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Research article

A review of unit continuous probability distributions

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Abstract: Unit continuous probability distributions play a fundamental role in modeling variables bounded within the interval $[0, 1]$, such as proportions and probabilities. In recent decades, there has been a significant increase in the development of new parametric families of these distributions. In this work, we present a comprehensive and up-to-date review of more than one hundred unit continuous distributions, including classical models, such as the beta and Kumaraswamy distributions, along with their various extensions. We examined key statistical properties such as moments and demonstrated the practical effectiveness of twelve selected distributions through applications to nine distinct datasets, thereby highlighting their flexibility in modeling a wide range of data types. To the best of our knowledge, this is the most extensive review focused specifically on unit distributions and is a valuable reference for researchers and practitioners.

Keywords: cumulative distribution function; moments; probability density function; unit distributions
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1. Introduction

Probability distributions defined on the interval $[0, 1]$ [37] have numerous practical applications across fields. One fundamental distribution in this domain is the uniform distribution, commonly used as a non-informative prior in Bayesian statistics. Due to its properties, it assumes that all outcomes within the interval are equally probable, which is essential when there is no prior knowledge about the parameter of interest. Applications in clinical trials frequently leverage this uniformity to model uncertainty and formulate inferential statistics [37]. For instance, when assessing treatment effects in randomized controlled trials, researchers can utilize uniform priors to represent their lack of prior beliefs about the likelihood of particular outcomes. Furthermore, uniform distributions play a significant role in reliability analysis and lifetime data modeling, providing a foundational base for the construction of other distributions, such as the beta distribution, which is commonly utilized for modeling proportions [116].

Another critical application of probability distributions on $[0, 1]$ [37] is evident in optimization and machine learning. In the realm of sensitivity analysis, the uniform distribution supports algorithms to explore parameter spaces, enhancing the understanding of system behavior [40]. This is particularly vital in complex systems where the relationships between variables are non-linear and intricate. Moreover, the use of the uniform distribution is central in the development of methods for generating random samples in Monte Carlo simulations, which have diverse applications ranging from financial modeling to environmental studies [82]. The generation of random numbers uniformly distributed over $[0, 1]$ [37] is foundational for sampling-based methods, including Bayesian computation and Latin hypercube sampling, which enable effective exploration of multidimensional input spaces [84]. This breadth of applications underscores the integral role that probability distributions on the interval $[0, 1]$ [37] play in modern statistical practice.

The beta distribution is defined by two shape parameters that control the skewness, and kurtosis, making it adaptable to various types of data. Other notable unit distributions include the Kumaraswamy distribution, which, like the beta distribution, is defined on the [0,1] interval but offers simpler cumulative distribution function (CDF) and probability density function (PDF) expressions, and the Dirichlet distribution, a multivariate generalization of the beta distribution used in Bayesian statistics. In recent years, several generalizations and modifications of these traditional distributions have been proposed to better model complex data. Some of these distributions include the generalized five parameter beta distribution in [143], the rectangular beta distribution in [75], the log gamma distribution in [48], the modified power function distribution in [138], the unit generalized half-normal distribution in [99], the unit log-Xgamma distribution in [18], and the unit gamma distribution in [71], among others. Each of these distributions offers unique advantages and greater flexibility in fitting data with specific characteristics.

Given the growing interest and recent development in unit continuous probability distributions and their applications, we believe it is timely to provide a comprehensive review of these distributions and their modifications. We also believe that such a review could be an important reference, promote broader application of unit probability distributions, and encourage further developments in this field. Hence, we aim to provide a comprehensive overview of the most widely used unit continuous probability distributions.

Thus, in this paper, we systematically review the properties, including the PDF, CDF, moments, and other relevant characteristics of various unit continuous probability distributions. In doing so, we focus exclusively on the basic unit distributions and deliberately exclude their numerous generalizations (for example, inflated forms, transmuted versions, power-type modifications, etc.). Although such extensions are mathematically valid and interesting, each unit distribution can, in principle, be generalized in uncountably many ways. Including them would broaden the scope beyond the stated purpose of this review and make the exposition unnecessarily long and less clear. Our aim is therefore to provide a clear and coherent review of the foundational unit distributions, leaving aside those arising from transmutation, inflation, or other modification techniques.

The remainder of this paper is organized as follows: In Section 2, we provide the collection of these unit distributions and their modifications. In Section 3, we discuss a real data application that compares several of the reviewed distributions. Finally, the concluding remarks are given in Section 4, and potential areas for future research are suggested. A collection of special functions used in Section 2 is provided in the appendix.

2. The collection of unit models

2.1. Beta distribution

This distribution is one of the most popular probability distributions in statistical literature with a wide range of applications in different areas. For example, in Bayesian inference, this distribution is often employed as a conjugate prior to the following probability distributions: Bernoulli, binomial, negative binomial, and geometric distributions [29, 50, 51, 90, 114]. The invention of this distribution dates back to 1676 in a letter from Sir Isaac Newton to Henry Oldenbeg. A standard PDF of the beta distribution is defined by

$$f(x) = \frac{x^{\alpha}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \tag{2.1}$$

for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$ are the shape parameters. The corresponding CDF is given by

$$\begin{aligned} F(x) &= \frac{1}{B(\alpha,\beta)} \int_0^x w^{\alpha-1}(1-w)^{\beta-1} dw \\ &= \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)} \\ &= I_x(\alpha,\beta). \end{aligned}$$

We note here that if α and β are integers, the CDF can be evaluated as the binomial sum given by

$$F(x) = 1 - \sum_{i=0}^{\alpha-1} \binom{\alpha+\beta-1}{i} x^i (1-x)^{\alpha+\beta-1-i}.$$

In addition, the mean, variance, and kth moment associated with the beta distribution are

$$E(X) = \frac{\alpha}{\alpha+\beta},$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

and

$$E\left(X^k\right) = \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+k)} = \frac{(\alpha)_k}{(\alpha+\beta)_k},$$

respectively, for $k \geq 1$. The PDF can be monotonically increasing, monotonically decreasing, bathtub shaped, or unimodal. The distribution has zero skewness if $\alpha = \beta$, positive skewness $\alpha < \beta$, and negative skewness if $\alpha > \beta$.

