summing the above equations and integrating over $(0,t)$, the condition $w''(b) > m_0^{-1}$ yields: for all $t > 0$,

$$\begin{aligned} &\|(\sqrt{\overline{\varrho}}\,\overline{\mathbf{v}}(t), \nabla\overline{b})\|_{L^\infty(0,t;L^2)}^2 + \|w(\overline{b})\|_{L^\infty(0,t;L^1)} + \|(\nabla\overline{\mathbf{v}}, \nabla\overline{b}, \Delta\overline{b}, w'(\overline{b}))\|_{L^2(0,t;L^2)}^2 \\ &\le C(\|\sqrt{\overline{\varrho}_0}\,\overline{\mathbf{v}}_0\|_{L^2}^2 + \|\nabla\overline{b}_0\|_{L^2}^2 + \|w(\overline{b}_0)\|_{L^1}), \end{aligned} \tag{4.4}$$

which, together with the interpolation theorem, implies that

$$\int_0^t \|\overline{\mathbf{v}}\|_{\dot{B}_{2,1}^{\frac{1}{2}}}^4 dt' \le C\int_0^t \|\overline{\mathbf{v}}\|_{L^2}^2\|\nabla\overline{\mathbf{v}}\|_{L^2}^2 dt' \le C(\|\overline{\mathbf{v}}_0\|_{L^2}^4 + \|\nabla\overline{b}_0\|_{L^2}^4 + \|w(\overline{b}_0)\|_{L^1}^2),$$

$$\int_0^t \|\nabla\overline{b}\|_{\dot{B}_{2,1}^{\frac{1}{2}}}^4 dt' \le C\int_0^t \|\nabla\overline{b}\|_{L^2}^2\|\nabla^2\overline{b}\|_{L^2}^2 dt' \le C(\|\overline{\mathbf{v}}_0\|_{L^2}^4 + \|\nabla\overline{b}_0\|_{L^2}^4 + \|w(\overline{b}_0)\|_{L^1}^2).$$

Thus, for any $\varepsilon > 0$, there exists $T_1(\varepsilon) > 0$ (simplified as T_1 hereafter) such that

$$\|\bar{\mathbf{v}}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\nabla\bar{b}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} < \varepsilon. \tag{4.5}$$

On the other hand, applying (2.7) to Eq $(3.2)_1$ over $(0, T_1)$ gives

$$\|\bar{a}\|_{\tilde{L}^{\infty}_{T_1}(\dot{B}^{\frac{3}{2}}_{2,1})} \le \|\bar{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \exp\{c_2 \int_0^{T_1} \|\nabla\bar{\mathbf{v}}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} d\tau\}. \tag{4.6}$$

For $t > T_1$, from (2.6), we deduce that

$$\|\bar{a}(t)\|_{\tilde{L}^{\infty}(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})} \le \|\bar{a}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + C\|\bar{a}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\|\bar{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})}. \tag{4.7}$$

For the momentum equation, by Lemma 2.8 and the product laws in Besov spaces, it follows that

$$\begin{aligned}
&\|\bar{\mathbf{v}}\|_{\tilde{L}^{\infty}(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})} + \|\bar{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\bar{P}\|_{L^1(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})}\\
&\le \|\bar{\mathbf{v}}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C\|\bar{\mathbf{v}}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})}\|\bar{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})} + C\|\bar{b}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\|\bar{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}\\
&\quad + C\|\bar{a}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\|\bar{b}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\|\bar{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}\\
&\quad + C\|\bar{a}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}(\|\nabla\bar{P}\|_{L^1(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})} + \|\bar{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})}).
\end{aligned} \tag{4.8}$$

Next, we analyze the $\bar{b}$ equation. Applying the block operator $\dot{\Delta}_j$ to $(3.1)_3$ and then the gradient operator ∇ yields

$$\partial_t\dot{\Delta}_j\nabla\bar{b} - \dot{\Delta}_j\nabla\Delta\bar{b} + w''(\bar{b})\dot{\Delta}_j\nabla\bar{b} = -\nabla([\dot{\Delta}_j, \bar{\mathbf{v}}\cdot\nabla]\bar{b}) - \nabla(\bar{\mathbf{v}}\cdot\dot{\Delta}_j\nabla\bar{b}) - [\dot{\Delta}_j, w''(\bar{b})]\nabla\bar{b}. \tag{4.9}$$

Taking the L^2 inner product of (4.9) with $\dot{\Delta}_j\nabla\bar{b}$ gives the fundamental energy inequality directly:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\nabla\bar{b}\|^2_{L^2} + m2^{2j}\|\dot{\Delta}_j\nabla\bar{b}\|^2_{L^2} + m_0^{-1}\|\dot{\Delta}_j\nabla\bar{b}\|^2_{L^2} \le C2^j\|[\dot{\Delta}_j, \bar{\mathbf{v}}\cdot\nabla]\bar{b}\|_{L^2}\|\dot{\Delta}_j\nabla\bar{b}\|_{L^2}\\
&+ \Big|\int_{\mathbb{R}^3}\nabla(\bar{\mathbf{v}}\cdot\dot{\Delta}_j\nabla\bar{b})\cdot\dot{\Delta}_j\nabla\bar{b}dx\Big| + \|[\dot{\Delta}_j, w''(\bar{b})]\nabla\bar{b}\|_{L^2}\|\dot{\Delta}_j\nabla\bar{b}\|_{L^2}.
\end{aligned} \tag{4.10}$$

Applying Lemmas 2.1 and 2.5 to the first term on the righthand side of the inequality (4.10) gives that

$$2^j\|[\dot{\Delta}_j, \bar{\mathbf{v}}\cdot\nabla]\bar{b}\|_{L^2}\|\dot{\Delta}_j\nabla\bar{b}\|_{L^2} \le C2^{-\frac{1}{2}j}\|\bar{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\nabla\bar{b}\|_{\dot{B}^{\frac{1}{2}}_{2,r}}\|\dot{\Delta}_j\nabla\bar{b}\|_{L^2}.$$

The second term vanishes after integrating by parts together with $\operatorname{div}\bar{\mathbf{v}} = 0$:

$$\int_{\mathbb{R}^3}\nabla(\bar{\mathbf{v}}\cdot\dot{\Delta}_j\nabla\bar{b})\cdot\dot{\Delta}_j\nabla\bar{b}dx = -\int_{\mathbb{R}^3}\bar{\mathbf{v}}\cdot\dot{\Delta}_j\nabla\bar{b}\nabla(\dot{\Delta}_j\nabla\bar{b})dx = -\frac{1}{2}\int_{\mathbb{R}^3}\bar{\mathbf{v}}\cdot\nabla(\dot{\Delta}_j\nabla\bar{b})^2dx = 0.$$

Integrating (4.10) over (T_1, t) and combining the above estimates, the following is obtained:

$$\begin{aligned}
&\|\dot{\Delta}_j\nabla\bar{b}\|_{L^{\infty}(T_1,t;L^2)} + m2^{2j}\|\dot{\Delta}_j\nabla\bar{b}\|_{L^1(T_1,t;L^2)} + m_0^{-1}\|\dot{\Delta}_j\nabla\bar{b}\|_{L^1(T_1,t;L^2)}\\
&\lesssim \|\dot{\Delta}_j\nabla\bar{b}(T_1)\|_{L^2} + C2^{-\frac{1}{2}j}\|\bar{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})}\|\nabla\bar{b}\|_{L^{\infty}(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})} + \|[\dot{\Delta}_j, w''(\bar{b})]\nabla\bar{b}\|_{L^1(T_1,t;L^2)}.
\end{aligned} \tag{4.11}$$

Plugging the commutator operator (2.5) into (4.11) and utilizing the definition of Besov spaces, we arrive at

$$\begin{aligned}&\|\overline{b}\|_{\tilde{L}^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}+\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}+\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\\&\le\|\overline{b}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+C\|\overline{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})}\|\overline{b}(t)\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\\&\quad+C\|\overline{b}\|_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}+C\|\overline{b}\|^3_{L^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}.\end{aligned}\tag{4.12}$$

On the other hand, define the function $\overline{Z}(t)$ as

$$\overline{Z}(t)\triangleq\|\overline{a}\|_{\tilde{L}^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}+\|\overline{\mathbf{v}}\|_{\tilde{L}^\infty(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})}+\|\overline{\mathbf{v}}\|_{L^1(T_1,t;\dot{B}^{\frac{5}{2}}_{2,1})}+\|\nabla\overline{P}\|_{L^1(T_1,t;\dot{B}^{\frac{1}{2}}_{2,1})}+\|\overline{b}\|_{\tilde{L}^\infty(T_1,t;\dot{B}^{\frac{3}{2}}_{2,1})}+\|\overline{b}\|_{L^1(T_1,t;\dot{B}^{\frac{7}{2}}_{2,1})}.\tag{4.13}$$

This, together with the inequalities (4.7), (4.8), and (4.12), yields

$$\overline{Z}(t)\le\|\overline{a}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+\|\overline{\mathbf{v}}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}+\|\overline{b}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+C_2(\overline{Z}^2+\overline{Z}^3+\overline{Z}^4).\tag{4.14}$$

We subsequently define a time T' as follows:

$$T'\triangleq\sup_{t>T_1}\{t:\overline{Z}\le 3(\|\overline{a}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+\|\overline{\mathbf{v}}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}+\|\overline{b}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})\}<\infty.\tag{4.15}$$

Assuming c_1 (introduced in (3.4)) and ε (introduced in (4.5)) are chosen sufficiently small, it follows directly from (4.6) and (4.14) that for all $T_1\le t\le T'$,

$$\overline{Z}(t)\le(\|\overline{a}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+\|\overline{\mathbf{v}}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}+\|\overline{b}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})(1+9C_2\overline{Z}).$$

Furthermore, choose $c_1\le\frac{1}{54C_2}$ and $\varepsilon\le\frac{1}{54C_2}$, while ensuring that both c_1 and ε are sufficiently small, then we get

$$\overline{Z}(t)\le 2(\|\overline{a}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}+\|\overline{\mathbf{v}}(T_1)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}+\|\overline{b}(T_1)\|_{\dot{B}^{\frac{3}{2}}_{2,1}}),$$

which contracts with the assumption (4.15). Thus, one can deduce that $T'=\infty$ and $\overline{Z}(t)$ remains bounded over $\mathbb{R}^+$, namely,

$$\|\overline{\mathbf{v}}\|_{\tilde{L}^\infty(0,+\infty;\dot{B}^{\frac{1}{2}}_{2,1})}+\|\overline{\mathbf{v}}\|_{L^1(0,+\infty;\dot{B}^{\frac{5}{2}}_{2,1})}+\|\nabla\overline{P}\|_{L^1(0,+\infty;\dot{B}^{\frac{1}{2}}_{2,1})}+\|\overline{a}\|_{\tilde{L}^\infty(0,+\infty;\dot{B}^{\frac{3}{2}}_{2,1})}+\|\overline{b}\|_{\tilde{L}^\infty(0,+\infty;\dot{B}^{\frac{3}{2}}_{2,1})}+\|\overline{b}\|_{L^1(0,+\infty;\dot{B}^{\frac{7}{2}}_{2,1})}\le C.\tag{4.16}$$

What's more, we have

$$\|\overline{a}\|_{\tilde{L}^\infty(0,+\infty;\dot{B}^{\frac{3}{2}}_{2,1})}\le 2(\|\overline{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\exp\{c_2\int_0^{T_1}\|\nabla\overline{\mathbf{v}}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}d\tau\}+\varepsilon)\le 2(c_1+\varepsilon).\tag{4.17}$$

Building on these estimates, we proceed to investigate the global well-posedness of system (3.2) with initial data $(a_0,\mathbf{v}_0,b_0)$. The initial data is decomposed as $(a_0,\mathbf{v}_0,b_0)=(\overline{a}_0+\tilde{a}_0,\overline{\mathbf{v}}_0+\tilde{\mathbf{v}}_0,\overline{b}_0+\tilde{b}_0)$, where $(\tilde{a}_0,\tilde{\mathbf{v}}_0,\tilde{b}_0)$ are sufficiently small perturbations. With $\tilde{\mathbf{v}}\triangleq\mathbf{v}-\overline{\mathbf{v}}$ and $\tilde{b}\triangleq b-\overline{b}$ defined, the variables

$(a, \tilde{\mathbf{v}}, \tilde{b})$ satisfy

$$\begin{cases} a_t + (\bar{\mathbf{v}} + \tilde{\mathbf{v}}) \cdot \nabla a = 0, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \tilde{\mathbf{v}}_t - \Delta \tilde{\mathbf{v}} + \nabla \tilde{P} = -(\bar{\mathbf{v}} + \tilde{\mathbf{v}}) \nabla \tilde{\mathbf{v}} - \tilde{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} - (1 + a)(\nabla \bar{b} \Delta \tilde{b} + \nabla \tilde{b} \Delta \bar{b} + \nabla \tilde{b} \Delta \tilde{b}) \\ \qquad\qquad + (a - \bar{a})(\Delta \bar{\mathbf{v}} - \nabla \bar{P} - \nabla \bar{b} \Delta \bar{b}) + a(\Delta \tilde{\mathbf{v}} - \nabla \tilde{P}), \\ \tilde{b}_t + \bar{\mathbf{v}} \cdot \nabla \tilde{b} - \Delta \tilde{b} = -\tilde{\mathbf{v}} \cdot \nabla \tilde{b} - \tilde{\mathbf{v}} \cdot \nabla \bar{b} - w''(\bar{b}) \tilde{b} - o(|\tilde{b}|^2), \\ \operatorname{div} \tilde{\mathbf{v}} = 0, \\ (a, \tilde{\mathbf{v}}, \tilde{b})(x, t)|_{t=0} = (a_0, \tilde{\mathbf{v}}_0, \tilde{b}_0)(x). \end{cases} \tag{4.18}$$

Perturbations of the single-frequency term generate the higher-order nonlinear term $o(|\tilde{b}|^2)$, but it barely affects the stability and decay estimates of the equation, thus it can be neglected. For computational convenience, we will remove it in the subsequent energy estimates.

An application of Lemma 2.7 to the density Eq $(4.18)_1$ yields

$$\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \le \|a_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + C \int_0^t \|\bar{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|a\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \, d\tau + C \|a\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\tilde{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})}.$$

Then, from (4.16), one further derives

$$\|a\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \le C\Big(\|a_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|a\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\tilde{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} \Big). \tag{4.19}$$

Adopting the method of (4.8) provides the estimate for the velocity field $\tilde{\mathbf{v}}$ as follows:

$$\begin{aligned} \|\tilde{\mathbf{v}}\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{1}{2}}_{2,1})} &+ \|\tilde{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla \tilde{P}\|_{L^1_t(\dot{B}^{\frac{1}{2}}_{2,1})} \le \|\tilde{\mathbf{v}}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C \int_0^t \|\bar{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|\tilde{\mathbf{v}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \, d\tau \\ &+ C \|\tilde{\mathbf{v}}\|_{L^\infty_t(\dot{B}^{\frac{1}{2}}_{2,1})} \|\tilde{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + C \int_0^t \|\bar{b}\|_{\dot{B}^{\frac{7}{2}}_{2,1}} \|\tilde{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \, d\tau + C \|\tilde{b}\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\tilde{b}\|_{L^1_t(\dot{B}^{\frac{7}{2}}_{2,1})} \\ &+ C \|a\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \Big(\|\bar{b}\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\tilde{b}\|_{L^1_t(\dot{B}^{\frac{7}{2}}_{2,1})} + \|\tilde{b}\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\tilde{b}\|_{L^1_t(\dot{B}^{\frac{7}{2}}_{2,1})} \Big) \\ &+ C(\|a\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\bar{a}\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})}) \Big(\|\bar{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla \bar{P}\|_{L^1_t(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\bar{b}\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\bar{b}\|_{L^1_t(\dot{B}^{\frac{7}{2}}_{2,1})} \Big) \\ &+ C \|a\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \Big(\|\tilde{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla \tilde{P}\|_{L^1_t(\dot{B}^{\frac{1}{2}}_{2,1})} \Big). \end{aligned}$$

Similarly, the $\tilde{b}$ equation implies that

$$\begin{aligned} \|\tilde{b}\|_{\tilde{L}^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} &+ \|\tilde{b}\|_{L^1_t(\dot{B}^{\frac{7}{2}}_{2,1})} \le \|\tilde{b}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + C \int_0^t \Big(\|\bar{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|\tilde{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\bar{b}\|_{\dot{B}^{\frac{7}{2}}_{2,1}} \|\tilde{\mathbf{v}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \Big) d\tau \\ &+ \int_0^t (1 + \|\bar{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}})^2 \|\bar{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|\tilde{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \, d\tau + C \|\tilde{b}\|_{L^\infty_t(\dot{B}^{\frac{3}{2}}_{2,1})} \|\tilde{\mathbf{v}}\|_{L^1_t(\dot{B}^{\frac{5}{2}}_{2,1})}, \end{aligned}$$

where we used the fact that

$$\|w''(\bar{b})\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \le C(1 + \|\bar{b}\|_{L^\infty})^2 \|\bar{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \le C(1 + \|\bar{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}})^2 \|\bar{b}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}.$$

Summing the two inequalities above, it follows from (4.16) that

$$\begin{aligned}
&\|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\tilde{P}\|_{L_t^1(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\tilde{b}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\tilde{b}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} \\
\lesssim&\|\tilde{\mathbf{v}}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\tilde{b}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\tilde{b}\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\tilde{\mathbf{v}}\|_{L_t^\infty(\dot{B}^{\frac{1}{2}}_{2,1})}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} \\
&+ \|\tilde{b}\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}\|\tilde{b}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} + \|a\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}\|\overline{b}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} + \|a\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}\|\tilde{b}\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}\|\tilde{b}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} \\
&+ m_2'(\|\overline{a}\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \|a\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}) + \|a\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}(\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\tilde{P}\|_{L_t^1(\dot{B}^{\frac{1}{2}}_{2,1})}).
\end{aligned} \tag{4.20}$$

Let

$$\tilde{Z}(t) \triangleq \|a\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\tilde{P}\|_{L_t^1(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\tilde{b}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\tilde{b}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})}. \tag{4.21}$$

By multiplying (4.19) with a suitable coefficient (e.g., $m_2' + 1$), we can eliminate the $\|a\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}$ term from the righthand side of (4.20). Subsequent addition of the resulting equation to (4.20) yields

$$\tilde{Z}(t) \leq m_2\Big(\|\tilde{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\tilde{\mathbf{v}}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\tilde{b}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\overline{a}\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \tilde{Z}^2(t) + \tilde{Z}^3(t)\Big).$$

Along the same line of $\overline{Z}(t)$ and by invoking (4.17), we conclude that if $\|\tilde{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\tilde{\mathbf{v}}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\tilde{b}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + 2c_1 + 2\varepsilon$ is sufficiently small, then for all $t > 0$, the following holds:

$$\begin{aligned}
&\|a\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\tilde{P}\|_{L_t^1(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\tilde{b}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} \\
&+ \|\tilde{b}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} \leq 3m_2(\|\tilde{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\overline{a}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\tilde{\mathbf{v}}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\tilde{b}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}}).
\end{aligned} \tag{4.22}$$

Next, we turn to enhancing the regularity propagation for smoother initial data. Upon applying Lemmas 2.7 and 2.8 and Corollary 2.1 to the equations for a and b in system (3.2), it follows that

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{5}{2}}_{2,1})} \lesssim \|a_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \exp\{\int_0^t (\|\overline{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\tilde{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}})d\tau\} \lesssim \|a_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}},$$

and by using the estimates (4.16) and (4.22),

$$\begin{aligned}
\|b\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{5}{2}}_{2,1})} + \|b\|_{L_t^1(\dot{B}^{\frac{9}{2}}_{2,1})} &\lesssim (\|b_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|w'(b)\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})}) \exp\{\int_0^t (\|\overline{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\tilde{\mathbf{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}})d\tau\} \\
&\lesssim \|b_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + (1 + \|b\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})})^3 \|b\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} \leq C.
\end{aligned}$$

With regard to the velocity field $\mathbf{v}$, a standard energy estimate applied to the solution of $(3.1)_3$ leads to

$$\begin{aligned}
&\|\mathbf{v}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})} + \|\mathbf{v}\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} + \|\nabla P\|_{L_t^1(\dot{B}^{\frac{3}{2}}_{2,1})} \\
\lesssim& \|\mathbf{v}_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \int_0^t \|\mathbf{v}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\mathbf{v}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} d\tau + \|a\|_{L_t^\infty(\dot{B}^{\frac{5}{2}}_{2,1})}\big(\|\mathbf{v}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla P\|_{L_t^1(\dot{B}^{\frac{1}{2}}_{2,1})}\big) \\
&+ \|b\|_{L_t^\infty(\dot{B}^{\frac{5}{2}}_{2,1})}\|b\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})} + \|a\|_{L_t^\infty(\dot{B}^{\frac{3}{2}}_{2,1})}\|b\|_{L_t^\infty(\dot{B}^{\frac{5}{2}}_{2,1})}\|b\|_{L_t^1(\dot{B}^{\frac{7}{2}}_{2,1})}.
\end{aligned} \tag{4.23}$$

Using (4.16), (4.22), and (4.23), and adopting the same proof method as in (2.7), we deduce

$$\|\mathbf{v}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\mathbf{v}\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} + \|\nabla P\|_{L_t^1(\dot{B}_{2,1}^{\frac{3}{2}})} \leq C. \tag{4.24}$$

Finally, following the approach of [1], these a priori bounds combined with a classical regularity argument establish the following improved regularity result:

$$a \in C_b(0,\infty; B_{2,1}^{\frac{5}{2}}), \quad \mathbf{v} \in C_b(0,\infty; B_{2,1}^{\frac{3}{2}}) \cap L^1(0,\infty; \dot{B}_{2,1}^{\frac{7}{2}}), b \in C_b(0,\infty; B_{2,1}^{\frac{5}{2}}) \cap L^1(0,\infty; \dot{B}_{2,1}^{\frac{9}{2}}).$$

Hence, Theorem 3.1 is now established.

