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Research article

Maximum amplitude interval solution of a given solution in min-product system

Xiaobin Yang[1], **Guocheng Zhu**[2] **and Xiaopeng Yang**[3,*]

[1] Asset Management Department, Hanshan Normal University, Chaozhou 521041, China

[2] School of Humanities Education, Guangdong Innovative Technical College, Dongguan 523960, China

[3] School of Mathematics and Statistics, Hanshan Normal University, Chaozhou 521041, China

* **Correspondence:** Email: 706697032@qq.com; Tel: +8615976337822.

Abstract: In a supply chain system, the price of the goods should satisfy some requirements of the consumer. These requirements have been exactly characterized by a group of min-product fuzzy relation inequalities (FRIs). Correspondingly, any feasible pricing scheme is characterized by a solution of the min-product FRIs system. For a given pricing scheme, or say a given solution in the min-product FRIs, the flexibility is reflected by the maximum amplitude, or equivalently the maximum amplitude interval solution (MAIS). Motivated by such an application background, this work attempts to study the MAIS in min-product FRIs. An efficient algorithm is discovered for computing the MAIS of a solution within the given min-product system. The MAIS will help the system manager be aware of the flexibility of a given pricing scheme and produce better decision-making.

Keywords: min-product composition; fuzzy relation inequality; pricing scheme; supply chain; stability
Mathematics Subject Classification: 90C70, 90C90

1. Introduction

The fuzzy relation equation (FRE) or fuzzy relation inequality (FRI) has been widely investigated and explored in these few decades. The common composition employed in the systems of FREs or FRIs is the max-t-norm (max-triangular-norm), including the famous max-min and max-product composed operations. The max and min operators are denoted by $\vee$ and $\wedge$, respectively, where

$$\alpha \vee \beta = \max(\alpha,\beta), \quad \alpha \wedge \beta = \min(\alpha,\beta),$$

for any $\alpha, \beta \in [0, 1]$. In such types of FREs or FRIs, when the systems are consistent, they have only one maximum solution but finitely many minimal solutions. In the past few decades, some scholars focused on exploring the resolution methods for various kinds of FREs and FRIs. Since the maximum solution is unique and can be directly solved by the specific formula, solving the FREs and FRIs turns out to be solving its minimal solution set. Searching for all minimal solutions has been formally proved to be highly related to some covering problems, belonging to NP-hard problems.

Addition-min is another interesting composition operation. It was first introduced to the FRIs in 2012 by J.-X. Li et al. [1] to characterize constraints in P2P network systems. As the same, for the addition-min systems, their solution sets depend on the minimal solutions [2]. Note that a max-min system (either the equation system or the inequality system) always has a finite number of minimal solutions [3,4]. However, a consistent addition-min system has an infinite number of minimal solutions in most cases [2]. Moreover, the solution set of a max-min system is non-convex if there are at least two minimal solutions. But in an addition-min system, its solution set should be convex [5]. Due to these two different features, the method for dealing with the addition-min system is distinct from that for solving the max-min (or max-t-norm) system [6]. It is usually not easy to find out all minimal solutions in an addition-min system. In [2, 7, 8], an interesting method has been discovered for searching some specific minimal solutions, but not all the whole minimal solutions.

The FRIs with min-product composition, i.e.,

$$\begin{cases} \bigwedge_{j=1}^{n} a_{ij}x_j \le b_i, \quad i = 1, \cdots, m, \\ \underline{x}_j \le x_j \le \bigwedge_{i=1}^{m} \frac{b_i}{a_{ij}}, \quad j = 1, \cdots, n, \end{cases} \tag{1.1}$$

were constructed by H. Guo et al. [9] in 2018 for the first time. In [9], a simple method was proposed to examine the consistency of system (1.1). The authors also studied some properties of system (1.1). Based on these properties, the structure of the whole solution set to (1.1) was discovered. It was found that for the consistent system (1.1), there should be a unique minimum solution, i.e., x^L, and a finite number of maximal solutions. All these solutions jointly determined the whole solution set. In [11], X. Yang further distinguished the general solution and the strong solution to a system of min-product FRIs. Two effective algorithms were designed for solving the solution set and the strong solution set, respectively. In fact, it is not always necessary to obtain the complete solution set of the min-product FRIs. As a consequence, X.-G. Zhou et al. [10] devoted themselves to the lexicographic maximum solution in system (1.1). A resolution algorithm was proposed in [10] based on m auxiliary vectors. Each auxiliary vector should be checked in the above system (1.1) for its feasibility. To reduce the repeated inspection procedures in the algorithm presented in [10], Y. Wu et al. [12] further developed an improved algorithm for searching the lexicographic maximum solution in system (1.1). Some interesting properties were investigated for system (1.1), which contributed significantly to the resolution algorithm. The lexicographic maximum solution was employed to embody the optimal pricing with a fixed priority in a supply chain system [10, 12]. However, if all the suppliers were treated equally, then the corresponding optimal pricing was indeed the optimal solution of the maximin optimization problem subject to the min-product FRIs, i.e., system (1.1) [13].

The FRIs, or FREs, have been introduced to model several real-world systems. For example, the above-mentioned addition-min FRIs were applied in the P2P network system, and the min-product FRIs were applied in the supply chain system. In [14], the max-product FRIs were adopted to characterize the wireless server-to-client (S2C) network system. The concept of (maximum) deviation of a given solution, which reflected the stability of a feasible scheme in the corresponding S2C network system, was defined and investigated for the first time [14]. The author designed an effective algorithm to find the maximum deviation with polynomial computational complexity. The maximum deviation/amplitude was also introduced to the addition-min FRIs [15]. The maximum amplitude of any solution in the addition-min FRIs system could also be obtained within a polynomial time. [18] further improved the method for finding the maximum-interval solution of a given solution in the addition-min FRIs system.

Motivated by the idea presented in [14, 15], Y. Chen et al. first defined and studied the interval solution of a known solution in a max-min FRIs system [16]. M. Chen et al. [17] distinguished different types of interval solutions, i.e., the lower interval solution, the upper, and the including ones. Furthermore, the widest interval solution, meaning the solution having the maximum width, was defined and investigated in a max-min FRIs system [18]. In fact, the widest interval solution embodied the most stable feasible scheme in the instructional resource allocation system [18]. L. Zhang [20] introduced the so-called optimal symmetric interval solution to the max-min FRIs system, considering the symmetry to the interval solution. In such an optimal symmetric interval solution, the fluctuations among its components might not be identical.

As mentioned above, the min-product FRIs have been applied to model the supply chain system. The feasible schemes in the supply chain system are represented by the solutions of the min-product FRIs. In this work, considering the stability of the feasible schemes, we aim to introduce the concept of maximum amplitude interval solution of the solutions in the min-product FRIs, i.e., system (2.3).

In what follows: Section 2 will provide the foundation for our studied system (2.3), with min-product composition. Section 3 is the application background and research motivation of this work. Section 4 displays the definition and property of MAIS. In Section 5, we discuss the resolution of the MAIS in two cases. Detailed resolution procedures and algorithms, together with some illustrative examples, are arranged in Section 6. Section 7 briefly summarizes this work.

2. Application background and research motivation

2.1. Application background

As mentioned in the previous section, the min-product FRIs system was introduced in [9]. Here, we further review its application background. In [9], the authors attempted to model a supply chain system, having m retailers and n suppliers (see Figure 1).