The beta distribution is widely used in fields that involve modeling probabilities and proportions because it is defined on the interval $[0, 1]$ and can take on a variety of shapes depending on its parameters. In Bayesian statistics, it is a conjugate prior to the binomial and Bernoulli distributions, enabling for the updating of beliefs about probabilities as new data is observed. In project management, it is applied in PERT (Program Evaluation and Review Technique) to model the uncertainty in task completion times. The distribution also finds use in reliability engineering to represent the probability of system success, in finance for modeling asset return probabilities or risk, and in quality control to model the proportion of defective items. Additionally, it is employed in machine learning and A/B testing to estimate success probabilities and in population genetics to describe allele frequencies.

2.2. Arcsine distribution

The arcsine distribution is a notable model in statistics and probability theory and has been studied for over a century. It a special case of the beta distribution with $\alpha = \beta = \frac{1}{2}$. This implies that if X is distributed according the arcsine distribution, then $X \sim B\left(\frac{1}{2}, \frac{1}{2}\right)$. The name of this distribution arises from the fact that the inverse of sin or hyperbolic sine function is a component of the CDF of this model. There are numerous applications for the arcsine distribution across disciplines, including actuarial sciences, Brownian motion, fiducial inference, statistical linguistics, stochastic processes, thermal calibration systems. The PDF and CDF of the standard arcsine distribution are

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}},$$

and

$$F(x) = \frac{2}{\pi}\arcsin\left(\sqrt{x}\right),$$

respectively, for $0 < x < 1$. In addition, the kth moment is

$$E\left(X^k\right) = \frac{1}{\pi}B\left(k+\frac{1}{2}, \frac{1}{2}\right),$$

for $k \geq 1$. In particular, the mean and variance are

$$E(X) = \frac{1}{2},$$

and

$$Var\left(X\right)=\frac{1}{8},$$

respectively. The PDF is symmetric around 0. The skewness is 0, and excess kurtosis is $-\frac{3}{2}$.

The arcsine distribution is applied in fields where probabilities are concentrated near the extremes of an interval than around the mean. It is commonly used in physics to model random processes such as the fraction of time a particle spends on one side of a barrier or the proportion of time a Brownian particle remains positive. In finance, it can describe the distribution of time an asset's price stays above or below a reference level. Additionally, it appears in reliability analysis, signal processing, and queuing theory, particularly in scenarios involving extreme events, persistence times, or boundary-crossing phenomena, where outcomes near the endpoints are more probable than those near the center.

2.3. Type I noncentral beta distribution

A type I noncentral beta distribution is a distribution that arises from the ratio $X=\frac{S}{S+T}$, where S is distributed according to noncentral χ^2 with 2α degrees of freedom and a positive noncentrality parameter δ, and T is distributed according to the central χ^2 distribution with 2β degrees of freedom. This model is a generalization of the beta distribution and this extension is advantageous in many ways. For example, unlike the standard beta distribution, it properly describes portions of the data that have values close to zero and one. The PDF and CDF of X, represented as a Poisson mixture in [177], are

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty}\frac{\Gamma\left(\alpha+\beta+i\right)}{\Gamma\left(\alpha+i\right)}\frac{x^{i+\alpha-1}(1-x)^{\beta-1}u_i}{\Gamma(\beta)} \\ &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\beta)}\sum_{i=0}^{\infty}\frac{\Gamma\left(\alpha+\beta+i\right)}{\Gamma\left(\alpha+i\right)}u_i x^i, \end{aligned} \tag{2.2}$$

and

$$F\left(x\right)=\sum_{i=0}^{\infty}u_i I_x(\alpha+i,\beta), \tag{2.3}$$

respectively, where $0<x<1, \beta>0$ and $\alpha>0$ are positive shape parameters, and

$$u_i=\frac{\left(\frac{\delta}{2}\right)^i\exp\left(-\frac{\delta}{2}\right)}{i!},$$

represents the probability mass function (PMF) of the Poisson distribution with $\frac{\delta}{2}$ as the parameter. Furthermore, the mean and variance associated with (2.2) and (2.3) are

$$E(X)=\exp\left(-\frac{\delta}{2}\right)\sum_{i=0}^{\infty}\frac{\delta^i}{2^i i!}\frac{\alpha+i}{\alpha+\beta+i},$$

and

$$Var\left(X\right)=\exp\left(-\frac{\delta}{2}\right)\sum_{i=1}^{\infty}\frac{\delta^i}{2^i i!}\frac{(\alpha+i)(\alpha+i+1)}{(\alpha+\beta+i)(\alpha+\beta+i+1)}-\left[E(X)\right]^2,$$

respectively. These can be more elegantly expressed in terms of the hypergeometric identities as

$$E(X) = \frac{\alpha \exp\left(-\frac{\delta}{2}\right)}{\alpha+\beta}\, {}_2F_2\left(\alpha+1, \alpha+\beta; \alpha, \alpha+\beta+1; \frac{\delta}{2}\right),$$

and

$$Var(X) = \frac{\alpha(\alpha+1)\exp\left(-\frac{\delta}{2}\right)}{(\alpha+\beta)(\alpha+\beta+1)}\, {}_2F_2\left(\alpha+2, \alpha+\beta; \alpha, \alpha+\beta+2; \frac{\delta}{2}\right) - \left[E(X)\right]^2.$$

In addition, the kth moment about zero associated with (2.2) is

$$E\left(X^k\right) = \frac{(\alpha)_k}{(\alpha+\beta)_k} \exp\left(-\frac{\delta}{2}\right) {}_2F_2\left(\alpha+k, \alpha+\beta; \alpha, \alpha+\beta+k; \frac{\delta}{2}\right),$$

for $k \geq 1$.