5. Proof of Theorem 3.2

As noted in Section 3, the reference system's global estimate (3.16) relies on small initial data. To overcome this limitation, we use Schonbek's approach (systematically developed in [24]) to address it here. This section focuses on proving the large-time decay of global solution to system (3.2) (Theorem 3.2), with the proof structured into five propositions for clarity.

Proposition 5.1. *Under the conditions of Theorem 3.2, there exist a time $t_2 > 0$ and some positive constants c_i' $(i = 1, 2, 3, 4)$ such that for all $t > t_2$, the following inequality holds*

$$\frac{d}{dt}\|(\sqrt{\varrho}\mathbf{v}, \nabla\mathbf{v}, \nabla b, \nabla^2 b)\|_{L^2}^2 + c_1'\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + c_2'\|\nabla b_t\|_{L^2}^2 + c_3'\|\nabla^2\mathbf{v}\|_{L^2}^2 + c_4'\|\nabla^3 b\|_{L^2}^2 \leq 0, \tag{5.1}$$

or, consequently,

$$\begin{aligned} &\sup_{t\geq t_2}\|(\sqrt{\varrho}\mathbf{v}(t), \nabla\mathbf{v}(t), \nabla b(t), \nabla^2 b(t))\|_{L^2}^2 + \int_{t_2}^\infty \Big(c_1'\|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 + c_2'\|\nabla b_\tau\|_{L^2}^2 \\ &+ c_3'\|\nabla^2\mathbf{v}\|_{L^2}^2 + c_4'\|\nabla^3 b\|_{L^2}^2\Big)d\tau \lesssim \|(\sqrt{\varrho}\mathbf{v}, \nabla\mathbf{v}, \nabla b, \nabla^2 b)(t_2)\|_{L^2}^2. \end{aligned} \tag{5.2}$$

Proof. Motivated by the method in [24], we multiply $(3.1)_2$ by $-\frac{1}{\varrho}\Delta\mathbf{v}$ and integrate over $\mathbb{R}^3$, obtaining

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_{L^2}^2 + m\|\Delta\mathbf{v}\|_{L^2}^2 \lesssim \int_{\mathbb{R}^3}\Big|(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Delta\mathbf{v} + \frac{1}{\varrho}\nabla P\cdot\Delta\mathbf{v} + \frac{1}{\varrho}\nabla b\Delta b\cdot\Delta\mathbf{v}\Big|dx \\ &\lesssim \|\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}\|_{L^6}\|\Delta\mathbf{v}\|_{L^2} + \|1+a\|_{L^\infty}(\|\nabla P\|_{L^2}\|\Delta\mathbf{v}\|_{L^2} + \|\nabla b\|_{L^3}\|\Delta b\|_{L^6}\|\Delta\mathbf{v}\|_{L^2}). \end{aligned} \tag{5.3}$$

Regarding the pressure gradient term $\|\nabla P\|_{L^2}^2$, elliptic regularity estimates in conjunction with the divergence-free constraint lead to

$$\begin{aligned} \|\nabla P\|_{L^2}^2 &\lesssim \|\varrho\mathbf{v}_t + \varrho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla b\Delta b\|_{L^2}^2 \\ &\lesssim \|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}\|\nabla^2\mathbf{v}\|_{L^2}^2 + \|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^2. \end{aligned} \tag{5.4}$$

Then, substituting (5.4) into (5.3) shows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_{L^2}^2 + \frac{m}{2}\|\Delta\mathbf{v}\|_{L^2}^2 \lesssim& \|\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{v}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}\|\nabla^2\mathbf{v}\|_{L^2}^2 \\ &+ \|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + \|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^2. \end{aligned} \tag{5.5}$$

On the other hand, testing $(3.1)_2$ with $\mathbf{v}_t$ and integrating by parts gives rise to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|_{L^2}^2+\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2\lesssim\int_{\mathbb{R}^3}|\varrho\mathbf{v}\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t+\nabla b\Delta b\cdot\mathbf{v}_t|dx\\
&\lesssim\|\sqrt{\varrho}\|_{L^\infty}\|\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}\|_{L^6}\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}+\|\nabla b\|_{L^3}\|\nabla^2 b\|_{L^6}\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}\\
&\lesssim\frac{1}{2}\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2+\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}\|\nabla^2\mathbf{v}\|_{L^2}^2+\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^2.
\end{aligned}\tag{5.6}$$

Subsequently, by applying the gradient operator ∇ to Eq $(3.1)_3$, we obtain

$$\nabla b_t+\nabla\mathbf{v}\nabla b+\mathbf{v}\nabla^2 b+w''(b)\nabla b-\nabla^3 b=0.\tag{5.7}$$

Taking the L^2 inner product of (5.7) with $-\nabla^3 b$ and using the bounds $m_0^{-1}\le w''(b)\le m_0$, $|w'''(b)|\le m_1$ leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 b\|_{L^2}^2+\|\nabla^3 b\|_{L^2}^2\\
&\lesssim\int_{\mathbb{R}^3}\Big|\nabla\mathbf{v}\cdot\nabla b\nabla^3 b+\mathbf{v}\nabla^2 b\nabla^3 b+w''(b)|\nabla^2 b|^2+w'''(b)|\nabla b|^2\nabla^2 b\Big|dx\\
&\lesssim\|\nabla\mathbf{v}\|_{L^6}\|\nabla b\|_{L^3}\|\nabla^3 b\|_{L^2}+\|\mathbf{v}\|_{L^3}\|\nabla^2 b\|_{L^6}\|\nabla^3 b\|_{L^2}+\|\nabla b\|_{L^2}\|\nabla b\|_{L^6}\|\nabla^2 b\|_{L^3}\\
&\le\frac{1}{2}\|\nabla^3 b\|_{L^2}^2+C\|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}+C\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}\|\nabla^3 b\|_{L^2}^2\\
&\quad+C\|\nabla b\|_{L^2}^{\frac{4}{3}}\|\nabla^2 b\|_{L^2}^2+C_1\|\nabla^2 b\|_{L^2}^2.
\end{aligned}\tag{5.8}$$

Additionally, taking the L^2 inner product of with $-\Delta b_t$ and integrating by parts yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 b\|_{L^2}^2+\frac{1}{2}w''(b)\frac{d}{dt}\|\nabla b\|_{L^2}^2+\|\nabla b_t\|_{L^2}^2\\
&\lesssim\|\nabla\mathbf{v}\|_{L^6}\|\nabla b\|_{L^3}\|\nabla b_t\|_{L^2}+\|\mathbf{v}\|_{L^3}\|\nabla^2 b\|_{L^6}\|\nabla b_t\|_{L^2}\\
&\le\frac{1}{2}\|\nabla b_t\|_{L^2}^2+C(\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^2\mathbf{v}\|_{L^2}^2+\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}\|\nabla^3 b\|_{L^2}^2).
\end{aligned}\tag{5.9}$$

To eliminate the last term in (5.8), we sum Eqs (4.2) and (4.3) and multiply the result by constant C_1, yielding

$$\frac{C_1}{2}\frac{d}{dt}(\|\sqrt{\varrho}\mathbf{v}\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+C_1\|\nabla^2 b\|_{L^2}^2\le 0.\tag{5.10}$$

Multiplying (5.9) by a sufficiently small $\varepsilon>0$ and using $m_0^{-1}\le w''(b)\le m_0$, we combine this modified equation with (5.5), (5.6), (5.8), and (5.10). This yields suitable constants $c_1'',...,c_4''>0$ such that

$$\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\varrho}\mathbf{v}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)+c_1''\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2+c_2''\|\nabla b_t\|_{L^2}^2\\
&+(c_3''-\|\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}^{\frac{1}{2}}-\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}-\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2})\|\nabla^2\mathbf{v}\|_{L^2}^2\\
&+(c_4''-\|\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}^{\frac{1}{2}}-\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2}-\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2})\|\nabla^3 b\|_{L^2}^2\le 0.
\end{aligned}\tag{5.11}$$

Moreover, estimate (4.4) ensures that for any $\gamma>0$, there exists a time $t_2(\gamma)>0$ such that

$$\|\nabla\mathbf{v}(t_2)\|_{L^2}+\|\nabla^2 b(t_2)\|_{L^2}\le\gamma.\tag{5.12}$$

We choose γ sufficiently small, so that

$$\|\mathbf{v}_0\|_{L^2}^{\frac{1}{2}}\gamma^{\frac{1}{2}} + \|\mathbf{v}_0\|_{L^2}\gamma + \|\nabla b_0\|_{L^2}\gamma \le \min\{\frac{c_3''}{2}, \frac{c_4''}{2}\}, \tag{5.13}$$

and define

$$T'' \triangleq \sup_{t\ge t_2}\{\|\nabla \mathbf{v}(t)\|_{L^2}^2 + \|\nabla^2 b(t)\|_{L^2}^2 \le 4\|(\sqrt{\varrho}\mathbf{v}, \nabla b)(t_2)\|_{L^2}^2 + 4\gamma^2\}. \tag{5.14}$$

The objective is now to prove that $T'' = \infty$. If $T'' < \infty$, for any $t \ge t_2$, (5.11) and (5.13) can be recast as

$$\frac{d}{dt}\|(\sqrt{\varrho}\mathbf{v}, \nabla\mathbf{v}, \nabla b, \nabla^2 b)\|_{L^2}^2 + c_1'\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + c_2'\|\nabla b_t\|_{L^2}^2 + c_3'\|\nabla^2\mathbf{v}\|_{L^2}^2 + c_4'\|\nabla^3 b\|_{L^2}^2 \le 0, \tag{5.15}$$

which implies

$$\begin{aligned}&\|(\sqrt{\varrho}\mathbf{v}, \nabla\mathbf{v}, \nabla b, \nabla^2 b)(T'')\|_{L^2}^2 + \int_{t_2}^{T''}(c_1'\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + c_2'\|\nabla b_t\|_{L^2}^2 + c_3'\|\nabla^2\mathbf{v}\|_{L^2}^2 + c_4'\|\nabla^3 b\|_{L^2}^2)dt\\ &\le \|(\sqrt{\varrho}\mathbf{v}, \nabla b)(t_2)\|_{L^2}^2 + \|\nabla\mathbf{v}(t_2)\|_{L^2}^2 + \|\nabla^2 b(t_2)\|_{L^2}^2 \le \|(\sqrt{\varrho}\mathbf{v}, \nabla b)(t_2)\|_{L^2}^2 + \gamma^2,\end{aligned}$$

contradicting the definition of T'' in (5.14). Thus, $T'' = \infty$, and (5.2) is established. Moreover, (5.2) and (5.4) directly give

$$\int_{t_2}^{\infty}\|\nabla P(\tau)\|_{L^2}^2 d\tau \le C. \tag{5.16}$$

□

Proposition 5.2. *Under the conditions of Theorem 3.2, for any $q \in (1, \frac{6}{5})$, we have* $\mathbf{v}(t) \in C(0,\infty; L^q)$.

Proof. Multiplying the component equations of the velocity field by $|\mathbf{v}^j|^{q-1}sign(\mathbf{v}^j)$ and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\|\varrho^{\frac{1}{q}}\mathbf{v}^j\|_{L^q}^q + \frac{4(q-1)}{q}\int_{\mathbb{R}^3}|\nabla|\mathbf{v}^j|^{\frac{q}{2}}|^2 dx &= -q\int_{\mathbb{R}^3}(\nabla P + \nabla b\Delta b)|\mathbf{v}^j|^{q-1}sign(\mathbf{v}^j)dx\\ &\le C(\|\nabla P\|_{L^q} + \|\nabla b\Delta b\|_{L^q})\|\mathbf{v}^j\|_{L^q}^{q-1}.\end{aligned}$$

Simplifying and integrating over time yields

$$\|\mathbf{v}(t)\|_{L^\infty(0,t;L^q)} \le C(\|\mathbf{v}_0\|_{L^q} + \|\nabla P\|_{L^1(0,t;L^q)} + \|\nabla b\Delta b\|_{L^1(0,t;L^q)}). \tag{5.17}$$

To estimate the pressure gradient term, we take the divergence of $(3.2)_2$ and use div $\mathbf{v} = 0$, leading to

$$\Delta P = \mathrm{div}[-\mathbf{v}\cdot\nabla\mathbf{v} - a(\nabla P - \Delta\mathbf{v}) - (1+a)\nabla b\Delta b]. \tag{5.18}$$

For $1 < q < \frac{6}{5}$, using the embedding $L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3) \hookrightarrow L^{\frac{2q}{2-q}}(\mathbb{R}^3)$ and standard elliptic estimates, we get

$$\begin{aligned}\|\nabla P\|_{L^q} &\lesssim \|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^q} + \|a(\nabla P - \Delta\mathbf{v})\|_{L^q} + \|\nabla b\Delta b\|_{L^q} + \|a\nabla b\Delta b\|_{L^q}\\ &\lesssim \|\mathbf{v}\|_{L^{\frac{2q}{2-q}}}\|\nabla\mathbf{v}\|_{L^2} + \|a\|_{L^{\frac{2q}{2-q}}}\|\nabla P - \Delta\mathbf{v}\|_{L^2} + \|\nabla b\|_{L^{\frac{2q}{2-q}}}\|\nabla^2 b\|_{L^2}\\ &\quad + \|a\|_{L^{\frac{2q}{2-q}}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b\|_{L^2}\\ &\lesssim \|\mathbf{v}\|_{L^2\cap L^3}\|\nabla\mathbf{v}\|_{L^2} + \|a\|_{L^2\cap L^3}\|\nabla P - \Delta\mathbf{v}\|_{L^2} + \|\nabla b\|_{L^2\cap L^3}\|\nabla^2 b\|_{L^2}\\ &\quad + \|a\|_{L^2\cap L^3}^2\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2} + \|\nabla^3 b\|_{L^2}^2.\end{aligned} \tag{5.19}$$

Moreover, using the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, and estimates (5.2), we find

$$
\begin{aligned}
\|\nabla P\|_{L^1_t(L^q)} \le & C\Big(\|\mathbf{v}\|_{L^2_t(H^1)}\|\nabla \mathbf{v}\|_{L^2_t(L^2)} + \|\nabla P - \Delta \mathbf{v}\|_{L^1_t(L^2)} \\
& + \|\nabla b\|_{L^2_t(H^1)}\|\nabla^2 b\|_{L^2_t(L^2)} + \|\nabla^3 b\|^2_{L^2_t(L^2)}\Big) \\
\le & C(t) + C(t)(\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2_t(L^2)} + \|\varrho \mathbf{v}\cdot\nabla \mathbf{v}\|_{L^2_t(L^2)} + \|\nabla b \Delta b\|_{L^2_t(L^2)}) \\
\le & C(t) + \|\sqrt{\varrho}\mathbf{v}\|_{L^\infty_t(L^3)}\|\nabla \mathbf{v}\|_{L^2_t(L^6)} + \|\nabla b\|_{L^\infty_t(L^3)}\|\nabla^2 b\|_{L^2_t(L^6)}) \le C(t).
\end{aligned}
\tag{5.20}
$$

As for the last term in (5.17), similar arguments lead to

$$
\|\nabla b \Delta b\|_{L^1(0,t;L^q)} \le C\int_0^t \|\nabla b\|_{L^{\frac{2q}{2-q}}}\|\nabla^2 b\|_{L^2} d\tau \le C\|\nabla b\|_{L^2_t(H^1)}\|\nabla^2 b\|_{L^2_t(L^2)} \le C. \tag{5.21}
$$

Substituting these two estimates into (5.17) gives

$$
\|\mathbf{v}\|_{L^\infty(0,t;L^q)} \le C(t). \tag{5.22}
$$

By applying the Aubin-Lions lemma to (5.21) and using the estimate $\|\mathbf{v}_t\|_{L^2(0,t;L^2)} \le C$ from (5.2), we immediately conclude that $\mathbf{v} \in C(0,\infty; L^q)$. □

Proposition 5.3. *Under the conditions of Theorem 3.2,* (3.7) *and* (3.8) *hold.*

Proof. To enhance readability, the proof is structured into four steps.
Step 1: Decay rates of $\|\nabla b\|_{L^2}$ and $\|\nabla^2 b\|_{L^2}$. First, we analyze the decay rate of $\|\nabla b\|_{L^2}$. Taking the L^2 inner product of Eq $(3.1)_3$ with $-\Delta b$ and using the bound $m_0^{-1} \le w''(b) \le m_0$, we derive the energy inequality:

$$
\frac{d}{dt}\|\nabla b\|^2_{L^2} + 2m_0^{-1}\|\nabla b\|^2_{L^2} + 2\|\Delta b\|^2_{L^2} \le 2\int_{\mathbb{R}^3} |\mathbf{v}\nabla b \Delta b| dx \le 2\|\nabla \mathbf{v}\|_{L^2}\|\nabla b\|^{\frac12}_{L^2}\|\Delta b\|^{\frac32}_{L^2}.
$$

Then multiplication by $e^{2m_0^{-1}t}$ gives

$$
\begin{aligned}
\frac{d}{dt}\|e^{m_0^{-1}t}\nabla b\|^2_{L^2} + 2e^{2m_0^{-1}t}\|\Delta b\|^2_{L^2} &\le 2\|\nabla \mathbf{v}\|_{L^2}\|e^{m_0^{-1}t}\nabla b\|^{\frac12}_{L^2}\cdot e^{\frac32 m_0^{-1}t}\|\Delta b\|^{\frac32}_{L^2} \\
&\le e^{2m_0^{-1}t}\|\Delta b\|^2_{L^2} + C\|\nabla \mathbf{v}\|^4_{L^2}\|e^{m_0^{-1}t}\nabla b\|^2_{L^2}.
\end{aligned}
$$

Using the a priori bound (5.2), we get by integrating over the interval $[t_2, t]$ (with t_2 from Proposition 5.1) that

$$
\|e^{m_0^{-1}t}\nabla b(t)\|^2_{L^2} + \int_{t_2}^t e^{2m_0^{-1}\tau}\|\Delta b\|^2_{L^2} d\tau \le Ce^{m_0^{-1}t_2}\|\nabla b(t_2)\|^2_{L^2}\exp\{C\int_{t_2}^t \|\nabla \mathbf{v}\|^4_{L^2} d\tau\}.
$$

Thus, for any $t \ge t_2$, the exponential decay results are derived as follows:

$$
\|\nabla b(t)\|_{L^2} \lesssim e^{-m_0^{-1}t}, \quad \int_{t_2}^t e^{2m_0^{-1}\tau}\|\Delta b(\tau)\|^2_{L^2} d\tau \le C. \tag{5.23}
$$

Similarly, taking the L^2 inner product of $(3.1)_3$ with $-\Delta b_t$, combined with integration by parts, leads to

$$\begin{aligned}\frac{d}{dt}\|\nabla^2 b\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2 &\lesssim \int_{\mathbb{R}^3} |w''(b)\nabla b\cdot\nabla b_t + (\nabla\mathbf{v}\cdot\nabla)b\cdot\nabla b_t + (\mathbf{v}\cdot\nabla^2)b\cdot\nabla b_t|dx\\ &\lesssim \|\nabla b\|_{L^2}\|\nabla b_t\|_{L^2} + \|\nabla\mathbf{v}\|_{L^6}\|\nabla b\|_{L^3}\|\nabla b_t\|_{L^2} + \|\mathbf{v}\|_{L^\infty}\|\nabla^2 b\|_{L^2}\|\nabla b_t\|_{L^2}\\ &\leq \frac{1}{2}\|\nabla b_t\|_{L^2}^2 + C\Big(\|\nabla b\|_{L^2}^2 + \|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla b\|_{L^2}^2 + \|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2\Big).\end{aligned}$$

Multiplying the above by $e^{m_0^{-1}t}$ yields

$$\begin{aligned}\frac{d}{dt}\|e^{\frac{1}{2}m_0^{-1}t}\nabla^2 b\|_{L^2}^2 + e^{m_0^{-1}t}\|\nabla b_t\|_{L^2}^2 &\lesssim e^{m_0^{-1}t}\|\nabla^2 b\|_{L^2}^2 + e^{m_0^{-1}t}\|\nabla b\|_{L^2}^2 + e^{m_0^{-1}t}\|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla b\|_{L^2}^2\\ &\quad + (\|\nabla^2\mathbf{v}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2)\|e^{\frac{1}{2}m_0^{-1}t}\nabla^2 b\|_{L^2}^2.\end{aligned}$$