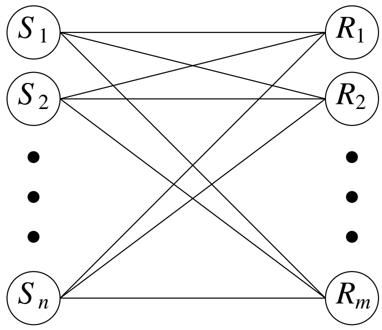

Figure 1. Supply chain system with n suppliers and m retailers.

A kind of goods will be supplied by the suppliers to the retailers. To ensure the goods are salable at the ith retailer, the acceptable price of the goods for the ith retailer should be less than or equal to b_i. Here, $b_i > 0$ and

$$i \in \mathbb{I} = \{1, 2, \cdots, m\}. \tag{2.1}$$

On the other hand, to avoid the suppliers in a state of loss, the supply price of the jth supplier, denoted by x_j, must satisfy $x_j \geq x_j^L$. Here, $x_j^L > 0$ and

$$j \in \mathbb{J} = \{1, 2, \cdots, n\}. \tag{2.2}$$

When the goods are transported from the suppliers to the retailers, there will be transportation fees and cargo damage fees, etc. These fees will push up the price of the goods received by the retailers. Adding these fees, suppose the price of the goods received by the ith retailer and supplied by the jth supplier is $a_{ij}x_j$. It is obvious that $a_{ij}x_j > x_j$, i.e., $a_{ij} > 1$. To ensure the goods sales at the ith retailer R_i, there is at least one supplier satisfying the requirement of R_i, i.e.,

$$a_{i1}x_1 \wedge a_{i2}x_2 \wedge \cdots \wedge a_{in}x_n \leq b_i.$$

On the other hand, for each supplier, it is also requested to satisfy at least one retailer. That is to say, for each $j \in \mathbb{J}$, in the following m inequalities:

$$a_{ij}x_j \leq b_i, \ \ i = 1, 2, \cdots, m,$$

there is at least one inequality that holds. It is equivalent to

$$x_j \leq \bigvee_{i\in\mathbb{I}} \frac{b_i}{a_{ij}}, \ \ \forall j \in \mathbb{J}.$$

All the above constraints in the supply chain system could be formulated as

$$\begin{cases} a_{i1}x_1 \wedge a_{i2}x_2 \wedge \cdots \wedge a_{in}x_n \leq b_i, \\ i = 1, 2, \cdots, m, \\ x^L \leq x \leq x^U, \end{cases} \tag{2.3}$$

where $x = (x_1, \cdots, x_n)$, $x^U = (x_1^U, \cdots, x_n^U)$ with

$$x_j^U = \bigvee_{i\in\mathbb{I}} \frac{b_i}{a_{ij}}, \tag{2.4}$$

and $x^L = (x_1^L, \cdots, x_n^L)$ is a known vector with $x^L \le x^U$. After unitization on the parameters and variables, we always stipulate that $a_{ij}, x_j, b_i, x_j^L, x_j^U \in (0, 1]$, for any $i \in \mathbb{I}$, $j \in \mathbb{J}$. Represent the coefficient matrix and the right-hand vector of system (2.3) by

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \text{ and } b^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

with $A_i = (a_{i1}, \cdots, a_{in})$. Let $A \otimes x^T = [A_1 \otimes x^T, \cdots, A_m \otimes x^T]^T$, where

$$A_i \otimes x^T = (a_{i1}, \cdots, a_{in}) \otimes (x_1, \cdots, x_n)^T = a_{i1}x_1 \wedge a_{i2}x_2 \wedge \cdots \wedge a_{in}x_n.$$

Then system (2.3) could be written as

$$A \otimes x^T \le b^T, \quad x^L \le x \le x^U. \tag{2.5}$$

2.2. Research motivation

As mentioned above, applying the min-product FRIs, i.e., system (2.3), to model the supply chain system, x_j represents the price of the goods supplied by the jth supplier S_j. As a result, each solution x in system (2.3) indeed means a feasible pricing scheme of the goods. Accordingly, a given solution corresponds to a preset pricing scheme provided by the system manager. Considering the sales volume and profit, the suppliers might change the preset pricing scheme, i.e., the given solution. Therefore, for a given solution, we focus on how much the components of it can withstand changes within a certain range, ensuring it is still a solution.

Given a solution $x^G = (x_1^G, \cdots, x^n)$, let

$$x^G \pm \alpha = (x_1^G \pm \alpha, \cdots, x^n \pm \alpha).$$

where α reflects the change occurring in the ith component x_j^G. If α^G is the maximum value of α, ensuring both $x^G + \alpha$ and $x^G - \alpha$ are still solutions of (2.3), then we call α^G the maximum amplitude. Of course, if a solution has a bigger maximum amplitude, then it is considered to be more flexible, because it can bear a bigger change occurring in its components. In other words, the maximum amplitude of a solution embodies its flexibility.

The purpose of this work is to find the maximum amplitude, or equivalently the maximum amplitude interval solution, for a given solution in (2.3).

Table 1. Main notations used in this work.

I	Index set defined by (2.1)
J	Index set defined by (2.2)
A	Coefficient matrix of system (2.3)
b	Right-hand vector of system (2.3)
x	Decision variable vector of system (2.3)
x^L	Lower bound of the vector x
x^U	Upper bound of the vector x
$X(A, b, x^L, x^U)$	Solution set of system (2.3)
$\hat{x}$	Maximal solution of (2.3)
$\check{x}$	Minimum solution of (2.3)
$\hat{X}(A, b, x^L, x^U)$	Maximal solution set of system (2.3)
x^G	A given solution in (2.3)
$[x^G - \alpha^G, x^G + \alpha^G]$	The MAIS of x^G in system (2.3)
α^L	Defined by (4.1)
$\mathbb{I}^U$	Index set defined by (5.2)
α^U	Defined by (5.3) when $\mathbb{I}^U = \emptyset$, and by (5.9) when $\mathbb{I}^U \neq \emptyset$

3. Foundation on the min-product system (2.3)

In what follows, we denote the set of all solutions for system (2.3) by

$$X(A, b, x^L, x^U) = \{x \in [x^L, x^U] \mid A \otimes x^T \leq b^T\}.$$

As a consequence, $X(A, b, x^L, x^U) \neq \emptyset$ means that system (2.3) is consistent or, say, solvable. On the contrary, if $X(A, b, x^L, x^U) \neq \emptyset$, system (2.3) is said to be inconsistent. The maximal and minimum solutions of (2.3) are defined in Definition 1 below.

Definition 1. A solution $\hat{x} \in X(A, b, x^L, x^U)$ is named *maximal solution*, if for any $x \in X(A, b, x^L, x^U)$, $x \geq \hat{x}$ always implies $x = \hat{x}$. A solution $\check{x} \in X(A, b, x^L, x^U)$ is named *minimum solution*, if for any $x \in X(A, b, x^L, x^U)$, it always holds $\check{x} \leq x$.

The following Theorem 1 could be used for checking whether system (2.3) is consistent.

Theorem 1. *[9–13] System* (2.3) *is consistent iff* $x^L \in X(A, b, x^L, x^U)$. *Moreover, when* (2.3) *is consistent,* x^L *is its minimum solution.*

Proposition 1. *[9–13] If* $x^1, x^2 \in X(A, b, x^L, x^U)$ *with* $x^1 \leq x^2$*, then there is* $[x^1, x^2] \subseteq X(A, b, x^L, x^U)$.

