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Research article

Asymptotic performance of nonoscillatory solutions of functional differential equations involving a delayed damping term

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Abstract: This work investigated the asymptotic performance of nonoscillatory solutions to the functional differential equation (FDE) $\left(\alpha(u)\left|y'(u)\right|^{\kappa-1}y'(u)\right)' + p(u)F(y'(\delta(u))) + q(u)G(y(\rho(u))) = 0$, which involves a delayed damping term. Using Riccati and comparison methods, we extended the previous results to the nonlinear case of the considered equation. Furthermore, the new criteria improved upon the previous ones by removing some constraints on the delay functions. Then, for the linear case, we derived new criteria that take into account all parameters of the equation. The examples and comparisons provided illustrate the importance and novelty of our results.

Keywords: differential equations; second-order equation; delayed damping term; asymptotic performance of nonoscillatory solutions; oscillatory properties
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

Qualitative theory is of immense importance and vitality due to its many applications in most branches of science. This theory answers many questions surrounding non-linear mathematical models and provides information about the stability of solutions, their oscillation, periodicity, synchrony, symmetry, and others.

Oscillation theory is one of the main areas of qualitative theory, which primarily examines the oscillatory characteristics of differential equation solutions, see [1–3]. The study of the oscillatory properties of differential equations has advanced significantly over the last two decades, see [4–6].

Oscillation and asymptotic properties of delay equations have important applications in various real-world mathematical models. Furthermore, there are many intriguing theoretical issues in oscillation theory that require sophisticated mathematical analysis methods; see, for example, [7–10]. Second-order nonlinear differential equations appear in the modeling of many natural phenomena, see, for example, [11–13].

A key idea in the analytical study of the behavior of solutions to differential equations is damping. In actuality, "damping" describes the state of the solution that arises from the addition of this term, in which the solution's amplitude gradually decreases. Investigating how non-delay damping influences the oscillatory behavior of solutions to differential equations presents an intriguing and meaningful direction for further analysis; see [14–16]. Numerous realistic systems, such as mechanical vibrations [17], biological rhythms [18], and electrical RLC circuits (Resistance, Inductance, and Capacitance) [19], are modeled using second-order damped differential equations.

In this study, we consider the functional differential equation (FDE)

$$\left(\alpha(u)\,|y'(u)|^{\kappa-1}\,y'(u)\right)' + p(u)\,F\left(y'(\delta(u))\right) + q(u)\,G\left(y(\rho(u))\right) = 0, \tag{1.1}$$

where $u \geq u_0$ and $\kappa > 1$ is a real number. Furthermore, we use the following assumptions:

(H1) $\alpha, p, q \in \mathbf{C}(\mathbb{I}_{u_0}, [0, \infty))$ and $\alpha(u) > 0$, where $\mathbb{I}_\varrho = [\varrho, \infty)$;
(H2) $\delta, \rho \in \mathbf{C}^1(\mathbb{I}_{u_0}, \mathbb{R})$, $\delta(u) \leq u$, $\rho(u) \leq u$, $\delta'(u) \geq 0$, $\rho'(u) \geq 0$, $\delta(u) \to \infty$, and $\rho(u) \to \infty$ as $u \to \infty$;
(H3) $F \in \mathbf{C}(\mathbb{R}, \mathbb{R})$ and there is a $\ell > 0$ such that $F(z)/z \geq \ell$, for $z \neq 0$;
(H4) $G \in \mathbf{C}(\mathbb{R}, \mathbb{R})$ and there is a $k > 0$ such that $G(y)/y^\kappa \geq k$, for $y \neq 0$.

We say that a function $y \in \mathbf{C}\left(\mathbb{I}_{u_y}, \mathbb{R}\right)$, $u_y \in \mathbb{I}_{u_0}$, is a solution of (1.1) if y satisfies (1.1) for $u \in \mathbb{I}_{u_y}$, and $\alpha\,(y')^\kappa \in \mathbf{C}^1\left(\mathbb{I}_{u_y}, \mathbb{R}\right)$. We only consider those solutions of (1.1) which satisfy $\sup\{|y(u)| : u \geq u_1\} > 0$ for every $u_1 \in \mathbb{I}_{u_y}$.

A solution y is called oscillatory if it has a sequence of zeros $\{u_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} u_n = \infty$. Stating that all solutions of a given equation are oscillatory or converge to zero implies that any solution—if it exists—must exhibit oscillatory or asymptotic behavior.

Definition 1.1. *Equation (1.1) satisfies Property $\mathcal{A}$ if all of its solutions oscillate or converge to zero.*

There has been a recent active research movement on the oscillatory and asymptotic performance of solutions to FDEs. Oscillation criteria for first-order equations have been developed by several methods; see, for example, [20–23]. Studies [24–27] developed well-known techniques to obtain more efficient oscillatory criteria for testing solutions of second-order equations. This development has also been reflected in the study of the oscillatory properties of even-order equations; see, for example, [28–30]. Odd-order equations have received less attention, but their oscillatory testing methods have also been improved in several studies (see [31–34]).