The type I noncentral beta distribution is applied in statistical and engineering contexts where modeling ratios of dependent or non-independent quantities is required. It is commonly used in reliability analysis, quality control, and signal processing, particularly in cases involving noncentral F or t statistics, where nonzero means or effects are present. In Bayesian statistics, it is a prior or posterior for proportions when prior information suggests a deviation from a purely central distribution. Additionally, it appears in econometrics and biometrics for modeling proportions or rates that are influenced by underlying noncentrality parameters, enabling more flexibility than the standard beta distribution in capturing skewness and asymmetry caused by external factors or prior effects.

2.4. *Type II noncentral beta distribution*

A type II noncentral beta distribution is a distribution that arises from the ratio $X = \frac{S}{S+T}$, where S is distributed according to the central χ^2 with 2α degrees of freedom, and T is distributed according to the noncentral χ^2 distribution with 2β degrees of freedom and a positive noncentrality parameter δ. Using the Poisson mixture representation in Tang [177], the CDF of X can be expressed as

$$F(x) = \sum_{i=0}^{\infty} u_i I_x(\alpha, \beta+i), \tag{2.4}$$

for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$ are shape parameters, and

$$u_i = \frac{\left(\frac{\delta}{2}\right)^i \exp\left(-\frac{\delta}{2}\right)}{i!}.$$

Then, the PDF of $(1-X)$ corresponding to (2.4) takes the same form as (2.2) with the shape parameters, or the degrees of freedom reversed. The kth moment about zero associated is

$$E\left(X^k\right) = \frac{(\alpha)_k}{(\alpha+\beta)_k} \exp\left(-\frac{\delta}{2}\right) {}_1F_1\left(\alpha+\beta; \alpha+\beta+k; \frac{\delta}{2}\right).$$

In particular, the mean and variance are

$$E\left(X\right)=\frac{\alpha}{\alpha+\beta}\exp\left(-\frac{\delta}{2}\right){}_1F_1\left(\alpha+\beta;\alpha+\beta+1;\frac{\delta}{2}\right),$$

and

$$Var\left(X\right)=\frac{(\alpha)_2}{(\alpha+\beta)_2}\exp\left(-\frac{\delta}{2}\right){}_1F_1\left(\alpha+\beta;\alpha+\beta+2;\frac{\delta}{2}\right)-\left[E(X)\right]^2,$$

respectively.

The type II noncentral beta distribution is applied in several advanced statistical and engineering contexts where modeling ratios of random variables with noncentrality is required. It is particularly useful in multivariate statistical analysis, such as in the distributions of eigenvalues of certain random matrices and in hypothesis testing, including the distribution of test statistics in multivariate analysis of variance (MANOVA), and generalized likelihood ratio tests. In signal processing and communications, it arises in modeling the signal-to-noise ratio under noncentral chi-square assumptions, relevant for radar, wireless communication, and reliability analysis. Additionally, it is applied in Bayesian inference for prior and posterior modeling when dealing with ratios of noncentral chi-square variables, and in quality control and reliability engineering, where it can describe performance ratios or failure probabilities under noncentral conditions. Its flexibility in handling noncentrality makes it valuable wherever asymmetry or shifted distributions of ratios are present.

2.5. Doubly noncentral beta distribution

Suppose S ~noncentral-$\chi^2_{(2\alpha)}$ with noncentrality parameter δ_1, and T ~noncentral-$\chi^2_{(2\beta)}$ with noncentrality parameter δ_2, then

$$X=\frac{S}{S+T},$$

is said to have the doubly noncentral beta distribution with shape parameters α,β, and noncenrality parameters δ_1,δ_2. Using the Poisson mixture representation, we can write the PDF of the doubly noncentral beta distribution as

$$f(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_iv_j\frac{x^{\alpha+i-1}(1-x)^{\beta+j-1}}{B\left(\alpha+i,\beta+j\right)},\tag{2.5}$$

for $0<x<1$, where $\alpha>0$ and $\beta>0$ are shape parameters. Using the perturbation representation in Ongaro and Orsi [140], we can rewrite Eq (2.5) as

$$f(x)=\text{Beta}(x;\alpha,\beta).\exp\left(\frac{\delta_1+\delta_2}{2}\right)\Psi_2\left(\alpha+\beta,\alpha,\beta;\frac{\delta_1}{2}x,\frac{\delta_2}{2}(1-x)\right).\tag{2.6}$$

The corresponding CDF is given by

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_iv_jI_x\left(\alpha+i,\beta+j\right),$$

for $0 < x < 1$, where

$$u_i = \frac{\left(\frac{\delta_1}{2}\right)^i \exp\left(-\frac{\delta_1}{2}\right)}{i!}, \qquad v_j = \frac{\left(\frac{\delta_2}{2}\right)^j \exp\left(-\frac{\delta_2}{2}\right)}{j!},$$

and

$$\mathrm{Beta}(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}.$$

The kth moment corresponding to (2.6) is

$$E\left(X^k\right) = \frac{(\alpha)_k}{(\eta^*)_k} \exp\left(-\frac{\delta^*}{2}\right) \sum_{i=0}^{k} \frac{\binom{k}{i} (\eta^*)_i \left(\frac{\delta_1}{2}\right)^i}{(\alpha)_i (\eta^*+k)_i}\, {}_1F_1\left(\eta^*+i;\eta^*+k+i;\frac{\delta^*}{2}\right),$$

for $k \geq 1$, where $\delta^* = \delta_1 + \delta_2$ and $\eta^* = \alpha + \beta$. Specifically, the mean and variance are [141]

$$E(X) = \frac{\alpha}{\eta^*} \exp\left(-\frac{\delta^*}{2}\right)\left[{}_1F_1\left(\eta^*;\eta^*+1;\frac{\delta^*}{2}\right) + \frac{\eta^* \frac{\delta_1}{2}}{\alpha(\eta^*+1)}\, {}_1F_1\left(\eta^*+1;\eta^*+2;\frac{\delta^*}{2}\right)\right],$$

and

$$Var\,(X) = \frac{(\alpha)_2}{(\eta^*)_2} \exp\left(-\frac{\delta^*}{2}\right)\left[{}_1F_1\left(\eta^*;\eta^*+2;\frac{\delta^*}{2}\right) + \frac{\eta^*\delta_1}{\alpha(\eta^*+2)}\, {}_1F_1\left(\eta^*+1;\eta^*+3;\frac{\delta^*}{2}\right)\right.$$
$$\left. + \frac{(\eta^*)_2 \left(\frac{\delta_1}{2}\right)^2}{(\alpha)_2\,(\eta^*+2)_2}\, {}_1F_1\left(\eta^*+2;\eta^*+4;\frac{\delta^*}{2}\right)\right] - [E(X)]^2,$$

respectively.