In a consistent system (2.3), there might exist a minimum solution and some maximal solutions. Moreover, for a consistent system (2.3), the whole solution set could be constructed by its minimum solution and all maximal solutions. The structure to characterize the solution set is shown in Theorem 2 below.

Theorem 2. *[9–13] Assume that system* (2.3) *is consistent with the minimum solution x^L. The solution set can be written as*

$$X(A, b, x^L, x^U) = \bigcup_{\hat{x}\in\hat{X}(A,b,x^L,x^U)} [x^L, \hat{x}]. \tag{3.1}$$

Here, the set $\hat{X}(A, b, x^L, x^U)$ contains all the maximal solutions for (2.3).

In [11], the author has designed a detailed approach for obtaining the above solution set $X(A, b, x^L, x^U)$. In fact, it is equivalent to deriving all the maximal solutions. Moreover, the number of maximal solutions should be finite. That is to say, $\hat{X}(A, b, x^L, x^U)$ is actually a finite set, i.e., $|\hat{X}(A, b, x^L, x^U)| < \infty$.

4. Definition and brief property of the maximum amplitude interval solution

Given the solution x^G of system (2.3), i.e., $x^G \in X(A, b, x^L, x^U)$.

Definition 2. (*Maximum amplitude interval solution* & *Maximum amplitude*) Let $\alpha^G \in [0, 1]$ be a real number. The interval solution $[x^G - \alpha^G, x^G + \alpha^G]$ is called maximum amplitude interval solution (MAIS) of x^G in system (2.3), if it fulfills:

(i) $[x^G - \alpha^G, x^G + \alpha^G] \subseteq X(A, b, x^L, x^U)$;

(ii) $[x^G - \alpha, x^G + \alpha] \nsubseteq X(A, b, x^L, x^U)$ for arbitrary real number α with $\alpha > \alpha^G$.

Correspondingly, α^G is called the maximum amplitude.

Next, we try to illustrate the existence and uniqueness of the MAIS. First, denote

$$\alpha^L = \min_{j\in\mathbb{J}}\{x_j^G - x_j^L\}. \tag{4.1}$$

The following Proposition 2 shows the simple property of the above-denoted number α^L.

Proposition 2. *Let α^L be defined as* (4.1). *Then the following statements hold:*

(i) $\alpha^L \in [0, 1]$;

(ii) $x^G - \alpha^L \in X(A, b, x^L, x^U)$;

(iii) if $\alpha > \alpha^L$, then it holds $x^G - \alpha \notin X(A, b, x^L, x^U)$.

Proof 1. (i) Since x^L is the minimum solution, while x^G a general one, we have $x^L \le x^G$, i.e.,

$$x_j^G - x_j^L \ge 0, \quad \forall j \in \mathbb{J}. \tag{4.2}$$

Thus by (4.1), $\alpha^L = \min_{j\in\mathbb{J}}\{x_j^G - x_j^L\} \ge 0$. Moreover,

$$\alpha^L \le x_j^G - x_j^L \le x_j^G \le 1, \quad \forall j \in \mathbb{J}. \tag{4.3}$$

Hence, $\alpha^L \in [0, 1]$.

(ii) Since $\alpha^L \ge 0$, by (4.3) we also have

$$x_j^L \le x_j^G - \alpha^L \le x_j^G, \quad \forall j \in \mathbb{J}, \tag{4.4}$$

i.e., $x^L \le x^G - \alpha^L \le x^G$. According to Proposition 1, $x^L, x^G \in X(A, b, x^L, x^U)$ implies $x^G - \alpha^L \in X(A, b, x^L, x^U)$.

(iii) According to (4.1), there is $j^* \in \mathbb{J}$ such that $\alpha^L = x^G_{j^*} - x^L_{j^*}$. If $\alpha > \alpha^L$, then we have

$$x^L_{j^*} > x^G_{j^*} - \alpha. \tag{4.5}$$

Since x^L the minimum solution, we know $x^G - \alpha \notin X(A, b, x^L, x^U)$. (Otherwise, there should be $x^L \leq x^G - \alpha$, which conflicts with (4.5)) □

Proposition 3. *Let α', α'' be two real numbers satisfying the following four points:*

(i) $\alpha', \alpha'' \in [0, 1]$;
(ii) $x^G - \alpha', x^G + \alpha'' \in X(A, b, x^L, x^U)$;
(iii) if $\alpha > \alpha'$, then it holds $x^G - \alpha \notin X(A, b, x^L, x^U)$.
(iv) if $\alpha > \alpha''$, then it holds $x^G + \alpha \notin X(A, b, x^L, x^U)$.
Then, $[x^G - \alpha' \wedge \alpha'', x^G + \alpha' \wedge \alpha'']$ is the MAIS of x^G in system (2.3).

Proof 2. Denote $\alpha^G = \alpha' \wedge \alpha''$. By (i), $\alpha', \alpha'' \in [0, 1]$ implies $\alpha^G = \alpha' \wedge \alpha'' \in [0, 1]$.

Since $\alpha', \alpha'' \in [0, 1]$, it is obvious that $x^G - \alpha' \leq x^G + \alpha''$. Considering $x^G - \alpha', x^G + \alpha'' \in X(A, b, x^L, x^U)$ in (ii), we follow from Proposition 1 that $[x^G - \alpha', x^G + \alpha''] \subseteq X(A, b, x^L, x^U)$. Considering $0 < \alpha^G = \alpha' \wedge \alpha'' \leq \alpha', \alpha''$, we have

$$x^G - \alpha' \leq x^G - \alpha^G \leq x^G + \alpha^G \leq x^G + \alpha''.$$

Thus, $[x^G - \alpha^G, x^G + \alpha^G] \subseteq [x^G - \alpha', x^G + \alpha''] \subseteq X(A, b, x^L, x^U)$.

Assume that α is an arbitrary number in $[0, 1]$ with $\alpha > \alpha^G$. Since $\alpha^G = \alpha' \wedge \alpha''$, it holds either $\alpha^G = \alpha'$ or $\alpha^G = \alpha''$. That is to say, either $\alpha > \alpha'$ or $\alpha > \alpha''$ holds. If $\alpha > \alpha'$, then by (iii), it holds

$$x^G - \alpha \notin X(A, b, x^L, x^U),$$

and

$$[x^G - \alpha, x^G + \alpha] \nsubseteq X(A, b, x^L, x^U).$$

If $\alpha > \alpha''$, then by (iv), it holds

$$x^G + \alpha \notin X(A, b, x^L, x^U),$$

and

$$[x^G - \alpha, x^G + \alpha] \nsubseteq X(A, b, x^L, x^U).$$

According to Definition 2, we know $[x^G - \alpha^G, x^G + \alpha^G] = [x^G - \alpha' \wedge \alpha'', x^G + \alpha' \wedge \alpha'']$ is the MAIS of x^G in system (2.3). □

Theorem 3. *(Existence and uniqueness of the MAIS) If system* (2.3) *is consistent and x^G is one of its solutions, then MAIS of x^G always exists. Moreover, when the MAIS of x^G exists, it should be unique.*

Proof 3. (Existence) It is previously pointed out $|\hat{X}(A, b, x^L, x^U)| < \infty$. Denote

$$\hat{X}^G = \{\hat{x} \in \hat{X}(A, b, x^L, x^U) \mid x^G \leq \hat{x}\}. \tag{4.6}$$

Since $x^G \in X(A, b, x^L, x^U)$, by Theorem 2, we know that there is $\hat{x} \in \hat{X}(A, b, x^L, x^U)$, with

$$x^G \in [x^L, \hat{x}], \text{ i.e., } x^L \leq x^G \leq \hat{x}.$$

This indicates $\hat{X}^G \neq \emptyset$. As a subset of $\hat{X}(A, b, x^L, x^U)$, $\hat{X}^G$ is also a finite set. Assume that

$$\hat{X}^G = \{\hat{x}^1, \cdots, \hat{x}^s\}, \tag{4.7}$$

where s is a positive integer. Denote

$$\alpha^l = \min_{j \in \mathbb{J}} \{\hat{x}^l_j - x^G_j\}, \quad l = 1, \cdots, s, \tag{4.8}$$

and

$$\alpha^U = \max_{1 \leq l \leq s} \alpha^l. \tag{4.9}$$

It could be directly checked that $0 \leq \alpha^U \leq 1$.