On the other hand, numerous studies discussed the effect of adding a damping term on the oscillatory performance of solutions to FDEs; see [35, 36] for second-order equations and [37–39] for higher even-order equations. The analytical problems resulting from this addition were often addressed in one of two ways: first, ignoring one of the equation's terms during the investigation, and second, incorporating the damping term into the higher-order term. However, the drawbacks of the criteria, which are obtained with these approaches, are that they are not affected by some coefficients of the equation, and some of them are difficult to apply.

For FDEs with a delayed damping term

$$y''(u) + p(u) y'(\delta(u)) + q(u) G(y(\rho(u))) = 0,$$

Grace [40] tested the oscillatory properties of this equation, where $yG(y) > 0$ and G is nondecreasing. He used the comparison principle to exclude increasing positive solutions by the condition

$$\liminf_{u\to\infty} \int_{\delta(u)}^{u} p(l)\,\mathrm{d}l > \frac{1}{\mathrm{e}}.$$

Kamenev criteria are conditions applied to ascertain whether the solutions of specific kinds of differential equations, especially second-order equations, oscillate. These criteria usually entail using methods that analyze the attributes of the solutions to identify oscillation conditions, such as the Riccati transform and integral averaging (see [41]). Saker et al. [42] presented the Kamenev-type criteria for the oscillatory properties of the FDE

$$(\alpha(u) y'(u))' + p(u) y'(\delta(u)) + q(u) G(y(\rho(u))) = 0, \tag{1.2}$$

where $\delta \geq \rho$ and G satisfies (H4) when $\kappa = 1$.

Very recently, Moaaz and Ramos [43] studied the oscillatory behavior of the FDE (1.2). Using a new approach, they were able to obtain a criterion that guarantees the oscillation of all solutions.

When studying increasing positive solutions, the technique used in Grace's results [40] depends on neglecting the last term of the equation (which is positive) and thus obtaining the inequality

$$y''(u) + p(u) y'(\delta(u)) \leq 0.$$

As a result, the effects of $q(u)$ and $\rho(u)$ are not considered by the criterion that excludes increasing positive solutions, while the results in [42] neglected the damping term and obtained the inequality

$$(\alpha(u) y'(u))' + q(u) G(y(\rho(u))) \leq 0.$$

Therefore, the criteria for this case are clearly not affected by $p(u)$ and $\delta(u)$. On the other hand, we noted that previous results considered only the linear case of the equation under study. Therefore, the motivations for this study lie in extending previous results to the nonlinear case and improving previous results by obtaining criteria that take all parameters into account.

Among the studies that appeared recently and contributed to the development of oscillation theory for neutral differential equations are Li et al. [44] and Alarfaj and Muhib [45]. They studied the oscillatory behavior of mixed neutral delay differential equations. Alqhtani et al. [46] and Hou and Sun [47] are also among the authors that improved the oscillation conditions for the neutral equations.

The main objective of this study is to investigate the oscillatory behavior of solutions to FDE (1.1). Our results extend and improve the results in [40, 42, 43] to the nonlinear case.

The following section is divided into three parts: the first introduces some auxiliary lemmas, followed by theorems for testing the asymptotic behavior in the nonlinear case, and finally, improved criteria for the linear case of FDE (1.1).

2. Main results

In this section, the category of all eventually positive solutions to (1.1) is denoted by $\mathbb{Y}^+$ for convenience. Also, we define $\widetilde{\kappa} := 1/(\kappa+1)^{\kappa+1}$ and

$$\varphi(u) := \int_{u_1}^{u} \frac{\mathrm{d}l}{\alpha^{1/\kappa}(l)},$$

for $u \in \mathbb{I}_{u_1}$, where $u_1 \geq u_0$ is large enough.

2.1. Auxiliary lemmas

Here, we conclude some significant facts and inequalities for studying the oscillatory performance of solutions. Minutely, we work on:

– Determining the behavior of the first derivative of a positive solution;
– Finding the conditions that ensure that the decreasing positive solutions converge to zero;
– Deducing relationships between the derivatives of the increasing positive solutions.