The doubly noncentral beta distribution is applied in fields where ratios of noncentral chi-square variables naturally arise, particularly in statistics, engineering, and reliability analysis. It is commonly used in the study of the distribution of the F statistic in the presence of noncentrality parameters, making it relevant for power analysis and hypothesis testing under non-null conditions. In reliability engineering, it models the proportion of life consumed or stress experienced when system components follow noncentral chi-square distributions. Additionally, it appears in Bayesian analysis as a prior or posterior distribution for parameters constrained between 0 and 1, especially when incorporating noncentrality effects, and in quality control, signal processing, and risk assessment where variability and asymmetry need to be captured more accurately than the standard beta distribution enables.

2.6. Libby and Novick's generalized beta distribution

Libby and Novick [113] generalized three parameter beta distribution is specified by the PDF and CDF

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)\,[1-(1-\lambda)x]^{\alpha+\beta}}, \tag{2.7}$$

and

$$F(x) = I_{\frac{\lambda x}{1+\lambda x - x}}(\alpha, \beta),$$

respectively, for $0 < x < 1$, where $\alpha > 0, \beta > 0$ are shape parameters, and $\lambda > 0$ is a scale parameter. It is important to mention that (2.7) reduces to (2.1) when $\lambda = 1$. In addition, the kth moment of the generalized three parameter beta distribution is

$$E\left(X^k\right) = \frac{\lambda^\alpha \Gamma(\alpha+\beta)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(\alpha+\beta+k)}\,{}_2F_1(\alpha+k, \alpha+\beta; \alpha+\beta+k; 1-\lambda).$$

The kth moment can be rewritten by utilising features of the Gauss hypergeometric function as

$$E\left(X^k\right) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)\Gamma(\alpha+\beta+k)}\,{}_2F_1\left(\alpha+k, k; \alpha+\beta+k; 1-\frac{1}{\lambda}\right),$$

if $\lambda \geq 1$ and

$$E\left(X^k\right) = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)\Gamma(\alpha+\beta+k)}\,{}_2F_1(\beta, k; \alpha+\beta+k; 1-\lambda),$$

if $\lambda < 1$. In particular, the mean and variance are

$$E(X) = \frac{\lambda^\alpha \Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\,{}_2F_1(\alpha+1, \alpha+\beta; \alpha+\beta+1; 1-\lambda),$$

and

$$Var(X) = \frac{\lambda^\alpha \Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)}\,{}_2F_1(\alpha+2, \alpha+\beta; \alpha+\beta+2; 1-\lambda) - [E(X)]^2,$$

respectively. Libby and Novick's [113] generalized beta distribution is applied in fields requiring flexible probability models due to its ability to capture a broad range of distributional shapes, including skewness, and kurtosis variations. It is commonly used in Bayesian statistics as a prior distribution, particularly in modeling proportions, rates, and reliability data. In survival analysis and reliability engineering, it provides a versatile model for lifetimes and failure times, accommodating increasing and decreasing hazard rates. In economics and finance, it has been applied to model income distributions, risk assessment, and asset returns, while in environmental sciences and hydrology, it is useful for modeling rainfall, flood frequencies, and other naturally bounded phenomena. Its adaptability to diverse data patterns makes it a powerful tool in theoretical research and applied statistical modeling.

2.7. Gauss hypergeometric distribution

A random variable X is said to have the Gauss hypergeometric distribution suggested by Armero and Bayarri [23] if the PDF is given by

$$f(x) = C\frac{x^\alpha(1-x)^{\beta-1}}{(1+zx)^\gamma}, \tag{2.8}$$

for $0 < x < 1$, where $\alpha > 0, \beta > 0, -\infty < \gamma < \infty$ are shape parameters, and $z > -1$ is a scale parameter. C is the proportionality constant given by

$$\frac{1}{C} = B(\alpha,\beta)\ {}_2F_1(\gamma,\alpha,\alpha+\beta;-z).$$

If $\gamma = 0$ or $z = 0$, then (2.8) will deplete to the standard beta distribution. The CDF corresponding to (2.8) is

$$F(x) = \int_0^x f(t)dt = \int_0^x C\frac{t^\alpha(1-t)^{\beta-1}}{(1+zt)^\gamma}dt.$$

The kth moment of X is

$$E\left(X^k\right) = \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}\frac{{}_2F_1(\gamma,\alpha+k;\alpha+\beta+k;-z)}{{}_2F_1(\gamma,\alpha;\alpha+\beta;-z)}.$$

In particular, the mean and variance are

$$E(X) = \frac{B(\alpha+,\beta)}{B(\alpha,\beta)}\frac{{}_2F_1(\gamma,\alpha+k;\alpha+\beta+1;-z)}{{}_2F_1(\gamma,\alpha;\alpha+\beta;-z)},$$

and

$$Var(X) = \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)}\frac{{}_2F_1(\gamma,\alpha+k;\alpha+\beta+2;-z)}{{}_2F_1(\gamma,\alpha;\alpha+\beta;-z)} - [E(X)]^2,$$

respectively. The PDF can be bathtub shaped, unimodal, decreasing-increasing-decreasing, or monotonically increasing. The Gauss hypergeometric distribution is applied mostly in Bayesian reliability and survival analysis, particularly for modeling failure times and repairable systems where prior information is incorporated in a flexible way. It is useful in predictive inference for survival data, Bayesian testing of exponentiality, and in constructing objective priors for hazard rates in reliability studies. The distribution's tractability and connection with the beta and negative binomial families make it valuable for lifetime modeling, system reliability assessment, and medical survival studies, especially in contexts requiring a balance between informative priors and noninformative (objective) Bayesian approaches.