We now prove that $x^G + \alpha^U \in X(A, b, x^L, x^U)$. By (4.8) and (4.9), there exists $l^* \in \{1, \cdots, s\}$ such that

$$\alpha^U = \alpha^{l^*} = \min_{j \in \mathbb{J}} \{\hat{x}^{l^*}_j - x^G_j\}.$$

This indicates

$$\alpha^U \leq \hat{x}^{l^*}_j - x^G_j, \quad \forall j \in \mathbb{J},$$

i.e.,

$$x^G_j + \alpha^U \leq \hat{x}^{l^*}_j, \quad \forall j \in \mathbb{J}.$$

As a result, $x^G \leq x^G + \alpha^U \leq \hat{x}^{l^*}$. Following Proposition 1, we know $x^G + \alpha^U \in X(A, b, x^L, x^U)$.

Next, we aim to prove, by contradiction, that for any $\alpha > \alpha^U$, there is $x^G + \alpha \notin X(A, b, x^L, x^U)$. Otherwise, if $x^G + \alpha \in X(A, b, x^L, x^U)$, then by Theorem 2, we can find

$$\hat{x}^\# \in \hat{X}(A, b, x^L, x^U),$$

so that $x^G + \alpha \in [x^L, \hat{x}^\#]$, i.e.,

$$x^L \leq x^G + \alpha \leq \hat{x}^\#. \tag{4.10}$$

Since $\alpha > \alpha^U \geq 0$, it is clear $x^G \leq x^G + \alpha \leq \hat{x}^\#$. Following (4.6), we find $\hat{x}^\# \in \hat{X}^G$. Observing (4.7), there exists $t \in \{1, \cdots, s\}$ such that $\hat{x}^\# = \hat{x}^t$. Inequality (4.10) becomes $x^G + \alpha \leq \hat{x}^\# = \hat{x}^t$. Thus,

$$x^G_j + \alpha \leq \hat{x}^t_j, \quad \forall j \in \mathbb{J}. \tag{4.11}$$

Considering $\alpha^t \leq \max_{1 \leq l \leq s} \alpha^l = \alpha^U < \alpha$, we further have

$$\hat{x}^t_j - x^G_j \geq \alpha > \alpha^t, \quad \forall j \in \mathbb{J}. \tag{4.12}$$

Besides, since $\alpha^t = \min_{j \in \mathbb{J}} \{\hat{x}^t_j - x^G_j\}$, there is $j_0 \in \mathbb{J}$, with

$$\alpha^t = \hat{x}^t_{j_0} - x^G_{j_0}. \tag{4.13}$$

This is in contradiction with Inequality (4.12).

Until now, we have proved that the number defined by (4.8) and (4.9) fulfills:

$$\begin{cases} (i) \;\; \alpha^U \in [0, 1], \;\; x^G + \alpha^U \in X(A, b, x^L, x^U), \\ (ii) \;\; \text{if } \alpha > \alpha^U, \;\; \text{then } x^G + \alpha \notin X(A, b, x^L, x^U). \end{cases} \tag{4.14}$$

On the other hand, it has been shown in Proposition 2 that number α^L fulfills:

$$\begin{cases} (iii)\ \ \alpha^L \in [0,1],\ \ x^G - \alpha^L \in X(A,b,x^L,x^U), \\ (iv)\ \ \text{if } \alpha > \alpha^L,\ \text{then } x^G - \alpha \notin X(A,b,x^L,x^U). \end{cases} \tag{4.15}$$

It follows from Proposition 3 that $[x^G - \alpha^L \wedge \alpha^G,\ x^G + \alpha^L \wedge \alpha^G]$ is the MAIS of x^G in system (2.3).

(Uniqueness) Suppose there are two MAISes of x^G in system (2.3), denoted by $[x^G - \alpha^{G_1}, x^G + \alpha^{G_1}]$ and $[x^G - \alpha^{G_2}, x^G + \alpha^{G_2}]$. Then by (i) in Definition 2, we know

$$[x^G - \alpha^{G_1}, x^G + \alpha^{G_1}] \subseteq X(A,b,x^L,x^U),$$

and

$$[x^G - \alpha^{G_2}, x^G + \alpha^{G_2}] \subseteq X(A,b,x^L,x^U). \tag{4.16}$$

We assert that $\alpha^{G_1} = \alpha^{G_2}$. Otherwise, without loss of generality, we assume that $\alpha^{G_1} < \alpha^{G_2}$. Since α^{G_1} is the MAIS of x^G, by (ii) in Definition 2, we know

$$[x^G - \alpha^{G_2}, x^G + \alpha^{G_2}] \nsubseteq X(A,b,x^L,x^U).$$

This is contradictory with the inclusion indicated in (4.16). As a consequence, there is $\alpha^{G_1} = \alpha^{G_2}$, i.e., $[x^G - \alpha^{G_1}, x^G + \alpha^{G_1}] = [x^G - \alpha^{G_2}, x^G + \alpha^{G_2}]$, meaning the MAIS of the identical given solution x^G is unique. □

5. Resolution of the maximum amplitude interval solution of a given solution

Previously, we have defined the concept of MAIS of x^G. Immediately following, we will try to exploit the method for solving the MAIS of x^G. Here, x^G is given as a preset solution of system (2.3).

Based on the upper bound vector x^U, define

$$\mathbb{J}_i^U = \{j \in \mathbb{J} \mid a_{ij}x_j^U \le b_i\},\ \ i \in \mathbb{I}. \tag{5.1}$$

Moreover, based on these index sets, denote

$$\mathbb{I}^U = \{i \in \mathbb{I} \mid \mathbb{J}_i^U = \emptyset\}. \tag{5.2}$$

In fact, it is easy to check $\mathbb{I}^U \neq \mathbb{I}$. The index set $\mathbb{I}^U$ helps us to classify the situations in solving the MAIS of x^G.

Theorem 4. *For the upper bound x^U, there is $x^U \in X(A,b,x^L,x^U) \Leftrightarrow \mathbb{I}^U = \emptyset$.*

Proof 4. Considering $x^L \le x^U \le x^U$, there is

$$\begin{aligned} x^U \in X(A,b,x^L,x^U) &\Leftrightarrow a_{i1}x_1^U \wedge a_{i2}x_2^U \wedge \cdots \wedge a_{in}x_n^U \le b_i,\ \ \forall i \in \mathbb{I} \\ &\Leftrightarrow \text{for every } i \in \mathbb{I}, \text{there is } j_i \in \mathbb{J}, \text{fulfilling } a_{ij_i}x_{j_i}^U \le b_i \\ &\Leftrightarrow \mathbb{J}_i^U \neq \emptyset,\ \ \forall i \in \mathbb{I} \\ &\Leftrightarrow \mathbb{I}^U = \emptyset. \end{aligned}$$

The proof is complete. □

5.1. *In the case that* $\mathbb{I}^U = \emptyset$

This subsection provides the MAIS of the given solution x^G in the case that $\mathbb{I}^U = \emptyset$. In such a case, denote

$$\alpha^U = \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}. \tag{5.3}$$

Proposition 4. *The above-defined α^U satisfies the following statements:*

(i) $\alpha^U \in [0, 1]$;
(ii) $x^G + \alpha^U \in X(A, b, x^L, x^U)$;
(iii) if $\alpha > \alpha^U$, then it holds $x^G + \alpha \notin X(A, b, x^L, x^U)$.