Lemma 2.1. *The first derivative of eventually positive solutions to Eq (1.1) has a constant sign.*

Proof. Let $y \in \mathbb{Y}^+$. Thus, $y(\rho(u)) > 0$ for $u \in \mathbb{I}_{u_1}$, where $u_1 \geq u_0$ is large enough. Using (H3) and (H4), Eq (1.1) takes the form

$$(\alpha(u)[y'(u)]^{\kappa})' + \ell\, p(u)\, y'(\delta(u)) + k\, q(u)\, y^{\kappa}(\rho(u)) \leq 0. \tag{2.1}$$

Suppose the contrary that y' is oscillatory. Therefore, there is $u_2 \in \mathbb{I}_{u_1}$ such that $y'(\delta(u_2)) = 0$. It follows from (1.1) that

$$(\alpha(u)[y'(u)]^{\kappa})'\big|_{u=u_2} \leq -k\, q(u_2)\, y^{\kappa}(\rho(u_2)) \leq 0.$$

This necessitates that y' cannot have any zeros after the first one, which goes against its oscillatory feature. So, y' is of one sign. □

Lemma 2.2. *All decreasing positive solutions to Eq (1.1) converge to zero if there is a function* $\mu \in \mathbf{C}^1(\mathbb{I}_{u_0}, (0,\infty))$ *that satisfies the following conditions:*

$$\mu'(u) \geq 0 \ \text{ and } \ \left(\frac{\mu(u)\, p(u)}{\delta'(u)}\right)' \leq 0, \tag{2.2}$$

$$\int_{u_0}^{\infty} \mu(l)\, q(l)\, \mathrm{d}l = \infty, \tag{2.3}$$

and

$$\int_{u_0}^{\infty} \left(\frac{1}{\mu(x)\,\alpha(x)} \int_{u_0}^{x} \mu(l)\, q(l)\, \mathrm{d}l\right)^{1/\kappa} \mathrm{d}x = \infty. \tag{2.4}$$

Proof. Let $y \in \mathbb{Y}^+$ and $y'(u) < 0$ for $u \geq u_1$. Then, there is $y_0 \geq 0$ such that $y \to y_0$ as $u \to \infty$, and so

$$y(u) \geq y_0, \tag{2.5}$$

for $u \geq u_2 \geq u_1$.

Suppose the contrary that $y_0 > 0$. Now, we define $\Omega := \mu\,\alpha\,[y']^{\kappa}$. Using (1.1), we obtain

$$\begin{aligned} \Omega' &= \mu\,(\alpha\,[y']^{\kappa})' + \mu'\,\alpha\,[y']^{\kappa} \\ &\leq -\ell\,\mu\,p\,y'(\delta) - k\,\mu\,q\,y^{\kappa}(\rho) + \mu'\,\alpha\,[y']^{\kappa}, \end{aligned}$$

which with (2.2) and (2.5) gives

$$\Omega' \leq -\ell\,\mu\,p\,y'(\delta) - k\,y_0^{\kappa}\,\mu\,q.$$

Integrating the last inequality implies that

$$\Omega(u) \leq \Omega(u_2) - \ell\int_{u_2}^{u}\mu(l)\,p(l)\,y'(\delta(l))\,\mathrm{d}l - k\,y_0^{\kappa}\int_{u_2}^{u}\mu(l)\,q(l)\,\mathrm{d}l. \tag{2.6}$$

Using the Bonnet theorem, there is a $u^* \in [u_2, u]$ that satisfies

$$\begin{aligned} \int_{u_2}^{u}\mu(l)\,p(l)\,[-y'(\delta(l))]\,\mathrm{d}l &= -\frac{\mu(u_2)\,p(u_2)}{\delta'(u_2)}\int_{u_2}^{u^*}\delta'(l)\,y'(\delta(l))\,\mathrm{d}l \\ &= \frac{\mu(u_2)\,p(u_2)}{\delta'(u_2)}\,[y(\delta(u_2)) - y(\delta(u^*))] \\ &\leq \frac{\mu(u_2)\,p(u_2)}{\delta'(u_2)}\,y(\delta(u_2)). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we arrive at

$$\Omega(u) \leq \Omega(u_2) + \ell\frac{\mu(u_2)\,p(u_2)}{\delta'(u_2)}\,y(\delta(u_2)) - k\,y_0^{\kappa}\int_{u_2}^{u}\mu(l)\,q(l)\,\mathrm{d}l.$$

In view of (2.3), this inequality becomes

$$\mu(u)\,\alpha(u)\,[y'(u)]^{\kappa} \leq -\frac{k y_0^{\kappa}}{2}\int_{u_2}^{u}\mu(l)\,q(l)\,\mathrm{d}l,$$

or

$$y'(u) \leq -\frac{k^{1/\kappa}y_0}{2^{1/\kappa}}\left(\frac{1}{\mu(u)\,\alpha(u)}\int_{u_2}^{u}\mu(l)\,q(l)\,\mathrm{d}l\right)^{1/\kappa}.$$