Utilizing the a priori estimates (5.2) and (5.23), we easily obtain

$$\begin{aligned}\|e^{\frac{1}{2}m_0^{-1}t}\nabla^2 b(t)\|_{L^2}^2 + \int_{t_2}^t e^{m_0^{-1}\tau}\|\nabla b_\tau\|_{L^2}^2 d\tau &\lesssim \Big(\|e^{\frac{1}{2}m_0^{-1}t_2}\nabla^2 b(t_2)\|_{L^2}^2 + \int_{t_2}^t e^{m_0^{-1}\tau}\|\nabla^2 b\|_{L^2}^2 d\tau\\ &+ \int_{t_2}^t e^{-m_0^{-1}\tau}d\tau + \int_{t_2}^t e^{-m_0^{-1}\tau}\|\nabla^2\mathbf{v}\|_{L^2}^2 d\tau\Big)\exp\{C\int_{t_2}^t (\|\nabla^2\mathbf{v}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2)d\tau\} \leq C.\end{aligned}$$

Thus, for $t > t_2$, the following holds:

$$\|\nabla^2 b(t)\|_{L^2} \lesssim e^{-\frac{1}{2}m_0^{-1}t}, \qquad \int_{t_2}^t e^{m_0^{-1}\tau}\|\nabla b_\tau(\tau)\|_{L^2}^2 d\tau \leq C. \tag{5.24}$$

To estimate $\|\nabla^3 b\|_{L^2}$, the gradient operator is applied to $(3.1)_3$, producing

$$\|\nabla^3 b\|_{L^2}^2 \lesssim \|\nabla b_t\|_{L^2}^2 + \|\nabla\mathbf{v}\cdot\nabla b\|_{L^2}^2 + \|\mathbf{v}\cdot\nabla^2 b\|_{L^2}^2 + \|\nabla w'(b)\|_{L^2}^2. \tag{5.25}$$

The righthand side terms admit the following bounds:

$$\begin{aligned}&\|\nabla\mathbf{v}\cdot\nabla b\|_{L^2}^2 \lesssim \|\nabla\mathbf{v}\|_{L^6}^2\nabla b\|_{L^3}^2 \lesssim \|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2},\\ &\|\mathbf{v}\cdot\nabla^2 b\|_{L^2}^2 \lesssim \|\nabla\mathbf{v}\|_{L^2}^2\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2} \leq \frac{1}{2}\|\nabla^3 b\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^4\|\nabla^2 b\|_{L^2}^2,\\ &\|\nabla w'(b)\|_{L^2}^2 \lesssim \|w''(b)\nabla b\|_{L^2}^2 \lesssim \|\nabla b\|_{L^2}^2.\end{aligned}$$

Combining these estimates gives

$$\|\nabla^3 b\|_{L^2}^2 \lesssim \|\nabla b_t\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^4\|\nabla^2 b\|_{L^2}^2 + \|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2} + \|\nabla b\|_{L^2}^2.$$

Multiplying by $e^{m_0^{-1}t}$ and integrating over (t_2, t), using (5.2), (5.23), and (5.24), we get

$$\int_{t_2}^t e^{m_0^{-1}\tau}\|\nabla^3 b\|_{L^2}^2 d\tau \leq C. \tag{5.26}$$

Step 2: Decay rate of $\|\mathbf{v}\|_{L^2}$. Following [2], we decompose $\mathbb{R}^3$ into two time-dependent regions:

$$A_1(t) \triangleq \{\zeta \mid |\zeta| \leq \sqrt{\varrho_s}h(t)\}, \qquad A_1^c(t) \triangleq \mathbb{R}^3\backslash A_1(t),$$

where $h(t) \leq C(1+t)^{-\frac{1}{2}}$ and

$$\varrho_s \triangleq \sup_{x\in\mathbb{R}^3,\, t\in\mathbb{R}^+} \varrho(x,t) = \sup_{x\in\mathbb{R}^3} \varrho_0(x).$$

Using Fourier transform properties, we split $\|\nabla \mathbf{v}\|_{L^2}$ into low and high-frequency components:

$$\|\nabla \mathbf{v}(t)\|_{L^2}^2 = \int_{A_1(t)} |\zeta|^2 |\hat{\mathbf{v}}(\zeta,t)|^2 d\zeta + \int_{A_1^c(t)} |\zeta|^2 |\hat{\mathbf{v}}(\zeta,t)|^2 d\zeta. \tag{5.27}$$

Thus, (4.2) becomes

$$\frac{d}{dt}\|\sqrt{\varrho}\mathbf{v}(t)\|_{L^2}^2 + 2h^2(t)\|\sqrt{\varrho}\mathbf{v}(t)\|_{L^2}^2 \leq 2\varrho_s h^2 \int_{A_1(t)} |\hat{\mathbf{v}}(\zeta,t)|^2 d\zeta + 2\|\mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2}^2. \tag{5.28}$$

For the second righthand term, using (5.2) and (5.24) gives, for all $t > t_2$,

$$\|\mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2}^2(t) \leq Ce^{-m_0^{-1}t}.$$

Next, we analyze the first righthand term in (5.28) (the low-frequency component of $\mathbf{v}(x,t)$). Let $\mathbb{P}$ denote the Leray projection operator; applying heat kernel theory to $(3.2)_1$ and using t_2 from Proposition 5.1, we express $\mathbf{v}(x,t)$ as

$$\mathbf{v}(x,t) = e^{(t-t_2)\Delta}\mathbf{v}(x,t_2) + \int_{t_2}^{t} e^{(t-\tau)\Delta}\mathbb{P}\Big[\nabla\cdot(-\mathbf{v}\otimes\mathbf{v}) + a(\Delta\mathbf{v} - \nabla P) + (1+a)\nabla b\Delta b\Big] d\tau. \tag{5.29}$$

Fourier transforming in x yields

$$\begin{aligned} |\hat{\mathbf{v}}(\zeta,t)| \leq & e^{-(t-t_2)|\zeta|^2}|\hat{\mathbf{v}}(\zeta,t_2)| + \int_{t_2}^{t} e^{-(t-\tau)|\zeta|^2}\Big(|\zeta||\mathcal{F}(\mathbf{v}\otimes\mathbf{v})| \\ & + |\mathcal{F}(a(\Delta\mathbf{v}-\nabla P))| + |\mathcal{F}(\nabla b\Delta b)| + |\mathcal{F}(a\nabla b\Delta b)|\Big) d\tau. \end{aligned} \tag{5.30}$$

Integrating over $A_1(t)$ gives

$$\begin{aligned} \int_{A_1(t)} |\hat{\mathbf{v}}(\zeta,t)|^2 d\zeta \leq & \int_{A_1(t)} e^{-2(t-t_2)|\zeta|^2}|\hat{\mathbf{v}}(\zeta,t_2)|^2 d\zeta \\ & + Ch^5\Big(\int_{t_2}^{t} \|\mathcal{F}(\mathbf{v}\otimes\mathbf{v})\|_{L^\infty_\zeta} d\tau\Big)^2 + Ch^3\Big(\int_{t_2}^{t} \|\mathcal{F}(a(\Delta\mathbf{v}-\nabla P))\|_{L^\infty_\zeta} d\tau\Big)^2 \\ & + Ch^3\Big(\int_{t_2}^{t} \|\mathcal{F}(\nabla b\Delta b)\|_{L^\infty_\zeta} d\tau\Big)^2 + Ch^3\Big(\int_{t_2}^{t} \|\mathcal{F}(a\nabla b\Delta b)\|_{L^\infty_\zeta} d\tau\Big)^2. \end{aligned} \tag{5.31}$$

For the first term of (5.31), introduce p and p', with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p'} \triangleq \frac{4}{3}\delta(q) = \frac{2}{q} - 1$. By Proposition 5.2, $\mathbf{v}(t_2) \in L^q(\mathbb{R}^3)$ for $q \in (1, \frac{6}{5})$, combined with the Hausdorff-Young inequality, we obtain

$$\int_{A_1(t)} e^{-2(t-t_2)|\zeta|^2}|\hat{\mathbf{v}}(\zeta,t_2)|^2 d\zeta \lesssim \int_{A_1(t)} e^{-2p'(t-t_2)|\zeta|^2} d\zeta)^{\frac{1}{p'}} \|\hat{\mathbf{v}}(t_2,\zeta)\|_{L^p}^2 \lesssim (1+t)^{-\frac{3}{2p'}}\|\mathbf{v}(t_2)\|_{L^q}^2. \tag{5.32}$$

For the remaining terms, from (5.2) and (5.16),

$$h^5\Big(\int_{t_2}^t \|\mathcal{F}(\mathbf{v}\otimes\mathbf{v})\|_{L^\infty_\zeta}d\tau\Big)^2 \lesssim h^5\Big(\int_{t_2}^t \|\mathbf{v}(\tau)\|_{L^2}^2 d\tau\Big)^2 \lesssim (t-t_2)^{-\frac{1}{2}},$$

$$h^3\Big(\int_{t_2}^t \|\mathcal{F}(a(\Delta\mathbf{v}-\nabla P))\|_{L^\infty_\zeta}d\tau\Big)^2 \lesssim h^3\Big[\int_{t_2}^t \|a\|_{L^2}\|(\Delta\mathbf{v},\nabla P)\|_{L^2}d\tau\Big]^2 \lesssim (t-t_2)^{-\frac{1}{2}},$$

$$\begin{aligned} h^3\Big(\int_{t_2}^t \|\mathcal{F}(a\nabla b\Delta b)\|_{L^\infty_\zeta}d\tau\Big)^2 &\lesssim h^3\Big(\int_{t_2}^t \|a\|_{L^2}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b\|_{L^2}d\tau\Big)^2 \\ &\lesssim h^3\Big(\int_{t_2}^t \|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}d\tau\Big)\Big(\int_{t_2}^t \|\nabla^3 b\|_{L^2}^2 d\tau\Big) \\ &\lesssim h^3\int_{t_2}^t e^{-\frac{3}{2}m_0^{-1}\tau}d\tau \lesssim (1+t)^{-\frac{3}{2}}, \end{aligned}$$

$$h^3\Big(\int_{t_2}^t \|\mathcal{F}(\nabla b\Delta b)\|_{L^\infty_\zeta}d\tau\Big)^2 \lesssim (1+t)^{-\frac{3}{2}}.$$

Substituting these into (5.31) and using $1 < q < \frac{6}{5}$ (implying $1 < 2\delta(q) < \frac{3}{2}$) gives, for all $t > t_2$,

$$\int_{A_1(t)} |\hat{\mathbf{v}}(\zeta,t)|^2 d\zeta \lesssim (1+t)^{-2\delta(q)} + (1+t)^{-\frac{1}{2}} \lesssim (1+t)^{-\frac{1}{2}}. \tag{5.33}$$

Thus, (5.28) simplifies to

$$\frac{d}{dt}\|\sqrt{\varrho}\mathbf{v}(t)\|_{L^2}^2 + h^2(t)\|\sqrt{\varrho}\mathbf{v}(t)\|_{L^2}^2 \lesssim (1+t)^{-\frac{3}{2}} + e^{-m_0^{-1}t} \lesssim (1+t)^{-\frac{3}{2}}.$$

Integrating over time yields

$$e^{\int_{t_2}^t h^2(\tau)d\tau}\|\sqrt{\varrho}\mathbf{v}(t)\|_{L^2}^2 \lesssim \|\sqrt{\varrho}\mathbf{v}(t_2)\|_{L^2}^2 + \int_{t_2}^t e^{\int_{t_2}^\tau h^2(t')dt'}(1+\tau)^{-\frac{3}{2}}d\tau.$$

Due to $h(t) \triangleq \sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $\theta > \frac{1}{2}$, substituting it into the above gives

$$\|\sqrt{\varrho}\mathbf{v}(t)\|_{L^2}^2(1+t)^\theta \lesssim 1 + \int_{t_2}^t (1+\tau)^{\theta-\frac{3}{2}}d\tau \lesssim 1 + (1+t)^{\theta-\frac{1}{2}}.$$

Hence, taking $\theta = 2$, for example,

$$\|\mathbf{v}(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}}. \tag{5.34}$$

Step 3: Decay rate of $\|\nabla\mathbf{v}\|_{L^2}$. First, adding energy inequalities (5.5) and (5.6) and applying (5.13) gives

$$\frac{d}{dt}\|\nabla\mathbf{v}\|_{L^2}^2 + m_3\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + m_4\|\nabla^2\mathbf{v}\|_{L^2}^2 \lesssim \|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^2 \lesssim e^{-\frac{3}{2}m_0^{-1}t}\|\nabla^3 b\|_{L^2}^2. \tag{5.35}$$

Using the same method as above, split the phase space into two time-dependent regions:

$$A_2(t) \triangleq \{\zeta \big| |\zeta| \le \sqrt{\frac{1}{m_4}}h(t)\}, \qquad A_2^c(t) \triangleq \mathbb{R}^3 \backslash A_2(t),$$

where $h(t) \leq C(1+t)^{-\frac{1}{2}}$ (explicit form specified later). Decompose $\|\nabla^2 \mathbf{v}\|_{L^2}^2$ as:

$$\|\nabla^2 \mathbf{v}\|_{L^2}^2 = \int_{A_2(t)} |\zeta|^4 |\hat{\mathbf{v}}(\zeta,t)|^2 d\zeta + \int_{A_2^c(t)} |\zeta|^2 |\widehat{\nabla \mathbf{v}}(\zeta,t)|^2 d\zeta. \tag{5.36}$$

Substituting into (5.35) gives:

$$\frac{d}{dt}\|\nabla \mathbf{v}(t)\|_{L^2}^2 + h^2(t)\|\nabla \mathbf{v}\|_{L^2}^2 + m_3\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 \lesssim h^4 \int_{A_2(t)} |\hat{\mathbf{v}}|^2 d\zeta + e^{-\frac{3}{2}m_0^{-1}t}\|\nabla^3 b\|_{L^2}^2. \tag{5.37}$$

Integrating over (t_2, t) and using (5.34) yields

$$e^{\int_{t_2}^t h^2(\tau)d\tau}\|\nabla \mathbf{v}(t)\|_{L^2}^2 + m_3 \int_{t_2}^t e^{\int_{t_2}^\tau h^2(t')dt'}\|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 d\tau \lesssim \|\nabla \mathbf{v}(t_2)\|_{L^2}^2 + \int_{t_2}^t e^{\int_{t_2}^\tau h^2(t')dt'}\Big[(1+\tau)^{-\frac{5}{2}} + e^{-\frac{3}{2}m_0^{-1}\tau}\|\nabla^3 b\|_{L^2}^2\Big]d\tau.$$

Let $h(t) \triangleq \sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $\theta > \frac{3}{2}$. By (5.26),

$$\|\nabla \mathbf{v}\|_{L^2}^2(1+t)^\theta + m_3 \int_{t_2}^t (1+\tau)^\theta \|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 d\tau \lesssim 1 + \int_{t_2}^t (1+\tau)^{\theta-\frac{5}{2}} d\tau \lesssim (1+t)^{\theta-\frac{3}{2}}.$$

This gives the rough decay estimate

$$\|\nabla \mathbf{v}(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}, \tag{5.38}$$

and for all $0 \leq l \leq \frac{3}{2}$:

$$\int_{t_2}^t (1+\tau)^l \|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 d\tau \leq C. \tag{5.39}$$

Step 4: Improved decay rate for $\|\mathbf{v}\|_{L^2}$ and $\|\nabla \mathbf{v}\|_{L^2}$. To improve the decay rate of $\|\mathbf{v}\|_{L^2}$, we refine the estimate for (5.31) by sharpening the bounds on its second and third righthand terms. Utilizing (5.34), we get

$$\int_{t_2}^t \|\mathcal{F}(\mathbf{v}\otimes\mathbf{v})\|_{L^\infty_\zeta} d\tau \lesssim \int_{t_2}^t \|\mathbf{v}(\tau)\|_{L^2}^2 d\tau \lesssim \int_{t_2}^t (1+\tau)^{-\frac{1}{2}} d\tau \lesssim (1+t)^{\frac{1}{2}}. \tag{5.40}$$

Meanwhile, it is straightforward that

$$\begin{aligned} \int_{t_2}^t \|\mathcal{F}(a(\Delta \mathbf{v} - \nabla P))\|_{L^\infty_\zeta} d\tau &\lesssim \int_{t_2}^t \|\Delta \mathbf{v} - \nabla P\|_{L^2} d\tau \|a\|_{L^\infty(L^2_\zeta)} \\ &\lesssim \int_{t_2}^t \|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2} d\tau + \int_{t_2}^t \|\varrho \mathbf{v}\cdot\nabla \mathbf{v}(\tau)\|_{L^2} d\tau + \int_{t_2}^t \|\nabla b \Delta b\|_{L^2} d\tau. \end{aligned} \tag{5.41}$$

For the first righthand term in the above inequality, using (5.39) yields

$$\Big(\int_{t_2}^t \|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2} d\tau\Big)^2 = \Big(\int_{t_2}^t (1+\tau)^{-\frac{5}{8}}(1+\tau)^{\frac{5}{8}}\|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2} d\tau\Big)^2 \lesssim \int_{t_2}^t (1+\tau)^{-\frac{5}{4}} d\tau \int_{t_2}^t (1+\tau)^{\frac{5}{4}}\|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 d\tau \leq C.$$

Using the previously derived decay estimates, the remaining terms in (5.41) are bounded as follows,

$$\begin{aligned} \Big(\int_{t_2}^t \|\varrho \mathbf{v}\cdot\nabla \mathbf{v}(\tau)\|_{L^2} d\tau\Big)^2 &\lesssim \Big(\int_{t_2}^t \|\nabla \mathbf{v}\|_{L^2}^{\frac{3}{2}} \|\nabla^2 \mathbf{v}\|_{L^2}^{\frac{1}{2}} d\tau\Big)^2 \lesssim \Big(\int_{t_2}^t (1+\tau)^{-\frac{9}{8}} \|\nabla^2 \mathbf{v}\|_{L^2}^{\frac{1}{2}} d\tau\Big)^2 \\ &\lesssim (1+t)^{-\frac{3}{4}} \|\nabla^2 \mathbf{v}\|_{L^2(t_2,t;L^2)} \leq C, \end{aligned}$$

$$\Big(\int_{t_2}^{t}\|\nabla b\Delta b\|_{L^2}d\tau\Big)^2 \lesssim \Big(\int_{t_2}^{t}\|\nabla^2 b\|_{L^2}^{\frac{3}{2}}\|\nabla^3 b\|_{L^2}^{\frac{1}{2}}d\tau\Big)^2 \lesssim \|\nabla^2 b\|^3_{L^2(t_2,t;L^2)}\|\nabla^3 b\|_{L^2(t_2,t;L^2)} \le C.$$

Thus, (5.41) is uniformly bounded:

$$\int_{t_2}^{t}\|\mathcal{F}\big(a(\Delta\mathbf{v}-\nabla P)\big)\|_{L^\infty_\zeta}d\tau \le C. \tag{5.42}$$

Substituting (5.32), (5.40)–(5.42) into (5.31) gives

$$\int_{A_2(t)}|\hat{\mathbf{v}}(\zeta,t)|^2d\zeta \lesssim (1+t)^{-2\delta(q)}+(1+t)^{-\frac{3}{2}} \lesssim (1+t)^{-2\delta(q)}. \tag{5.43}$$

Substituting this result into (5.28) yields

$$\frac{d}{dt}\|\sqrt{\varrho}\mathbf{v}(t)\|^2_{L^2}+2h^2(t)\|\sqrt{\varrho}\mathbf{v}(t)\|^2_{L^2} \lesssim (1+t)^{-1-2\delta(q)}+e^{-m_0^{-1}t} \lesssim (1+t)^{-1-2\delta(q)}.$$

Multiplying both sides by $e^{\int_{t_2}^{t}2h^2(\tau)d\tau}$ leads to

$$\frac{d}{dt}\Big(e^{\int_{t_2}^{t}2h^2(\tau)d\tau}\|\sqrt{\varrho}\mathbf{v}(t)\|^2_{L^2}\Big) \lesssim e^{\int_{t_2}^{t}2h^2(\tau)d\tau}(1+t)^{-1-2\delta(q)}.$$

Let $h(t)\triangleq\sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $\theta>2\delta(q)$. Integrating over (t_2,t) gives the key energy estimate

$$\|\sqrt{\varrho}\mathbf{v}(t)\|^2_{L^2}(1+t)^{\theta} \lesssim \|\sqrt{\varrho}\mathbf{v}(t_2)\|^2_{L^2}+(1+t)^{\theta-2\delta(q)}, \tag{5.44}$$

from which the decay rate follows:

$$\|\mathbf{v}(t)\|_{L^2} \le C(1+t)^{-\delta(q)}. \tag{5.45}$$

On the other hand, substituting (5.43) into (5.37) gives

$$\frac{d}{dt}\|\nabla\mathbf{v}\|^2_{L^2}+h^2(t)\|\nabla\mathbf{v}\|^2_{L^2}+m_3\|\sqrt{\varrho}\mathbf{v}_t\|^2_{L^2} \lesssim (1+t)^{-2\delta(q)-2}+e^{-\frac{3}{2}m_0^{-1}t}\|\nabla^3 b\|^2_{L^2}.$$

