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Research article

A novel approach to refining discrete Jensen's inequality and its applications

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Abstract: The aim of this article is to introduce a novel and logically motivated approach that successfully yields a new refinement of the discrete form of Jensen's inequality. The proposed refinement is further applied to several important areas, including inequalities involving means, Hölder's inequality, and information theory, thereby demonstrating both the theoretical strength and practical utility of the result.

Keywords: Jensen's inequality; refinement; convex function; power mean; information theory; Hölder inequality
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1. Introduction

In 1905, Jensen proved the following inequality:

$$\Psi\left(\bar{L}\right) \leq \frac{1}{F} \sum_{k=1}^{m} \xi_k \Psi(\varrho_k), \tag{1.1}$$

if $\Psi : [\delta_1, \delta_2] \to \mathbb{R}$ is a convex function, $\varrho_k \in [\delta_1, \delta_2]$, $\xi_k \geq 0$ with $F := \sum_{k=1}^{m} \xi_k > 0$ and $\bar{L} := \frac{1}{F} \sum_{k=1}^{m} \xi_k \varrho_k$.

This inequality has attracted the attention of numerous mathematicians, and extensive research has been conducted on its generalizations, extensions, and refinements. There are several applications of this inequality in various directions. Some recent related results on generalized convexity can be found in [2, 19], while several useful inequalities and their applications are discussed in [14, 20]. One of the most interesting aspects of research on the Jensen inequality is the pursuit of its refinements. In 1989, Pečarić and Dragomir obtained the refinement of (1.1) containing the means of certain points of $\varrho_k's$ [15]. Some related results are derived in [7, 8]. In [12], the delay-dependent stability of linear systems with time-varying delays is investigated, with particular emphasis on refining Jensen's inequality, which has been widely used in stability analysis via Lyapunov-Krasovskii functionals. A novel integral inequality, formulated as an infinite series and developed without relying on the Wirtinger inequality, is proposed to enhance the classical Jensen-based approach. The sharpness results for the discrete as well as integral Jensen inequality were derived using the Green function approach, and applied the results for different divergence measures and the Zipf-Mandelbrot law [16]. The refinement derived in [6], based on functionals constructed from indexing sets, offers a clear and easily understandable proof. As a consequence, the more complex refinement obtained earlier in [9] can be deduced directly. A further generalization with respect to arbitrary n indexing sets is presented in [13]. The use of signed measures and majorization inequalities has been employed to derive and refine Jensen's inequality through a novel approach. In particular, a refinement of the Jensen-Steffensen inequality has also been deduced in [11]. Some more related results are given in [10]. An interesting approach was introduced in [17] for deriving results related to Jensen's inequality. This method focuses on establishing an integral identity involving the difference form of Jensen's inequality. By applying various classes of convex functions, along with the Hölder inequality and the integral form of Jensen's inequality, several refined results were obtained. These results were further utilized to derive applications in classical inequalities and information theory. In 2008, Azar [1] demonstrated the economic and statistical significance of Jensen's inequality through its variants in financial economics. These variants of Jensen's inequality arise in contexts such as foreign exchange pricing, forward market efficiency, and expected utility theory. Using simulations, the study confirmed their practical relevance despite the presence of sampling error. The findings underscore that Jensen's inequality serves as a foundational, rather than merely theoretical, tool in financial analysis. Denny [5] highlighted the significance of Jensen's inequality by providing a clear graphical interpretation of its effects and examining its implications across atomic, molecular, organismal, and ecological levels. In 2023, Vivas-Cortez et al. [18] derived novel discrete and integral versions of Jensen' s inequality and extended several classical fractional integral inequalities for their newly introduced class of interval-valued functions, termed CR-γ-convex functions. Their findings were further supported through illustrative applications and graphical representations. By employing a weighted time scales form of Jensen's inequality, Ansari et al. [3] established novel inequalities related to various divergences and distances within the framework of time scales. Their work significantly advanced the theory by introducing new results in h-discrete and quantum calculus, while also generalizing several existing inequalities. Basir et al. [4] explored new forms of weighted majorization inequality by utilizing Jensen's inequality. To show the value of their work, they also demonstrated how these results can be applied in areas like information theory.

A careful study of the existing literature on Jensen's inequality reveals that deriving new and meaningful refinements has become a highly challenging task due to the depth and maturity of prior results. In this article, we successfully present a new refinement of Jensen's inequality by employing a novel and logically structured approach. Specifically, we express both sides of the inequality in a carefully formulated manner that enables the effective application of the Hermite-Hadamard inequality, leading to an improved version of the classical Jensen inequality. Building upon this refinement, we further derive enhanced forms of Hölder's inequality and inequalities involving means. Finally, we apply the derived results to obtain several applications in the framework of information theory.

2. Refinement

We contribute to the literature by presenting the following refinement of Jensen's inequality.

Theorem 2.1. *Suppose $\varrho_k \in [\delta_1, \delta_2]$, $\xi_k \geq 0$ for each $k \in \{1, 2, \ldots, m\}$ with $F := \sum_{k=1}^{m} \xi_k > 0$, and let the function $\Psi : [\delta_1, \delta_2] \to \mathbb{R}$ be convex. Then*

$$\begin{aligned}\Psi\Big(\frac{1}{F}\sum_{k=1}^{m}\xi_k\varrho_k\Big) &\leq \frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)\\ &\leq \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+2\sum_{k<l}\xi_k\xi_l\frac{1}{\varrho_l-\varrho_k}\int_{\varrho_k}^{\varrho_l}\Psi(\sigma)d\sigma\Big)\leq\frac{1}{F}\sum_{k=1}^{m}\xi_k\Psi(\varrho_k). \qquad (2.1)\end{aligned}$$

The inequality (2.1) *reversed whenever function Ψ is concave.*

Proof. Clearly, the following identity holds:

$$\begin{aligned}\frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big) &= \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+\sum_{k\neq l}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)\Big)\\ &= \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+\sum_{k<l}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)+\sum_{k>l}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)\Big)\\ &= \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+2\sum_{k<l}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)\Big). \qquad (2.2)\end{aligned}$$

With the function Ψ being convex, the below Hermite-Hadamard inequality is valid:

$$\Psi\Big(\frac{\delta_1+\delta_2}{2}\Big)\leq\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Psi(\sigma)d\sigma\leq\frac{\Psi(\delta_1)+\Psi(\delta_2)}{2}. \qquad (2.3)$$

Without compromising the generality, let's consider that all ϱ_k are distinct for $k = 1, 2, \ldots, m$. Applying (2.3) with δ_1 as ϱ_k and δ_2 as ϱ_l, the inequality obtained is as follows:

$$\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)\leq\frac{1}{\varrho_l-\varrho_k}\int_{\varrho_k}^{\varrho_l}\Psi(\sigma)d\sigma\leq\frac{\Psi(\varrho_k)+\Psi(\varrho_l)}{2}. \qquad (2.4)$$

The inequality below is deduced by substituting (2.4) into (2.2) :

$$\begin{aligned}\frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big) &\leq \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+2\sum_{k<l}\xi_k\xi_l\frac{1}{\varrho_l-\varrho_k}\int_{\varrho_k}^{\varrho_l}\Psi(\sigma)d\sigma\Big)\\ &\leq \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+2\sum_{k<l}\xi_k\xi_l\Big(\frac{\Psi(\varrho_k)+\Psi(\varrho_l)}{2}\Big)\Big). \qquad (2.5)\end{aligned}$$

Because of the convexity of function Ψ, we can employ the Jensen inequality, which allows us to establish the following inequality:

$$\begin{aligned}
\Psi\Big(\frac{1}{F}\sum_{k=1}^{m}\xi_k\varrho_k\Big) &= \Psi\Big(\frac{1}{F_n^2}\Big(\frac{2F\sum_{k=1}^{m}\xi_k\varrho_k}{2}\Big)\Big) = \Psi\Big(\frac{1}{F^2}\Big(\frac{F\sum_{k=1}^{m}\xi_k\varrho_k + F\sum_{l=1}^{m}\xi_l\varrho_l}{2}\Big)\Big)\\
&= \Psi\Big(\frac{1}{F^2}\Big(\frac{\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\varrho_k + \sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\varrho_l}{2}\Big)\Big)\\
&\leq \frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big). \qquad (2.6)
\end{aligned}$$

The right-hand side of (2.5) can be expressed as

$$\begin{aligned}
&\frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k) + 2\sum_{k<l}\xi_k\xi_l\Big(\frac{\Psi(\varrho_k)+\Psi(\varrho_l)}{2}\Big)\Big)\\
=\ &\frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k) + \sum_{k<l}\xi_k\xi_l\Big(\frac{\Psi(\varrho_k)+\Psi(\varrho_l)}{2}\Big) + \sum_{k>l}\xi_k\xi_l\Big(\frac{\Psi(\varrho_k)+\Psi(\varrho_l)}{2}\Big)\Big)\\
=\ &\frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k) + \sum_{i\neq j}\xi_k\xi_l\Big(\frac{\Psi(\varrho_k)+\Psi(\varrho_l)}{2}\Big)\Big)\\
=\ &\frac{1}{2F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l(\Psi(\varrho_k)+\Psi(\varrho_l))\\
=\ &\frac{1}{2F^2}\sum_{k=1}^{m}\xi_k\Big(F\Psi(\varrho_k)+\sum_{l=1}^{m}\xi_l\Psi(\varrho_l)\Big)\\
=\ &\frac{1}{2F^2}\Big(F\sum_{k=1}^{m}\xi_k\Psi(\varrho_k)+F\sum_{l=1}^{m}\xi_l\Psi(\varrho_l)\Big)\\
=\ &\frac{1}{F}\sum_{k=1}^{m}\xi_k\Psi(\varrho_k). \qquad (2.7)
\end{aligned}$$

Upon amalgamating (2.5)–(2.7), we attain (2.1).