Proof 5. (i) Given the solution x^G, there is $x^L \le x^G \le x^U$. Thus,

$$\alpha^U = \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\} \ge 0.$$

On the other hand, $x^L, x^U \in [0, 1]^n$ indicates $x^G \in [0, 1]^n$. Thus,

$$\alpha^U = \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\} \le \min_{j\in\mathbb{J}}\{x_j^U\} \le 1.$$

As a consequence, there is $\alpha^U \in [0, 1]$.

(ii) According to (5.3),

$$\alpha^U = \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\} \le x_j^U - x_j^G, \quad \forall j \in \mathbb{J}, \tag{5.4}$$

i.e.,

$$x_j^G + \alpha^U \le x_j^U, \quad \forall j \in \mathbb{J}. \tag{5.5}$$

This indicates $x^G + \alpha^U \le x^U$. Since $x^L \le x^G \le x^U$, we have

$$x^L \le x^G \le x^G + \alpha^U \le x^U. \tag{5.6}$$

On the other hand, since $\mathbb{I}^U = \emptyset$, by Theorem 4, we find

$$x^U \in X(A, b, x^L, x^U).$$

Besides, by Theorem 1, we find

$$x^L \in X(A, b, x^L, x^U).$$

Considering (5.6) and Proposition 1, it holds $x^G + \alpha^U \in X(A, b, x^L, x^U)$.

(iii) Since $\alpha^U = \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}$ by (5.3), there is $j^0 \in \mathbb{J}$ such that $\alpha^U = x_{j^0}^U - x_{j^0}^G$. If $\alpha > \alpha^U = x_{j^0}^U - x_{j^0}^G$, then $x_{j^0}^G + \alpha > x_{j^0}^U$. That is to say, $x^G + \alpha \le x^U$ doesn't hold. Thus, $x^G + \alpha \notin X(A, b, x^L, x^U)$. □

Theorem 5. *When $\mathbb{I}^U = \emptyset$, the MAIS of the given solution x^G is $[x^G - \alpha^G, x^G + \alpha^G]$, where $\alpha^G = \alpha^L \wedge \alpha^U$, α^L and α^U are as defined by* (4.1) *and* (5.3).

Proof 6. The results in the theorem could be directly deduced from Propositions 2–4. □

5.2. In case that $\mathbb{I}^U \neq \emptyset$ and $\mathbb{I}^U \neq \mathbb{I}$

In order to find the MAIS of x^G in case that $\mathbb{I}^U \neq \emptyset$ and $\mathbb{I}^U \neq \mathbb{I}$, we further define

$$\mathbb{J}_i^G = \{j \in \mathbb{J} \mid x_j^G \leq \frac{b_i}{a_{ij}}\}, \quad \forall i \in \mathbb{I}^U, \tag{5.7}$$

and

$$\alpha_i^U = \max_{j\in\mathbb{J}_i^G}\{\frac{b_i}{a_{ij}} - x_j^G\}, \quad \forall i \in \mathbb{I}^U. \tag{5.8}$$

Moreover, let

$$\alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\}. \tag{5.9}$$

Proposition 5. *The above-defined α^U satisfies the following statements:*

(i) $\alpha^U \in [0, 1]$;
(ii) $x^G + \alpha^U \in X(A, b, x^L, x^U)$;
(iii) if $\alpha > \alpha^U$, then it holds $x^G + \alpha \notin X(A, b, x^L, x^U)$.

Proof 7. (i) By (5.7), we know that

$$x_j^G \leq \frac{b_i}{a_{ij}}, \quad \forall j \in \mathbb{J}_i^G, i \in \mathbb{I}^U, \tag{5.10}$$

i.e.,

$$\frac{b_i}{a_{ij}} - x_j^G \geq 0, \quad \forall j \in \mathbb{J}_i^G, i \in \mathbb{I}^U. \tag{5.11}$$

So, we have

$$\alpha_i^U = \max_{j\in\mathbb{J}_i^G}\{\frac{b_i}{a_{ij}} - x_j^G\} \geq 0, \quad \forall i \in \mathbb{I}^U. \tag{5.12}$$

Thus,

$$\min_{i\in\mathbb{I}^U}\{\alpha_i^U\} \geq 0. \tag{5.13}$$

Besides, $x^G \leq x^U$ indicates $x_j^U - x_j^G \geq 0$. Thus,

$$\min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}. \tag{5.14}$$

Inequalities (5.13) and (5.14) contribute to

$$\alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\} \geq 0. \tag{5.15}$$

On the other hand,

$$\alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\} \leq \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\} \leq x_1^U - x_1^G \leq x_1^U \leq 1. \tag{5.16}$$

Combining (5.15) and (5.16), we obtain $\alpha^U \in [0, 1]$.

(ii) Since $x^L \leq x^G \leq x^U$ and $\alpha^U \in [0,1]$, we have

$$x^L \leq x^G \leq x^G + \alpha^U. \tag{5.17}$$

On the other hand, according to (5.9), it holds

$$\alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\} \leq \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}. \tag{5.18}$$

As a result, $\alpha^U \leq x_j^U - x_j^G$, $\forall j \in \mathbb{J}$, i.e.,

$$x_j^G + \alpha^U \leq x_j^U, \quad \forall j \in \mathbb{J}. \tag{5.19}$$

That is $x^G + \alpha^U \leq x^U$. Considering (5.17), we have $x^L \leq x^G + \alpha^U \leq x^U$.

Take arbitrarily $i' \in \mathbb{I}$.

Case 1. If $i' \in \mathbb{I}^U$, then by (5.2), $\mathbb{J}_{i'}^U = \emptyset$. Considering (5.8), there is $j' \in \mathbb{J}_i^G$, such that $\alpha_{i'}^U = \frac{b_{i'}}{a_{i'j'}} - x_{j'}^G$. Thus,

$$\alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\} \leq \min_{i\in\mathbb{I}^U}\{\alpha_i^U\} \leq \alpha_{i'}^U = \frac{b_{i'}}{a_{i'j'}} - x_{j'}^G,$$

i.e.,

$$a_{i'j'}(x_{j'}^G + \alpha^U) \leq b_{i'}. \tag{5.20}$$

This implies that

$$a_{i'1}(x_1 + \alpha^U) \wedge \cdots \wedge a_{i'n}(x_n + \alpha^U) \leq a_{i'j'}(x_{j'}^G + \alpha^U) \leq b_{i'}. \tag{5.21}$$

Case 2. If $i' \notin \mathbb{I}^U$, then by (5.2), $\mathbb{J}_{i'}^U \neq \emptyset$. Take arbitrarily $j' \in \mathbb{J}_{i'}^U$. By (5.1), we have

$$a_{i'j'}x_{j'}^U \leq b_{i'}. \tag{5.22}$$

Since

$$\alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\} \leq \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\} \leq x_{j'}^U - x_{j'}^G. \tag{5.23}$$