Consequently,

$$y(u) \leq y(u_2) - \frac{k^{1/\kappa}y_0}{2^{1/\kappa}}\int_{u_2}^{u}\left(\frac{1}{\mu(x)\,\alpha(x)}\int_{u_2}^{x}\mu(l)\,q(l)\,\mathrm{d}l\right)^{1/\kappa}\mathrm{d}x.$$

This, with (2.4), leads to a contradiction. □

Lemma 2.3. *Every increasing positive solution y to Eq (1.1) satisfies the following:*

$$(\alpha(u)\,[y'(u)]^{\kappa})' \leq 0, \tag{2.8}$$

$$y(u) \geq \alpha^{1/\kappa}(u)\,y'(u)\,\varphi(u), \tag{2.9}$$

and

$$\left(\frac{y(u)}{\varphi(u)}\right)' \leq 0. \tag{2.10}$$

Proof. Let $y \in \mathbb{Y}^{+}$ and $y'(u) > 0$ for $u \geq u_1$. By virtue of (H3) and (H4), Eq (1.1) becomes

$$(\alpha\ [y']^{\kappa})' \leq -\ell\, p\, y'(\delta) - k\, q\, y^{\kappa}(\rho) \leq 0.$$

So, we have

$$y(u) \geq \int_{u_1}^{u} \frac{\left[\alpha(l)\,(y'(l))^{\kappa}\right]^{1/\kappa}}{\alpha^{1/\kappa}(l)} \mathrm{d}l \geq \alpha^{1/\kappa}(u)\, y'(u)\, \varphi(u),$$

which yields

$$\left(\frac{y}{\varphi}\right)' = \frac{\varphi\, y' - \alpha^{1/\kappa} y}{\varphi^{2}} \leq 0.$$

□

2.2. New criteria

By simple and direct approaches, this section derives some criteria that exclude increasing positive solutions. Thus, by combining these criteria with the conditions in Lemma 2.2, we can guarantee Property $\mathcal{A}$.

Theorem 2.1. *Equation (1.1) satisfies Property* $\mathcal{A}$ *if there is* $\mu \in \mathbf{C}^{1}(\mathbb{I}_{u_0}, (0,\infty))$ *such that one of the following cases is satisfied:*
(i) *Conditions (2.2), (2.4), and*

$$\int_{u_0}^{\infty} q(l)\, \mathrm{d}l = \infty; \tag{2.11}$$

(ii) *Conditions (2.2)–(2.4), and there is* $\beta \in \mathbf{C}^{1}(\mathbb{I}_{u_0}, (0,\infty))$ *such that*

$$\limsup_{u\to\infty} \int_{u_0}^{u} \left(k\beta(l)\, q(l) \frac{\varphi^{\kappa}(\rho(l))}{\varphi^{\kappa}(l)} - \widetilde{\kappa} \frac{\alpha(l)\,[\beta'(l)]^{\kappa+1}}{\beta^{\kappa}(l)} \right) \mathrm{d}l = \infty. \tag{2.12}$$

Proof. Suppose that $y \in \mathbb{Y}^{+}$. In view of Lemma 2.1, y' has a constant sign. We note that condition (2.11), along with the fact that μ is nondecreasing, leads to the fulfillment of condition (2.3). So, in the case where $y' < 0$, Lemma 2.2 asserts that $y \to 0$ as $u \to \infty$.

Now, we assume that $y'(u) > 0$ for $u \geq u_1$. Thus, for all $u \geq u_1$, we obtain $y(u) \geq y(u_1) := y_1 > 0$. Ignoring the middle term of Eq (1.1), we get

$$(\alpha\ [y']^{\kappa})' \leq -k\, q\, y^{\kappa}(\rho) \leq -k\, y_1^{\kappa}\, q. \tag{2.13}$$

Integration of (2.13) produces

$$\int_{u_1}^{\infty} q(l)\, \mathrm{d}l \leq \frac{\alpha(u_1)\,[y'(u_1)]^{\kappa}}{k\, y_1^{\kappa}}.$$

This leads to a contradiction with (2.7).

For case (ii), we define

$$\omega := \beta \frac{\alpha\ [y']^{\kappa}}{y^{\kappa}} > 0. \tag{2.14}$$

Using (2.10), (2.13), and (2.14), we obtain

$$\omega' = \frac{\beta'}{\beta}\omega + \beta \frac{(\alpha\ [y']^{\kappa})'}{y^{\kappa}} - \kappa\beta \frac{\alpha\ [y']^{\kappa+1}}{y^{\kappa+1}} \tag{2.15}$$

$$\leq \quad -k\beta q\frac{\varphi^{\kappa}(\rho)}{\varphi^{\kappa}}+\frac{\beta'}{\beta}\omega-\frac{\kappa}{\alpha^{1/\kappa}\beta^{1/\kappa}}\omega^{1+1/\kappa}$$

$$\leq \quad -k\beta q\frac{\varphi^{\kappa}(\rho)}{\varphi^{\kappa}}+\widetilde{\kappa}\frac{\alpha\ [\beta']^{\kappa+1}}{\beta^{\kappa}}, \tag{2.16}$$

which depends on the known inequality (see [48])

$$A\omega-B\omega^{1+1/\kappa}\leq\kappa^{\kappa}\widetilde{\kappa}\frac{A^{\kappa+1}}{B^{\kappa}}.$$