The following result refines the simple form of Jensen's inequality.

Corollary 2.1. *Let the function* $\Psi : [\delta_1, \delta_2] \to \mathbb{R}$ *be convex and* $\varrho_k \in [\delta_1, \delta_2]$. *Then*

$$\begin{aligned}
\Psi\Big(\frac{1}{m}\sum_{k=1}^{m}\varrho_k\Big) &\leq \frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\Psi\Big(\frac{\varrho_k+\varrho_l}{2}\Big)\\
&\leq \frac{1}{m^2}\Big(\sum_{k=1}^{m}\Psi(\varrho_k) + 2\sum_{k<l}\frac{1}{\varrho_l-\varrho_k}\int_{\varrho_k}^{\varrho_l}\Psi(\sigma)d\sigma\Big)\\
&\leq \frac{1}{m}\sum_{k=1}^{m}\Psi(\varrho_k). \qquad (2.8)
\end{aligned}$$

Proof. Substituting $\xi_k = 1$ for $k = 1, 2, \ldots, m$ in (2.1) leads us to derive (2.8).

3. Applications for the means inequality

This section focuses on the applications of established refinement to power and quasi-arithmetic means. We present results that provide improved inequalities refining the relationships between these means. To proceed, we begin with the definition of power means:

Definition 3.1. *Let $\boldsymbol{p} = (\xi_1, \xi_2, \cdots, \xi_m)$ and $\boldsymbol{x} = (\varrho_1, \varrho_2, \cdots, \varrho_m)$ be positive tuples with $F := \sum\limits_{k=1}^{m} \xi_k$. The power mean of order $\zeta_1 \in \mathbb{R}$ is then defined as follows:*

$$M_{\zeta_1}(\boldsymbol{p}, \boldsymbol{x}) = \begin{cases} \left(\frac{1}{F}\sum\limits_{k=1}^{m} \xi_k \varrho_k^{\zeta_1}\right)^{\frac{1}{\zeta_1}}, & \zeta_1 \neq 0, \\ \left(\prod\limits_{k=1}^{m} \varrho_k^{\xi_k}\right)^{\frac{1}{F}}, & \zeta_1 = 0. \end{cases} \tag{3.1}$$

The following corollary is a direct application of Theorem 2.1, providing a refined relation for the power means.

Corollary 3.1. *Let ξ_k, $\varrho_k > 0$ for $k = 1, 2, \ldots, m$ with $F := \sum_{k=1}^{m} \xi_k$ and $\zeta_1, \zeta_2 \in \mathbb{R}$ such that $\zeta_1 \geq \zeta_2$. Then*
(i)

$$\begin{aligned} M_{\zeta_2}(\boldsymbol{p}, \boldsymbol{x}) &\leq \left(\frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m} \xi_k \xi_l \Big(\frac{\varrho_k^{\zeta_2} + \varrho_l^{\zeta_2}}{2}\Big)^{\frac{\zeta_1}{\zeta_2}}\right)^{\frac{1}{\zeta_1}} \\ &\leq \left(\frac{1}{F^2}\Big(\sum_{k=1}^{m} \xi_k^2 \varrho_k^{\zeta_1} + \frac{2\zeta_2}{\zeta_1+\zeta_2}\sum_{k<l} \frac{\xi_k \xi_l (\varrho_l^{\zeta_1+\zeta_2} - \varrho_k^{\zeta_1+\zeta_2})}{\varrho_l^{\zeta_2} - \varrho_k^{\zeta_2}}\Big)\right)^{\frac{1}{\zeta_1}} \\ &\leq M_{\zeta_1}(\boldsymbol{p}, \boldsymbol{x}), \qquad \zeta_1, \zeta_2 \neq 0. \end{aligned} \tag{3.2}$$

(ii)

$$\begin{aligned} M_{\zeta_1}(\boldsymbol{p}, \boldsymbol{x}) &\geq \left(\frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m} \xi_k \xi_l \Big(\frac{\varrho_k^{\zeta_1} + \varrho_l^{\zeta_1}}{2}\Big)^{\frac{\zeta_2}{\zeta_1}}\right)^{\frac{1}{\zeta_2}} \\ &\geq \left(\frac{1}{F^2}\Big(\sum_{k=1}^{m} \xi_k^2 \varrho_k^{\zeta_2} + \frac{2\zeta_1}{\zeta_2+\zeta_1}\sum_{k<l} \frac{\xi_k \xi_l (\varrho_l^{\zeta_2+\zeta_1} - \varrho_k^{\zeta_2+\zeta_1})}{\varrho_l^{\zeta_1} - \varrho_k^{\zeta_1}}\Big)\right)^{\frac{1}{\zeta_2}} \\ &\geq M_{\zeta_2}(\boldsymbol{p}, \boldsymbol{x}), \qquad \zeta_1, \zeta_2 \neq 0. \end{aligned} \tag{3.3}$$

Proof. (i) In the situation, where ζ_1 and ζ_2 both hold positive values, or in the scenario where ζ_1 remains positive while ζ_2 takes negative values, the function $\Psi(\sigma) = \sigma^{\frac{\zeta_1}{\zeta_2}}$ is convex, while obeying the condition $\zeta_1 \geq \zeta_2$. Consequently, by utilizing inequality (2.1) with the choices $\Psi(\sigma) = \sigma^{\frac{\zeta_1}{\zeta_2}}$ and $\varrho_k = \varrho_k^{\zeta_2}$, we arrive at

$$\frac{1}{F}\sum_{k=1}^{m} \xi_k (\varrho_k^{\zeta_2})^{\frac{\zeta_1}{\zeta_2}} \geq \frac{1}{F^2}\sum_{k=1}^{m} \xi_k^2 (\varrho_k^{\zeta_2})^{\frac{\zeta_1}{\zeta_2}} + \frac{2}{F^2}\sum_{k<l} \frac{\xi_k \xi_l}{\varrho_l^{\zeta_2} - \varrho_k^{\zeta_2}} \int_{\varrho_k^{\zeta_2}}^{\varrho_l^{\zeta_2}} \sigma^{\frac{\zeta_1}{\zeta_2}} d\sigma$$

$$\geq \frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Big(\frac{\varrho_k^{\zeta_2}+\varrho_l^{\zeta_2}}{2}\Big)^{\frac{\zeta_1}{\zeta_2}}$$
$$\geq \Big(\frac{1}{F}\sum_{k=1}^{m}\xi_k\varrho_k^{\zeta_2}\Big)^{\frac{\zeta_1}{\zeta_2}}. \tag{3.4}$$

Upon performing some calculations in inequality (3.4), and then taking the exponent as $\frac{1}{\zeta_1}$, we led to the following inequality:

$$\begin{aligned}\Big(\frac{1}{F}\sum_{k=1}^{m}\xi_k\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}} &\geq \Big(\frac{1}{F^2}\sum_{k=1}^{m}\xi_k^2\varrho_k^{\zeta_1}+\frac{2r}{(\zeta_1+\zeta_2)F^2}\sum_{k<l}\frac{\xi_k\xi_l}{\varrho_l^{\zeta_2}-\varrho_k^{\zeta_2}}\big(\varrho_l^{\zeta_1+\zeta_2}-\varrho_k^{\zeta_1+\zeta_2}\big)\Big)^{\frac{1}{\zeta_1}}\\ &\geq \Big(\frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Big(\frac{\varrho_k^{\zeta_2}+\varrho_l^{\zeta_2}}{2}\Big)^{\frac{\zeta_1}{\zeta_2}}\Big)^{\frac{1}{\zeta_1}}\\ &\geq \Big(\frac{1}{F}\sum_{k=1}^{m}\xi_k\varrho_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}.\end{aligned} \tag{3.5}$$

Without a doubt, the inequality (3.5) corresponds to the inequality (3.2).

The case when both ζ_1 and ζ_2 are negative with $\zeta_1 \geq \zeta_2$, then the function $\Psi(\sigma) = \varrho^{\frac{\zeta_1}{\zeta_2}}$ demonstrates concave behavior. Hence, by employing the reverse inequality in (2.1) and following the same procedure, we derive (3.2).