We have $x_{j'}^G + \alpha^U \leq x_{j'}^U$. Considering Inequality (5.22), there is

$$a_{i'j'}(x_{j'}^G + \alpha^U) \leq a_{i'j'}x_{j'}^U \leq b_{i'}. \tag{5.24}$$

This implies that

$$a_{i'1}(x_1 + \alpha^U) \wedge a_{i'2}(x_2 + \alpha^U) \wedge \cdots \wedge a_{i'n}(x_n + \alpha^U) \leq a_{i'j'}(x_{j'}^G + \alpha^U) \leq b_{i'}. \tag{5.25}$$

Combining Inequalities (5.21) and (5.25), there is

$$a_{i'1}(x_1 + \alpha^U) \wedge \cdots \wedge a_{i'n}(x_n + \alpha^U) \leq a_{i'j'}(x_{j'}^G + \alpha^U) \leq b_{i'}, \quad \forall i' \in \mathbb{I}. \tag{5.26}$$

As a result, $x^G + \alpha^U \in X(A, b, x^L, x^U)$.

(iii) Suppose $\alpha > \alpha^U = \min\{\min_{i\in\mathbb{I}^U}\{\alpha_i^U\}, \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}\}$. Then either

$$\alpha > \min_{i\in\mathbb{I}^U}\{\alpha_i^U\},$$

or

$$\alpha > \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}$$

holds.

Case 1. If $\alpha > \min_{i\in\mathbb{I}^U}\{\alpha_i^U\}$, then there is $i' \in \mathbb{I}^U$ such that

$$\alpha > \min_{i\in\mathbb{I}^U}\{\alpha_i^U\} = \alpha_{i'}^U. \tag{5.27}$$

According to (5.8), $\alpha_{i'}^U = \max_{j\in\mathbb{J}_{i'}^G}\{\frac{b_{i'}}{a_{i'j}} - x_j^G\}$. There is $j' \in \mathbb{J}_{i'}^G$ such that

$$\alpha_{i'}^U = \max_{j\in\mathbb{J}_{i'}^G}\{\frac{b_{i'}}{a_{i'j}} - x_j^G\} = \frac{b_{i'}}{a_{i'j'}} - x_{j'}^G \geq \frac{b_{i'}}{a_{i'k}} - x_k^G, \ \ \forall k \in \mathbb{J}_{i'}^G. \tag{5.28}$$

It follows from (5.27) and (5.28) that

$$\alpha > \alpha_{i'}^U \geq \frac{b_{i'}}{a_{i'k}} - x_k^G, \ \ \forall k \in \mathbb{J}_{i'}^G,$$

i.e.,

$$a_{i'k}(x_k^G + \alpha) > b_{i'}, \ \ \forall k \in \mathbb{J}_{i'}^G. \tag{5.29}$$

On the other hand, according to (5.7), we have

$$x_k^G > \frac{b_{i'}}{a_{i'k}}, \ \ \forall k \notin \mathbb{J}_{i'}^G.$$

Note that $\alpha > \alpha^U \geq 0$. There is

$$x_k^G + \alpha > x_k^G > \frac{b_{i'}}{a_{i'k}}, \ \ \forall k \notin \mathbb{J}_{i'}^G,$$

i.e.,

$$a_{i'k}(x_k^G + \alpha) > b_{i'}, \ \ \forall k \notin \mathbb{J}_{i'}^G. \tag{5.30}$$

Inequalities (5.29) and (5.30) contribute to

$$a_{i'k}(x_k^G + \alpha) > b_{i'}, \ \ \forall k \notin \mathbb{J}. \tag{5.31}$$

Thus,

$$a_{i'1}(x_1^G + \alpha) \wedge a_{i'2}(x_2^G + \alpha) \wedge \cdots \wedge a_{i'n}(x_n^G + \alpha) > b_{i'}.$$

This indicates $x^G + \alpha \notin X(A, b, x^L, x^U)$.

Case 2. If $\alpha > \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\}$, then there is $j' \in \mathbb{J}$ such that $\alpha > \min_{j\in\mathbb{J}}\{x_j^U - x_j^G\} = x_{j'}^U - x_{j'}^G$, i.e.,

$$x_{j'}^G + \alpha > x_{j'}^U.$$

It is directly seen that the inequality $x^G + \alpha \leq x^U$ doesn't hold. As a consequence, $x^G + \alpha \notin X(A, b, x^L, x^U)$. □

Theorem 6. *When $\mathbb{I}^U \neq \emptyset$ and $\mathbb{I}^U \neq \mathbb{I}$, the MAIS of the given solution x^G is $[x^G - \alpha^G, x^G + \alpha^G]$, and the maximum amplitude is $\alpha^G = \alpha^L \wedge \alpha^U$, where α^L and α^U are as defined by* (4.1) *and* (5.9).

Proof 8. The results in the theorem could be directly deduced from Propositions 2, 3, and 5. □

Theorem 7. *There is a one-to-one correspondence between $X(A, b, x^L, x^U)$ and the set of all maximum amplitude interval solutions in system* (2.3).

Proof 9. Take arbitrarily $x^1, x^2 \in X(A, b, x^L, x^U)$. Suppose the MAIS of x^1 is $[x^1 - \alpha^1, x^1 + \alpha^1]$ while that of x^2 is $[x^2 - \alpha^2, x^2 + \alpha^2]$.

(i) If $x^1 = x^2$, then by Theorem 3, there is $[x^1 - \alpha^1, x^1 + \alpha^1] = [x^2 - \alpha^2, x^2 + \alpha^2]$.

(ii) If $[x^1 - \alpha^1, x^1 + \alpha^1] = [x^2 - \alpha^2, x^2 + \alpha^2]$, then

$$x^1 - \alpha^1 = x^2 - \alpha^2, \;\; x^1 + \alpha^1 = x^2 + \alpha^2.$$

As a result, $x^1 - x^2 = \alpha^1 - \alpha^2 = \alpha^2 - \alpha^1$. This indicates $\alpha^1 = \alpha^2$ and $x^1 = x^2$.

The above points (i) and (ii) show that the correspondence between the set $X(A, b, x^L, x^U)$ and the set of all maximum amplitude interval solutions is one-to-one. □

6. Resolution algorithm and illustrative example

Following the idea provided in the previous section, a resolution algorithm is designed to derive the MAIS in this section. Moreover, some detailed examples will be provided to illustrate the efficiency of the resolution algorithm.

Algorithm: for obtaining the MAIS of the given solution x^G

Step 1. Compute value of α^L by (4.1).

Step 2. Construct the index sets $\{\mathbb{J}_i^U \mid i \in \mathbb{I}\}$, by (5.1).

Step 3. Construct the index set $\mathbb{I}^U$ by (5.2). Check whether $\mathbb{I}^U = \emptyset$. If $\mathbb{I}^U = \emptyset$, turn to Step 4. While if $\mathbb{I}^U \neq \emptyset$ turn to Step 6.

Step 4. Compute value of α^U by (5.3).

Step 5. Compute $\alpha^G = \alpha^L \wedge \alpha^U$. As presented in Theorem 5, when $\mathbb{I}^U = \emptyset$, the MAIS of x^G in system (2.3) is $[x^G - \alpha^G, x^G + \alpha^G]$. Terminate!