Integration of (2.16) gives

$$\int_{u_1}^{u}\left(k\beta(l)\,q(l)\,\frac{\varphi^{\kappa}(\rho(l))}{\varphi^{\kappa}(l)}-\widetilde{\kappa}\frac{\alpha(l)\,[\beta'(l)]^{\kappa+1}}{\beta^{\kappa}(l)}\right)\mathrm{d}l\leq\omega(u_1),$$

which contradicts (2.12).

This completes the proof. □

Using the comparison method with first-order equations, we present the following theorem:

Theorem 2.2. *Equation (1.1) satisfies Property $\mathcal{A}$ if there is $\mu\in\mathbf{C}^1(\mathbb{I}_{u_0},(0,\infty))$ such that (2.2)–(2.4) hold, and one of the following FDE is oscillatory:*

$$w'(u)+k\,q(u)\,\varphi^{\kappa}(\rho(u))\,w(\rho(u))=0 \tag{2.17}$$

or

$$w'(u)+\frac{\ell\,p(u)}{\alpha^{1/\kappa}(\delta(u))}w^{1/\kappa}(\delta(u))=0. \tag{2.18}$$

Proof. Suppose that $y\in\mathbb{Y}^{+}$. In view of Lemma 2.1, y' has a constant sign. So, in the case where $y'<0$, Lemma 2.2 asserts that $y\to0$ as $u\to\infty$.

Now, we assume that $y'(u)>0$ for $u\geq u_1$. From (2.9) and (2.13), we have

$$(\alpha\ [y']^{\kappa})'\leq-k\,q\,y^{\kappa}(\rho)\leq-k\,q\,\varphi^{\kappa}(\rho)\,\alpha(\rho)\,[y'(\rho)]^{\kappa}.$$

Letting

$$w:=\alpha\ [y']^{\kappa}, \tag{2.19}$$

we see that w is a positive solution of

$$w'+k\,q\,\varphi^{\kappa}(\rho)\,w(\rho)\leq0.$$

Using Theorem 1 in [49], the associated FDE (2.17) has a positive solution, which is a contradiction.

On the other hand, ignoring the last term of Eq (1.1), we get

$$(\alpha\ [y']^{\kappa})'+\ell\,p\,y'(\delta)\leq0.$$

From (2.19), we have that w is a positive solution of

$$w'+\frac{\ell\,p}{\alpha^{1/\kappa}(\delta)}w^{1/\kappa}(\delta)\leq0.$$

By virtue of Lemma 1 in [50], the associated FDE (2.18) has a positive solution, which is a contradiction.

This completes the proof. □

If $\kappa \geq 1$, the following corollary uses known results in the literature to provide oscillation criteria for the first-order equations in the previous theorem.

Corollary 2.1. *Equation (1.1) satisfies Property* $\mathcal{A}$ *if there is* $\mu \in \mathbf{C}^1(\mathbb{I}_{u_0}, (0,\infty))$ *such that (2.2)–(2.4) hold, and one of the following conditions is satisfied:*

$$\liminf_{u\to\infty} \int_{\rho(u)}^{u} q(l)\,\varphi^{\kappa}(\rho(l))\,\mathrm{d}l > \frac{1}{k\,\mathrm{e}} \tag{2.20}$$

or

$$\liminf_{u\to\infty} \int_{\delta(u)}^{u} \frac{p(l)}{\alpha(\delta(l))}\mathrm{d}l > L, \tag{2.21}$$

where

$$L := \begin{cases} 1/(\ell\,\mathrm{e}) & \text{if } \kappa = 1; \\ 0 & \text{if } \kappa > 1. \end{cases}$$

Proof. According to Theorems 2.1.1 and 3.1.2 in [51], conditions (2.20) and (2.21) guarantee that the solutions to (2.17) and (2.18) oscillate. □

Remark 2.1. *Consider the case where* $\kappa = 1$ *and* $F(z) = G(z) = z$*. Theorem 2.1 (case (ii)) is similar to Theorem 3.1 in [42] with a slight improvement: Theorem 2.1 does not require the constraint* $\rho(u) \leq \delta(u)$*. Furthermore, if* $\alpha(u) = 1$*, then Corollary 2.1 (condition (2.21) reduces to Theorem 1 in [40]. Therefore, the results in this section extend and slightly improve the results in [40, 42].*