(ii) In this part, we consider the function $\Psi(\sigma) = \sigma^{\frac{\zeta_2}{\zeta_1}}$. By applying the same procedure as in the first part, we arrive at (3.3).

We now recall the definition of the quasi-arithmetic mean:

Definition 3.2. *Let* $\boldsymbol{p} = (\xi_1, \xi_2, \cdots, \xi_m)$ *and* $\boldsymbol{x} = (\varrho_1, \varrho_2, \cdots, \varrho_m)$ *be tuples with positive components and h be a continuous and strictly monotonic function. The quasi-arithmetic mean associated with h is then defined as follows:*

$$M_h(\boldsymbol{p},\boldsymbol{x}) = h^{-1}\Big(\frac{1}{\sum_{k=1}^{m}\xi_k}\sum_{k=1}^{m}\xi_k h(\varrho_k)\Big). \tag{3.6}$$

The next result establishes a relation for quasi-arithmetic means as a direct application of Theorem 2.1.

Corollary 3.2. *Assuming that the function h is continuous and strictly monotonic such that* $\Psi \circ h^{-1}$ *is convex on* $(0,\infty)$*, and* ξ_k*,* $\varrho_k > 0$ *for* $k = 1, 2, \ldots, m$ *with* $F := \sum_{k=1}^{m}\xi_k$*, then*

$$\begin{aligned}\Psi(M_h(\boldsymbol{p},\boldsymbol{x})) &\leq \frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi\circ h^{-1}\Big(\frac{h(\varrho_k)+h(\varrho_l)}{2}\Big)\\ &\leq \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi(\varrho_k)+2\sum_{k<l}\frac{\xi_k\xi_l}{h(\varrho_l)-h(\varrho_k)}\int_{h(\varrho_k)}^{h(\varrho_l)}\Psi(\sigma)d\sigma\Big)\\ &\leq \frac{1}{F}\sum_{k=1}^{m}\xi_k\Psi(\varrho_k).\end{aligned} \tag{3.7}$$

Proof. By replacing ϱ_k by $h(\varrho_k)$ and Ψ by $\Psi \circ h^{-1}$ in inequality (2.1), we deduce

$$\begin{aligned}\Psi \circ h^{-1}\Big(\frac{1}{F}\sum_{k=1}^{m}\xi_k h(\varrho_k)\Big) &\leq \frac{1}{F^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi \circ h^{-1}\Big(\frac{h(\varrho_k)+h(\varrho_l)}{2}\Big)\\ &\leq \frac{1}{F^2}\Big(\sum_{k=1}^{m}\xi_k^2\Psi \circ h^{-1}(h(\varrho_k)) + 2\sum_{k<l}\frac{\xi_k\xi_l}{h(\varrho_l)-h(\varrho_k)}\int_{h(\varrho_k)}^{h(\varrho_l)}\Psi \circ h^{-1}(h(\sigma))d\sigma\Big)\\ &\leq \frac{1}{F}\sum_{k=1}^{m}\xi_k\Psi \circ h^{-1}(h(\varrho_k)). \qquad (3.8)\end{aligned}$$

Instantly, simplifying (3.8), we obtain (3.7).

4. Applications for the Hölder inequality

In this section, we present several refinements of Hölder's inequality as applications of the main results. We begin with the following corollary, which offers a refinement of the classical inequality.

Proposition 4.1. *Let ξ_k and ϱ_k be positive real numbers, where $k = 1, 2, \ldots, m$.*
(i) If $\zeta_1, \zeta_2 > 1$ with $\frac{1}{\zeta_1} + \frac{1}{\zeta_2} = 1$, then

$$\begin{aligned}\sum_{k=1}^{m}\xi_k\varrho_k &\leq \Big(\frac{1}{\big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{-\frac{\zeta_2}{\zeta_1}}\varrho_k + \xi_l^{-\frac{\zeta_2}{\zeta_1}}\varrho_l}{2}\Big)^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)\\ &\leq \Big[\frac{1}{\big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\big)^2}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\varrho_k^{\zeta_1} + \frac{2}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{-\frac{\zeta_2}{\zeta_1}}\varrho_l - \xi_k^{-\frac{\zeta_2}{\zeta_1}}\varrho_k}\\ &\quad\times\Big(\Big(\xi_l^{-\frac{\zeta_2}{\zeta_1}}\varrho_l\Big)^{\zeta_1+1} - \Big(\xi_k^{-\frac{\zeta_2}{\zeta_1}}\varrho_k\Big)^{\zeta_1+1}\Big)\Big)\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)\\ &\leq \Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}. \qquad (4.1)\end{aligned}$$

(ii) If $0 < \zeta_1 < 1$, with $\zeta_2 = \frac{\zeta_1}{\zeta_1 - 1}$, then

$$\begin{aligned}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}} &\leq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)} + \varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}}{2}\Big)^{\frac{1}{\zeta_1}}\\ &\leq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k\xi_k^{(1-\zeta_2)} + \frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)} - \varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}}\\ &\quad\times\Big(\Big(\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}\Big)^{\frac{\zeta_1+1}{\zeta_1}} - \Big(\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}\Big)^{\frac{\zeta_1+1}{\zeta_1}}\Big)\Big)\\ &\leq \sum_{k=1}^{m}\xi_k\varrho_k. \qquad (4.2)\end{aligned}$$

(*iii*) *If* $\zeta_1 < 0$ *and* $\zeta_2 = \frac{\zeta_1}{\zeta_1 - 1}$, *then*

$$\begin{aligned}
\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}} &\leq \frac{1}{\sum\limits_{k=1}^{m}\varrho_k^{\zeta_1}}\sum_{k=1}^{m}\sum_{l=1}^{m}\varrho_k^{\zeta_2}\varrho_l^{\zeta_2}\Big(\frac{\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}+\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}}{2}\Big)^{\frac{1}{\zeta_2}} \\
&\leq \frac{1}{\sum\limits_{k=1}^{m}\varrho_k^{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k\varrho_k^{(\zeta_1+1)}+\frac{2\zeta_2}{\zeta_2+1}\sum_{k<l}\frac{\varrho_k^{\zeta_2}\varrho_l^{\zeta_2}}{\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}-\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}} \\
&\quad\times\Big(\Big(\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}\Big)^{\frac{\zeta_2+1}{\zeta_2}}-\Big(\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}\Big)^{\frac{\zeta_2+1}{\zeta_2}}\Big)\Big) \\
&\leq \sum_{k=1}^{m}\xi_k\varrho_k. \qquad (4.3)
\end{aligned}$$

Proof. (*i*) Let us consider the function $\Psi(\varrho) = \varrho^{\zeta_1}$, where $\varrho \in (0,\infty)$. Then $\Psi''(\varrho) = \zeta_1(\zeta_1 - 1)\varrho^{\zeta_1-2}$. It is evident that $\Psi''(\varrho) > 0$ for $\varrho > 0$ and $\zeta_1 > 1$. This observation solidifies the assertion regarding the convex nature of Ψ. Hence, apply (2.1) for $\Psi(\varrho) = \varrho^{\zeta_1}$, $\xi_k = \xi_k^{\zeta_2}$, and $\varrho_k = \varrho_k\xi_k^{-\frac{\zeta_2}{\zeta_1}}$, we arrive at

$$\begin{aligned}
\Bigg(\frac{\sum\limits_{k=1}^{m}\xi_k^{\zeta_2}\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Bigg)^{\zeta_1} &\leq \frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k+\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l}{2}\Big)^{\zeta_1} \\
&\leq \frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\Big(\sum_{k=1}^{m}\xi_k^{2q}\Big(\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k\Big)^{\zeta_1}+2\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l-\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k}\int_{\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k}^{\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l}\varrho^{\zeta_1}d\varrho\Big) \\
&\leq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big(\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k\Big)^{\zeta_1}. \qquad (4.4)
\end{aligned}$$

Now, taking power as $\frac{1}{\zeta_1}$ of (4.4) and then doing some calculation, we gain

$$\begin{aligned}
\frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\xi_k\varrho_k &\leq \Big[\frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k+\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l}{2}\Big)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}} \\
&\leq \Big[\frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\varrho_k^{\zeta_1}+\frac{2}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l-\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k} \\
&\quad\times\Big(\Big(\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l\Big)^{\zeta_1+1}-\Big(\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k\Big)^{\zeta_1+1}\Big)\Big)\Big]^{\zeta_1} \\
&\leq \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{-\frac{1}{\zeta_1}}. \qquad (4.5)
\end{aligned}$$

Multiplying (4.5) by $\sum_{k=1}^{m}\xi_k^{\zeta_2}$, we receive

$$\begin{aligned}\sum_{k=1}^{m}\xi_k\varrho_k &\le \Big[\frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k+\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l}{2}\Big)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)\\
&\le \Big[\frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\varrho_k^{\zeta_1}+\frac{2}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l-\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k}\\
&\quad\times\Big(\Big(\xi_l^{\frac{-\zeta_2}{\zeta_1}}\varrho_l\Big)^{\zeta_1+1}-\Big(\xi_k^{\frac{-\zeta_2}{\zeta_1}}\varrho_k\Big)^{\zeta_1+1}\Big)\Big)\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)\\
&\le \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{1-\frac{1}{\zeta_1}}. \qquad (4.6)\end{aligned}$$

Obviously, (4.6) is the same as that of (4.1).