Step 6. Construct the index sets $\{\mathbb{J}_i^G \mid i \in \mathbb{I}^U\}$, by (5.7).

Step 7. Compute value of α_i^U, for each $i \in \mathbb{I}^U$, by (5.8).

Step 8. Compute value of α^U by (5.9).

Step 9. Compute $\alpha^G = \alpha^L \wedge \alpha^U$. As presented in Theorem 6, when $\mathbb{I}^U \neq \emptyset$, the MAIS of x^G in system (2.3) is $[x^G - \alpha^G, x^G + \alpha^G]$. Terminate!

Computational complexity

In our proposed algorithm, Steps 1-3 take $2n$, $2mn$, and $m + 1$ operations, respectively. After Step 3, there are two possible ways to continue the algorithm, i.e., go to Steps 4 and 5 or go to Steps 6–9. Running Steps 4 and 5, these two steps cost $2n$ and $1 + 2n$ operations, respectively. While running Steps 6–9, these four steps cost $2mn$, $3mn$, $m + 2n$, and $1 + 2n$ operations, respectively. Note that

$$2n + 1 + 2n = 4n + 1 < 2mn + 3mn + m + 2n + 1 + 2n = 5mn + m + 4n + 1.$$

As a result, in the worst case, all the steps in the above algorithm cost

$$2n + 2mn + m + 1 + 5mn + m + 4n + 1 = 7mn + 2m + 6n + 2$$

operations. Hence, the computational complexity is $O(mn)$.

Example 1. Given a supply chain, which has been reduced into the following FRIs with min-product:

$$\begin{cases} 0.8x_1 \wedge 0.75x_2 \wedge 0.85x_3 \wedge 0.6x_4 \wedge 0.7x_5 \wedge 0.8x_6 \wedge 0.75x_7 \wedge 0.9x_8 \leq 0.6, \\ 0.75x_1 \wedge 0.9x_2 \wedge 0.8x_3 \wedge 0.7x_4 \wedge 0.9x_5 \wedge 0.75x_6 \wedge 0.7x_7 \wedge 0.8x_8 \leq 0.64, \\ 0.7x_1 \wedge 0.8x_2 \wedge 0.6x_3 \wedge 0.9x_4 \wedge 0.8x_5 \wedge 0.75x_6 \wedge 0.87x_7 \wedge 0.72x_8 \leq 0.56, \\ 0.81x_1 \wedge 0.78x_2 \wedge 0.75x_3 \wedge 0.88x_4 \wedge 0.65x_5 \wedge 0.7x_6 \wedge 0.8x_7 \wedge 0.9x_8 \leq 0.6, \\ x \geq x^L, \end{cases} \tag{6.1}$$

where $x = (x_1, \cdots, x_8)$ and $x^L = (0.5, 0.45, 0.48, 0.51, 0.49, 0.52, 0.43, 0.4)$. According to Eq. (2.4), we can directly compute the upper bound of system (6.1) as

$$x^U = (0.8533, 0.8, 0.9333, 1, 0.9230, 0.8571, 0.9142, 0.8).$$

Applying our proposed algorithm, try to compute the MAIS of the provided solution

$$x^G = (0.67, 0.63, 0.7, 0.75, 0.7, 0.68, 0.65, 0.6)$$

in system (6.1).

Solution:
Step 1. Compute α^L by (4.1) as follows. Since

$$\begin{aligned} x^G - x^L =& (0.67, 0.63, 0.7, 0.75, 0.7, 0.68, 0.65, 0.6) - (0.5, 0.45, 0.48, 0.51, 0.49, 0.52, 0.43, 0.4) \\ =& (0.17, 0.18, 0.22, 0.24, 0.21, 0.16, 0.22, 0.2), \end{aligned}$$

we have

$$\alpha^L = \min_{1 \leq j \leq 8} \{x_j^G - x_j^L\} = \{0.17, 0.18, 0.22, 0.24, 0.21, 0.16, 0.22, 0.2\} = 0.16.$$

Step 2. Construct the index sets $\{\mathbb{J}_i^U \mid 1 \leq i \leq 4\}$, by (5.1), as follows:

$$\begin{cases} \mathbb{J}_1^U = \{1 \leq j \leq 8 \mid a_{1j}x_j^U \leq b_1\} = \{2, 4\}, \\ \mathbb{J}_2^U = \{1 \leq j \leq 8 \mid a_{2j}x_j^U \leq b_2\} = \{1, 7, 8\}, \\ \mathbb{J}_3^U = \{1 \leq j \leq 8 \mid a_{3j}x_j^U \leq b_3\} = \{3\}, \\ \mathbb{J}_4^U = \{1 \leq j \leq 8 \mid a_{4j}x_j^U \leq b_4\} = \{5, 6\}. \end{cases}$$

Step 3. Obviously, for $i = 1, 2, 3, 4$, it always holds $\mathbb{J}_i^U \neq \emptyset$. According to (5.2), there is $\mathbb{I}^U = \emptyset$. Thus, we turn to Step 4.
Step 4. Compute value of α^U by (5.3):

$$\begin{aligned} &\alpha^U \\ =& \min\{0.8533 - 0.67, 0.8 - 0.63, 0.9333 - 0.7, 1 - 0.75, \\ &\quad 0.923 - 0.7, 0.8571 - 0.68, 0.9142 - 0.65, 0.8 - 0.6\} \\ =& \min\{0.1833, 0.17, 0.2333, 0.25, 0.223, 0.1771, 0.2642, 0.2\} \\ =& 0.17. \end{aligned}$$

Step 5. Based on the above-obtained values of α^L and α^U, we gain $\alpha^G = \alpha^L \wedge \alpha^U = 0.16 \wedge 0.17 = 0.16$. Since $\mathbb{I}^U = \emptyset$, it follows from Theorem 5 that the MAIS of x^G in system (6.1) is

$$\begin{aligned}[x^G - \alpha^G, x^G + \alpha^G] = ([0.51, 0.83], [0.47, 0.79], [0.54, 0.86], [0.59, 0.91], \\ [0.54, 0.86], [0.52, 0.84], [0.49, 0.81], [0.44, 0.76]).\end{aligned} \tag{6.2}$$

Example 2. Given a supply chain, which has been reduced into the following FRIs with min-product:

$$\begin{cases} 0.8x_1 \wedge 0.75x_2 \wedge 0.85x_3 \wedge 0.6x_4 \wedge 0.7x_5 \wedge 0.8x_6 \wedge 0.75x_7 \wedge 0.9x_8 \leq 0.6, \\ 0.75x_1 \wedge 0.9x_2 \wedge 0.8x_3 \wedge 0.7x_4 \wedge 0.9x_5 \wedge 0.75x_6 \wedge 0.7x_7 \wedge 0.8x_8 \leq 0.64, \\ 0.7x_1 \wedge 0.8x_2 \wedge 0.6x_3 \wedge 0.9x_4 \wedge 0.8x_5 \wedge 0.75x_6 \wedge 0.87x_7 \wedge 0.72x_8 \leq 0.56, \\ 0.81x_1 \wedge 0.78x_2 \wedge 0.75x_3 \wedge 0.88x_4 \wedge 0.65x_5 \wedge 0.7x_6 \wedge 0.8x_7 \wedge 0.9x_8 \leq 0.6, \\ 0.8x_1 \wedge 0.75x_2 \wedge 0.85x_3 \wedge 0.78x_4 \wedge 0.9x_5 \wedge 0.7x_6 \wedge 0.82x_7 \wedge 0.65x_8 \leq 0.6, \\ 0.9x_1 \wedge 0.7x_2 \wedge 0.8x_3 \wedge 0.8x_4 \wedge 0.75x_5 \wedge 0.85x_6 \wedge 0.85x_7 \wedge 0.75x_8 \leq 0.62, \\ 0.78x_1 \wedge 0.85x_2 \wedge 0.75x_3 \wedge 0.65x_4 \wedge 0.85x_5 \wedge 0.8x_6 \wedge 0.7x_7 \wedge 0.85x_8 \leq 0.58, \\ x \geq x^L, \end{cases} \tag{6.3}$$