Example 2.1. *Consider the FDE*

$$([y'(u)]^{\kappa})' + \frac{p_0}{u^{\kappa}} y'(\delta_0 u) + \frac{q_0}{u^{\kappa+1}} y^{\kappa}(\rho_0 u) = 0, \tag{2.22}$$

where $p_0 \geq 0$*,* $q_0 > 0$*,* $\delta_0, \rho_0 \in (0,1]$*. By choosing* $\mu(u) = u^{\kappa}$*, conditions (2.2)–(2.4) are satisfied. It is not difficult to see that condition (2.11) is not fulfilled. However, Eq (2.22) satisfies Property* $\mathcal{A}$ *in one of the following cases:*

$$\rho_0^{\kappa} q_0 > \kappa^{\kappa+1}\widetilde{\kappa}, \text{ using (2.12) with } \beta(u) = u^{\kappa}; \tag{C1}$$

$$\rho_0^{\kappa} q_0 \ln(1/\rho_0) > 1/\mathrm{e}, \text{ using (2.20)}; \tag{C2}$$

$$p_0 \ln(1/\delta_0) > 1/\mathrm{e} \text{ for } \kappa = 1, \text{ using (2.21)}. \tag{C3}$$

Remark 2.2. *The significant differences between the conditions in (C1)–(C3) are evident. In particular, the conditions in (C1) and (C2) do not depend on* p_0 *and* δ_0*, while the condition in (C3) does not depend on* q_0 *and* ρ_0*. Moreover, Figure 1 compares two cases of conditions in (C1) and (C2), which clearly show that no condition is absolutely better than the other.*

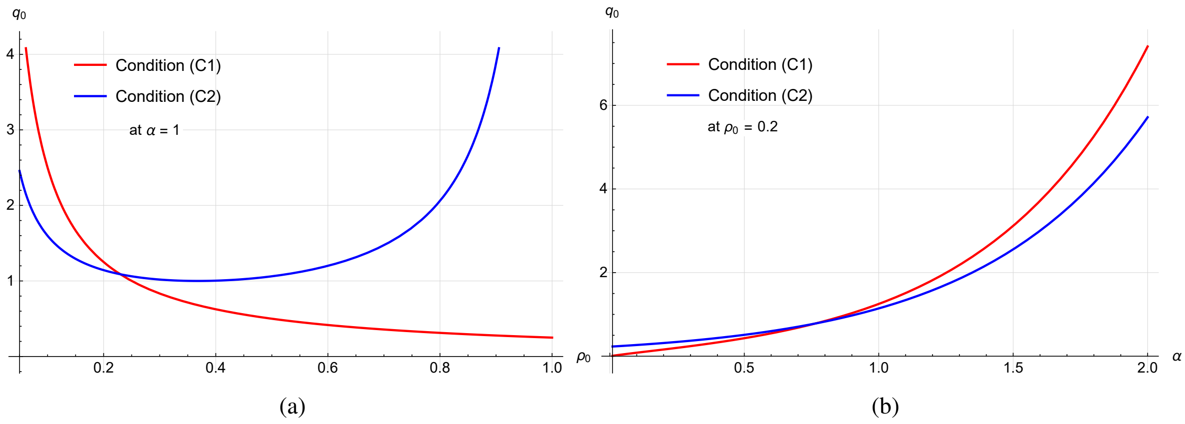

Figure 1. Comparison between the conditions in (C1) and (C2).

Remark 2.3. *The weaknesses in the criteria of this section result from the fact that the approaches used are always based on ignoring one of the second or third terms of Eq (1.1). Thus, conditions (2.11), (2.12), (2.20), and (2.21) lack the effect of either p and δ, or q and ρ.*

2.3. Improved criteria

This section derives improved conditions for testing Property $\mathcal{A}$ without ignoring any terms of Eq (1.1), when $\kappa = 1$. The following theorem uses Riccati substitution to test the Property $\mathcal{A}$ of the solutions of Eq (1.1).

Theorem 2.3. *Equation (1.1) satisfies Property $\mathcal{A}$ if there is $\mu, \beta \in \mathbf{C}^1(\mathbb{I}_{u_0}, (0,\infty))$ such that (2.2)–(2.4) hold, and*

$$\limsup_{u\to\infty} \int_{u_0}^{u} \left(k\beta(l)\eta(l)q(l)\frac{\varphi(\rho(l))}{\varphi(l)} - \frac{1}{4}\frac{\eta(l)\alpha(l)[\beta'(l)]^2}{\beta(l)} \right) \mathrm{d}l = \infty, \tag{2.23}$$

where

$$\eta(u) := \exp\left(\ell \int_{u_0}^{u} \frac{p(l)}{\alpha(\delta(l))} \mathrm{d}l\right).$$

Proof. Suppose that $y \in \mathbb{Y}^+$. In view of Lemma 2.1, y' has a constant sign. So, in the case where $y' < 0$, Lemma 2.2 asserts that $y \to 0$ as $u \to \infty$.