2.8. Confluent hypergeometric distribution

A random variable X is said to have the confluent hypergeometric distribution proposed by Gordy [69] if its PDF is given by

$$f(x) = Cx^{a-1}(1-x)^{b-1}\exp(-cx),$$

for $0 < x < 1$, where $a > 0, b > 0$ are shape parameters, and $c > 0$ is a scale parameter. C is the proportionality constant given by

$$\frac{1}{C} = B(a,b)\ {}_1F_1(a;a+b;-c).$$

The kth moment of X is

$$E\left(X^k\right) = \frac{B(k+a,b)\ {}_1F_1\,(k+a;k+a+b;-c)}{B(a,b)\ {}_1F_1\,(a;a+b;-c)}.$$

In particular, the mean and variance are

$$E\,(X) = \frac{a}{a+b}\frac{{}_1F_1\,(1+a;1+a+b;-c)}{{}_1F_1\,(a;a+b;-c)},$$

and

$$Var\,(X) = \frac{a(a+1)}{(a+b)(a+b+1)}\frac{{}_1F_1\,(2+a;2+a+b;-c)}{{}_1F_1\,(a;a+b;-c)} - [E(X)]^2\,,$$

respectively. The PDF can be bathtub shaped, unimodal, increasing-decreasing-increasing, or decreasing-increasing-decreasing. The confluent hypergeometric distribution has application areas mostly in problems where overdispersion, or clustering arises beyond what the classical hypergeometric, or binomial models can capture. It has been used in statistical modeling of rare events, reliability studies, genetics, epidemiology, and ecology, particularly when dealing with sampling without replacement under conditions that involve unequal selection probabilities, or population heterogeneity. In reliability and survival analysis, the confluent hypergeometric distribution is useful for modeling lifetimes with varying failure rates; in genetics and ecology, it captures the distribution of alleles, or species counts in heterogeneous environments; and in epidemiology, it helps describe disease incidence where clustering occurs. Its flexibility makes it valuable for situations where classical discrete distributions fail to adequately represent the data structure.

2.9. Lagrangian beta distribution

A random variable X is said to have the Lagrangian beta distribution if its PDF and CDF are

$$f(x) = \sum_{k=0}^{r-1} \frac{n}{n+\beta k}\binom{n+\beta k}{k}\alpha(\alpha x)^k(1-\alpha x)^{n+\beta k-k-1}, \tag{2.9}$$

and

$$F(x) = 1 - \sum_{k=0}^{r-1} \frac{n}{n+\beta k}\binom{n+\beta k}{k}\alpha(\alpha x)^k(1-\alpha x)^{n+\beta k-k},$$

respectively, for $0 < x < 1$, where n, r are positive integers, $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. It is not difficult to see that (2.9) is a linear sum of the standard beta PDFs, which arise as the inter-arrival distribution in a binomial process and has been studied in detail by Patel and Khatri [142]. It encompasses some other distributions; for example, if $r = 1$ and $n \to \infty$, then for any value of β, (2.9) reduces to the standard exponential with the PDF

$$f(x;n,\alpha,\beta,1) = \vartheta \exp\left(-\vartheta x\right),$$

where $\vartheta = n\alpha$. In the same vein, if $\beta = 1$, (2.9) reduces to

$$f(x; n, \alpha, 1, r) = \frac{\alpha}{B(n,r)}(\alpha x)^{r-1}(1-\alpha x)^{n-1}.$$

The mean and variance are

$$E(X) = \frac{n}{\alpha}\sum_{k=0}^{r-1}\frac{1}{(n+\beta k)(n+\beta k+1)},$$

and

$$Var\,(X) = \frac{2}{n\alpha}E(X) + \frac{2}{\alpha^2}\sum_{k=0}^{r-1}\frac{k(n-\beta)-2}{(n+\beta k)(n+\beta k+1)(n+\beta k+2)} - [E(X)]^2,$$

respectively. The Lagrangian beta distribution is useful in reliability engineering (for modeling component lifetimes and failure probabilities), survival analysis (to represent time-to-event data with bounded support), and environmental and hydrological studies (such as rainfall proportions, river flow shares, or pollutant concentration ratios). It is also applied in finance and risk management, where asset returns, or risk measures are naturally constrained within an interval, and in biostatistics and population genetics, for modeling proportions or rates like gene frequencies and disease prevalence. Additionally, its role in Bayesian statistics as a prior distribution for parameters restricted to [0, 1] further broadens its utility in complex inference problems.

2.10. Binomial-beta mixture distribution

The binomial-beta mixture distribution is found in [157]. Its PDF is

$$f(x) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\frac{x^{\frac{\alpha}{2}+k-1}(1-x)^{\frac{\beta}{2}-1}}{B\left(\frac{\alpha}{2}+k, \frac{\beta}{2}\right)}, \tag{2.10}$$

for $0 < x < 1$, where n is a positive integer and $\alpha > 0, \beta > 0$ are shape parameters. The rth moment corresponding to (2.10) is

$$E\left(X^k\right) = \sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}\frac{\Gamma\left(\frac{\alpha}{2}+k+i\right)\Gamma\left(\frac{\alpha+\beta}{2}+i\right)}{\Gamma\left(\frac{\alpha+\beta}{2}+k+i\right)\Gamma\left(\frac{\alpha}{2}+i\right)},$$

for $k \geq 1$. The mean and variance are

$$E(X) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\frac{\alpha+2k}{\alpha+\beta+2k},$$

and

$$Var\,(X) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\frac{(\alpha+2k)(\alpha+2k+2)}{(\alpha+\beta+2k)(\alpha+\beta+2k+2)} - [E(X)]^2,$$

respectively.