Multiplying by $e^{\int_{t_2}^{t}h^2(\tau)d\tau}$ and integrating over time yields

$$\begin{aligned}&e^{\int_{t_2}^{t}h^2(\tau)d\tau}\|\nabla\mathbf{v}(t)\|^2_{L^2}+m_3\int_{t_2}^{t}e^{\int_{t_2}^{\tau}h^2(t')dt'}\|\sqrt{\varrho}\mathbf{v}_\tau\|^2_{L^2}d\tau\\ \lesssim&\|\nabla\mathbf{v}(t_2)\|^2_{L^2}+\int_{t_2}^{t}e^{\int_{t_2}^{\tau}h^2(\tau)d\tau}\Big[(1+\tau)^{-2\delta(q)-2}+e^{-\frac{3}{2}m_0^{-1}\tau}\|\nabla^3 b\|^2_{L^2}\Big]d\tau.\end{aligned}$$

Taking $h(t)\triangleq\sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $\theta>1+2\delta(q)$ gives

$$\|\nabla\mathbf{v}\|^2_{L^2}(1+t)^{\theta}+m_3\int_{t_2}^{t}(1+\tau)^{\theta}\|\sqrt{\varrho}\mathbf{v}_\tau\|^2_{L^2}d\tau \lesssim 1+\int_{t_2}^{t}(1+\tau)^{\theta-2-2\delta(q)}d\tau \lesssim (1+t)^{\theta-1-2\delta(q)},$$

leading to

$$\|\nabla \mathbf{v}\|_{L^2} \le C(1+t)^{-\frac{1+2\delta(q)}{2}}. \tag{5.46}$$

Moreover, with the choice $0 \le \theta < 1 + 2\delta(q)$, one obtains

$$\int_{t_2}^{t} (1+\tau)^{\theta} \|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 d\tau \le C,$$

which implies that for any $0 \le l \le 1 + 2\delta(q)$,

$$\int_{t_2}^{t} (1+\tau)^{l} \|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 d\tau \le C. \tag{5.47}$$

On the other hand, for any $0 \le l < 1 + 2\delta(q)$, there holds

$$\int_{t_2}^{t} (1+\tau)^{l} (\|\Delta \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) d\tau \lesssim \int_{t_2}^{t} (1+\tau)^{l} \Big(\|\sqrt{\varrho}\mathbf{v}_\tau\|_{L^2}^2 + \|\varrho \mathbf{v}\cdot\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla b \Delta b\|_{L^2}^2 \Big) d\tau.$$

A direct application of (5.47) bounds the first righthand term; while the improved decay rates of $\|\mathbf{v}\|_{L^2}$ and $\|\nabla \mathbf{v}\|_{L^2}$ provide estimates for the other terms:

$$\begin{aligned}
\int_{t_2}^{t} (1+\tau)^{l} \|\varrho \mathbf{v}\cdot\nabla \mathbf{v}\|_{L^2}^2 d\tau &\lesssim \int_{t_2}^{t} (1+\tau)^{l} \|\nabla \mathbf{v}\|_{L^2}^3 \|\nabla^2 \mathbf{v}\|_{L^2} d\tau \lesssim \int_{t_2}^{t} (1+\tau)^{-\frac{1+2\delta}{2}} \|\nabla^2 \mathbf{v}\|_{L^2} d\tau \\
&\lesssim \Big[\int_{t_2}^{t} (1+\tau)^{-(1+2\delta(q))} d\tau \Big]^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2(t_2,t;L^2)} \le C; \\
\int_{t_2}^{t} (1+\tau)^{l} \|\|\nabla b \Delta b\|_{L^2}^2\|_{L^2}^2 d\tau &\lesssim \int_{t_2}^{t} (1+\tau)^{l} \|\nabla b\|_{L^2} \|\nabla^2 b\|_{L^2} \|\nabla^3 b\|_{L^2}^2 d\tau \\
&\lesssim \int_{t_2}^{t} (1+\tau)^{1+2\delta(q)} e^{-\frac{3}{2}m_1\tau} \|\nabla^3 b\|_{L^2}^2 d\tau \lesssim \|\nabla^3 b\|_{L^2(t_2,t;L^2)}^2 \le C.
\end{aligned}$$

Substitution of these estimates into the above inequality yields the final result

$$\int_{t_2}^{t} (1+\tau)^{l} (\|\Delta \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) d\tau \le C. \tag{5.48}$$

□

Remark 5.1. *By* (5.48)*, using the same method as in [1], for any* $l \in (1+2\delta, 2+6\delta)$*, we have*

$$\int_{0}^{\infty} (1+\tau)^{l} (\|\Delta \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2) d\tau \lesssim (1+t)^{l-1-2\delta}. \tag{5.49}$$

Proposition 5.4. *Under the conditions of Theorem 3.2, for any* $\frac{2}{3} \le \alpha \le 2$*,*

$$\int_{0}^{\infty} \Big(\|\Delta \mathbf{v}\|_{L^2}^{\alpha} + \|\nabla P\|_{L^2}^{\alpha} + \|\nabla^3 b\|_{L^2}^{\alpha} \Big) dt \le C. \tag{5.50}$$

Proof. **Step 1:** Estimates for $\|\mathbf{v}_t\|_{L^2}$, $\|\nabla\mathbf{v}_t\|_{L^2}$, $\|\nabla b_t\|_{L^2}$, $\|\nabla^2 b_t\|_{L^2}$, $\|\nabla^2\mathbf{v}\|_{L^2}$, and $\|\nabla^3 b\|_{L^2}$. First, we estimate $\|\mathbf{v}_t\|_{L^2}$ and $\|\nabla\mathbf{v}_t\|_{L^2}$. Differentiating $(3.1)_1$ with respect to t and taking the L^2 inner product with $\mathbf{v}_t$ gives

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + 2\|\nabla\mathbf{v}_t\|_{L^2}^2 &= -\int_{\mathbb{R}^3}\varrho_t|\mathbf{v}_t|^2dx - 2\int_{\mathbb{R}^3}\varrho_t\mathbf{v}\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t dx - 2\int_{\mathbb{R}^3}\varrho|\mathbf{v}_t|^2\nabla\mathbf{v}dx\\ &\quad - 2\int_{\mathbb{R}^3}\varrho\mathbf{v}\cdot\mathbf{v}_t\cdot\nabla\mathbf{v}_t dx - 2\int_{\mathbb{R}^3}\nabla b_t\Delta b\cdot\mathbf{v}_t dx - 2\int_{\mathbb{R}^3}\nabla b\Delta b_t\cdot\mathbf{v}_t dx\\ &\triangleq \sum_{j=1}^{6} I_j.\end{aligned} \tag{5.51}$$

Using the continuity equation $\varrho_t = -\mathrm{div}(\varrho\mathbf{v})$ and integration by parts, the second term I_2 is rewritten as

$$I_2 = -2\int_{\mathbb{R}^3}\varrho\mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t)dx = -2\int_{\mathbb{R}^3}(\varrho\mathbf{v}|\nabla\mathbf{v}|^2\cdot\mathbf{v}_t + \varrho|\mathbf{v}|^2\nabla^2\mathbf{v}\cdot\mathbf{v}_t + \varrho|\mathbf{v}|^2\nabla\mathbf{v}\nabla\cdot\mathbf{v}_t)dx.$$

The Gagliardo–Nirenberg inequality then yields the estimate for I_2:

$$|I_2| \lesssim \|\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}\|_{L^2}\|\mathbf{v}_t\|_{L^6} + \|\mathbf{v}\|_{L^6}^2\|\nabla^2\mathbf{v}\|_{L^2}\|\mathbf{v}_t\|_{L^6} + \|\mathbf{v}\|_{L^6}^2\|\nabla\mathbf{v}\|_{L^6}\|\nabla\mathbf{v}_t\|_{L^2} \le \frac{1}{6}\|\nabla\mathbf{v}_t\|_{L^2}^2 + C\|\nabla^2\mathbf{v}\|_{L^2}^2\|\nabla\mathbf{v}\|_{L^2}^4.$$

Similarly, the remaining terms are bounded as

$$\begin{aligned}|I_1| &\lesssim \int_{\mathbb{R}^3}\left|\varrho\mathbf{v}\cdot\nabla|\mathbf{v}_t|^2\right|dx \lesssim \|\mathbf{v}\|_{L^6}\|\mathbf{v}_t\|_{L^3}\|\nabla\mathbf{v}_t\|_{L^2}\\ &\lesssim \|\nabla\mathbf{v}\|_{L^2}\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}_t\|_{L^2}^{\frac{3}{2}} \le \frac{1}{6}\|\nabla\mathbf{v}_t\|_{L^2}^2 + C\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2\|\nabla\mathbf{v}\|_{L^2}^4,\\ |I_3| + |I_4| &\lesssim \int_{\mathbb{R}^3}\varrho|\mathbf{v}_t|^2|\nabla\mathbf{v}|dx + \int_{\mathbb{R}^3}|\varrho\mathbf{v}\cdot\mathbf{v}_t\cdot\nabla\mathbf{v}_t|dx\\ &\lesssim \|\mathbf{v}_t\|_{L^3}\|\mathbf{v}_t\|_{L^6}\|\nabla\mathbf{v}\|_{L^2} + \|\mathbf{v}\|_{L^6}\|\mathbf{v}_t\|_{L^3}\|\nabla\mathbf{v}_t\|_{L^2} \le \frac{1}{3}\|\nabla\mathbf{v}_t\|_{L^2}^2 + C\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2\|\nabla\mathbf{v}\|_{L^2}^4,\end{aligned}$$

and

$$\begin{aligned}|I_5| + |I_6| &\lesssim \|\nabla b_t\|_{L^6}\|\Delta b\|_{L^3}\|\mathbf{v}_t\|_{L^2} + \|\nabla b\|_{L^6}\|\Delta b_t\|_{L^2}\|\mathbf{v}_t\|_{L^3}\\ &\le \frac{1}{2}\|\nabla^2 b_t\|_{L^2}^2 + \frac{1}{6}\|\nabla\mathbf{v}_t\|_{L^2}^2 + C\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2 + C\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2\|\nabla^3 b\|_{L^2}^2 + C\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2\|\nabla^2 b\|_{L^2}^4.\end{aligned}$$

Substituting the bounds for I_1 to I_6 into the original equation yields

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + \frac{7}{6}\|\nabla\mathbf{v}_t\|_{L^2}^2 &\le \frac{1}{2}\|\nabla^2 b_t\|_{L^2}^2 + C\|\nabla\mathbf{v}\|_{L^2}^4\|\nabla^2\mathbf{v}\|_{L^2}^2\\ &\quad + C\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2(\|\nabla\mathbf{v}\|_{L^2}^4 + \|\nabla^2 b\|_{L^2}^2 + \|\nabla^3 b\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^4).\end{aligned} \tag{5.52}$$

On the other hand, differentiating $(3.1)_3$ with respect to time t, taking the L^2 inner product with $-\Delta b_t$, and using $|w'''(b)| \le m_1$, we obtain

$$\frac{d}{dt}\|\nabla b_t\|_{L^2}^2 + 2\|\nabla^2 b_t\|_{L^2}^2 + 2m_0^{-1}\|\nabla b_t\|_{L^2}^2 \lesssim \int_{\mathbb{R}^3}|\mathbf{v}_t\cdot\nabla b\cdot\Delta b_t + (\mathbf{v}\cdot\nabla)b_t\cdot\Delta b_t + w'''(b)b_t\nabla b\cdot\nabla b_t|dx.$$

Estimates for the righthand side terms proceed as

$$
\begin{aligned}
|\int_{\mathbb{R}^3} \mathbf{v}_t \cdot \nabla b \cdot \Delta b_t dx| &\lesssim \|\mathbf{v}_t\|_{L^3}\|\nabla b\|_{L^6}\|\nabla^2 b_t\|_{L^2} \le \frac{1}{6}\|\nabla^2 b_t\|_{L^2}^2 + \frac{1}{6}\|\nabla \mathbf{v}_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^4\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2,\\
|\int_{\mathbb{R}^3} (\mathbf{v}\cdot\nabla) b_t \cdot \Delta b_t dx| &\lesssim \|\mathbf{v}\|_{L^6}\|\nabla b_t\|_{L^3}\|\nabla^2 b_t\|_{L^2} \le \frac{1}{6}\|\nabla^2 b_t\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^4\|\nabla b_t\|_{L^2}^2,\\
|\int_{\mathbb{R}^3} w'''(b) b_t \nabla b \cdot \nabla b_t dx| &\lesssim \|b_t\|_{L^\infty}\|\nabla b\|_{L^2}\|\nabla b_t\|_{L^2} \le \frac{1}{6}\|\nabla^2 b_t\|_{L^2}^2 + \|\nabla b\|_{L^2}^{\frac{4}{3}}\|\nabla b_t\|_{L^2}^2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla b_t\|_{L^2}^2 + \frac{3}{2}\|\nabla^2 b_t\|_{L^2}^2 + 2m_0^{-1}\|\nabla b_t\|_{L^2}^2 \\
&\le \frac{1}{6}\|\nabla \mathbf{v}_t\|_{L^2}^2 + C(\|\nabla^2 b\|_{L^2}^4\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^4\|\nabla b_t\|_{L^2}^2 + \|\nabla b\|_{L^2}^{\frac{4}{3}}\|\nabla b_t\|_{L^2}^2).
\end{aligned}
\tag{5.53}
$$

Summing these two inequalities gives

$$
\begin{aligned}
\frac{d}{dt}\|(\sqrt{\varrho}\mathbf{v}_t, \nabla b_t)\|_{L^2}^2 + \|(\nabla \mathbf{v}_t, \nabla^2 b_t)\|_{L^2}^2 &\lesssim \|(\sqrt{\varrho}\mathbf{v}_t, \nabla b_t)\|_{L^2}^2\Big(\|(\nabla \mathbf{v}, \nabla^2 b)\|_{L^2}^4 \\
&\quad + \|\nabla^2 b\|_{L^2}^2 + \|\nabla^3 b\|_{L^2}^2 + \|\nabla b\|_{L^2}^{\frac{4}{3}}\Big) + \|\nabla \mathbf{v}\|_{L^2}^4\|\nabla^2 \mathbf{v}\|_{L^2}^2.
\end{aligned}
\tag{5.54}
$$

Using (5.2), (5.23), and combined with some routine calculations, we find

$$
\begin{aligned}
&\sup_{t\ge t_2}\|(\sqrt{\varrho}\mathbf{v}_t, \nabla b_t)\|_{L^2}^2 + \int_{t_2}^t \|(\nabla \mathbf{v}_\tau, \nabla^2 b_\tau)\|_{L^2}^2 d\tau \lesssim \Big(\|(\sqrt{\varrho}\mathbf{v}_t, \nabla b_t)(t_2)\|_{L^2}^2 \\
&+ \int_{t_2}^t \|\nabla \mathbf{v}\|_{L^2}^4\|\nabla^2 \mathbf{v}\|_{L^2}^2 d\tau\Big)\exp\Big\{\int_{t_2}^t \Big(\|(\nabla \mathbf{v}, \nabla^2 b)\|_{L^2}^4 + \|(\nabla^2 b, \nabla^3 b)\|_{L^2}^2 + \|\nabla b\|_{L^2}^{\frac{4}{3}}\Big)d\tau\Big\} \le C.
\end{aligned}
$$

Thus, it holds that

$$
\begin{aligned}
&\sqrt{\varrho}\mathbf{v}_t \in L^\infty(t_2, t; L^2),\ \nabla b_t \in L^\infty(t_2, t; L^2),\\
&\nabla \mathbf{v}_t \in L^2(t_2, t; L^2),\ \nabla^2 b_t \in L^2(t_2, t; L^2).
\end{aligned}
\tag{5.55}
$$

Finally, using these estimates, we analyze the regularity of $\nabla^2\mathbf{v}$, ∇P, and $\nabla^3 b$. An application of elliptic regularity to $(3.1)_2$, with the condition $\operatorname{div}\mathbf{v} = 0$, yields

$$
\begin{aligned}
\|\nabla^2 \mathbf{v}\|_{L^2} + \|\nabla P\|_{L^2} &\lesssim \|\sqrt{\varrho}\mathbf{v}_t\|_{L^2} + \|\varrho \mathbf{v}\cdot\nabla \mathbf{v}\|_{L^2} + \|\nabla b \Delta b\|_{L^2}\\
&\lesssim \|\sqrt{\varrho}\mathbf{v}_t\|_{L^2} + \|\varrho \mathbf{v}\|_{L^6}\|\nabla \mathbf{v}\|_{L^3} + \|\nabla b\|_{L^6}\|\nabla^2 b\|_{L^3}\\
&\le \frac{1}{4}(\|\nabla^3 b\|_{L^2} + \|\nabla^2 \mathbf{v}\|_{L^2}) + C\Big(\|\sqrt{\varrho}\mathbf{v}_t\|_{L^2} + \|\nabla \mathbf{v}\|_{L^2}^3 + \|\nabla^2 b\|_{L^2}^3\Big).
\end{aligned}
$$

For the elastic variable b, applying the gradient operator ∇ to $(3.1)_3$, gives

$$
\begin{aligned}
\|\nabla^3 b\|_{L^2} &\lesssim \|\nabla b_t\|_{L^2} + \|\nabla \mathbf{v}\nabla b\|_{L^2} + \|\mathbf{v}\cdot\nabla^2 b\|_{L^2} + \|\nabla w'(b)\|_{L^2}\\
&\lesssim \|\nabla b_t\|_{L^2} + \|\nabla \mathbf{v}\|_{L^2}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b\|_{L^2}^{\frac{1}{2}} + \|\nabla \mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla^2 \mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2} + C\|\nabla b\|_{L^2}
\end{aligned}
$$

$$\leq \frac{1}{4}\|(\nabla^3 b, \nabla^2 \mathbf{v})\|_{L^2} + C(\|\nabla b_t\|_{L^2} + \|\nabla \mathbf{v}\|_{L^2}^2\|\nabla^2 b\|_{L^2}) + C(\|\nabla \mathbf{v}\|_{L^2}\|\nabla^2 b\|_{L^2}^2 + \|\nabla b\|_{L^2}).$$

Summing these and using (5.2) and (5.55), we obtain

$$\sup_{t\geq t_2}\{\|\nabla^2 \mathbf{v}(t)\|_{L^2} + \|\nabla P(t)\|_{L^2} + \|\nabla^3 b(t)\|_{L^2}\} \leq C. \tag{5.56}$$

Step 2: Estimates for $\|\Delta \mathbf{v}\|_{L^{\frac{2}{3}}(t_2,t;L^2)}$, $\|\nabla P\|_{L^{\frac{2}{3}}(t_2,t;L^2)}$, and $\|\nabla^3 b\|_{L^{\frac{2}{3}}(t_2,t;L^2)}$. Setting $l = \frac{3}{2} + \delta(q)$ in (5.48) yields

$$\begin{aligned}
&\int_{t_2}^{t}(\|\Delta \mathbf{v}(\tau)\|_{L^2}^{\frac{2}{3}} + \|\nabla P(\tau)\|_{L^2}^{\frac{2}{3}})d\tau\\
\lesssim&\int_{t_2}^{t}(1+\tau)^{-\left(\frac{1}{2}+\frac{1}{3}\delta(q)\right)}(1+\tau)^{\frac{1}{2}+\frac{1}{3}\delta(q)}(\|\Delta \mathbf{v}(\tau)\|_{L^2}^{\frac{2}{3}} + \|\nabla P(\tau)\|_{L^2}^{\frac{2}{3}})d\tau\\
\lesssim&[\int_{t_2}^{t}(1+\tau)^{-(\frac{3}{4}+\frac{1}{2}\delta(q))}d\tau]^{\frac{2}{3}}[\int_{t_2}^{t}(1+\tau)^{\frac{3}{2}+\delta(q)}(\|\Delta \mathbf{v}(\tau)\|_{L^2}^{2} + \|\nabla P(\tau)\|_{L^2}^{2})d\tau]^{\frac{1}{3}} \leq C,
\end{aligned} \tag{5.57}$$

where $\frac{1}{2} < \delta < \frac{3}{4}$ ensures $\frac{3}{2} + \delta(q) < 1 + 2\delta(q)$ and $\frac{3}{4} + \frac{1}{2}\delta(q) > 1$ (guaranteeing integral convergence). A direct consequence of (5.26) is

$$\begin{aligned}
\int_{t_2}^{t}\|\nabla^3 b\|_{L^2}^{\frac{2}{3}}d\tau &\leq \int_{t_2}^{t} e^{-\frac{1}{3}m_0^{-1}\tau}e^{\frac{1}{3}m_0^{-1}\tau}\|\nabla^3 b\|_{L^2}^{\frac{2}{3}}d\tau\\
&\lesssim (\int_{t_2}^{t} e^{-\frac{1}{2}m_0^{-1}\tau}d\tau)^{\frac{2}{3}}(\int_{t_2}^{t} e^{m_0^{-1}\tau}\|\nabla^3 b\|_{L^2}^{2}d\tau)^{\frac{1}{3}} \leq C.
\end{aligned} \tag{5.58}$$

The combination of (5.2) and (5.16) gives

$$\int_{t_2}^{t}\left(\|\Delta \mathbf{v}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^3 b\|_{L^2}^2\right)(\tau)d\tau \leq C. \tag{5.59}$$

Interpolating (5.57)–(5.59) directly yields (5.50), proving Proposition 5.4. □

Proposition 5.5. *Under the conditions of Theorem 3.2, there holds*

$$\int_{t_2}^{\infty}\|(b, \nabla b, \nabla^2 b)(\tau)\|_{L^\infty} + \|\nabla b(\tau)\|_{L^\infty}^2 + \|(\mathbf{v}, \nabla \mathbf{v})(\tau)\|_{L^\infty}d\tau \leq C. \tag{5.60}$$