Now, we assume that $y'(u) > 0$ for $u \geq u_1$. In view of the facts $(\alpha y')' \leq 0$ and $\delta(u) \leq u$, Eq (1.1) becomes

$$\begin{aligned} 0 &= (\alpha y')' + \ell p y'(\delta) + k q y(\rho) \\ &\geq (\alpha y')' + \ell \frac{p\alpha}{\alpha(\delta)} y' + k q y(\rho) \\ &= \frac{1}{\eta(u)}[\eta \alpha y'] + k q y(\rho). \end{aligned} \tag{2.24}$$

We define

$$\omega := \beta \frac{\eta \alpha y'}{y}. \tag{2.25}$$

Using (2.10), (2.24), and (2.25), we find

$$\begin{aligned}\omega' &\le -k\beta\eta\, q\frac{y(\rho)}{y}+\frac{\beta'}{\beta}\omega+-\frac{1}{\beta\eta\,\alpha}\omega^2\\ &\le -k\beta\eta\, q\frac{\varphi(\rho)}{\varphi}+\frac{1}{4}\frac{\eta\,\alpha\,[\beta']^2}{\beta}. \end{aligned}\tag{2.26}$$

Integration of (2.26) gives

$$\int_{u_1}^{u}\left(k\beta(l)\,\eta(l)\,q(l)\,\frac{\varphi(\rho(l))}{\varphi(l)}-\frac{1}{4}\frac{\eta(l)\,\alpha(l)\,[\beta'(l)]^2}{\beta(l)}\right)\mathrm{d}l\le\omega(u_1),$$

which contradicts (2.23).

This completes the proof. □

Next, we use an improved comparison approach with first-order to obtain new criteria.

Theorem 2.4. *Equation (1.1) satisfies Property* $\mathcal{A}$ *if there is* $\mu\in\mathbf{C}^1(\mathbb{I}_{u_0},(0,\infty))$ *such that (2.2)–(2.4) hold, and*

$$\liminf_{u\to\infty}\int_{\sigma(u)}^{u}\left(\frac{\ell p(l)}{\alpha(\delta(l))}+k\varphi(\rho(l))\;q(l)\right)\mathrm{d}l>\frac{1}{\mathrm{e}},\tag{2.27}$$

where

$$\sigma(u)=\max\{\delta(u),\rho(u)\}.$$

Proof. Suppose that $y\in\mathbb{Y}^{+}$. In view of Lemma 2.1, y' has a constant sign. So, in the case where $y'<0$, Lemma 2.2 asserts that $y\to 0$ as $u\to\infty$.

Now, we assume that $y'(u)>0$ for $u\ge u_1$. In view of the fact (2.9) and that $(\alpha\, y')'\le 0$, Eq (1.1) becomes

$$\begin{aligned}0 &\ge (\alpha\, y')'+\ell\, p\, y'(\delta)+k\, q\,\varphi(\rho)\,\alpha(\rho)\, y'(\rho)\\ &\ge (\alpha\, y')'+\left(\ell\frac{p}{\alpha(\delta)}+k\varphi(\rho)\;q\right)\alpha(\sigma)\, y'(\sigma).\end{aligned}$$

Letting $w:=\alpha\, y'$, we see that w is a positive solution of

$$w'+\left(\ell\frac{p}{\alpha(\delta)}+k\varphi(\rho)\;q\right)w(\sigma)\le 0.\tag{2.28}$$

From Theorem 2.1.1 in [51], the conditions (2.27) guarantees that the solutions to (2.28) oscillate, which is a contradiction.

This completes the proof. □

Example 2.2. *Consider the FDE (2.22) with* $\kappa=1$ *(Euler type). Equation (2.22) satisfies Property* $\mathcal{A}$ *in one of the following cases:*

$$\rho_0 q_0>\frac{1}{4}(1-p_0)^2,\; p_0\le 1,\; \textit{using (2.23) with}\;\beta(u)=u^{1-p_0};\tag{C4}$$

$$(p_0+\rho_0\, q_0)\ln(1/\sigma_0)>1/\mathrm{e},\; \textit{using (2.27), where}\;\sigma_0=\max\{\delta_0,\rho_0\}.\tag{C5}$$

Remark 2.4. *We notice that the condition in (C4) is affected by the parameters ρ_0, q_0, and p_0, while the condition in (C5) is affected by all parameters of Eq (2.22). Figure 2 shows the efficiency of criteria in (C4) and (C5) compared to criteria in (C1) and (C2).*