2.11. Complementary beta distribution

The complementary beta distribution is in Jones [89]. Its PDF and CDF are specified by

$$f(x) = \frac{B(\alpha,\beta)}{\left[I_x^{-1}(\alpha,\beta)\right]^{\alpha-1}\left[1-I_x^{-1}(\alpha,\beta)\right]^{\beta-1}}, \tag{2.11}$$

and

$$F(x) = I_x^{-1}(\alpha,\beta),$$

respectively, for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$ are shape parameters, which are symmetrically related by

$$f(x,\alpha,\beta) = f(1-x,\beta,\alpha).$$

$I_x^{-1}(\alpha,\beta)$ represents the inverse of the incomplete beta function ratio. We note that (2.11) encompasses some other notable distributions such as the power function distribution when $\beta = 1$ with parameter $\frac{1}{\alpha}$; the standard beta distribution when $\alpha = 1$ with parameter $\frac{1}{\beta}$; the uniform distribution when $\alpha = \beta = 1$. Also, it is not difficult to see that when $\alpha = \beta = \frac{1}{2}$ and $\alpha = \beta = 2$, (2.11) reduces to

$$f(x) = \frac{\pi}{2}\sin(\pi, x),$$

and

$$f(x) = \frac{1}{3\sqrt{x(1-x)}}\cos\left[\frac{\arcsin(2x-1)}{3}\right],$$

respectively. The mean and variance corresponding to the complementary beta distribution are

$$E(X) = \frac{\beta}{\alpha+\beta},$$

and

$$Var(X) = \frac{2}{\alpha}\frac{B(2\alpha, 2\beta+1)}{B^2(\alpha,\beta)}\, {}_3F_2(\alpha+\beta, 1, 2\alpha; \alpha+1, 2(\alpha+\beta)+1; 1) - [E(X)]^2,$$

respectively. The PDF is unimodal with mode at $x = F^{-1}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)$ if $\alpha < 1, \beta < 1$ and U shaped with antimode at $x = F^{-1}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)$ if $\alpha > 1, \beta >$. If $\alpha < 1$ and $\beta > 1$ the PDF is J shaped, and if $a > 1$ and $b < 1$, the PDF is reverse J shaped.

2.12. Triangular distribution

Triangular distribution is regarded as a proxy to the beta distribution in the statistical literature [88]. As the name implies, the PDF is shaped like a triangle. A random variable X is said to have the triangular distribution if its PDF and CDF are specified as

$$f(x) = \begin{cases} \dfrac{2(x-\alpha)}{(\beta-\alpha)(c-\alpha)}, & \text{for} \quad \alpha \leq x < c, \\ \\ \dfrac{2}{\beta-\alpha}, & \text{for} \quad x = c, \\ \\ \dfrac{2(\beta-x)}{(\beta-\alpha)(\beta-c)}, & \text{for} \quad c < x \leq \beta, \end{cases} \tag{2.12}$$

and

$$F(x) = \begin{cases} \dfrac{(x-\alpha)^2}{(\beta-\alpha)(c-\alpha)}, & \text{for} \quad \alpha < x \leq c, \\ \\ 1 - \dfrac{(\beta-x)^2}{(\beta-\alpha)(\beta-c)}, & \text{for} \quad c < x < \beta, \\ \\ 1, & \text{for} \quad \beta \leq x, \end{cases} \tag{2.13}$$

respectively, where $\alpha > 0$, and $\beta > 0$ are scale parameters. In general, this distribution is governed by three parameters: The minimum parameter α, the maximum parameter β, and the peak parameter c. By changing these parameters, we can obtain a variety of unit triangular distributions. For example,

2.12.1. Case I

If $\alpha = 0, \beta = 1$ and $c = 1$ (2.12) and (2.13) reduces to

$$\left.\begin{array}{r} f(x) = 2x, \\ F(x) = x^2, \end{array}\right\} \tag{2.14}$$

for $0 < x < 1$. For this case, the mean and variance corresponding to (2.14) are

$$\begin{aligned} E(X) &= \frac{2}{3}, \\ Var(X) &= \frac{1}{18}. \end{aligned}$$

2.12.2. Case II

If $S_i, i = 1, 2$ are independent uniform random variables, then the absolute difference

$$X = |S_1 - S_2|,$$

has a triangular distribution with $\alpha = 0, \beta = 1$ and $c = 0$. The PDF and CDF of X become

$$\left.\begin{array}{r} f(x) = 2 - 2x, \\ F(x) = 2x - x^2, \end{array}\right\} \tag{2.15}$$

for $0 < x < 1$. For this case, the mean and variance corresponding to (2.15) are

$$\begin{aligned} E(X) &= \frac{1}{3}, \\ Var(X) &= \frac{1}{18}. \end{aligned}$$

2.12.3. Case III

If $S_i, i = 1, 2$ are independent uniform random variables on $[0, 1]$, then the distribution of the mean

$$X = \frac{S_1 + S_2}{2},$$

has a triangular distribution with $\alpha = 0, \beta = 1$ and $c = \frac{1}{2}$. The PDF and CDF of X are

$$f(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{2}, \\ 4(1 - x), & \frac{1}{2} \le x \le 1, \end{cases} \tag{2.16}$$

and

$$F(x) = \begin{cases} 2x^2, & 0 \le x < \frac{1}{2}, \\ 2x^2 - (2x - 1)^2, & \frac{1}{2} \le x \le 1, \end{cases}$$

respectively. For this case, the mean and variance corresponding to (2.16) are

$$\begin{aligned} E(X) &= \frac{1}{2}, \\ Var(X) &= \frac{1}{24}. \end{aligned}$$

The triangular distribution is widely applied in situations where data is limited but reasonable estimates of minimum, maximum, and most likely values are available. It is commonly used in project management (for example, PERT and Monte Carlo simulations) for modeling uncertain activity times and costs, in risk analysis for estimating outcomes with sparse data, and in inventory and demand forecasting when only rough estimates of ranges and a likely demand point are known. Additionally, it is applied in reliability engineering, quality control, and decision analysis, where it provides a simple yet flexible way to approximate probability distributions when empirical data is scarce, or incomplete.