(ii) When $0<\zeta_1<1$, then $\frac{1}{\zeta_1}>1$. Therefore, replacing ζ_1 by $\frac{1}{\zeta_1}$, ζ_2 by $\frac{1}{1-\zeta_1}$, ξ_k by $\xi_k^{-\zeta_1}$, and ϱ_k by $(\xi_k\varrho_k)^{\zeta_1}$ in (4.1), we receive

$$\begin{aligned}\sum_{k=1}^{m}\xi_k^{-\zeta_1}(\xi_k\varrho_k)^{\zeta_1} &\le \Big[\frac{1}{\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}(\xi_l^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\\
&\quad\times\Big(\frac{(\xi_k^{-\zeta_1})^{\frac{-\zeta_1}{1-\zeta_1}}(\xi_k\varrho_k)^{\zeta_1}+(p_l^{-\zeta_1})^{\frac{-\zeta_1}{1-\zeta_1}}(\xi_l\varrho_l)^{\zeta_1}}{2}\Big)^{\frac{1}{\zeta_1}}\Big]^{\zeta_1}\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big)\\
&\le \Big[\frac{1}{\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big)^2}\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big[(\xi_k\varrho_k)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}\\
&\quad+\frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}(\xi_l^{-\zeta_1})^{\frac{1}{1-\zeta_1}}}{(\xi_l^{-\zeta_1})^{\frac{-\zeta_1}{1-\zeta_1}}(\xi_l\varrho_l)^{\zeta_1}-(\xi_k^{-\zeta_1})^{\frac{-\zeta_1}{1-\zeta_1}}(\xi_k\varrho_k)^{\zeta_1}}\\
&\quad\times\Big(\Big((\xi_l^{-\zeta_1})^{\frac{-\zeta_1}{1-\zeta_1}}(\xi_l\varrho_l)^{\zeta_1}\Big)^{\frac{\zeta_1+1}{\zeta_1}}-\Big((\xi_k^{-\zeta_1})^{\frac{-\zeta_1}{1-\zeta_1}}(\xi_k\varrho_k)^{\zeta_1}\Big)^{\frac{\zeta_1+1}{\zeta_1}}\Big)\Big)\Big]^{\zeta_1}\times\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big)\\
&\le \Big(\sum_{k=1}^{m}(\varrho_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big)^{1-\zeta_1}\Big(\sum_{k=1}^{m}\Big[(\xi_k\varrho_k)^{\frac{1}{\zeta_1}}\Big]^{\zeta_1}\Big)^{\zeta_1}. \qquad (4.7)\end{aligned}$$

By simplifying (4.7), and then taking exponent as $\frac{1}{\zeta_1}$, we deduce

$$\begin{aligned}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}} &\le \frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}+\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}}{2}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_1}}\\
&\le \frac{1}{\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^2}\Big(\sum_{k=1}^{m}\varrho_k\xi_k^{(1-\zeta_2)}+\frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}-\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}}\end{aligned}$$

$$\times\Big(\Big((\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}\Big)^{\frac{\zeta_1+1}{\zeta_1}}-\Big(\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}\Big)^{\frac{\zeta_1+1}{\zeta_1}}\Big)\Big)\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_1}}$$
$$\leq \Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1-\zeta_1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k\varrho_k\Big). \tag{4.8}$$

Instantly multiplying (4.8) by $\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{\zeta_1-1}{\zeta_1}}$, we get

$$\begin{aligned}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{\zeta_1-1}{\zeta_1}} &\leq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}+\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}}{2}\Big)^{\frac{1}{\zeta_1}}\\
&\leq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k\xi_k^{(1-\zeta_2)}+\frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}-\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}}\\
&\quad\times\Big(\Big((\varrho_l^{\zeta_1}\xi_l^{\zeta_1(1-\zeta_2)}\Big)^{\frac{\zeta_1+1}{\zeta_1}}-\Big(\varrho_k^{\zeta_1}\xi_k^{\zeta_1(1-\zeta_2)}\Big)^{\frac{\zeta_1+1}{\zeta_1}}\Big)\Big)\\
&\leq \sum_{k=1}^{m}\xi_k\varrho_k.\end{aligned}\tag{4.9}$$

The inequalities stated in (4.9) are the same as those of the inequalities given in (4.2). Consequently, (4.9) proves (4.2) for the case, when $\zeta_1 \in (0,1)$.

(*iii*) Now, we prove the last case. For $\zeta_1 < 0$, then certainly, $\zeta_2 = \frac{1}{1-\zeta_1} \in (0,1)$. Replacing ζ_1, ζ_2, ϱ_k, and ξ_k by $\frac{1}{\zeta_2}$, $\frac{1}{1-\zeta_2}$, $\varrho_k^{-\zeta_2}$, and $(\xi_k\varrho_k)^{\zeta_2}$ respectively in (4.1), we acquire

$$\begin{aligned}\sum_{k=1}^{m}\varrho_k^{-\zeta_2}(\xi_k\varrho_k)^{\zeta_2} &\leq \Big[\frac{1}{\Big(\sum_{k=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\big[\varrho_l^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\\
&\quad\times\Big(\frac{\big[\varrho_k^{-\zeta_2}\big]^{\frac{\frac{-1}{1-q}}{\frac{1}{\zeta_2}}}(\xi_k\varrho_k)^{\zeta_2}+\big[\varrho_l^{-\zeta_2}\big]^{\frac{\frac{-1}{1-\zeta_2}}{\frac{1}{\zeta_2}}}(\xi_l\varrho_l)^{\zeta_2}}{2}\Big)^{\frac{1}{\zeta_2}}\Big]^{\zeta_2}\times\Big(\sum_{k=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\Big)\\
&\leq \Big[\frac{1}{\Big(\sum_{k=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\Big)^2}\Big(\sum_{k=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\big[(\xi_k\varrho_k)^{\zeta_2}\big]^{\frac{1}{\zeta_2}}\\
&\quad+\frac{1}{\frac{1}{\zeta_2}+1}\sum_{k<l}\frac{\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\big[\varrho_l^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}}{\big[\varrho_l^{-\zeta_2}\big]^{\frac{\frac{-1}{1-\zeta_2}}{\frac{1}{\zeta_2}}}(\xi_l\varrho_l)^{\zeta_2}-\big[\varrho_k^{-\zeta_2}\big]^{\frac{\frac{-1}{1-\zeta_2}}{\frac{1}{\zeta_2}}}(\xi_k\varrho_k)^{\zeta_2}}\\
&\quad\times\Big[\Big(\Big(\big[\varrho_l^{-\zeta_2}\big]^{\frac{\frac{-1}{1-\zeta_2}}{\frac{1}{\zeta_2}}}(\xi_l\varrho_l)^{\zeta_2}\Big)^{\frac{1}{\zeta_2}+1}-\Big(\big[\varrho_k^{-\zeta_2}\big]^{\frac{\frac{-1}{1-\zeta_2}}{\frac{1}{\zeta_2}}}(\xi_k\varrho_k)^{\zeta_2}\Big)^{\frac{1}{\zeta_2}+1}\Big]^{\zeta_2}\times\Big(\sum_{k=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\Big)\\
&\leq \Big(\sum_{k=1}^{m}\big[\varrho_k^{-\zeta_2}\big]^{\frac{1}{1-\zeta_2}}\Big)^{(1-\zeta_2)}\Big(\sum_{k=1}^{m}\xi_k\varrho_k\Big)^{\zeta_2}.\end{aligned}\tag{4.10}$$

By simplifying (4.10), and then taking exponent $\frac{1}{\zeta_2}$, we deduce

$$\begin{aligned}
\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}} &\leq \frac{1}{\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\varrho_k^{\zeta_2}\varrho_l^{\zeta_2}\Big(\frac{\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}+\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}}{2}\Big)^{\frac{1}{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_2}}\\
&\leq \frac{1}{\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^2}\Big(\sum_{k=1}^{m}\xi_k\varrho_k^{(\zeta_1+1)}+\frac{2\zeta_2}{\zeta_2+1}\sum_{k<l}\frac{\varrho_k^{\zeta_2}\varrho_l^{\zeta_2}}{\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}-\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}}\\
&\quad\times\Big(\Big(\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}\Big)^{\frac{\zeta_2+1}{\zeta_2}}-\Big(\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}\Big)^{\frac{\zeta_2+1}{\zeta_2}}\Big)\Big)\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_2}}\\
&\leq \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1-\zeta_2}{\zeta_2}}\Big(\sum_{k=1}^{m}\xi_k\varrho_k\Big). \qquad (4.11)
\end{aligned}$$