Proof. The previous decay estimates, combined with (5.2) and (5.50), implies these bounds:

$$\int_{t_2}^{t}\|b(\tau)\|_{L^\infty}d\tau \lesssim \int_{t_2}^{t}\|\nabla b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b(\tau)\|_{L^2}^{\frac{1}{2}}d\tau \lesssim \int_{t_2}^{\infty} e^{-\frac{1}{2}m_0^{-1}\tau}e^{-\frac{1}{4}m_0^{-1}\tau}d\tau \leq C;$$

$$\int_{t_2}^{t}\|\nabla b(\tau)\|_{L^\infty}d\tau \lesssim \int_{t_2}^{t}\|\nabla^2 b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b\|_{L^2}^{\frac{1}{2}}d\tau \lesssim (\int_{t_2}^{t}\|\nabla^2 b\|_{L^2}d\tau)^{\frac{1}{2}}\|\nabla^3 b\|_{L^1(t_2,t;L^2)}^{\frac{1}{2}} \lesssim \int_{t_2}^{t} e^{-\frac{1}{2}m_0^{-1}\tau}d\tau)^{\frac{1}{2}} \leq C;$$

$$\int_{t_2}^{t}\|\nabla b(\tau)\|_{L^\infty}^2 d\tau \lesssim \int_{t_2}^{t}\|\nabla^2 b(\tau)\|_{L^2}\|\nabla^3 b(\tau)\|_{L^2}d\tau \lesssim \int_{t_2}^{t} e^{-\frac{1}{2}m_0^{-1}\tau}\|\nabla^3 b\|_{L^2}d\tau \leq C.$$

Similar to the proof process for b, we obtain the estimate for $\mathbf{v}$ as follows:

$$\int_{t_2}^{t}\|\mathbf{v}(\tau)\|_{L^\infty}d\tau + \int_{t_2}^{t}\|\mathbf{v}(\tau)\|_{L^\infty}^2 d\tau \leq C. \tag{5.61}$$

Next, we estimate $\int_{t_2}^t \|\nabla^2 b(\tau)\|_{L^\infty} d\tau$. It follows from (5.7) that

$$\int_{t_2}^t \|\nabla^3 b\|_{L^6}^2 d\tau \le \int_{t_2}^t \Big(\|\nabla b_\tau\|_{L^6}^2 + \|\nabla \mathbf{v}\cdot\nabla b\|_{L^6}^2 + \|\mathbf{v}\cdot\nabla^2 b\|_{L^6}^2 + \|w''(b)\nabla b\|_{L^6}^2\Big) d\tau. \tag{5.62}$$

Thanks to (5.55), the first righthand term satisfies

$$\int_{t_2}^t \|\nabla b_\tau\|_{L^6}^2 d\tau \le \int_{t_2}^t \|\nabla^2 b_\tau\|_{L^2}^2 d\tau \le C.$$

Applying the Gagliardo-Nirenberg inequality, (5.24), and (5.56) to the second term gives

$$\begin{aligned}\int_{t_2}^t \|\nabla \mathbf{v}\cdot\nabla b\|_{L^6}^2 d\tau &\lesssim \int_{t_2}^t (\|\nabla \mathbf{v}\cdot\nabla^2 b\|_{L^2}^2 + \|\nabla^2 \mathbf{v}\cdot\nabla b\|_{L^2}^2) d\tau \\ &\lesssim \int_{t_2}^t (\|\nabla \mathbf{v}\|_{L^6}^2 \|\nabla^2 b\|_{L^3}^2 + \|\nabla^2 \mathbf{v}\|_{L^2}^2 \|\nabla b\|_{L^\infty}^2) d\tau \\ &\lesssim \int_{t_2}^t \|\nabla^2 \mathbf{v}\|_{L^2}^2 \|\nabla^2 b\|_{L^2} \|\nabla^3 b\|_{L^2} d\tau \le C.\end{aligned}$$

Likewise, the remaining terms are estimated as follows:

$$\begin{aligned}\int_{t_2}^t \|\mathbf{v}\cdot\nabla^2 b\|_{L^6}^2 d\tau &\lesssim \int_{t_2}^t \|\nabla \mathbf{v}\|_{L^6}^2 \|\nabla^2 b\|_{L^3}^2 + \|\mathbf{v}\|_{L^\infty} \|\nabla^3 b\|_{L^2}^2 d\tau \\ &\lesssim \int_{t_2}^t (\|\nabla^2 \mathbf{v}\|_{L^2}^2 \|\nabla^2 b\|_{L^2} \|\nabla^3 b\|_{L^2} + \|\nabla \mathbf{v}\|_{L^2} \|\nabla^2 \mathbf{v}\|_{L^2} \|\nabla^3 b\|_{L^2}^2) d\tau \le C, \\ \int_{t_2}^t \|w''(b)\nabla b\|_{L^6}^2 d\tau &\lesssim \int_{t_2}^t (\|w''(b)\nabla^2 b\|_{L^2}^2 + \|w'''(b)\nabla b\cdot\nabla b\|_{L^2}^2) d\tau \\ &\lesssim \int_{t_2}^t (\|\nabla^2 b\|_{L^2}^2 + \|\nabla b\|_{L^2} \|\nabla^2 b\|_{L^2}^3) d\tau \le C,\end{aligned}$$

where the boundedness of $w''(b)$ and $w'''(b)$ is used. As a consequence, we deduce

$$\|\nabla^3 b\|_{L^2(t_2,t;L^6)} \le C, \tag{5.63}$$

which combined with (5.58) yields

$$\int_{t_2}^t \|\nabla^2 b(\tau)\|_{L^\infty} d\tau \lesssim \int_{t_2}^t \|\nabla^3 b\|_{L^2}^{\frac{1}{2}} \|\nabla^3 b\|_{L^6}^{\frac{1}{2}} d\tau \lesssim \|\nabla^3 b\|_{L^{\frac{2}{3}}(t_2,t;L^2)}^{\frac{1}{2}} \|\nabla^3 b\|_{L^2(t_2,t;L^6)}^{\frac{1}{2}} \le C. \tag{5.64}$$

To estimate $\int_{t_2}^t \|\nabla \mathbf{v}(\tau)\|_{L^\infty} d\tau$, the momentum Eq $(3.1)_2$ takes the form

$$-\Delta \mathbf{v} + \nabla P = -\varrho \mathbf{v}_t - \varrho \mathbf{v}\cdot\nabla \mathbf{v} - \nabla b \Delta b.$$

By the divergence-free condition, we have

$$\nabla P = \nabla(-\Delta)^{-1} \mathrm{div}(\varrho \mathbf{v}_t + \varrho \mathbf{v}\cdot\nabla \mathbf{v} + \nabla b \Delta b), \tag{5.65}$$

which implies that

$$\int_{t_2}^{t} (\|\Delta \mathbf{v}\|_{L^6}^2 + \|\nabla P\|_{L^6}^2) d\tau \lesssim \int_{t_2}^{t} (\|\varrho \mathbf{v}_\tau\|_{L^6}^2 + \|\varrho \mathbf{v} \cdot \nabla \mathbf{v}\|_{L^6}^2 + \|\nabla b \Delta b\|_{L^6}^2) d\tau.$$

It follows from (5.55) that

$$\int_{t_2}^{t} \|\varrho \mathbf{v}_\tau\|_{L^6}^2 d\tau \leq C \int_{t_2}^{t} \|\nabla \mathbf{v}_\tau\|_{L^2}^2 d\tau \leq C.$$

Moreover, the Gagliardo-Nirenberg inequality and (5.56) show that

$$\begin{aligned}
\int_{t_2}^{t} \|\varrho \mathbf{v} \cdot \nabla \mathbf{v}\|_{L^6}^2 d\tau &\lesssim \int_{t_2}^{t} (\|\nabla \mathbf{v} \cdot \nabla \mathbf{v}\|_{L^2}^2 + \|\mathbf{v} \cdot \nabla^2 \mathbf{v}\|_{L^2}^2) d\tau \\
&\lesssim \int_{t_2}^{t} (\|\nabla \mathbf{v}\|_{L^6}^2 \|\nabla \mathbf{v}\|_{L^3}^2 + \|\mathbf{v}\|_{L^\infty}^2 \|\nabla^2 \mathbf{v}\|_{L^2}^2) d\tau \\
&\lesssim \int_{t_2}^{t} \|\nabla \mathbf{v}\|_{L^2} \|\nabla^2 \mathbf{v}\|_{L^2}^3 d\tau \leq C,
\end{aligned}$$

A similar argument gives

$$\int_{t_2}^{t} \|\nabla b \Delta b\|_{L^6}^2 d\tau \leq C.$$

Therefore, we arrive at

$$\int_{t_2}^{t} (\|\Delta \mathbf{v}\|_{L^6}^2 + \|\nabla P\|_{L^6}^2) d\tau \leq C. \tag{5.66}$$

which together with (5.50) implies that

$$\int_{t_2}^{t} \|\nabla \mathbf{v}(\tau)\|_{L^\infty} d\tau \lesssim \|\nabla^2 \mathbf{v}\|_{L^{\frac{2}{3}}(t_2,t;L^2)}^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2(t_2,t;L^6)}^{\frac{1}{2}} \leq C. \tag{5.67}$$

This completes the proof of Theorem 3.2. □

These decay estimates not only reveal the dynamic evolution rate of system perturbations but also provide a crucial quantitative tool for rigorously proving the asymptotic stability of the equilibrium state. Indeed, the decay of $\tilde{\mathbf{v}}$ and $\nabla \tilde{b}$ in the L^2-norm directly ensures the integrability and convergence to zero of the energy functional. Based on this, we will complete the stability proof in the next section.

6. Proof of Theorem 3.3

This section investigates the global stability of system (3.2) under small initial perturbations. Specifically, we prove that for slightly perturbed initial data $(\tilde{a}_0, \tilde{\mathbf{v}}_0, \tilde{b}_0)$, the system admits a unique global smooth solution that remains close to the reference solution. For clarity, the section is divided into two parts:

- We first rigorously derive global-in-time estimates for the reference solution, ensuring its regularity and boundedness.
- Using the above results, we prove the regularity of the perturbed components $(\tilde{a}, \tilde{\mathbf{v}}, \tilde{b})$ and establish global existence via a bootstrap argument. Notably, the L^2 decay rates of $\tilde{\mathbf{v}}$ and $\nabla \tilde{b}$ are required, so we also analyze their decay behaviors to complete the stability proof.

6.1. The global-in-time estimate for reference solutions

This part is devoted to establishing global-in-time estimates for the reference solutions of system (3.2), analogous to inequality (4.16).

Proposition 6.1. *Under the assumptions of Theorem 3.3, we have*

$$\|\overline{\mathbf{v}}\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{1}{2}}_{2,1})} + \|\overline{b}\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{5}{2}}_{2,1})} + \|\overline{\mathbf{v}}\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\overline{P}\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{1}{2}}_{2,1})} + \|\overline{b}\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{9}{2}}_{2,1})} \le C, \tag{6.1}$$

$$\|\overline{a}\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{5}{2}}_{2,1})} \le C. \tag{6.2}$$

Proof. Applying the gradient operator ∇ to (3.2), then taking the L^2 inner product with $\nabla\overline{a}$, and using $\operatorname{div}\mathbf{v} = 0$, we derive

$$\frac{d}{dt}\|\nabla\overline{a}\|_{L^2}^2 \lesssim \int_{\mathbb{R}^3} |\nabla\overline{\mathbf{v}}||\nabla\overline{a}|^2 dx \lesssim \|\nabla\overline{\mathbf{v}}\|_{L^\infty}\|\nabla\overline{a}\|_{L^2}^2,$$

for any $t < \infty$, where integrating over $(0,t)$ and using (5.67) yields

$$\|\nabla\overline{a}(t)\|_{L^2} \lesssim \|\nabla\overline{a}_0\|_{L^2} \exp\{\int_0^t \|\nabla\overline{\mathbf{v}}\|_{L^\infty} d\tau\} \lesssim \|\nabla\overline{a}_0\|_{L^2}. \tag{6.3}$$

Next, applying the Laplace operator Δ to $(3.2)_1$ and taking the L^2 inner product with $\nabla^2\overline{a}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla^2\overline{a}\|_{L^2}^2 &\lesssim \int_{\mathbb{R}^3} (|\nabla^2\overline{\mathbf{v}}\cdot\nabla\overline{a}\nabla^2\overline{a}| + |\nabla\overline{\mathbf{v}}|\cdot|\nabla^2\overline{a}|^2)dx\\ &\lesssim \|\nabla^2\overline{\mathbf{v}}\|_{L^3}\|\nabla\overline{a}\|_{L^6}\|\nabla^2\overline{a}\|_{L^2} + \|\nabla\overline{\mathbf{v}}\|_{L^\infty}\|\nabla^2\overline{a}\|_{L^2}^2.\end{aligned}$$

Integrating this over time and employing (5.57) and (5.66) then gives

$$\begin{aligned}\|\nabla^2\overline{a}(t)\|_{L^2} &\lesssim \|\nabla^2\overline{a}_0\|_{L^2} \exp\Big\{\int_0^t (\|\nabla^2\overline{\mathbf{v}}\|_{L^3} + \|\nabla\overline{\mathbf{v}}\|_{L^\infty})d\tau\Big\}\\ &\lesssim \|\nabla^2\overline{a}_0\|_{L^2} \exp\{(\|\nabla^2\overline{\mathbf{v}}\|_{L_t^{\frac{2}{3}}(L^2)} + \|\nabla^2\overline{\mathbf{v}}\|_{L_t^2(L^6)})\Big\}\\ &\lesssim \|\nabla^2\overline{a}_0\|_{L^2}.\end{aligned} \tag{6.4}$$

We now analyze the equation for $\overline{\mathbf{v}}$ in system (3.2). Using Remark 2.3 and the interpolation inequalities:

$$\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}^2_{2,1})} \le \|\overline{\mathbf{v}}\|^{\frac{1}{3}}_{L_t^1(H^1)}\|\overline{\mathbf{v}}\|^{\frac{2}{3}}_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})},$$

combined with estimates (3.7), (3.8), (5.50), and (6.4), we derive

$$\begin{aligned}&\|\overline{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}^{\frac{1}{2}}_{2,1})} + \|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + \|\nabla\overline{P}\|_{L_t^1(\dot{B}^{\frac{1}{2}}_{2,1})}\\ \lesssim & e^{\int_0^t \|\nabla\overline{\mathbf{v}}\|_{L^\infty}d\tau}\Big[\|\overline{\mathbf{v}}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\nabla\overline{b}\|_{L_t^\infty(\dot{B}^{\frac{1}{2}}_{2,1})}\|\Delta\overline{b}\|_{L_t^1(L^\infty)} + \|\overline{a}\|_{L_t^\infty(H^2)}\|\nabla\overline{b}\|_{L_t^\infty(L^2)}\|\Delta\overline{b}\|_{L_t^1(L^\infty)}\\ &+ \|\overline{a}\|_{L_t^\infty(H^2)}(\|\nabla\overline{P}\|_{L_t^1(L^2)} + \|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}^2_{2,1})})\Big]\\ \lesssim & 1 + \|\overline{\mathbf{v}}\|^{\frac{1}{3}}_{L_t^1(H^1)}\|\overline{\mathbf{v}}\|^{\frac{2}{3}}_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} \le \frac{1}{2}\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}^{\frac{5}{2}}_{2,1})} + C,\end{aligned}$$

yielding

$$\|\overline{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\nabla\overline{P}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \leq C. \tag{6.5}$$

From this result, we improve the regularity of $\overline{a}$ by applying Lemma 2.7 to its equation

$$\|\overline{a}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})} \leq \|\overline{a}_0\|_{\dot{B}_{2,1}^{\frac{5}{2}}} \exp\{C\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\} \leq C. \tag{6.6}$$

For the $\overline{b}$ equation in (3.2), (6.5) and Corollary 2.1 imply that

$$\|\overline{b}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{9}{2}})} \leq C\exp\{C\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}(\|\overline{b}_0\|_{\dot{B}_{2,1}^{\frac{5}{2}}} + \|w'(\overline{b})\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}).$$

The last term is bounded by applying Lemma 2.3 to $\|w'(\overline{b})\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}$. Using interpolation inequalities and the decay rates of $\|\nabla\overline{b}\|_{L^2}$, $\|\nabla^2\overline{b}\|_{L^2}$, we obtain

$$\|w'(\overline{b})\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} \lesssim (1+\|\overline{b}\|_{L_t^\infty(L^\infty)})^3\|\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} \lesssim (1+\|\nabla\overline{b}\|_{L_t^\infty(L^2)}^{\frac{3}{2}}\|\nabla^2\overline{b}\|_{L_t^\infty(L^2)}^{\frac{3}{2}})\|\overline{b}\|_{L_t^1(\dot{H}^2)}^{\frac{1}{2}}\|\overline{b}\|_{L_t^1(\dot{H}^3)}^{\frac{1}{2}} \leq C.$$

Thus, the desired estimate is established that

$$\|\overline{b}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{9}{2}})} \leq C. \tag{6.7}$$

□

Proposition 6.2. *Under the assumptions of Theorem 3.3, the following holds:*

$$\|\overline{\mathbf{v}}\|_{L^\infty(0,\infty;L^q)} + \|\overline{\mathbf{v}}\|_{\tilde{L}^\infty(0,\infty;\dot{B}_{2,1}^{\frac{3}{2}})} + \|\overline{\mathbf{v}}\|_{L^1(0,\infty;\dot{B}_{2,1}^{\frac{7}{2}})} + \|\nabla\overline{P}\|_{L^1(0,\infty;\dot{B}_{2,1}^{\frac{3}{2}})} \leq C. \tag{6.8}$$

Proof. The L^q-estimate for $\overline{\mathbf{v}}$ is first established. Combining inequalities (5.17)–(5.21) and using the decay rates in (3.7), we derive

$$\begin{aligned}
\|\overline{\mathbf{v}}\|_{\tilde{L}_t^\infty(L^q)} &\lesssim \|\overline{\mathbf{v}}_0\|_{L^q} + \int_0^t (\|\overline{\mathbf{v}}\|_{H^1}\|\nabla\overline{\mathbf{v}}\|_{L^2} + \|\overline{a}\|_{H^1}\|\nabla\overline{P}-\Delta\overline{\mathbf{v}}\|_{L^2} + \|\nabla\overline{b}\|_{H^1}\|\nabla^2\overline{b}\|_{L^2} \\
&\quad + \|\overline{a}\|_{H^1}^2\|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2})d\tau + 1 \\
&\lesssim [\|\overline{\mathbf{v}}\|_{L_t^\infty(H^1)}\|\nabla\overline{\mathbf{v}}\|_{L_t^1(L^2)} + \|\overline{a}\|_{L_t^\infty(H^1)}(\|\nabla\overline{P}\|_{L_t^1(L^2)} + \|\Delta\overline{\mathbf{v}}\|_{L_t^1(L^2)}) \\
&\quad + \|\nabla\overline{b}\|_{L_t^1(H^1)}\|\nabla^2\overline{b}\|_{L_t^\infty(L^2)} + \|\overline{a}\|_{L_t^\infty(H^1)}^2\|\nabla\overline{b}\|_{L_t^2(L^2)}\|\nabla^2\overline{b}\|_{L_t^2(L^2)} + 1] \\
&\leq C.
\end{aligned}$$

Next, it follows from differentiating the $\overline{\mathbf{v}}$ equation in (3.2) with respect to x_i that

$$\partial_t\partial_i\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\partial_i\overline{\mathbf{v}} - (1+\overline{a})(\Delta\partial_i\overline{\mathbf{v}} - \partial_i\nabla\overline{P}) = -\partial_i\overline{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}} + \partial_i\overline{a}(\Delta\overline{\mathbf{v}} - \nabla\overline{P}) - (1+\overline{a})\partial_i(\nabla\overline{b}\Delta\overline{b}) - \partial_i\overline{a}\nabla\overline{b}\Delta\overline{b}. \tag{6.9}$$

Applying Lemma 2.8 gives the following estimate:

$$\begin{aligned}
&\|\partial_i\overline{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\partial_i\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\nabla\partial_i\overline{P}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \\
&\lesssim \|\nabla\overline{\mathbf{v}}_0\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \int_0^t \|\partial_i\overline{\mathbf{v}}\|_{\dot{B}_{2,1}^{\frac{1}{2}}}\|\overline{\mathbf{v}}\|_{\dot{B}_{2,1}^{\frac{5}{2}}}\,d\tau + \|\partial_i\overline{a}(\Delta\overline{\mathbf{v}} - \nabla\overline{P})\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|(1+\overline{a})\partial_i(\nabla\overline{b}\Delta\overline{b})\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \qquad (6.10)\\
&+ \|\partial_i\overline{a}\nabla\overline{b}\Delta\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\overline{a}\|_{L_t^\infty(H^2)}\|\nabla\partial_i\overline{P}\|_{L_t^1(L^2)} + \|\overline{a}\|_{L_t^\infty(H^2)}\|\partial_i\overline{\mathbf{v}}\|_{L_t^1(H^2)}.
\end{aligned}$$