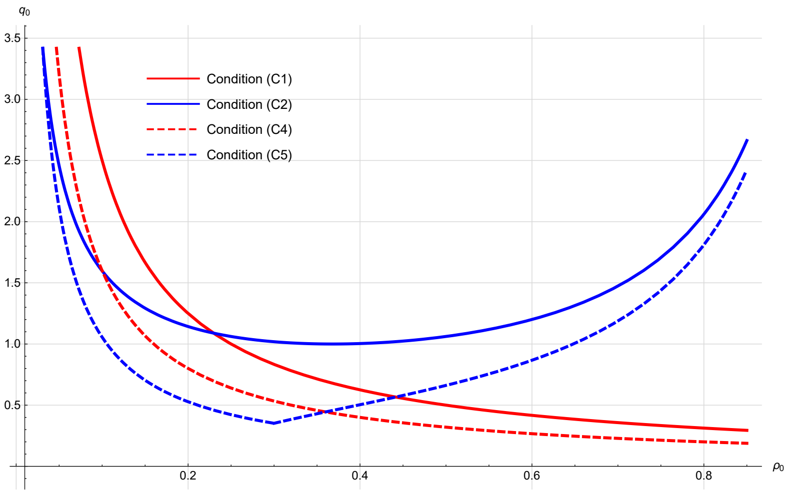

Figure 2. Comparison between conditions in (C1), (C2), (C4), and (C5).

Example 2.3. *Consider the FDE*

$$y''(u) + \frac{p_0}{u} y'(\delta_0 u)\left[1 + (y'(\delta_0 u))^2\right] + \frac{q_0}{u^2} y(\rho_0 u)\, e^{y^2(\rho_0 u)} = 0, \tag{2.29}$$

where $p_0 \geq 0$, $q_0 > 0$, $\delta_0, \rho_0 \in (0, 1]$. Note that $F(z) = z\left[1 + z^2\right]$ and $G(z) = z e^{z^2}$, and so $F(z)/z \geq 1$ and $G(z)/z \geq 1$. Thus, the condition in (C4) and (C5) (in Example 2.2) guarantees that Eq (2.29) satisfies Property $\mathcal{A}$. However, the results in [40, 42, 43] cannot be applied to this equation.

Remark 2.5. *Figure 3 shows some numerical solutions to the FDE*

$$y''(u) + \frac{p_0}{u} y'(u - c_1) + \frac{q_0}{u^2} y(u - c_2) = 0,$$

where p_0, q_0, c_1, and c_2 are nonnegative, and $q_0 > 0$.

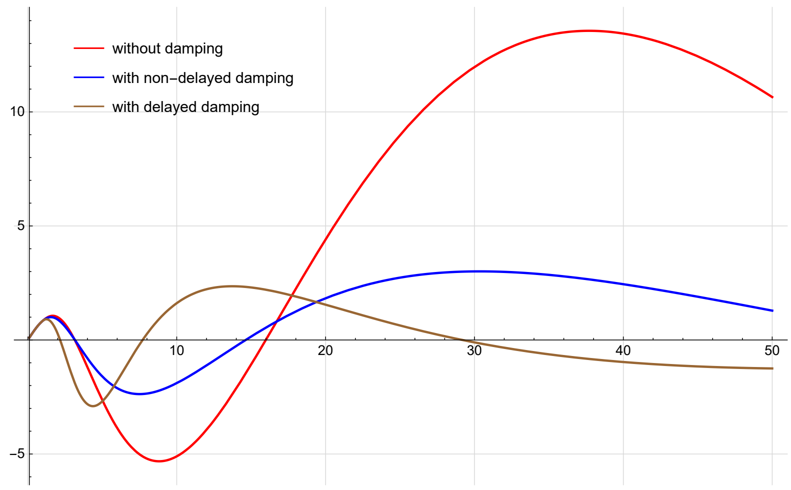

Figure 3. Numerical solutions for some special cases of the studied equation.

3. Conclusions

Asymptotic performance of nonoscillatory solutions to the FDE (1.1) is examined in this paper. The methods employed for equations with non-delayed damping are typically inappropriate for investigating this kind of equation. Therefore, related studies have overcome this problem by studying only the linear case and ignoring one of the terms of the equation when studying it. First, we verified Property $\mathcal{A}$ for the nonlinear case of Eq (1.1). These results extend and slightly improve upon the previous ones. Second, we established new criteria for the linear case that improve upon the previous findings and take into account the influence of all parameters of the considered equation. Finding criteria that ensure that every solution to FDE (1.1) oscillates in the nonlinear situation is an intriguing research point.

Author contributions

O. Moaaz: Conceptualization, formal analysis, investigation, writing – review and editing; A. Al-Jaser: Formal analysis, investigation, methodology, writing – original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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