Instantly multiplying (4.11) by $\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{\zeta_2-1}{\zeta_2}}$, we get

$$\begin{aligned}
\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{\zeta_2-1}{\zeta_2}} &\leq \frac{1}{\sum_{k=1}^{m}\varrho_k^{\zeta_1}}\sum_{k=1}^{m}\sum_{l=1}^{m}\varrho_k^{\zeta_2}\varrho_l^{\zeta_2}\Big(\frac{\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}+\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}}{2}\Big)^{\frac{1}{\zeta_2}}\\
&\leq \frac{1}{\sum_{k=1}^{m}\varrho_k^{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k\varrho_k^{(\zeta_1+1)}+\frac{2\zeta_2}{\zeta_2+1}\sum_{k<l}\frac{\varrho_k^{\zeta_2}\varrho_l^{\zeta_2}}{\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}-\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}}\\
&\quad\times\Big(\Big(\xi_l^{\zeta_2}\varrho_l^{\zeta_2(1-\zeta_1)}\Big)^{\frac{\zeta_2+1}{\zeta_2}}-\Big(\xi_k^{\zeta_2}\varrho_k^{\zeta_2(1-\zeta_1)}\Big)^{\frac{\zeta_2+1}{\zeta_2}}\Big)\Big)\\
&\leq \sum_{k=1}^{m}\xi_k\varrho_k. \qquad (4.12)
\end{aligned}$$

Hence, (4.12) proves the desired inequalities stated in (4.3) for the condition, when $\zeta_1 < 0$.

Proposition 4.2. *Assume that ξ_k and ϱ_k are both positive for $k = 1, 2, \ldots, m$.*
(i) If $\zeta_1 > 1$ and $\zeta_2 = \frac{\zeta_1}{\zeta_1-1}$, then

$$\begin{aligned}
\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}} &\geq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}+\xi_l^{-\zeta_2}\varrho_l^{\zeta_1}}{2}\Big)^{\frac{1}{\zeta_1}}\\
&\geq \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2\left(\frac{2\zeta_1-1}{\zeta_1}\right)}\varrho_k+\frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{-\zeta_2}\varrho_l^{\zeta_1}-\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}}\\
&\quad\times\Big(\Big(\xi_l^{-\zeta_2}\varrho_l^{\zeta_1}\Big)^{\frac{1}{\zeta_1}+1}-\Big(\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}+1}\Big)\Big)\\
&\geq \sum_{k=1}^{m}\xi_k\varrho_k. \qquad (4.13)
\end{aligned}$$

(ii) If $\zeta_1 \in (0, 1)$, and $\zeta_2 = \frac{\zeta_1}{\zeta_1-1}$, then

$$\sum_{k=1}^{m}\xi_k\varrho_k \geq \Big[\frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\varrho_k\xi_k^{(1-\zeta_2)}+\varrho_l\xi_l^{(1-\zeta_2)}}{2}\Big)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}$$

$$\geq \Big[\frac{1}{\sum_{k=1}^m \xi_k^{\zeta_2}}\Big(\sum_{k=1}^m \varrho_k^{\zeta_1}\xi_k^{\zeta_2(2-\zeta_1)+\zeta_1} + \frac{2}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\varrho_l\xi_l^{(1-\zeta_2)} - \varrho_k\xi_k^{(1-\zeta_2)}}$$

$$\times\Big(\Big(\varrho_l\xi_l^{(1-\zeta_2)}\Big)^{\zeta_1+1} - \Big(\varrho_k\xi_k^{(1-\zeta_2)}\Big)^{\zeta_1+1}\Big)\Big)\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^m \xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}$$

$$\geq \Big(\sum_{k=1}^m \varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^m \xi_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}. \tag{4.14}$$

(*iii*) *If* $\zeta_1 < 0$ *and* $\zeta_2 = \frac{\zeta_1}{\zeta_1 - 1}$, *then*

$$\sum_{k=1}^m \xi_k\varrho_k \geq \frac{1}{\sum\limits_{k=1}^m \xi_k^{\zeta_2}}\sum_{k=1}^m\sum_{l=1}^m \xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{(\zeta_2-\zeta_1)}\varrho_k^{\zeta_2} + \xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}}{2}\Big)^{\zeta_2}\Big(\sum_{k=1}^m \xi_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}$$

$$\geq \frac{1}{\sum_{k=1}^m \xi_k^{\zeta_2}}\Big(\sum_{k=1}^m \xi_k^{\zeta_1(2-\zeta_2)+\zeta_1}\varrho_k^{\zeta_2} + \frac{2}{\zeta_2+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2} - \xi_k^{(\zeta_2-\zeta_1)}\varrho_k^{\zeta_2}}$$

$$\times\Big(\Big(\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}\Big)^{\zeta_2+1} - \Big(\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}\Big)^{\zeta_2+1}\Big)\Big)\Big(\sum_{k=1}^m \xi_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}$$

$$\geq \Big(\sum_{k=1}^m \varrho_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}\Big(\sum_{k=1}^m \xi_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}. \tag{4.15}$$

Proof. (*i*) The function $\Psi(\varrho) = \varrho^{\frac{1}{\zeta_1}}$ exhibits concavity over $(0, \infty)$ for values of $\zeta_1 > 1$, because $\Psi''(x) = \frac{1}{\zeta_1}(\frac{1}{\zeta_1} - 1)\varrho^{\frac{1}{\zeta_1}-2} < 0$. Therefore, by employing the inequality (2.1) with the function $\Psi(\varrho) = \varrho^{\frac{1}{\zeta_1}}$, and setting $\xi_k = \xi_k^{\zeta_2}$ and $\varrho_k = \varrho_k^{\zeta_1}\xi_k^{-\zeta_2}$, we obtain

$$\Big(\frac{1}{\sum_{k=1}^m \xi_k^{\zeta_2}}\sum_{k=1}^m \xi_k^{\zeta_2}\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}} \geq \frac{1}{\Big(\sum_{k=1}^m \xi_k^{\zeta_2}\Big)^2}\sum_{k=1}^m\sum_{l=1}^m \xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{-\zeta_2}\varrho_k^{\zeta_1} + \xi_l^{-\zeta_2}\varrho_l^{\zeta_1}}{2}\Big)^{\frac{1}{\zeta_1}}$$

$$\geq \frac{1}{\Big(\sum_{k=1}^m \xi_k^{\zeta_2}\Big)^2}\Big(\sum_{k=1}^m \xi_k^{2\zeta_2}\Big(\varrho_k^{\zeta_1}\xi_k^{-\zeta_2}\Big)^{\frac{1}{\zeta_1}}$$

$$+\frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{-\zeta_2}\varrho_l^{\zeta_1} - \xi_k^{-\zeta_2}\varrho_k^{\zeta_1}}\int_{\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}}^{\xi_l^{-\zeta_2}\varrho_l^{\zeta_1}} \varrho^{\frac{1}{\zeta_1}}d\varrho\Big)$$

$$\geq \frac{1}{\sum_{k=1}^m \xi_k^{\zeta_2}}\sum_{k=1}^m \xi_k^{\zeta_2}\Big(\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}. \tag{4.16}$$

By simplifying (4.16), and subsequently multiplying by $\sum_{k=1}^m \xi_k^{\zeta_2}$, we attain

$$\Big(\sum_{k=1}^m \varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^m \xi_k^{\zeta_2}\Big)^{1-\frac{1}{\zeta_1}} \geq \frac{1}{\sum_{k=1}^m \xi_k^{\zeta_2}}\sum_{k=1}^m\sum_{l=1}^m \xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{-\zeta_2}\varrho_k^{\zeta_1} + \xi_l^{-\zeta_2}\varrho_l^{\zeta_1}}{2}\Big)^{\frac{1}{\zeta_1}}$$

$$\geq \frac{1}{\sum_{k=1}^m \xi_k^{\zeta_2}}\Big(\sum_{k=1}^m \xi_k^{\zeta_2\left(\frac{2\zeta_1-1}{\zeta_1}\right)}\varrho_k + \frac{2\zeta_1}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{-\zeta_2}\varrho_l^{\zeta_1} - \xi_k^{-\zeta_2}\varrho_k^{\zeta_1}}$$

$$
\begin{aligned}
&\times\Big(\Big(\xi_l^{-\zeta_2}\varrho_l^{\zeta_1}\Big)^{\frac{1}{\zeta_1}+1}-\Big(\xi_k^{-\zeta_2}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}+1}\Big)\Big) \\
\geq\ & \sum_{k=1}^{m}\xi_k\varrho_k. \qquad (4.17)
\end{aligned}
$$

Evidently, the inequality (4.17) is the same as that of (4.13).