Using product laws in Besov spaces, (6.4), and Proposition 6.1, each term on the righthand side is estimated as follows:

$$\begin{aligned}
&\|\partial_i\overline{a}(\Delta\overline{\mathbf{v}} - \nabla\overline{P})\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \lesssim \|\overline{a}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})}(\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\nabla\overline{P}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})});\\
&\|(1+\overline{a})\partial_i(\nabla\overline{b}\Delta\overline{b})\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\partial_i\overline{a}\nabla\overline{b}\Delta\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \lesssim \|\overline{a}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})}\|\overline{b}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})}\|\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})};\\
&\|\overline{a}\|_{L_t^\infty(H^2)}\|\nabla\partial_i\overline{P}\|_{L_t^1(L^2)} \lesssim \|\nabla\overline{P}\|^{\frac{1}{2}}_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})}\|\nabla\overline{P}\|^{\frac{1}{2}}_{L_t^1(\dot{B}_{2,1}^{\frac{3}{2}})} \le \frac{1}{2}\|\nabla\overline{P}\|_{L_t^1(\dot{B}_{2,1}^{\frac{3}{2}})} + C\|\nabla\overline{P}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})};\\
&\|\overline{a}\|_{L_t^\infty(H^2)}\|\partial_i\overline{\mathbf{v}}\|_{L_t^1(H^2)} \lesssim \|\overline{\mathbf{v}}\|^{\frac{1}{2}}_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\|\overline{\mathbf{v}}\|^{\frac{1}{2}}_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} \lesssim \frac{1}{2}\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} + C\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}.
\end{aligned}$$

Plugging all these estimates into (6.10), we get by some elementary derivation that

$$\|\overline{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} + \|\nabla\overline{P}\|_{L_t^1(\dot{B}_{2,1}^{\frac{3}{2}})} \le C. \qquad (6.11)$$

□

6.2. Stability of the global large solutions

The global of this section is to prove Theorem 3.3. Let $(\tilde{a}, \tilde{\mathbf{v}}, \tilde{b}) \triangleq (a-\overline{a}, \mathbf{v}-\overline{\mathbf{v}}, b-\overline{b})$. Then, $(\tilde{a}, \tilde{\mathbf{v}}, \tilde{b})$ satisfies the following system:

$$\begin{cases}
\tilde{a}_t + (\overline{\mathbf{v}}+\tilde{\mathbf{v}})\cdot\nabla\tilde{a} = -\tilde{\mathbf{v}}\cdot\nabla\overline{a},\\
\tilde{\mathbf{v}}_t + (\overline{\mathbf{v}}+\tilde{\mathbf{v}})\nabla\tilde{\mathbf{v}} + \tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}} - (1+\overline{a}+\tilde{a})(\Delta\tilde{\mathbf{v}} - \nabla\tilde{P} - \nabla\tilde{b}\Delta(\overline{b}+\tilde{b}) - \nabla\overline{b}\Delta\tilde{b})\\
\qquad\qquad = \tilde{a}(\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}),\\
\tilde{b}_t + (\overline{\mathbf{v}}+\tilde{\mathbf{v}})\cdot\nabla\tilde{b} + w''(\overline{b})\tilde{b} + o(|\tilde{b}|^2) - \Delta\tilde{b} = -\tilde{\mathbf{v}}\cdot\nabla\overline{b},\\
\operatorname{div}\tilde{\mathbf{v}} = 0,\\
(\tilde{a}, \tilde{\mathbf{v}}, \tilde{b})(x,t)|_{t=0} = (\tilde{a}_0, \tilde{\mathbf{v}}_0, \tilde{b}_0)(x).
\end{cases} \qquad (6.12)$$

Thus, the stability analysis of system (3.2) reduces to establishing the global well-posedness of the perturbation system (6.12). Following the methodology from [25] and employing the coupled parabolic-hyperbolic theory, there exists a time $T > 0$ such that the Cauchy problem for system (3.2) with initial data $(a_0, \mathbf{v}_0, b_0)$ has a unique solution $(a, \mathbf{v}, b)$ satisfying

$$a \in C(0,T; B_{2,1}^{\frac{5}{2}}),\ \mathbf{v} \in C(0,T; B_{2,1}^{\frac{3}{2}}) \cap L^1_{\mathrm{loc}}(0,T; \dot{B}_{2,1}^{\frac{7}{2}}),\ b \in C(0,T; B_{2,1}^{\frac{5}{2}}) \cap L^1_{\mathrm{loc}}(0,T; \dot{B}_{2,1}^{\frac{9}{2}}).$$

Consequently, $(\tilde{a}, \tilde{\mathbf{v}}, \tilde{b})$ satisfies

$$\tilde{a} \in C(0,T; B^{\frac{5}{2}}_{2,1}),\ \tilde{\mathbf{v}} \in C(0,T; B^{\frac{3}{2}}_{2,1}) \cap L^1_{\text{loc}}(0,T; \dot{B}^{\frac{7}{2}}_{2,1}),\ \tilde{b} \in C(0,T; B^{\frac{5}{2}}_{2,1}) \cap L^1_{\text{loc}}(0,T; \dot{B}^{\frac{9}{2}}_{2,1}).$$

Let T be the maximal existence time of $(a, \mathbf{v}, b)$. This section aims to show that $(\tilde{a}, \tilde{\mathbf{v}}, \tilde{b})$ remains small for all $t > 0$ and $T = \infty$. From the equation for $\tilde{a}$ in (6.12), controlling $\tilde{a}$ requires $\int_0^\infty \|\nabla\mathbf{v}\|_{L^\infty} dt$ to be small. This, in turn, necessitates decay estimates for $\|\mathbf{v}\|_{L^2}$ and $\|\nabla\mathbf{v}\|_{L^2}$, which are obtained through energy estimates and phase space analysis.

Proposition 6.3. *Define $U(t) \triangleq 2\int_0^t \|\nabla\overline{\mathbf{v}}\|_{L^\infty} d\tau$. Under the hypotheses of Theorem 3.3, for any $t < T$, there holds*

$$\begin{aligned}
&\frac{d}{dt}[e^{-U(t)}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|^2_{L^2}] + h^2 e^{-U(t)}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|^2_{L^2} \\
\lesssim\, & e^{-U(t)}\Big\{h^2 \int_{A_1(t)} e^{-2t|\zeta|^2}|\widehat{\tilde{\mathbf{v}}}_0|^2 d\zeta + h^7(\|\tilde{\mathbf{v}}\|^4_{L^2_t(L^2)} + \|\tilde{\mathbf{v}}\|^2_{L^2_t(L^2)}) \\
&+ h^5(\|\Delta\tilde{\mathbf{v}}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{P}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{b}\|_{L^2_t(L^2)}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)} \\
&+ \|\nabla\tilde{b}\|_{L^2_t(L^2)}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)}\|\nabla^3\tilde{b}\|^2_{L^2_t(L^2)} + \|\nabla^3\tilde{b}\|^2_{L^2_t(L^2)} + \|\tilde{\varrho}\|^2_{L^\infty_t(L^2)}) \\
&+ \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\Delta\overline{b}\|^2_{L^2} + \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|^3_{L^2} + \|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2}\|\Delta\tilde{b}\|^2_{L^2} \\
&+ \|\tilde{\varrho}\|^2_{L^3}\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|^2_{L^2}\Big\},
\end{aligned} \tag{6.13}$$

where the time-dependent phase space region $A_1(t)$ is given in the above section. In what follows, we denote $\varrho \triangleq (1 + \overline{a} + \tilde{a})^{-1}$, $\tilde{\varrho} \triangleq (1 + \tilde{a})^{-1}$, and $\overline{\varrho} \triangleq \varrho - \tilde{\varrho}$.

Proof. Substituting ϱ and $\tilde{\varrho}$ into (6.12), $(\varrho, \tilde{\mathbf{v}}, \tilde{b})$ satisfies

$$\begin{cases}
\varrho_t + \operatorname{div}[\varrho(\overline{\mathbf{v}} + \tilde{\mathbf{v}})] = 0, \\
\varrho\tilde{\mathbf{v}}_t + \varrho(\overline{\mathbf{v}} + \tilde{\mathbf{v}})\cdot\nabla\tilde{\mathbf{v}} + \varrho\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}} - \Delta\tilde{\mathbf{v}} + \nabla\tilde{P} + \nabla\tilde{b}\Delta(\overline{b} + \tilde{b}) + \nabla\overline{b}\Delta\tilde{b} = -\frac{\tilde{\varrho}}{\varrho}(\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}), \\
\tilde{b}_t + (\overline{\mathbf{v}} + \tilde{\mathbf{v}})\cdot\nabla\tilde{b} + w''(\overline{b})\tilde{b} + o(|\tilde{b}|^2) - \Delta\tilde{b} = -\tilde{\mathbf{v}}\cdot\nabla\overline{b}, \\
\operatorname{div}\tilde{\mathbf{v}} = 0, \\
(\varrho, \tilde{\mathbf{v}}, \tilde{b})(x,t)|_{t=0} = (\varrho_0, \tilde{\mathbf{v}}_0, \tilde{b}_0)(x).
\end{cases} \tag{6.14}$$

The higher-order term $o(|\tilde{b}|^2)$ is omitted for simplicity, as it has negligible effects on stability and decay.

From the multiplication of $(6.14)_2$ by $\tilde{\mathbf{v}}$ and subsequent spatial integration, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|^2_{L^2} + \|\nabla\tilde{\mathbf{v}}\|^2_{L^2} = \int_{\mathbb{R}^3} \varrho\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\cdot\tilde{\mathbf{v}} + \nabla\tilde{b}\Delta\overline{b}\cdot\tilde{\mathbf{v}} + \nabla\tilde{b}\Delta\tilde{b}\cdot\tilde{\mathbf{v}} + \nabla\overline{b}\Delta\tilde{b}\cdot\tilde{\mathbf{v}} + \frac{\tilde{\varrho}}{\varrho}(\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b})\cdot\tilde{\mathbf{v}} dx.$$

The terms on the righthand side can be estimated similarly, with the first two serving as representative cases:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \varrho\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\cdot\tilde{\mathbf{v}} dx\right| \le \|\nabla\overline{\mathbf{v}}\|_{L^\infty}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|^2_{L^2}; \\
&\left|\int_{\mathbb{R}^3} \nabla\tilde{b}\Delta\overline{b}\cdot\tilde{\mathbf{v}} dx\right| \le \|\nabla\tilde{b}\|^{\frac{1}{2}}_{L^2}\|\nabla^2\tilde{b}\|^{\frac{1}{2}}_{L^2}\|\Delta\overline{b}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2} \le \frac{1}{6}\|\nabla\tilde{\mathbf{v}}\|^2_{L^2} + \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\Delta\overline{b}\|^2_{L^2}.
\end{aligned}$$

Consequently, one has

$$\frac{d}{dt}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L^2}^2+\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2\lesssim\|\nabla\overline{\mathbf{v}}\|_{L^\infty}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\Delta\overline{b}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^3$$
$$+\|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^3}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2,$$

which further simplifies to

$$\frac{d}{dt}(e^{-U(t)}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L^2}^2)+e^{-U(t)}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2\lesssim e^{-U(t)}\Big(\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\Delta\overline{b}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^3$$
$$+\|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^3}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big).\tag{6.15}$$

Following the derivation of Eq (5.28), Eq (6.15) becomes

$$\frac{d}{dt}(e^{-U(t)}\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L^2}^2)+e^{-U(t)}h^2\|\tilde{\mathbf{v}}\|_{L^2}^2$$
$$\lesssim e^{-U(t)}\Big(h^2\int_{A_1(t)}|\widehat{\tilde{\mathbf{v}}}|^2d\zeta+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\Delta\overline{b}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^3\tag{6.16}$$
$$+\|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^3}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big).$$

The handling of the low-frequency component $\int_{A_1(t)}|\hat{\tilde{\mathbf{v}}}|^2d\zeta$ proceeds by applying heat kernel theory to $(6.14)_2$, which gives

$$\tilde{\mathbf{v}}(t)=e^{t\Delta}\tilde{\mathbf{v}}_0+\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\Big\{\nabla\cdot(\overline{\mathbf{v}}\otimes\tilde{\mathbf{v}}+\tilde{\mathbf{v}}\otimes\tilde{\mathbf{v}}+\tilde{\mathbf{v}}\otimes\overline{\mathbf{v}})+(1-\frac{1}{\varrho})(-\Delta\tilde{\mathbf{v}}+\nabla\tilde{P})$$
$$-\frac{1}{\varrho}(\nabla\tilde{b}\Delta\overline{b}+\nabla\tilde{b}\Delta\tilde{b}+\nabla\overline{b}\Delta\tilde{b})-\frac{\tilde{\varrho}}{\varrho\overline{\varrho}}(\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b})\Big\}d\tau.\tag{6.17}$$

After Fourier transformation and integration over $A_1(t)$, we obtain

$$\int_{A_1(t)}|\widehat{\tilde{\mathbf{v}}}|^2d\zeta\lesssim\int_{A_1(t)}e^{-2t|\zeta|^2}|\widehat{\tilde{\mathbf{v}}}_0|^2d\zeta+h^5\Big(\int_0^t(\|\mathcal{F}(\overline{\mathbf{v}}\otimes\tilde{\mathbf{v}})\|_{L^\infty_\zeta}+\|\mathcal{F}(\tilde{\mathbf{v}}\otimes\tilde{\mathbf{v}})\|_{L^\infty_\zeta})d\tau\Big)^2$$
$$+h^3\Big(\int_0^t(\|\mathcal{F}[(1-\frac{1}{\varrho})\Delta\tilde{\mathbf{v}}]\|_{L^\infty_\zeta}+\|\mathcal{F}[(1-\frac{1}{\varrho})\nabla\tilde{P}]\|_{L^\infty_\zeta}+\|\mathcal{F}(\frac{1}{\varrho}\nabla\tilde{b}\Delta\overline{b})\|_{L^\infty_\zeta})d\tau\tag{6.18}$$
$$+\int_0^t(\|\mathcal{F}(\frac{1}{\varrho}\nabla\tilde{b}\Delta\tilde{b})\|_{L^\infty_\zeta}+\|\mathcal{F}(\frac{1}{\varrho}\nabla\overline{b}\Delta\tilde{b})\|_{L^\infty_\zeta}+\|\mathcal{F}[\frac{\tilde{\varrho}}{\varrho\overline{\varrho}}(\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b})]\|_{L^\infty_\zeta})d\tau\Big)^2.$$

By applying the Gagliardo-Nirenberg inequality and (3.7), three representative terms are estimated, with the remainder following similarly:

$$\int_0^t\|\mathcal{F}(\overline{\mathbf{v}}\otimes\tilde{\mathbf{v}})\|_{L^\infty_\zeta}d\tau\lesssim\int_0^t\|\overline{\mathbf{v}}\|_{L^2}\|\tilde{\mathbf{v}}\|_{L^2}d\tau,$$
$$\int_0^t\|\mathcal{F}(\frac{1}{\varrho}\nabla\tilde{b}\Delta\tilde{b})\|_{L^\infty_\zeta}d\tau\lesssim\int_0^t\|\nabla\tilde{b}\Delta\tilde{b}\|_{L^2}d\tau\lesssim\int_0^t\|\nabla\tilde{b}\|_{L^2}^{\frac12}\|\nabla^2\tilde{b}\|_{L^2}^{\frac12}\|\nabla^3\tilde{b}\|_{L^2}d\tau$$
$$\lesssim\|\nabla\tilde{b}\|_{L^2_t(L^2)}^{\frac12}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)}^{\frac12}\|\nabla^3\tilde{b}\|_{L^2_t(L^2)},$$

$$\int_0^t \|\mathcal{F}[\frac{\tilde{\varrho}}{\varrho\bar{\varrho}}(\Delta\bar{\mathbf{v}} - \nabla\bar{P} - \nabla\bar{b}\Delta\bar{b})]\|_{L^\infty_\zeta} d\tau \lesssim \int_0^t \|\tilde{\varrho}\|_{L^2}\Big(\|\Delta\bar{\mathbf{v}}\|_{L^2} + \|\nabla\bar{P}\|_{L^2} + \|\nabla\bar{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\bar{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^3\bar{b}\|_{L^2}\Big)d\tau \lesssim \|\tilde{\varrho}\|_{L^\infty_t(L^2)}.$$

As a consequence, (6.18) can be rewritten as

$$\begin{aligned}\int_{A_1(t)} |\widehat{\tilde{\mathbf{v}}}|^2 d\zeta \lesssim& \int_{A_1(t)} e^{-2t|\zeta|^2}|\widehat{\tilde{\mathbf{v}}}_0|^2 d\zeta + h^5(\|\tilde{\mathbf{v}}\|^4_{L^2_t(L^2)} + \|\tilde{\mathbf{v}}\|^2_{L^2_t(L^2)})\\ &+ h^3\Big(\|\Delta\tilde{\mathbf{v}}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{P}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{b}\|_{L^2_t(L^2)}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)}\\ &+ \|\nabla\tilde{b}\|_{L^2_t(L^2)}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)}\|\nabla^3\tilde{b}\|^2_{L^2_t(L^2)} + \|\nabla^3\tilde{b}\|^2_{L^2_t(L^2)} + \|\tilde{\varrho}\|^2_{L^\infty_t(L^2)}\Big).\end{aligned} \tag{6.19}$$

Substituting (6.19) into (6.16) yields the desired estimate (6.13), completing the proof of Proposition 6.3. □

Proposition 6.4. *Define $V(t) \triangleq C\int_0^t \|\nabla\bar{\mathbf{v}}\|_{L^2}\|\nabla^2\bar{\mathbf{v}}\|_{L^2}d\tau$. Under the hypotheses of Theorem 3.3, suppose that there exist a time $t_3 \le T$ and a sufficiently small constant $c_5 > 0$ such that*

$$\sup_{t\in[0,t_3)}(\|\tilde{\mathbf{v}}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2} + \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}) \le c_5, \tag{6.20}$$

Then, there holds

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\tilde{\mathbf{v}}\|^2_{L^2} + \|\nabla^2\tilde{b}\|^2_{L^2}) + \|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|^2_{L^2} + \|\nabla\tilde{b}_t\|^2_{L^2} + m_4(\|\Delta\tilde{\mathbf{v}}\|^2_{L^2} + \|\nabla^3\tilde{b}\|^2_{L^2})\\ \lesssim&(\|\nabla\bar{\mathbf{v}}\|_{L^2}\|\nabla^2\bar{\mathbf{v}}\|_{L^2} + \|\nabla^2\bar{b}\|_{L^2}\|\nabla^3\bar{b}\|_{L^2})(\|\nabla\tilde{\mathbf{v}}\|^2_{L^2} + \|\nabla^2\tilde{b}\|^2_{L^2})\\ &+ \|\tilde{\varrho}\|^2_{L^\infty}\|\Delta\bar{\mathbf{v}} - \nabla\bar{P} - \nabla\bar{b}\Delta\bar{b}\|^2_{L^2} + \|\nabla\bar{b}\|_{L^2}\|\nabla^2\bar{b}\|_{L^2}\|\nabla\tilde{b}\|^2_{L^2} + \|\nabla^2\bar{b}\|^4_{L^2}\|\nabla\tilde{b}\|^2_{L^2}.\end{aligned} \tag{6.21}$$

In addition, we have

$$\begin{aligned}&\frac{d}{dt}(e^{-V(t)}\|\nabla\tilde{\mathbf{v}}\|^2_{L^2}) + e^{-V(t)}(\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|^2_{L^2} + h^2\|\nabla\tilde{\mathbf{v}}\|^2_{L^2})\\ \lesssim& e^{-V(t)}\Big[h^4\int_{A_2(t)} e^{-2t|\zeta|^2}|\widehat{\tilde{\mathbf{v}}}_0|^2 d\zeta + h^9(\|\tilde{\mathbf{v}}\|^4_{L^2_t(L^2)} + \|\tilde{\mathbf{v}}\|^2_{L^2_t(L^2)})\\ &+ h^7(\|\Delta\tilde{\mathbf{v}}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{P}\|^2_{L^1_t(L^2)} + \|\nabla\tilde{b}\|_{L^2_t(L^2)}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)}\\ &+ \|\nabla\tilde{b}\|_{L^2_t(L^2)}\|\nabla^2\tilde{b}\|_{L^2_t(L^2)}\|\nabla^3\tilde{b}\|^2_{L^2_t(L^2)} + \|\nabla^3\tilde{b}\|^2_{L^2_t(L^2)} + \|\tilde{\varrho}\|^2_{L^\infty_t(L^2)})\\ &+ |\nabla^2\bar{b}\|_{L^2}\|\nabla^3\bar{b}\|_{L^2}\|\nabla^2\tilde{b}\|^2_{L^2} + \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\nabla^3\tilde{b}\|^2_{L^2}\\ &+ \|\tilde{\varrho}\|^2_{L^\infty}\|\Delta\bar{\mathbf{v}} - \nabla\bar{P} - \nabla\bar{b}\Delta\bar{b}\|^2_{L^2}\Big].\end{aligned} \tag{6.22}$$