2.13. The uniform distribution

If $\alpha < \beta$, a random variable X is said to follow a continuous uniform distribution on the bound (α, β) if and only if the PDF of X is

$$f(x) = \frac{1}{\beta - \alpha},$$

and the corresponding CDF is

$$F(x) = \frac{x - \alpha}{\beta - \alpha},$$

for $0 < x < 1$, where $\beta > \alpha$ are scale parameters. The first moment, kth moment, and the variance of the general form of the uniform distribution are

$$E(X) = \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)},$$

$$E\left(X^k\right) = \frac{\beta^{k+1} - \alpha^{k+1}}{(k+1)(\beta - \alpha)},$$

and

$$Var(X) = \frac{(\beta - \alpha)^2}{12},$$

respectively, where α and β are the parameters. If $\alpha = 0$ and $\beta = 1$ the associated unit version of the model arises. This is also called the standard uniform distribution, i.e., $X \sim u[0, 1]$. An intriguing property of the standard uniform distribution is that $1 - u = v$ has a standard uniform distribution if u does. Among others, this characteristic can be used to create antithetic variates. This property is known as the inversion approach, which enables the generation of random numbers for any other continuous distribution using the continuous standard uniform distribution. Due to the simplicity of the functional form of the standard uniform distribution, it is easy to determine and compute the associated probabilities. As a result, this distribution has many applications and connections to other prominent distributions. For example,

- if $X \sim U(0, 1)$, then $Y = X^n \sim B\left(\frac{1}{n}, 1\right) \implies U(0, 1) \in B(\alpha = 1, \beta = 1)$,
- if $X \sim U(0, 1)$, then $Y = -\log(X) \sim \exp(1)$, or in general $Y = -\lambda^{-1} \log(X) \sim \exp(\lambda)$ with rate parameter λ,
- if $X \sim U(0, 1)$, then $Y = -2\log(X) \sim \chi_2^2$ with 2 degrees of freedom,
- if $X_i, i = 1, \cdots, n$ are independent samples from n different $U(0, 1)$ populations, then

$$P_n = \sum_{i=1}^{n} -2\log\left(X_i\right) \sim \chi_{2n}^2,$$

and is called the Pearson statistic in statistical inference.

The uniform distribution is widely applied in situations where all outcomes within a certain range are equally likely, making it useful for modeling randomness in bounded intervals. In simulation and Monte Carlo methods, it is often used to generate random numbers as a basis for more complex probability distributions. In quality control and reliability testing, it helps model measurement errors, or uncertainty when no outcome is favored over another within specified limits. It is also applied in computer science for randomized algorithms, gaming, and cryptography where fair selection within a range is required. Additionally, in operations research and decision-making, it is used to model uncertain but bounded parameters such as demand, lead times, or project durations when only the range of possible values is known but no further information is available.

2.14. The standard two-sided power distribution

Van Dorp and Kotz [181] proposed an extension of the triangular distribution popularly known as the standard two sided power (STSP) distribution. Its PDF and CDF are

$$f(x) = \begin{cases} n\left(\dfrac{x}{\theta}\right)^{n-1}, & 0 < x \leq \theta, \\ \\ n\left(\dfrac{1-x}{1-\theta}\right)^{n-1}, & \theta \leq x < 1, \end{cases} \tag{2.17}$$

and

$$F(x) = \begin{cases} \theta\left(\dfrac{x}{\theta}\right)^{n}, & 0 \leq x \leq \theta, \\ \\ 1-(1-\theta)\left(\dfrac{1-x}{1-\theta}\right)^{n}, & \theta \leq x \leq 1, \end{cases}$$

respectively, where $0 < \theta < 1$ is a cutoff point and $n > 0$ is a shape parameter. We note that n does not need be an integer. It is important to highlight that the standard two sided power reduces to the uniform $[0, 1]$ when $n = 1$ and simplifies to the triangular distribution on the interval $[0, 1]$ for $n = 2$. Also, if $\theta = 1$, then (2.17) reduces to the power function distribution. Furthermore, the k moment of a standard two sided power distribution is

$$E\left(X^k\right) = \frac{n\theta^{k+1}}{n+k} + \sum_{i=0}^{k}(-1)^i\binom{k}{k-i}\frac{n}{n+i}(1-\theta)^{i+1},$$

for $k \geq 1$. The mean and variance are

$$E(X) = \frac{(n-1)\theta+1}{n+1},$$

and

$$Var(X) = \frac{n-2(n-1)\theta(1-\theta)}{(n+2)(n+1)^2},$$

respectively. It is evident that the mean is a weighted average of the lower bound 0 and the upper bound 1 with the location parameter θ, where n is a single factor acting in the weights determination. In reference [181], Figures 1–3 showed that the PDF can be uniform, triangular, U shaped, positively skewed, negatively skewed, or J shaped. The two-sided power distribution is particularly useful in applications where data exhibit asymmetry, bounded support, and varying tail behaviors that cannot be well captured by standard symmetric distributions. Its major application areas include reliability engineering and risk analysis, where failure times, or risk measures often show skewness; finance and insurance, for modeling asset returns and claim sizes that are bounded yet skewed; environmental and hydrological studies, where variables such as rainfall, wind speeds, or pollutant concentrations display asymmetric bounded distributions; and project management/PERT analysis, as it generalizes the triangular and uniform distributions to better represent expert–elicited uncertain activity times. Overall, this distribution is applied wherever flexible modeling of bounded, skewed, and heavy/light-tailed data is required.

2.15. Uneven two-sided power distribution

For an uneven two-sided power random variable [107], the PDF and CDF are specified by

$$f(x)=\begin{cases} p\dfrac{n_1}{\theta}\left(\dfrac{x}{\theta}\right)^{n_1-1}, & 0\le x<\theta,\\ (1-p)\dfrac{n_3}{1-\theta}\left(\dfrac{1-x}{1-\theta}\right)^{n_3-1}, & \theta\le x\le 1,\end{cases} \tag{2.18}$$

and

$$F(x)=\begin{cases} 0, & x\le 0,\\ p\left(\dfrac{x}{\theta}\right)^{n_1}, & 0<x<\theta,\\ 1-(1-p)\left(\dfrac{1-x}{1-\theta}\right)^{n_3-1}, & \theta\le x<1,\\ 1, & x\ge 1,\end{cases}$$

for $n_1>0, n_3>0$ shape parameters, and $0<\theta<1$ a cutoff point, where p is a mixing probability defined by

$$p=\frac{\alpha\theta n_3}{\alpha\theta n_3+(1-\theta)n_1}=\frac{\theta\alpha n_3}{\theta\left(\alpha n_3-n_1\right)+n_1}.$$