(ii) When ζ_1 assumes values between 0 and 1, then $\frac{1}{\zeta_1}$ will have values greater than 1. Therefore, apply (4.13) with the replacement of ζ_1 by $\frac{1}{\zeta_1}$, ζ_2 by $\frac{1}{1-\zeta_1}$, ξ_k by $\xi_k^{-\zeta_1}$, and ϱ_k by $(\xi_k\varrho_k)^{\zeta_1}$, we reach

$$
\begin{aligned}
&\Big(\sum_{k=1}^{m}[(\xi_k\varrho_k)^{\zeta_1}]^{\frac{1}{\zeta_1}}\Big)^{\zeta_1}\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\Big)^{1-\zeta_1} \\
\geq\ & \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}(\xi_l^{-\zeta_1})^{\frac{1}{1-\zeta_1}}\times\Big(\frac{(\xi_k^{-\zeta_1})^{\frac{-1}{1-\zeta_1}}\Big[(\xi_k\varrho_k)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}+(\xi_l^{-\zeta_1})^{\frac{-1}{1-\zeta_1}}\Big[(\xi_l\varrho_l)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}}{2}\Big)^{\zeta_1} \\
\geq\ & \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_1})^{(\frac{1}{1-\zeta_1})(\frac{\frac{2}{\zeta_1}-1}{\frac{1}{\zeta_1}})}(\xi_k\varrho_k)^{\zeta_1}+\frac{2}{\zeta_1+1}\sum_{k<l}\frac{(\xi_k^{-\zeta_1})^{\frac{1}{1-\zeta_1}}(\xi_l^{-\zeta_1})^{\frac{1}{1-\zeta_1}}}{(\xi_l^{-\zeta_1})^{\frac{-1}{1-\zeta_1}}\Big[(\xi_l\varrho_l)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}-(\xi_k^{-\zeta_1})^{\frac{-1}{1-\zeta_1}}\Big[(\xi_k\varrho_k)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}} \\
&\times\Big(\Big((\xi_l^{-\zeta_1})^{\frac{-1}{1-\zeta_1}}\Big[(\xi_l\varrho_l)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}\Big)^{\zeta_1+1}-\Big((\xi_k^{-\zeta_1})^{\frac{-1}{1-\zeta_1}}\Big[(\xi_k\varrho_k)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}\Big)^{\zeta_1+1}\Big)\Big) \\
\geq\ & \sum_{k=1}^{m}(\xi_k\varrho_k)^{\zeta_1}\xi_k^{-\zeta_1}. \qquad (4.18)
\end{aligned}
$$

By doing the calculation in (4.18), and then taking power $\frac{1}{\zeta_1}$, we arrive at

$$
\begin{aligned}
\Big(\sum_{k=1}^{m}\xi_k\varrho_k\Big)\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{1-\zeta_1}{\zeta_1}} \geq\ & \Big[\frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\varrho_k\xi_k^{(1-\zeta_2)}+\varrho_l\xi_l^{(1-\zeta_2)}}{2}\Big)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}} \\
\geq\ & \Big[\frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\xi_k^{-\zeta_2(2-\zeta_1)+\zeta_1}+\frac{2}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l\xi_l^{(1-\zeta_2)}-\xi_k\xi_k^{(1-\zeta_2)}} \\
&\times\Big(\Big(\varrho_l\xi_l^{(1-\zeta_2)}\Big)^{\zeta_1+1}-\Big(\varrho_k\xi_k^{(1-\zeta_2)}\Big)^{\zeta_1+1}\Big)\Big)\Big]^{\frac{1}{\zeta_1}} \\
\geq\ & \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}. \qquad (4.19)
\end{aligned}
$$

The following inequality can be achieved by multiplying (4.19) by $\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{\zeta_1-1}{\zeta_1}}$:

$$
\begin{aligned}
\sum_{k=1}^{m}\xi_k\varrho_k \geq\ & \Big[\frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\varrho_k\xi_k^{(1-\zeta_2)}+\varrho_l\xi_l^{(1-\zeta_2)}}{2}\Big)^{\zeta_1}\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{\zeta_1-1}{\zeta_1}} \\
\geq\ & \Big[\frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\xi_k^{\zeta_2(2-\zeta_1)+\zeta_1}+\frac{2}{\zeta_1+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\varrho_l\xi_l^{(1-\zeta_2)}-\varrho_k\xi_k^{(1-\zeta_2)}}
\end{aligned}
$$

$$
\times\Big(\Big(\varrho_l\xi_l^{(1-\zeta_2)}\Big)^{\zeta_1+1}-\Big(\varrho_k\xi_k^{(1-\zeta_2)}\Big)^{\zeta_1+1}\Big)\Big)\Big]^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{\zeta_1-1}{\zeta_1}}
$$

$$
\geq \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_1}\Big)^{\frac{1}{\zeta_1}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_2}\Big)^{\frac{\zeta_1-1}{\zeta_1}}. \tag{4.20}
$$

Clearly, (4.20) is coincident to (4.14).

(*iii*) When $\zeta_1 < 0$, then $\zeta_2 = \frac{\zeta_1}{\zeta_1-1}$ will assume values in $(0,1)$. Therefore, by applying (4.13) while replacing ζ_1 by $\frac{1}{\zeta_2}$, ζ_2 by $\frac{1}{1-\zeta_2}$, ξ_k by $\varrho_k^{-\zeta_2}$, and ξ_k by $(\xi_k\varrho_k)^{\zeta_2}$, we deduce

$$
\begin{aligned}
&\Big(\sum_{k=1}^{m}[(\xi_k\varrho_k)^{\zeta_2}]^{\frac{1}{\zeta_2}}\Big)^{\zeta_2}\Big(\sum_{k=1}^{m}(\varrho_k^{-\zeta_2})^{\frac{1}{1-\zeta_2}}\Big)^{1-\zeta_2}\\
\geq\;& \frac{1}{\sum\limits_{k=1}^{m}(\xi_k^{-\zeta_2})^{\frac{1}{1-\zeta_2}}}\sum_{k=1}^{m}\sum_{l=1}^{m}(\xi_k^{-\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_l^{-\zeta_2})^{\frac{1}{1-\zeta_2}}\times\Big(\frac{(\xi_k^{\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_k\varrho_k)^{\zeta_2}+(\xi_l^{\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_l\varrho_l)^{\zeta_2}}{2}\Big)^{\zeta_2}\\
\geq\;& \frac{1}{\sum\limits_{k=1}^{m}(\xi_k^{-\zeta_2})^{\frac{1}{1-\zeta_2}}}\Big(\sum_{k=1}^{m}(\xi_k^{-\zeta_2})^{\frac{1}{1-\zeta_2}\left(\frac{\frac{2}{\zeta_2}-1}{\frac{1}{\zeta_2}}\right)}(\xi_k\varrho_k)^{\zeta_2}\\
&+\frac{\frac{2}{\zeta_2}}{\frac{1}{\zeta_2}+1}\sum_{k<l}\frac{(\xi_k^{-\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_l^{-\zeta_2})^{\frac{1}{1-\zeta_2}}}{(\xi_l^{\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_l\varrho_l)^{\zeta_2}-(\xi_k^{\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_k\varrho_k)^{\zeta_2}}\times\Big(\Big((\xi_l^{\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_l\varrho_l)^{\zeta_2}\Big)^{(\zeta_2+1)}-\Big((\xi_k^{\zeta_2})^{\frac{1}{1-\zeta_2}}(\xi_k\varrho_k)^{\zeta_2}\Big)^{(\zeta_2+1)}\Big)\Big)\\
\geq\;& \sum_{k=1}^{m}\xi_k^{-\zeta_2}(\xi_k\varrho_k)^{\zeta_2}.
\end{aligned} \tag{4.21}
$$

By doing calculation in (4.21), and then taking power $\frac{1}{\zeta_2}$, we arrive at

$$
\begin{aligned}
\Big(\sum_{k=1}^{m}\xi_k\varrho_k\Big)\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1}\Big)^{\frac{1-\zeta_2}{\zeta_2}} \;\geq\;& \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{(\zeta_2-\zeta_1)}\varrho_k^{\zeta_2}+\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}}{2}\Big)^{\zeta_2}\\
\geq\;& \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1(2-\zeta_2)+\zeta_1}\varrho_k^{\zeta_2}+\frac{2}{\zeta_2+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}-\xi_k^{(\zeta_2-\zeta_1)}\varrho_k^{\zeta_2}}\\
&\times\Big(\Big(\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}\Big)^{\zeta_2+1}-\Big(\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}\Big)^{\zeta_2+1}\Big)\Big)\\
\geq\;& \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}.
\end{aligned} \tag{4.22}
$$

The following inequality can be achieved by multiplying (4.22) by $\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1}\Big)^{\frac{\zeta_2-1}{\zeta_2}}$:

$$
\begin{aligned}
\sum_{k=1}^{m}\xi_k\varrho_k \;\geq\;& \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k^{\zeta_2}\xi_l^{\zeta_2}\Big(\frac{\xi_k^{(\zeta_2-\zeta_1)}\varrho_k^{\zeta_2}+\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}}{2}\Big)^{\zeta_2}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1}\Big)^{\frac{\zeta_2-1}{\zeta_2}}\\
\geq\;& \frac{1}{\sum_{k=1}^{m}\xi_k^{\zeta_2}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1(2-\zeta_2)+\zeta_1}\varrho_k^{\zeta_2}+\frac{2}{\zeta_2+1}\sum_{k<l}\frac{\xi_k^{\zeta_2}\xi_l^{\zeta_2}}{\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}-\xi_k^{(\zeta_2-\zeta_1)}\varrho_k^{\zeta_2}}
\end{aligned}
$$

$$\times\Big(\Big(\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}\Big)^{\zeta_2+1}-\Big(\xi_l^{(\zeta_2-\zeta_1)}\varrho_l^{\zeta_2}\Big)^{\zeta_2+1}\Big)\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1}\Big)^{\frac{\zeta_2-1}{\zeta_2}}$$

$$\geq \Big(\sum_{k=1}^{m}\varrho_k^{\zeta_2}\Big)^{\frac{1}{\zeta_2}}\Big(\sum_{k=1}^{m}\xi_k^{\zeta_1}\Big)^{\frac{\zeta_2-1}{\zeta_2}}. \tag{4.23}$$

Hence, (4.23) proves the required inequality.