Proof. Taking the L^2 inner product of Eq $(6.14)_2$ with $\tilde{\mathbf{v}}_t$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{\mathbf{v}}\|^2_{L^2} + \|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|^2_{L^2} \le& C\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}\Big(\|\bar{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2} + \|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2} + \|\tilde{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}\|_{L^2}\\ &+ \|\nabla\tilde{b}\Delta\bar{b}\|_{L^2} + \|\nabla\tilde{b}\Delta\tilde{b}\|_{L^2} + \|\nabla\bar{b}\Delta\tilde{b}\|_{L^2} + \|\frac{\tilde{\varrho}}{\bar{\varrho}\sqrt{\varrho}}\|_{L^\infty}\|\Delta\bar{\mathbf{v}} - \nabla\bar{P} - \nabla\bar{b}\Delta\bar{b}\|_{L^2}\Big)\\ \le& \frac{1}{2}\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|^2_{L^2} + C\Big(\|\bar{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|^2_{L^2} + \|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|^2_{L^2} + \|\tilde{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}\|^2_{L^2} + \|\nabla\tilde{b}\Delta\bar{b}\|^2_{L^2}\\ &+ \|\nabla\tilde{b}\Delta\tilde{b}\|^2_{L^2} + \|\nabla\bar{b}\Delta\tilde{b}\|^2_{L^2} + \|\tilde{\varrho}\|^2_{L^\infty}\|\Delta\bar{\mathbf{v}} - \nabla\bar{P} - \nabla\bar{b}\Delta\bar{b}\|^2_{L^2}\Big).\end{aligned} \tag{6.23}$$

Similarly, multiplying by $-\Delta\tilde{\mathbf{v}}$ and integrating over $\mathbb{R}^3$ yields

$$\begin{aligned}\frac{d}{dt}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\varrho\|_{L^\infty}^{-1}\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2 &\lesssim \frac{1}{2}\|\varrho\|_{L^\infty}^{-1}\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2+\|\overline{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}^2\\ &+\|\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\|_{L^2}^2+\|\nabla\tilde{P}\|_{L^2}^2+\|\nabla\tilde{b}\Delta\overline{b}\|_{L^2}^2+\|\nabla\tilde{b}\Delta\tilde{b}\|_{L^2}^2\\ &+\|\nabla\overline{b}\Delta\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2.\end{aligned}\tag{6.24}$$

Using the divergence-free condition and standard elliptic estimates to $(6.14)_2$, we obtain

$$\begin{aligned}&\|\nabla\tilde{P}\|_{L^2}+\|\Delta\tilde{\mathbf{v}}\|_{L^2}\lesssim\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}+\|\overline{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}+\|\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\|_{L^2}\\ &+\|\nabla\tilde{b}\Delta\overline{b}\|_{L^2}+\|\nabla\tilde{b}\Delta\tilde{b}\|_{L^2}+\|\nabla\overline{b}\Delta\tilde{b}\|_{L^2}+\|\tilde{\varrho}\|_{L^\infty}\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}.\end{aligned}\tag{6.25}$$

Then substituting (6.25) into (6.24) yields

$$\begin{aligned}&\frac{d}{dt}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+m_5'\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2\lesssim\|\overline{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\|_{L^2}^2+\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2\\ &+\|\nabla\tilde{b}\Delta\overline{b}\|_{L^2}^2+\|\nabla\tilde{b}\Delta\tilde{b}\|_{L^2}^2+\|\nabla\overline{b}\Delta\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2.\end{aligned}\tag{6.26}$$

Summing (6.23) and (6.26), and applying the Gagliardo-Nirenberg inequality, we derive

$$\begin{aligned}&\frac{d}{dt}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2+m_5'\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2\lesssim\|\overline{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\|_{L^2}^2\\ &\quad+\|\nabla\tilde{b}\Delta\overline{b}\|_{L^2}^2+\|\nabla\tilde{b}\Delta\tilde{b}\|_{L^2}^2+\|\nabla\overline{b}\Delta\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\\ &\quad\lesssim\|\nabla\overline{\mathbf{v}}\|_{L^2}\|\nabla^2\overline{\mathbf{v}}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\tilde{\mathbf{v}}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2}\|\nabla^2\tilde{\mathbf{v}}\|_{L^2}^2+\|\nabla^2\overline{b}\|_{L^2}\|\nabla^3\overline{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^2\\ &\quad+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\nabla^3\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2.\end{aligned}\tag{6.27}$$

Then, by taking the L^2 inner product of $(6.14)_3$ with $-\Delta\tilde{b}_t$ and $\nabla^4\tilde{b}$, respectively, summing the results, and applying the Gagliardo-Nirenberg inequality, it is obtained that

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2\tilde{b}\|_{L^2}^2+\|\nabla\tilde{b}_t\|_{L^2}^2+\|\nabla^3\tilde{b}\|_{L^2}^2\lesssim\|\nabla\overline{\mathbf{v}}\|_{L^2}\|\nabla^2\overline{\mathbf{v}}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^2\\ &+\|\nabla^2\tilde{\mathbf{v}}\|_{L^2}^2\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}+\|\tilde{\mathbf{v}}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2}\|\nabla^3\tilde{b}\|_{L^2}^2\\ &+\|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2}\|\nabla\tilde{b}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}^2+\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2\|\nabla^2\overline{b}\|_{L^2}\|\nabla^3\overline{b}\|_{L^2}.\end{aligned}\tag{6.28}$$

Hence, the combination of (6.27) and (6.28) gives

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\nabla^2\tilde{b}\|_{L^2}^2)+\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2+\|\nabla\tilde{b}_t\|_{L^2}^2\\ &+(m_5'-C\|\tilde{\mathbf{v}}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2}-C\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2})(\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2+\|\nabla^3\tilde{b}\|_{L^2}^2)\\ &\leq C(\|\nabla\overline{\mathbf{v}}\|_{L^2}\|\nabla^2\overline{\mathbf{v}}\|_{L^2}+\|\nabla^2\overline{b}\|_{L^2}\|\nabla^3\overline{b}\|_{L^2})(\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\nabla^2\tilde{b}\|_{L^2}^2)\\ &\quad+C\Big(\|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2+\|\nabla\overline{b}\|_{L^2}\|\nabla^2\overline{b}\|_{L^2}\|\nabla\tilde{b}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}^2\Big).\end{aligned}\tag{6.29}$$

Assuming c_5 in (6.20) is sufficiently small such that $c_5\leq\frac{m_5'}{2C}$, we establish (6.21). On the other hand, using (6.20), we rewrite (6.27) as follows:

$$\begin{aligned}&\frac{d}{dt}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2+\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2+m_5\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2\lesssim\|\nabla\overline{\mathbf{v}}\|_{L^2}\|\nabla^2\overline{\mathbf{v}}\|_{L^2}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2\\ &+\|\nabla^2\overline{b}\|_{L^2}\|\nabla^3\overline{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^2+\|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\nabla^3\tilde{b}\|_{L^2}^2+\|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}}-\nabla\overline{P}-\nabla\overline{b}\Delta\overline{b}\|_{L^2}^2,\end{aligned}$$

where m_5 denotes any positive constant satisfying $m_5' - c_5$. Multiplying by $e^{-V(t)}$, we obtain

$$\frac{d}{dt}(e^{-V(t)}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2) + e^{-V(t)}(\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2 + m_5\|\Delta\tilde{\mathbf{v}}\|_{L^2}^2)$$
$$\le Ce^{-V(t)}\Big(\|\nabla^2\overline{b}\|_{L^2}\|\nabla^3\overline{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\nabla^3\tilde{b}\|_{L^2}^2 + \|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big).$$

Then, application of (5.36) and definition of the frequency domain $A_2(t) \triangleq \zeta : |\zeta| \le \sqrt{\frac{1}{m_5}}h(t)$ lead to

$$\frac{d}{dt}(e^{-V(t)}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2) + e^{-V(t)}(\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2 + h^2\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2)$$
$$\le Ce^{-V(t)}\Big(h^4\int_{A_2(t)}|\widehat{\tilde{\mathbf{v}}}|^2 d\zeta + \|\nabla^2\overline{b}\|_{L^2}\|\nabla^3\overline{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}\|\nabla^2\tilde{b}\|_{L^2}\|\nabla^3\tilde{b}\|_{L^2}^2$$
$$+ \|\tilde{\varrho}\|_{L^\infty}^2\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big).$$

Substituting (6.19) completes the proof of (6.22). □

Proposition 6.5. *Under the hypotheses of Theorem 3.3, there exist positive constants C and c_4 such that if*

$$\beta_0 \triangleq \|\tilde{\mathbf{v}}_0\|_{H^1} + \|\tilde{\mathbf{v}}_0\|_{L^q} + \|\tilde{b}_0\|_{H^2} + \|\tilde{\varrho}_0\|_{L^2} + \|\tilde{\varrho}_0\|_{L^\infty} < c_4, \tag{6.30}$$

then, for all $t \le T$, we have

$$\begin{aligned}
&\|\tilde{\mathbf{v}}(t)\|_{L^2} \le C\beta_0(1+t)^{-\delta}, \quad \|\nabla\tilde{\mathbf{v}}(t)\|_{L^2} \le C\beta_0(1+t)^{-\frac{1}{2}(1+2\delta)},\\
&\|\nabla\tilde{b}(t)\|_{L^2} \le C\beta_0 e^{m_0^{-1}t}, \quad \|\nabla^2\tilde{b}(t)\|_{L^2} \le C\beta_0 e^{-\frac{1}{2}m_0^{-1}t},\\
&\int_0^T(\|\nabla\tilde{P}\|_{L^2} + \|\Delta\tilde{\mathbf{v}}\|_{L^2} + \|\nabla^3\tilde{b}\|_{L^2})dt \le C\beta_0, \quad \int_0^T(\|\tilde{\mathbf{v}}\|_{L^\infty} + \|\nabla\tilde{b}\|_{L^\infty})dt \le C\beta_0.
\end{aligned} \tag{6.31}$$

Moreover, for any $l \in [0, 1+2\delta)$, there holds

$$\int_0^T (1+t)^l(\|\partial_t\tilde{\mathbf{v}}\|_{L^2}^2 + \|\nabla\tilde{P}\|_{L^2}^2 + \|\Delta\tilde{\mathbf{v}}\|_{L^2}^2)dt \le C\beta_0. \tag{6.32}$$

Proof. **Step 1:** Preparatory estimates. Define

$$\eta(t) \triangleq \sup_{t'\in[0,t]}(\|\tilde{\varrho}\|_{L^2} + \|\tilde{\varrho}\|_{L^3} + \|\tilde{\varrho}\|_{L^\infty})(t'). \tag{6.33}$$

Then, taking the L^2 inner product of $(6.14)_2$ and $(6.14)_3$ with $\tilde{\mathbf{v}}$ and $-\Delta\tilde{b}$, respectively; summing the two results yields

$$\frac{d}{dt}(\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}^2) + \|\nabla\tilde{\mathbf{v}}\|_{L^2}^2 + \|\nabla^2\tilde{b}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}^2$$
$$\lesssim\Big(\|\nabla\overline{\mathbf{v}}\|_{L^\infty} + \|\nabla^2\overline{b}\|_{L^\infty} + \|\nabla\overline{\mathbf{v}}\|_{L^2}^4 + \|\nabla\overline{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\overline{b}\|_{L^2}^{\frac{1}{2}}\Big)(\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}^2)$$
$$+ \|\tilde{\varrho}\|_{L^3}^2\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2.$$

By utilizing Theorem 3.2 and the definitions of β_0, η, ensures

$$\|\sqrt{\varrho}\tilde{\mathbf{v}}\|_{L_t^\infty(L^2)} + \|\nabla\tilde{b}\|_{L_t^\infty(L^2)} + \|\nabla\tilde{\mathbf{v}}\|_{L_t^2(L^2)} + \|\nabla^2\tilde{b}\|_{L_t^2(L^2)} + \|\nabla\tilde{b}\|_{L_t^2(L^2)}^2 \le C(\beta_0+\eta). \tag{6.34}$$

Furthermore, we get by integrating (6.21) over $[0, t_3]$ that

$$\|(\nabla\tilde{\mathbf{v}}, \nabla^2\tilde{b})\|_{L_{t_3}^\infty(L^2)} + \|(\sqrt{\varrho}\tilde{\mathbf{v}}_t, \nabla\tilde{b}_t, \nabla^2\tilde{\mathbf{v}}, \nabla^3\tilde{b})\|_{L_{t_3}^2(L^2)} \le C(\beta_0+\eta). \tag{6.35}$$

Step 2: Decay rates for $\|\nabla\tilde{b}\|_{L^2}$ and $\|\nabla^2\tilde{b}\|_{L^2}$.

The exponential decay of $\|\nabla\tilde{b}\|_{L^2}$ and $\|\nabla^2\tilde{b}\|_{L^2}$ in (6.31) follows directly by applying the methodology from Proposition 5.3 for $\|\nabla b\|_{L^2}$ and $\|\nabla^2 b\|_{L^2}$.

Step 3: Rough decay rates for $\|\tilde{\mathbf{v}}\|_{L^2}$ and $\|\nabla\tilde{\mathbf{v}}\|_{L^2}$.

We once again employ Schonbek's approach. Multiplying both sides of (6.13) by $e^{\int_0^t h^2 d\tau}$, integrating over time, and leveraging (5.32) and (6.34) gives, for all $t \le t_3$,

$$\begin{aligned}
e^{\int_0^t h^2 d\tau} e^{-U(t)}\|\sqrt{\varrho}\tilde{\mathbf{v}}(t)\|_{L^2}^2 \lesssim \|\sqrt{\varrho_0}\tilde{\mathbf{v}}_0\|_{L^2}^2 + \int_0^t e^{\int_0^{t'} h^2 d\tau} e^{-U(t')}\Big\{h^2\beta_0^2(1+t')^{-2\delta(q)} \\
+ h^7(\beta_0+\eta)^2(1+t')^2 + h^5(\beta_0+\eta)^2(1+t') + e^{-m_0^{-1}t'}(\beta_0+\eta)^2 \\
+ \eta^2\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big\}dt'.
\end{aligned} \tag{6.36}$$

Define $h(t) \triangleq \sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $\frac{1}{2} < \theta < 1 + 2\delta$. Then, utilizing (3.7) and (3.8), we get

$$\int_0^t (1+t')^\theta \|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2 dt' \le C. \tag{6.37}$$

Consequently, inequality (6.36) simplifies to

$$\begin{aligned}
(1+t)^\theta\|\sqrt{\varrho}\tilde{\mathbf{v}}(t)\|_{L^2}^2 \lesssim& \beta_0^2 + \int_0^t \Big[(1+t')^{\theta-1-2\delta(q)}\beta_0^2 + (1+t')^{\theta-\frac{3}{2}}(\beta_0+\eta)^2 \\
&+ (1+t')^\theta e^{-m_0^{-1}t'}(\beta_0+\eta)^2\Big]dt' + \eta^2 \\
\lesssim& (\beta_0+\eta)^2(1+t)^{\theta-\frac{1}{2}},
\end{aligned}$$

which shows that

$$\|\tilde{\mathbf{v}}(t)\|_{L_t^\infty(L^2)} \le C(\beta_0+\eta)(1+t)^{-\frac{1}{4}}. \tag{6.38}$$

Similarly, the decay rate of $\|\nabla\tilde{\mathbf{v}}\|_{L^2}$ is computed. Multiplying both sides of (6.22) by $e^{\int_0^t h^2 d\tau}$ and integrating over time, then taking $h(t) \triangleq \sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $\frac{3}{2} < \theta < 1 + 2\delta(q)$, and using (6.37), we obtain

$$\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2(1+t)^\theta + \int_0^t (1+\tau)^\theta\|\partial_\tau\tilde{\mathbf{v}}\|_{L^2}^2 d\tau \le C(\beta_0+\eta)^2(1+t)^{\theta-\frac{3}{2}}.$$

Hence,

$$\|\nabla\tilde{\mathbf{v}}\|_{L_t^\infty(L^2)} \le (\beta_0+\eta)(1+t)^{-\frac{3}{4}}. \tag{6.39}$$

Evidently, if $1 \le \theta \le \frac{3}{2}$, then for any $l \in [0, \frac{3}{2})$, it holds that

$$\int_0^t (1+\tau)^l \|\partial_\tau \tilde{\mathbf{v}}\|_{L^2}^2 d\tau \le (\beta_0+\eta)^2.$$

Moreover, this estimate implies

$$\begin{aligned}\int_0^t \|\partial_\tau \tilde{\mathbf{v}}\|_{L^2} d\tau &\le \int_0^t (1+\tau)^{-\frac{5}{8}}(1+\tau)^{\frac{5}{8}}\|\partial_\tau \tilde{\mathbf{v}}\|_{L^2} d\tau \\ &\le \Big(\int_0^t (1+\tau)^{-\frac{5}{4}} d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t (1+\tau)^{\frac{5}{4}}\|\partial_\tau \tilde{\mathbf{v}}\|_{L^2}^2 d\tau\Big)^{\frac{1}{2}} \\ &\le C(\beta_0+\eta)^2.\end{aligned}$$

Step 4: Improving the decay rates of $\|\tilde{\mathbf{v}}\|_{L^2}$ and $\|\nabla\tilde{\mathbf{v}}\|_{L^2}$.

First, an application of the Gagliardo-Nirenberg inequality and (6.20) to (6.25) yields

$$\begin{aligned}\|\nabla \tilde{P}\|_{L^2} + \|\Delta\tilde{\mathbf{v}}\|_{L^2} &\lesssim \|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2} + \|\nabla\overline{\mathbf{v}}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\overline{\mathbf{v}}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{\mathbf{v}}\|_{L^2} \\ &+ \|\nabla^2\overline{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^3\overline{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\tilde{b}\|_{L^2} + \|\nabla\tilde{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\tilde{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^3\tilde{b}\|_{L^2} \\ &+ \|\tilde{\varrho}\|_{L^\infty}(\|\Delta\overline{\mathbf{v}}\|_{L^2} + \|\nabla\overline{P}\|_{L^2} + \|\nabla\overline{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\overline{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^3\overline{b}\|_{L^2}).\end{aligned} \tag{6.40}$$

Next, we get by integrating over time and using the decay estimates (3.7), (6.38), and (6.39) to (6.40) that

$$\int_0^t (\|\Delta\tilde{\mathbf{v}}\|_{L^2} + \|\nabla\tilde{P}\|_{L^2}) d\tau \le C(\beta_0+\eta)^2 \ln^{\frac{1}{2}}(1+t). \tag{6.41}$$

For the term $\|\tilde{\mathbf{v}}\|_{L_t^2(L^2)}^2$ in (6.22), by virtue of (6.38), we have

$$\|\tilde{\mathbf{v}}\|_{L_t^2(L^2)}^2 = \int_0^t \|\tilde{\mathbf{v}}\|_{L^2}^2 d\tau \le C(\beta_0+\eta)^2(1+t)^{\frac{1}{2}}. \tag{6.42}$$

Substitution of (6.41) and (6.42) into (6.22), followed by multiplication by $e^{\int_0^t h^2 d\tau}$, leads directly to

$$\begin{aligned}&\frac{d}{dt}\big(e^{-V(t)}e^{\int_0^t h^2 d\tau}\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2\big) + e^{-V(t)}e^{\int_0^t h^2 d\tau}\|\sqrt{\varrho}\tilde{\mathbf{v}}_t\|_{L^2}^2 \\ &\le Ce^{\int_0^t h^2 d\tau}\Big[h^4\beta_0^2(1+t)^{-2\delta(q)} + h^9(\beta_0+\eta)^2(1+t) + h^7(\beta_0+\eta)^2\ln(1+t) \\ &\quad + e^{-m_0^{-1}t}(\beta_0+\eta)^2(\|\nabla^3\overline{b}\|_{L^2} + \|\nabla^3\tilde{b}\|_{L^2}^2) + \eta^2\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big].\end{aligned} \tag{6.43}$$

Let $h(t) \triangleq \sqrt{\theta}(1+t)^{-\frac{1}{2}}$ with $1+2\delta(q) < \theta < 2+6\delta(q)$. Integrating over time and applying (5.49) yields

$$\begin{aligned}&(1+t)^\theta\|\nabla\tilde{\mathbf{v}}\|_{L^2}^2 + \int_0^t (1+\tau)^\theta\|\partial_\tau\tilde{\mathbf{v}}\|_{L^2}^2 d\tau \\ &\le \|\nabla\tilde{\mathbf{v}}_0\|_{L^2}^2 + C(\beta_0+\eta)^2\int_0^t \Big[(1+\tau)^{\theta-2-2\delta(q)} + (1+\tau)^{\theta-\frac{7}{2}}\ln(1+\tau)\Big]d\tau \\ &\le C(\beta_0+\eta)^2(1+t)^{\theta-1-2\delta(q)},\end{aligned} \tag{6.44}$$

which implies that for all $t \le t_3$,

$$\|\nabla\tilde{\mathbf{v}}(t)\|_{L^2} \le C(\beta_0+\eta)(1+t)^{-\frac{1}{2}(1+2\delta(q))}. \tag{6.45}$$

Moreover, by setting $0<\theta<1+2\delta(q)$ for $h(t)$, it follows that for any $l \in (0, 1+2\delta(q))$,

$$\int_0^t (1+\tau)^l \|\partial_\tau\tilde{\mathbf{v}}\|_{L^2}^2 d\tau \le C(\beta_0+\eta)^2. \tag{6.46}$$