where $x = (x_1, \cdots, x_8)$ and $x^L = (0.5, 0.45, 0.48, 0.51, 0.49, 0.52, 0.43, 0.4)$. According to Eq. (2.4), we can directly compute the upper bound of system (6.1) as

$$x^U = (0.8533, 0.8, 0.9333, 1, 0.9230, 0.8571, 0.9142, 0.8).$$

Applying our proposed algorithm, try to compute the MAIS of the provided solution

$$x^G = (0.7, 0.6, 0.8, 0.8, 0.7, 0.65, 0.75, 0.55)$$

in system (6.3).

Solution:
Step 1. Compute α^L by (4.1) as follows. Since

$$\begin{aligned} x^G - x^L =& (0.7, 0.6, 0.8, 0.8, 0.7, 0.65, 0.75, 0.55) - (0.5, 0.45, 0.48, 0.51, 0.49, 0.52, 0.43, 0.4) \\ =& (0.2, 0.15, 0.32, 0.39, 0.21, 0.13, 0.32, 0.15), \end{aligned}$$

we have

$$\alpha^L = \min_{1 \leq j \leq 8} \{x_j^G - x_j^L\} = \{0.2, 0.15, 0.32, 0.39, 0.21, 0.13, 0.32, 0.15\} = 0.13.$$

Step 2. Construct the index sets $\{\mathbb{J}_i^U \mid 1 \leq i \leq 7\}$, by (5.1):

$$\begin{cases} \mathbb{J}_1^U = \{1 \leq j \leq 8 \mid a_{1j}x_j^U \leq b_1\} = \{2, 4\}, \\ \mathbb{J}_2^U = \{1 \leq j \leq 8 \mid a_{2j}x_j^U \leq b_2\} = \{1, 7, 8\}, \\ \mathbb{J}_3^U = \{1 \leq j \leq 8 \mid a_{3j}x_j^U \leq b_3\} = \{3\}, \\ \mathbb{J}_4^U = \{1 \leq j \leq 8 \mid a_{4j}x_j^U \leq b_4\} = \{5, 6\}, \\ \mathbb{J}_5^U = \mathbb{J}_6^U = \mathbb{J}_7^U = \emptyset. \end{cases}$$

Step 3. According to the results obtained in step 2 and Eq (5.2), there is $\mathbb{I}^U = \{5, 6, 7\} \neq \emptyset$. Thus, we turn to Step 6.
Step 6. Construct the index sets $\{\mathbb{J}_i^G \mid i \in \mathbb{I}^U\} = \{\mathbb{J}_5^G, \mathbb{J}_6^G, \mathbb{J}_7^G\}$, by (5.7).

$$\begin{cases} \mathbb{J}_5^G = \{1 \leq j \leq 8 \mid x_j^G \leq \frac{b_5}{a_{5j}}\} = \{1, 2, 6, 8\}, \\ \mathbb{J}_6^G = \{1 \leq j \leq 8 \mid x_j^G \leq \frac{b_6}{a_{6j}}\} = \{2, 5, 6, 8\}, \\ \mathbb{J}_7^G = \{1 \leq j \leq 8 \mid x_j^G \leq \frac{b_7}{a_{7j}}\} = \{1, 2, 4, 6, 7, 8\}. \end{cases}$$

Step 7. Compute value of α_i^U, for each $i = 5, 6, 7$, by (5.8).

$$\begin{aligned} \alpha_5^U &= \max_{j \in \mathbb{J}_5^G} \{\frac{b_5}{a_{5j}} - x_j^G\} \\ &= \max\{\frac{0.6}{0.8} - 0.7, \frac{0.6}{0.75} - 0.6, \frac{0.6}{0.7} - 0.65, \frac{0.6}{0.65} - 0.55\} \\ &= \max\{0.05, 0.2, 0.2071, 0.3730\} \\ &=0.373. \end{aligned}$$

In the same way, we find $\alpha_6^U = 0.2857$ and $\alpha_7^U = 0.1323$ after calculation.
Step 8. Compute value of α^U by (5.9). Since

$$\begin{aligned} &\min_{1 \leq j \leq 8} \{x_j^U - x_j^G\} \\ =& \min\{0.8533 - 0.7, 0.8 - 0.6, 0.9333 - 0.8, 1 - 0.8, \\ &\quad 0.923 - 0.7, 0.8571 - 0.65, 0.9142 - 0.75, 0.8 - 0.55\} \\ =& \min\{0.1533, 0.2, 0.1333, 0.2, 0.223, 0.2071, 0.1642, 0.25\} \\ =&0.1333. \end{aligned}$$

by (5.9), we have $\alpha^U = \min\{\min_{i=5,6,7} \{\alpha_i^U\}, \min_{1 \leq j \leq 8} \{x_j^U - x_j^G\}\}$.

$$\begin{aligned} \alpha^U &= \min\{\min_{i=5,6,7} \{\alpha_i^U\}, \min_{1 \leq j \leq 8} \{x_j^U - x_j^G\}\} \\ &= \min\{0.373, 0.2857, 0.1323, 0.1333\} \\ &= 0.1323. \end{aligned}$$

Step 9. Since $\alpha^G = 0.13 \wedge 0.1323 = 0.13$, the MAIS of x^G in system (6.3) is $[x^G - \alpha^G, x^G + \alpha^G]$, where

$$\begin{cases} x^G - \alpha^G = (0.57, 0.47, 0.67, 0.67, 0.57, 0.52, 0.62, 0.42), \\ x^G + \alpha^G = (0.83, 0.73, 0.93, 0.93, 0.83, 0.78, 0.88, 0.68). \end{cases}$$

7. Conclusions

In several existing works, min-product FRIs have been defined and adopted to describe the constraints in a supply chain system. In fact, the constraints mean the requirements of price, which should be satisfied. Characterized by the min-product FRIs, represented by system (2.3), any pricing scheme turns out to be a solution in system (2.3). For a given pricing scheme, or say a solution in system (2.3), the biggest variability among the components allowed for ensuring its feasibility in system (2.3) is characterized by the so-called MAIS in this work. If the MAIS is wider, we think the solution is more stable. The contribution of this work is to develop the resolution algorithm in Section 4. Applying our proposed algorithm, the MAIS of any given solution could be directly computed step-by-step. Two numerical examples are enumerated to check the feasibility of the algorithm.

In the past few decades, the FRIs or FREs with different kinds of composition have been introduced to model various real-world systems. All these systems require considering the stability of the feasible scheme. As a consequence, extending the MAIS to some other kinds of FRIs or FREs becomes an important research direction. We will develop and compare some different resolution methods for the MAIS in such kinds of FRIs or FREs.

Author contributions

Xiaobin Yang: Validation, investigation, writing–original draft; Guocheng Zhu: Methodology, writing–review and editing; Xiaopeng Yang: Conceptualization, methodology, funding acquisition, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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