5. Applications in information theory

Definition 5.1. *Let $\boldsymbol{p} = (\xi_1, \xi_2, \cdots, \xi_m)$, $\boldsymbol{x} = (\varrho_1, \varrho_2, \cdots, \varrho_m)$ be probability distributions with strictly positive components, and $\Psi : [\delta_1, \delta_2] \to \mathbb{R}$ be a convex function satisfying $\Psi(1) = 0$. The Csiszár divergence between $\boldsymbol{p}$ and $\boldsymbol{x}$ is then defined as*

$$C_\Psi(\boldsymbol{p}\|\boldsymbol{x}) = \sum_{k=1}^{m}\xi_k\Psi\Big(\frac{\varrho_k}{\xi_k}\Big).$$

Theorem 5.1. *Let $\boldsymbol{p} = (\xi_1, \xi_2, \cdots, \xi_m)$, $\boldsymbol{x} = (\varrho_1, \varrho_2, \cdots, \varrho_m)$ be probability distributions with strictly positive components. Suppose that the function $\Psi : [\delta_1, \delta_2] \to \mathbb{R}$ is convex and satisfies $\Psi(1) = 0$. Then*

$$C_\Psi(\boldsymbol{p}\|\boldsymbol{x}) \geq \frac{1}{\sum\limits_{k=1}^{m}\xi_k}\Bigg(\sum_{k=1}^{m}\xi_k^2\Psi\Big(\frac{\varrho_k}{\xi_k}\Big)+2\sum_{k<l}\frac{\xi_k^2\xi_k^2}{\xi_k\varrho_l-\xi_l\varrho_k}\int_{\frac{\varrho_k}{\xi_k}}^{\frac{\varrho_l}{\xi_l}}\Psi(\varrho)d\varrho\Bigg)$$

$$\geq \frac{1}{\sum\limits_{k=1}^{m}\xi_k}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\Psi\Big(\frac{\xi_k\varrho_l+\xi_l\varrho_k}{2\xi_k\xi_l}\Big)\geq\sum_{k=1}^{m}\xi_k\Psi\Bigg(\frac{\sum\limits_{k=1}^{m}\varrho_k}{\sum\limits_{k=1}^{m}\xi_k}\Bigg). \tag{5.1}$$

Proof. By using $\varrho_k = \frac{\varrho_k}{\xi_k}$ in (2.1), we deduce (5.1).

Definition 5.2. *Let $\boldsymbol{p} = (\xi_1, \xi_2, \cdots, \xi_m)$, $\boldsymbol{x} = (\varrho_1, \varrho_2, \cdots, \varrho_m)$ be probability distributions with strictly positive components. Several important information-theoretic measures between these distributions are defined as follows:*

- ***Shannon entropy:*** *The Shannon entropy of $\boldsymbol{p}$ is defined as*

$$S(\boldsymbol{p}) = -\sum_{k=1}^{m}\xi_i\log\xi_i.$$

- ***Kullback-Leibler divergence***: *The Kullback-Leibler divergence between $\boldsymbol{p}$ and $\boldsymbol{x}$ is given by*

$$D_K(\boldsymbol{p},\boldsymbol{x}) = \sum_{k=1}^{m}\xi_i\log\left(\frac{\xi_i}{\varrho_i}\right).$$

- ***Jeffrey's distance:*** *Jeffrey's distance is defined as*

$$J(\boldsymbol{p},\boldsymbol{x}) = \sum_{k=1}^{m}(\xi_i-\varrho_i)\log\left(\frac{\xi_i}{\varrho_i}\right).$$

- ***Hellinger distance:*** *The Hellinger distance between* $\boldsymbol{p}$ *and* $\boldsymbol{x}$ *is given by*

$$H(\boldsymbol{p},\boldsymbol{x}) = \frac{1}{\sqrt{2}}\Big(\sum_{k=1}^{m}\big(\sqrt{\xi_i} - \sqrt{\varrho_i}\big)^2\Big)^{1/2}.$$

- ***Total variation distance:*** *The total variation distance between is defined as*

$$V(\boldsymbol{p},\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{m}|\xi_i - \varrho_i|.$$

- ***Bhattacharyya coefficient:*** *The Bhattacharyya coefficient between is given by*

$$B(\boldsymbol{p},\boldsymbol{x}) = \sum_{k=1}^{m}\sqrt{\xi_i\varrho_i}.$$

Corollary 5.1. *Assume that* $\xi_k > 0$ *for* $k = 1, 2, \ldots, m$ *with* $\sum_{k=1}^{m}\xi_k = 1$. *Then*

$$\begin{aligned} S(\boldsymbol{P}) &\leq \sum_{k=1}^{m}\xi_k^2\log\xi_k + 2\sum_{k<l}\frac{\xi_k\xi_l}{\xi_k-\xi_l}\Big(\log\Big(\frac{p_k^{\xi_l}}{\xi_l^{\xi_k}}\Big) + \xi_l - \xi_k\Big) \\ &\leq \sum_{k=1}^{m}\sum_{l=1}^{m}\xi_k\xi_l\log\Big(\frac{\xi_k+\xi_l}{2\xi_k\xi_l}\Big) \\ &\leq \log m. \end{aligned} \tag{5.2}$$

Proof. Taking $\Psi(\varrho) = -\log\varrho,\ \varrho > 0$ and $\varrho_k = 1$ $(k = 1, 2, \ldots, m)$ in (2.1), we deduce (5.2).

The next corollary is an application of Theorem 2.1 for the Kullback divergence.

Corollary 5.2. *Suppose that* $\xi_k, \varrho_k > 0$ *for* $k = 1, 2, \ldots, m$ *with* $\sum_{k=1}^{m}\xi_k = 1$ *and* $\sum_{k=1}^{m}\varrho_k = 1$. *Then*

$$\begin{aligned} D_K(\boldsymbol{p},\boldsymbol{x}) &\geq \sum_{k=1}^{m}\xi_k\varrho_k\log\Big(\frac{\varrho_k}{\xi_k}\Big) + \frac{1}{2}\sum_{k<l}\frac{1}{\xi_k\varrho_l - \xi_l\varrho_k} \\ &\quad\times\Big(\xi_k^2\varrho_l^2\Big(\log\Big(\frac{\varrho_l}{\xi_l} - 1\Big)\Big) - \xi_l^2\varrho_k^2\Big(\log\Big(\frac{\varrho_k}{\xi_k}\Big)\Big)\Big) \\ &\geq \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m}\big(\xi_l\varrho_k + \xi_k\varrho_l\big)\log\Big(\frac{\xi_l\varrho_k + \xi_k\varrho_l}{2\xi_k\xi_l}\Big) \geq 0. \end{aligned} \tag{5.3}$$

Proof. Applying inequality (2.1) for $\Psi(\varrho) = \varrho\log\varrho,\ \varrho > 0$, we acquire (5.4).

The below corollary presents a relation for the Jeffrey's distance as an application of Theorem 2.1.

Corollary 5.3. *Under the assumptions of Corollary* 5.2, *we have*

$$\begin{aligned} J(\boldsymbol{p},\boldsymbol{x}) &\geq \sum_{k=1}^{m}\xi_k(\varrho_k - \xi_k)\log\Big(\frac{\varrho_k}{\xi_k}\Big) + \sum_{k<l}\frac{1}{\xi_k\varrho_l - \xi_l\varrho_k} \\ &\quad\times\Big(\xi_k^2\varrho_l(\varrho_l - \xi_l)\Big(\log\Big(\frac{\varrho_l}{\xi_l}\Big) - 1\Big) - \xi_l^2\varrho_k(\varrho_k - \xi_k)\Big(\log\Big(\frac{\varrho_k}{\xi_k}\Big) - 1\Big)\Big) \\ &\geq \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m}\big(\xi_l\varrho_k + \xi_k\varrho_l - 2\xi_k\xi_l\big)\log\Big(\frac{\xi_l\varrho_k + \xi_k\varrho_l}{2\xi_k\xi_l}\Big) \geq 0. \end{aligned} \tag{5.4}$$

Proof. Taking $\Psi(\varrho) = (\varrho - 1)\log\varrho,\ \varrho > 0$ in (2.1), we acquire (5.4).