Repeating the proof steps of $\|\tilde{\mathbf{v}}\|_{L^2}$, multiplying both sides of the Eq (6.13) by $e^{\int_0^t h^2 d\tau}$, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(e^{-U(t)}e^{\int_0^t h^2 d\tau}\|\tilde{\mathbf{v}}\|_{L^2}^2\Big) \le Ce^{-U(t)}e^{\int_0^t h^2 d\tau}\Big\{&h^2\beta_0^2(1+t)^{-2\delta(q)} + h^7(\beta_0+\eta)^2(1+t)\\ &+h^5(\beta_0+\eta)^2\ln(1+t) + (\beta_0+\eta)^2 e^{-\frac{5}{2}m_0^{-1}t} + \eta^2\|\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b}\|_{L^2}^2\Big\}.\end{aligned}$$

Let $h(t) \triangleq \sqrt{\theta}(1+t)^{-\frac{1}{2}}$, where $1+2\delta(q)<\theta<2+6\delta(q)$. Integration then shows that for all $t \le t_3$,

$$\|\tilde{\mathbf{v}}(t)\|_{L^2} \le C(\beta_0+\eta)(1+t)^{-\delta(q)}. \tag{6.47}$$

Step 5: Estimates for $\int_0^t \|\nabla\tilde{P}\|_{L^2}d\tau$ and $\int_0^t \|\Delta\tilde{\mathbf{v}}\|_{L^2}d\tau$. For any $0<l<1+2\delta(q)$, it follows from (6.40) that

$$\begin{aligned}&\int_0^t (\|\nabla\tilde{P}\|_{L^2}^2 + \|\Delta\tilde{\mathbf{v}}\|_{L^2}^2)(1+\tau)^l d\tau\\ &\lesssim \int_0^t (1+\tau)^l\|\partial_\tau\tilde{\mathbf{v}}\|_{L^2}^2 d\tau + (\beta_0+\eta)^2\int_0^t \Big[(1+\tau)^{-\frac{3}{2}(1+2\delta)+l}\|\Delta\overline{\mathbf{v}}\|_{L^2}\\ &\quad + e^{-\frac{3}{2}m_0^{-1}\tau}(1+\tau)^l\|\nabla^3\overline{b}\|_{L^2} + e^{-\frac{3}{2}m_0^{-1}\tau}(1+\tau)^l\|\nabla^3\tilde{b}\|_{L^2}^2\Big]d\tau\\ &\quad + \eta^2\int_0^t (1+\tau)^l(\|\Delta\overline{\mathbf{v}}\|_{L^2}^2 + \|\nabla\overline{P}\|_{L^2}^2 + e^{-\frac{3}{2}m_0^{-1}\tau}\|\nabla^3\overline{b}\|_{L^2}^2)d\tau \le C(\beta_0+\eta)^2.\end{aligned}$$

Therefore, Hölder's inequality gives that

$$\begin{aligned}\int_0^t (\|\nabla\tilde{P}\|_{L^2} + \|\Delta\tilde{\mathbf{v}}\|_{L^2})d\tau &\le C\Big[\int_0^t (1+\tau)^{-1-\delta(q)}d\tau\Big]^{\frac{1}{2}}\Big[\int_0^t (1+\tau)^{1+\delta(q)}(\|\nabla\tilde{P}\|_{L^2}^2 + \|\Delta\tilde{\mathbf{v}}\|_{L^2}^2)d\tau\Big]^{\frac{1}{2}}\\ &\le C(\beta_0+\eta).\end{aligned} \tag{6.48}$$

Step 6: Estimates for $\int_0^t \|\tilde{\mathbf{v}}\|_{L^\infty}d\tau$ and $\int_0^t \|\nabla\tilde{b}\|_{L^\infty}d\tau$. First, the density equation in (6.12) implies

$$\partial_t\tilde{\varrho} + (\overline{\mathbf{v}}+\tilde{\mathbf{v}})\cdot\nabla\tilde{\varrho} = -\tilde{\mathbf{v}}\cdot\nabla\overline{\varrho}. \tag{6.49}$$

Applying L^p energy estimates and utilizing the divergence-free condition yield, for any $2 \le p \le \infty$,

$$\|\tilde{\varrho}\|_{L_t^\infty(L^p)} \le \|\tilde{\varrho}_0\|_{L^p} + \|\nabla\overline{\varrho}\|_{L_t^\infty(L^p)}\|\tilde{\mathbf{v}}\|_{L_t^1(L^\infty)}.$$

Furthermore, in view of the definitions of η, β_0 and estimate (6.2), this inequality becomes

$$\eta(t) \le \beta_0 + C\|\tilde{\mathbf{v}}\|_{L_t^1(L^\infty)}, \tag{6.50}$$

which, combined with (6.48), yields

$$\int_0^t \|\tilde{\mathbf{v}}\|_{L^\infty} d\tau \le \int_0^t \|\nabla\tilde{\mathbf{v}}\|_{L^2}^{\frac{1}{2}} \|\nabla^2\tilde{\mathbf{v}}\|_{L^2}^{\frac{1}{2}} d\tau \le \varepsilon\|\Delta\tilde{\mathbf{v}}\|_{L_t^1(L^2)} + C\|\nabla\tilde{\mathbf{v}}\|_{L_t^1(L^2)}$$
$$\le C\varepsilon(\beta_0+\eta) + C\int_0^t (\beta_0+\eta(\tau))(1+\tau)^{-\frac{1}{2}(1+2\delta(q))} d\tau$$
$$\le C\beta_0 + C\varepsilon\|\tilde{\mathbf{v}}\|_{L_t^1(L^\infty)} + C\int_0^t \|\tilde{\mathbf{v}}\|_{L_\tau^1(L^\infty)}(1+\tau)^{-\frac{1}{2}(1+2\delta(q))} d\tau,$$

where ε is an arbitrarily small positive constant. By taking $\varepsilon = \frac{1}{2C}$, it follows that

$$\|\tilde{\mathbf{v}}\|_{L_t^1(L^\infty)} \le C\beta_0 + C\int_0^t \|\tilde{\mathbf{v}}\|_{L_\tau^1(L^\infty)}(1+\tau)^{-\frac{1}{2}(1+2\delta(q))} d\tau, \tag{6.51}$$

this, in turn, implies that $\|\tilde{\mathbf{v}}\|_{L_t^1(L^\infty)} \le C\beta_0$. Consequently, substituting it into (6.50) leads to

$$\eta(t) \le C\beta_0. \tag{6.52}$$

Finally, we estimate $\|\nabla\tilde{b}\|_{L_t^1(L^\infty)}$. A direct application of (6.52) gives

$$\int_0^t \|\nabla\tilde{b}\|_{L^\infty} d\tau \le \int_0^t \|\nabla^2\tilde{b}\|_{L^2}^{\frac{1}{2}} \|\nabla^3\tilde{b}\|_{L^2}^{\frac{1}{2}} d\tau \le (\int_0^t \|\nabla^2\tilde{b}\|_{L^2} d\tau)^{\frac{1}{2}} (\int_0^t \|\nabla^3\tilde{b}\|_{L^2} d\tau)^{\frac{1}{2}} \le C(\beta_0+\eta) \le C\beta_0. \tag{6.53}$$

Step 7: Proof of Proposition 6.5.

By virtue of (6.34), (6.35), and (6.52), we establish for all $t \le t_3$:

$$\|\tilde{\mathbf{v}}(t)\|_{L_t^\infty(L^2)} + \|\nabla\tilde{\mathbf{v}}(t)\|_{L_t^\infty(L^2)} + \|\nabla\tilde{b}(t)\|_{L_t^\infty(L^2)} + \|\nabla^2\tilde{b}(t)\|_{L_t^\infty(L^2)} \le C\beta_0. \tag{6.54}$$

Given that the constant c_5 in (6.20) is sufficiently small, we select β_0 small enough such that for all $t \le t_3$,

$$\|\tilde{\mathbf{v}}(t)\|_{L^2}\|\nabla\tilde{\mathbf{v}}(t)\|_{L^2} + \|\nabla\tilde{b}(t)\|_{L^2}\|\nabla^2\tilde{b}(t)\|_{L^2} \le C\beta_0^2 \le \frac{c_5}{4}, \tag{6.55}$$

implying that t_3 can be extended to T. Thus, Proposition 6.5 is proved. □

Proposition 6.6. *Under the assumptions of Theorem 3.3, if there exists a positive constant c_4 such that*

$$G_0 \triangleq \|\tilde{\mathbf{v}}_0\|_{H^1} + \|\tilde{\mathbf{v}}_0\|_{L^q} + \|\tilde{b}_0\|_{H^2} + \|\tilde{a}_0\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \le c_4, \tag{6.56}$$

then for all $t < T$, there holds that

$$\|\tilde{a}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(L^q)} + \|\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} + \|\tilde{b}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \le CG_0. \tag{6.57}$$

Proof. Note that $\beta_0 \le CG_0$. Applying Lemma 2.7 to Eqs $(3.2)_1$ and $(6.12)_1$ separately and using (6.1) and (6.31), we deduce

$$\|a\|_{\tilde{L}_t^\infty(\dot{H}^2)} \lesssim \|a_0\|_{H^2} \exp\{\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\} \lesssim \|a_0\|_{H^2} \exp\{C\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\} \tag{6.58}$$

and

$$\begin{aligned}\|\tilde{a}\|_{\tilde{L}_t^\infty(B_{2,1}^{\frac{3}{2}})} \lesssim&\Big(\|\tilde{a}_0\|_{B_{2,1}^{\frac{3}{2}}} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{3}{2}})}\|\nabla\overline{a}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}\Big)\exp\{\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}\\ \lesssim&(G_0 + \|\tilde{\mathbf{v}}\|_{L_t^1(H^1)}^{\frac{1}{2}}\|\tilde{\mathbf{v}}\|_{L_t^1(H^2)}^{\frac{1}{2}})\exp\{C\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\} \lesssim G_0\exp\{C\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}.\end{aligned} \tag{6.59}$$

Next, applying Corollary 2.1 to $(6.12)_3$ yields

$$\begin{aligned}\|\tilde{b}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} \lesssim&\Big(\|\tilde{b}_0\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|w''(\overline{b})\tilde{b}\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\tilde{\mathbf{v}}\cdot\nabla\overline{b}\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{3}{2}})}\Big)\exp\{\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}\\ \lesssim&\Big(\|\tilde{b}_0\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|w''(\overline{b})\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}\|\tilde{b}\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\tilde{\mathbf{v}}\|_{\tilde{L}_t^1(\dot{H}^2)}\|\overline{b}\|_{\tilde{L}_t^\infty(\dot{H}^2)}\Big)\exp\{\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}\\ \lesssim&G_0\exp\{\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\},\end{aligned} \tag{6.60}$$

where we used the fact that

$$\|w''(\overline{b})\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \le C(1 + \|\overline{b}\|_{L_t^\infty(L^\infty)})^2\|\overline{b}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \le C. \tag{6.61}$$

Similarly, employing the same approach to Eq $(6.12)_2$ via Lemma 2.8, we arrive at

$$\begin{aligned}&\|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} \lesssim \exp\{\|\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}\Big(\|\tilde{\mathbf{v}}_0\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})}\\ &+ \|\tilde{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\nabla\tilde{b}\Delta\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|a\nabla\tilde{b}\Delta\overline{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\nabla\tilde{b}\Delta\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})}\\ &+ \|a\nabla\tilde{b}\Delta\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\nabla\overline{b}\Delta\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|a\nabla\overline{b}\Delta\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\tilde{a}(\Delta\overline{\mathbf{v}} - \nabla\overline{P} - \nabla\overline{b}\Delta\overline{b})\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})}\\ &+ \|a\|_{L_t^\infty(H^2)}\|\nabla\tilde{P}\|_{L_t^1(L^2)} + \|a\|_{L_t^\infty(H^2)}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^2)}\Big).\end{aligned}$$

For conciseness, we estimate three arbitrarily chosen terms on the righthand side as follows:

$$\begin{aligned}&\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \lesssim \|\tilde{\mathbf{v}}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} \lesssim G_0\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})},\\ &\|a\nabla\tilde{b}\Delta\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{1}{2}})} \lesssim \|a\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}\|\nabla\tilde{b}\|_{L_t^\infty(\dot{H}^1)}\|\nabla^2\tilde{b}\|_{L_t^1(\dot{H}^1)} \lesssim G_0\exp\{C\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\},\\ &\|a\|_{L_t^\infty(H^2)}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^2)} \lesssim \|a\|_{L_t^\infty(H^2)}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{H}^1)}^{\frac{1}{3}}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}^{\frac{2}{3}} \le \frac{1}{2}\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} + CG_0\exp\{C\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}.\end{aligned}$$

Consequently,

$$\|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} + (\frac{1}{2} - CG_0)\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} \lesssim G_0\exp\{C\|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})}\}. \tag{6.62}$$

If G_0 and c_4 are sufficiently small with $G_0 \le c_4$, we sum (6.59)–(6.62) and apply the bootstrap method to deduce

$$\|\tilde{a}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\tilde{b}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} + \|\tilde{b}\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}})} + \|\tilde{\mathbf{v}}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}})} + \|\tilde{\mathbf{v}}\|_{L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})} \le CG_0. \tag{6.63}$$

Finally, in order to estimate $\|\tilde{\mathbf{v}}\|_{L^q}$, the divergence operator is applied to both sides of $(6.12)_2$, which yields

$$\begin{aligned}\Delta\tilde{P} =&\mathrm{div}\Big[-\bar{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}-\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}-\tilde{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}+(\bar{a}+\tilde{a})(\Delta\tilde{\mathbf{v}}-\nabla\tilde{P})-(1+\bar{a}+\tilde{a})(\nabla\tilde{b}\Delta\bar{b}\\&+\nabla\tilde{b}\Delta\tilde{b}+\nabla\bar{b}\Delta\tilde{b})+\tilde{a}(\Delta\bar{\mathbf{v}}-\nabla\bar{P}-\nabla\bar{b}\Delta\bar{b})\Big].\end{aligned}\tag{6.64}$$

Classical elliptic estimates then give

$$\begin{aligned}\|\nabla\tilde{P}\|_{L^q}\lesssim&\|\bar{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^q}+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^q}+\|\tilde{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}\|_{L^q}+\|(\bar{a}+\tilde{a})(\Delta\tilde{\mathbf{v}}-\nabla\tilde{P})\|_{L^q}\\&+\|(1+\bar{a}+\tilde{a})(\nabla\tilde{b}\Delta\bar{b}+\nabla\tilde{b}\Delta\tilde{b}+\nabla\bar{b}\Delta\tilde{b})\|_{L^q}+\|\tilde{a}(\Delta\bar{\mathbf{v}}-\nabla\bar{P}-\nabla\bar{b}\Delta\bar{b})\|_{L^q}.\end{aligned}\tag{6.65}$$

Taking the L^2 inner product of Eq $(6.12)_2$ with $|\mathbf{v}^j|^{q-1}\mathrm{sgn}(\mathbf{v}^j)$ $(j=1,2,3)$ gives rise to

$$\begin{aligned}&\frac{d}{dt}\|\tilde{\mathbf{v}}\|_{L^q}^q\leq\|\tilde{\mathbf{v}}\|_{L^q}^{q-1}\Big(\|\bar{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^q}+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^q}+\|\tilde{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}\|_{L^q}+\|(\bar{a}+\tilde{a})(\Delta\tilde{\mathbf{v}}-\nabla\tilde{P})\|_{L^q}\\&+\|\nabla\tilde{P}\|_{L^q}+\|(1+\bar{a}+\tilde{a})(\nabla\tilde{b}\Delta\bar{b}+\nabla\tilde{b}\Delta\tilde{b}+\nabla\bar{b}\Delta\tilde{b})\|_{L^q}+\|\tilde{a}(\Delta\bar{\mathbf{v}}-\nabla\bar{P}-\nabla\bar{b}\Delta\bar{b})\|_{L^q}\Big).\end{aligned}$$

Substituting (6.65) into the above inequality and integrating over time leads to

$$\begin{aligned}\|\tilde{\mathbf{v}}\|_{L^\infty_t(L^q)}\leq&\|\tilde{\mathbf{v}}_0\|_{L^q}+\|\bar{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^1_t(L^q)}+\|\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}\|_{L^1_t(L^q)}+\|\tilde{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}}\|_{L^1_t(L^q)}\\&+\|(\bar{a}+\tilde{a})(\Delta\tilde{\mathbf{v}}-\nabla\tilde{P})\|_{L^1_t(L^q)}+\|\nabla\tilde{b}\Delta\bar{b}+\nabla\tilde{b}\Delta\tilde{b}+\nabla\bar{b}\Delta\tilde{b}\|_{L^1_t(L^q)}\\&+\|(\bar{a}+\tilde{a})(\nabla\tilde{b}\Delta\bar{b}+\nabla\tilde{b}\Delta\tilde{b}+\nabla\bar{b}\Delta\tilde{b})\|_{L^1_t(L^q)}+\|\tilde{a}(\Delta\bar{\mathbf{v}}-\nabla\bar{P}-\nabla\bar{b}\Delta\bar{b})\|_{L^1_t(L^q)}.\end{aligned}\tag{6.66}$$

For $1<q<\frac{6}{5}$, applying the Sobolev embedding inequality along with (3.7), (6.53), and (6.63) yields

$$\begin{aligned}&\|(\bar{a}+\tilde{a})(\nabla\tilde{b}\Delta\bar{b}+\nabla\tilde{b}\Delta\tilde{b}+\nabla\bar{b}\Delta\tilde{b})\|_{L^1_t(L^q)}\\&\lesssim(\|\bar{a}\|_{L^\infty_t(H^{\frac{1}{2}})}+\|\tilde{a}\|_{L^\infty_t(H^{\frac{1}{2}})})(\|\nabla\tilde{b}\Delta\bar{b}\|_{L^1_t(L^2)}+\|\nabla\tilde{b}\Delta\tilde{b}\|_{L^1_t(L^2)}+\|\nabla\bar{b}\Delta\tilde{b}\|_{L^1_t(L^2)})\\&\lesssim(1+G_0)(\|\nabla\tilde{b}\|_{L^1_t(L^\infty)}\|\nabla^2\bar{b}\|_{L^\infty_t(L^2)}+\|\nabla\tilde{b}\|_{L^1_t(L^\infty)}\|\nabla^2\tilde{b}\|_{L^\infty_t(L^2)}\\&\quad+\|\nabla\bar{b}\|_{L^\infty_t(L^2)}^{\frac{1}{2}}\|\nabla^2\bar{b}\|_{L^\infty_t(L^2)}^{\frac{1}{2}}\|\nabla^3\tilde{b}\|_{L^1_t(L^2)})\lesssim G_0.\end{aligned}$$

Similarly, the remaining terms on the righthand side of (6.66) can also be bounded by CG_0. Therefore, we have

$$\|\tilde{\mathbf{v}}(t)\|_{L^\infty_t(L^q)}\leq CG_0,\tag{6.67}$$

which completes the proof of Proposition 6.6. □

Proof of Theorem 3.3. By applying Theorem 3.2 together with Propositions 6.5 and 6.6, we adopt a strategy similar to that outlined in Subsection 6.1 and, using a bootstrap argument, establish that the time T is unbounded. As a result, solutions can be extended to $T=\infty$, thereby obtaining (3.9). Subsequently, the application of classical interpolation techniques to (3.9) and (6.57) rigorously yields the critical estimate (3.10), thus completing the proof of Theorem 3.3. □

7. Conclusions

This study investigates a simplified viscoelastic fluid model with stress diffusion and establishes the stability of the equation for any globally smooth solution when the initial density is close to one. In the course of the proof, the velocity field $\mathbf{v}$ is shown to decay faster than $(1 + t)^{-3/4}$, while $\nabla b(t)$ decays exponentially. This represents a finding of considerable physical significance. The exponential decay of $\nabla b(t)$ explains why some viscoelastic fluids can rapidly "relax" after the cessation of external disturbances and exhibit a sharp decline in their resistance to deformation. These decay properties confirm the presence of irreversible dissipative mechanisms—such as viscosity, elastic relaxation, and thermal diffusion—that continuously transform ordered mechanical energy into disordered thermal energy.

Nevertheless, our work has several limitations. First, although considering only the case of spherical elastic stress has helped us gain insight into more complex models, this simplification essentially reduces the system to a toy model, suitable only for studying idealized fluids. Second, the regularity assumptions imposed on the initial data may not be optimal, and the feasibility of relaxing these conditions remains to be thoroughly investigated. Third, the stability results established in this work rely on the assumption of "sufficiently small initial perturbations". While this serves as a common foundation for rigorous mathematical analysis and reveals key features of the local dynamics near equilibrium, it may fail to capture the full range of behaviors in far-from-equilibrium flows—such as those dominated by large elastic stresses. Therefore, the present model leaves ample room for further exploration. What's more, future studies could also examine solution properties under temperature-dependent or variable viscosity coefficients. Finally, it should be emphasized that numerical simulations can offer a more concrete illustration of the decay and stability phenomena. We are currently conducting numerical experiments based on this model, and the detailed procedures and results will be presented in a forthcoming article.

Author contributions

Xi Wang: Conceptualization, Methodology, Formal Analysis, Writing – Original Draft, Writing – Review & Editing; Xueli Ke: Conceptualization, Methodology, Formal Analysis, Writing – Original Draft, Writing – Review & Editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors have no relevant financial or non-financial interests to disclose. The authors have no competing interests to declare that are relevant to the content of this article.

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