Corollary 5.4 provides an estimate for variational distance as a direct application of Theorem 2.1.

Corollary 5.4. *Presume that Corollary* 5.2 *hypotheses are true. Then*

$$\begin{aligned} V(\boldsymbol{p},\boldsymbol{x}) &\geq \sum_{k=1}^{m} \xi_k \left|\varrho_k - \xi_k\right| + \frac{1}{2}\sum_{k<l} \frac{\xi_k \xi_l}{\xi_k \varrho_l - \xi_l \varrho_k} \\ &\quad \times \Big(\xi_k(\varrho_l - \xi_l)\left|\varrho_l - \xi_l\right| - \xi_l(\varrho_k - \xi_k)\left|\varrho_k - \xi_k\right| \Big) \\ &\geq \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m} \left|\xi_l \varrho_k + \xi_k \varrho_l - 2\xi_k \xi_l\right| \\ &\geq 0. \end{aligned} \tag{5.5}$$

Proof. Utilizing $\Psi(\varrho) = |\varrho - 1|,\ \varrho > 0$ in (2.1), we acquire (5.5).

The preceding corollary establishes an upper bound for the Hellinger distance as a direct analytical consequence of Theorem 2.1.

Corollary 5.5. *Assume that the conditions of Corollary* 5.2 *hold. Then*

$$\begin{aligned} H(\boldsymbol{p},\boldsymbol{x}) &\geq \Bigg(\sum_{k=1}^{m} \xi_k \left(\sqrt{\varrho_k} - \sqrt{\xi_k}\right)^2 + \frac{2}{3}\sum_{k<l} \frac{1}{\xi_k \varrho_l - \xi_l \varrho_k} \\ &\quad \times \Big(\xi_k^2 \left(\sqrt{\varrho_l} + \sqrt{\xi_l}\right)^3 \sqrt{\varrho_l} - \xi_l^2 \left(\sqrt{\varrho_k} + \sqrt{\xi_k}\right)^3 \sqrt{\varrho_k} \Big) \Bigg) \\ &\geq \sum_{k=1}^{m}\sum_{l=1}^{m} \left(\sqrt{\xi_k \varrho_l + \xi_l \varrho_k} - \sqrt{2\xi_l \xi_k} \right)^2 \\ &\geq 0. \end{aligned} \tag{5.6}$$

Proof. Applying inequality (2.1) with $\Psi(\varrho) = (\sqrt{\varrho} - 1)^2$, for $\varrho > 0$, yields (5.6).

An application of Theorem 2.1 for the Bhattacharyya coefficient is acquired in the following corollary.

Corollary 5.6. *Assume that all the suppositions of Corollary* 5.2 *are satisfied. Then*

$$\begin{aligned} B(\boldsymbol{p},\boldsymbol{x}) &\leq \sum_{k=1}^{m} \sqrt{p_k^3 \varrho_k} + \frac{4}{3}\sum_{k<l} \frac{\sqrt{\xi_k \xi_l}}{\xi_k \varrho_l - \xi_l \varrho_k} \left(\sqrt{(\xi_k \varrho_l)^3} - \sqrt{(\xi_l \varrho_k)^3} \right) \\ &\leq \sum_{k=1}^{m}\sum_{l=1}^{m} \sqrt{\xi_k \xi_l (\xi_k \varrho_l + \xi_l \varrho_k)} \\ &\leq 1. \end{aligned} \tag{5.7}$$

Proof. Using the function $\Psi(\varrho) = -\sqrt{\varrho},\ \varrho > 0$ in (2.1), we arrive at (5.7).

6. Conclusions

In this research work, we have refined Jensen's inequality using an interesting technique associated with convexity and the Hermite-Hadamard inequality. Utilizing these refinements, we established new relationships involving power and quasi-arithmetic mean inequalities. Furthermore, we extended our main results to obtain improved forms of Hölder inequality through appropriate substitutions within the obtained refinements. In addition, several applications of the proposed refinements were provided in the field of information theory, presenting new relations for the Csiszár and Kullback-Leibler divergences, Shannon entropy, and the Bhattacharyya coefficient. Using this approach, refinements of the integral version of Jensen's inequality, the Jensen-Mercer inequality, and other related inequalities may also be obtained.

Author contributions

M. A. Khan: Supervision, Methodology, Validation; S. Sahar and H. Ullah: Investigation, Writing–original draft; N. N. Fang and K. A. Alnowibet: Validation, Writing–review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. A. Azar, Jensen's inequality in finance, *Int. Adv. Econ. Res.*, **14** (2008), 433–440.
2. M. W. Akram, S. Iqbal, A. Fahad, Y. H. Wang, Hermite-Hadamard-type inequalities for h-Godunova-Levin convex fuzzy interval-valued functions via Riemann-Liouville fractional q-integrals, *Fractal Fract.*, **9** (2025), 1–23. https://doi.org/10.3390/fractalfract9090578
3. I. Ansari, K. A. Khan, A. Nosheen, Đ. Pečarić, J. Pečarić, Estimation of divergence measures via weighted Jensen inequality on time scales, *J. Inequal. Appl.*, **2021** (2021), 93.
4. A. Basir, M. A. Khan, H. Ullah, Y. Almalki, S. Chasreechai, T. Sitthiwirattham, Derivation of bounds for majorization differences by a novel method and its applications in information theory, *Axioms*, **12** (2023), 1–17. https://doi.org/10.3390/axioms12090885

5. M. Denny, The fallacy of the average: on the ubiquity, utility and continuing novelty of Jensen's inequality, *J. Exp. Biol.*, **220** (2017), 139–146. https://doi.org/10.1242/jeb.140368
6. S. S. Dragomir, A new refinement of Jensen's inequality in linear spaces with applications, *Math. Comput. Model.*, **52** (2010), 1497–1505. https://doi.org/10.1016/j.mcm.2010.05.035
7. S. S. Dragomir, An improvement of Jensen's inequality, *Bull. Math. Soc. Sci. Math. Roumanie*, **34** (1990), 291–298.
8. S. S. Dragomir, A further improvement of Jensen's inequality, *Tamkang J. Math.*, **25** (1994), 29–36. https://doi.org/10.5556/j.tkjm.25.1994.4422
9. S. S. Dragomir, A refinement of Jensen's inequality with applications for f-divergence measures, *Taiwanese J. Math.*, **14** (2010), 153–164. https://doi.org/10.11650/twjm/1500405733
10. L. Horváth, Integral Jensen-Mercer and related inequalities for signed measures with refinements, *Mathematics*, **13** (2025), 1–19. https://doi.org/10.3390/math13030539
11. L. Horváth, Refining the integral Jensen inequality for finite signed measures using majorization, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM)*, **118** (2024), 129. https://doi.org/10.1007/s13398-024-01627-7
12. J. H. Kim, Further improvement of Jensen inequality and application to stability of time-delayed systems, *Automatica*, **64** (2016), 121–125. https://doi.org/10.1016/j.automatica.2015.08.025
13. M. A. Khan, G. A. Khan, T. Ali, A. Kilicman, On the refinement of Jensen's inequality, *Appl. Math. Comput.*, **262** (2015), 128–135. https://doi.org/10.1016/j.amc.2015.04.012
14. T. X. Li, D. Acosta-Soba, A. Columbu, G. Viglialoro, Dissipative gradient nonlinearities prevent δ-formations in local and nonlocal attraction-repulsion chemotaxis model, *Stud. Appl. Math.*, **154** (2025), e70018. https://doi.org/10.1111/sapm.70018
15. T. X. Li, Y. V. Rogovchenko, Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations, *Monatsh. Math.*, **184** (2017), 489–500. https://doi.org/10.1007/s00605-017-1039-9
16. J. Pečarić, S. S. Dragomir, A refinements of Jensen inequality and applications, *Stud. Univ. Babes Bolyai Math.*, **24** (1989), 15–19.
17. Đ. Pečarić, J. Pečarić, M. Rodić, About the sharpness of the Jensen inequality, *J. Inequal. Appl.*, **2018** (2018), 337. https://doi.org/10.1186/s13660-018-1923-4
18. H. Ullah, M. Adil Khan, T. Saeed, Determination of bounds for the Jensen gap and its applications, *Mathematics*, **9** (2021), 1–29. https://doi.org/10.3390/math9233132
19. M. Vivas-Cortez, S. Ramzan, M. U. Awan, M. Z. Javed, A. G. Khan, M. A. Noor, I.V-CR-γ-convex functions and their application in fractional Hermite-Hadamard inequalities, *Symmetry*, **15** (2023), 1–25. https://doi.org/10.3390/sym15071405
20. Y. H. Wang, U. Asif, M. Z. Javed, M. U. Awan, A. Kashuri, O. M. Alsalami, Super-quadratic stochastic processes with fractional inequalities and their applications, *Fractal Fract.*, **9** (2025), 1–29. https://doi.org/10.3390/fractalfract9100627