Equation (2.18) reduces to (2.17) for $\alpha=1$. In addition, the kth moment of (2.18) is given by

$$E\left(X^k\right)=p\frac{n_1\theta^k}{n_1+k}+(1-p)\left[n_3\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{(1-\theta)^i}{n_3+i}\right],$$

for $k\ge 1$. In particular, the mean and variance are

$$E(X)=p\frac{n_1\theta}{n_1+1}+(1-p)\frac{n_3\theta+1}{n_3+1},$$

and

$$Var\left(X\right)=p\frac{n_1\left(n_1+1\right)\theta^2}{\left(n_1+2\right)\left(n_1+1\right)}+(1-p)\frac{2+2n_3\theta+n_3\left(n_3+1\right)\theta^2}{\left(n_3+2\right)\left(n_3+1\right)}-\left[E(X)\right]^2,$$

respectively. This distribution has been used to model monthly USA Certificate Deposit rates.

2.16. A new generalized beta distribution

A five parameter beta distribution in [143] has the PDF

$$f(x)=Cx^{\alpha-1}(1-x)^{\beta-1}(1-\sigma x)^{-\rho}\exp(-\eta x), \tag{2.19}$$

for $0 < x < 1$, $\alpha > 0, \beta > 0, -\infty < \rho < \infty$ shape parameters, and $0 \leq \sigma < 1, -\infty < \eta < \infty$ scale parameters, where

$$C^{-1} = B(\alpha,\beta)\Psi_1(\alpha,\rho,\alpha+\beta;\sigma,-\eta).$$

The corresponding CDF is

$$F(x) = Cx^{\alpha} F^{1:1;1;0}_{1:0;0;0}\left[\begin{matrix} (\alpha) : (1-\beta);(\rho);\_; \\ (\alpha+1) : \rule{2em}{0.4pt};\rule{2em}{0.4pt},\_; \end{matrix} x, \sigma x, -\eta x \right],$$

where $F^{1:1;1;0}_{1:0;0;0}[\cdots]$ denotes the generalized Lauricella function, see (Srivastava and Manocha [171, p. 65]). The kth moment of (2.19) is

$$E\left(X^k\right) = CB(\alpha+r,\beta)\Psi_1\left(\alpha+r,\rho,r+\alpha+\beta;\sigma,-\eta\right),$$

for $k \geq 1$. The generalization given by (2.19) encompasses several other distributions. For example,

- if $\eta = 0$, (2.19) reduces to

$$f(x) = C_1 x^{\alpha-1}(1-x)^{\beta-1}(1-\sigma x)^{-\rho}\exp(-\eta x),$$

  for $0 < x < 1$, $\alpha > 0, \beta > 0, -\infty < \rho < \infty$ shape parameters, and $0 \leq \sigma < 1$ a scale parameter, where $C^{-1} = B(\alpha,\beta)\ {}_2F_1\left(\alpha,\rho;\alpha+\beta;\sigma\right)$.
- If $\rho = 0$, or $\sigma = 0$ (2.19) reduces to

$$f(x) = C_2 x^{\alpha-1}(1-x)^{\beta-1}\exp(-\eta x),$$

  for $0 < x < 1$, $\alpha > 0, \beta > 0$ shape parameters, and $-\infty < \eta < \infty$ a scale parameter, where $C_2^{-1} = B(\alpha,\beta)\ {}_1F_1\left(\alpha;\alpha+\beta;-\eta\right)$.
- If $\beta = 1$ (2.19) becomes

$$f(x) = C_3 x^{\alpha-1}(1-\sigma x)^{-\rho}\exp(-\eta x),$$

  for $0 < x < 1$, $\alpha > 0, -\infty < \rho < \infty$ shape parameters, and $0 \leq \sigma < 1, -\infty < \eta < \infty$ scale parameters, where $C_3^{-1} = \frac{1}{\alpha}\ \Psi_1\left(\alpha,\rho,\alpha+1;\sigma,-\eta\right)$.
- If $\beta = 1$ and $\eta = 0$, (2.19) becomes

$$f(x) = C_4 x^{\alpha-1}(1-\sigma x)^{-\rho},$$

  for $0 < x < 1$, $\alpha > 0, -\infty < \rho < \infty$ shape parameters, and $0 \leq \sigma < 1$ a scale parameter, where $C_4^{-1} = \frac{1}{\alpha}\ {}_2F_1\left(\alpha,\rho;\alpha+1;\sigma\right)$.

2.17. Power function distribution

The standard power function distribution has its PDF and CDF specified by

$$f(x) = \beta x^{\beta-1},$$

and

$$F(x) = x^{\beta},$$

respectively, for $0 < x < 1$, where $\beta > 0$ is the shape parameter. The power function distribution has been widely employed to model different data sets arising from real life scenarios, such as income distribution data, lifetime data, and failure processes. Given the simplicity of this model, it has also been generalized in many ways (see [22,109] and references therein). The reliability (survival) function and the hazard rate function are

$$R(x) = 1 - x^{\beta} \quad \text{and} \quad h(x) = \frac{\beta x^{\beta-1}}{1 - x^{\beta}}.$$

The kth moment of the standard power function distribution is given by

$$E\left(X^k\right) = \frac{\beta}{\beta + k},$$

for $k \geq 1$. Consequently,

$$E(X) = \frac{\beta}{\beta + 1},$$

and

$$Var(X) = \frac{\beta}{(\beta + 2)(\beta + 1)^2}.$$

It is important to mention that there are other versions of the power function distribution in the literature. For example,

$$f(x) = \frac{\nu}{1 - \nu} x^{\frac{2\nu - 1}{1 - \nu}},$$

for $0 < x < 1$ and $0 < \nu < 1$. The mean of this version is ν, and the corresponding variance is $\frac{\nu(1-\nu)^2}{2-\nu}$. This version of the power function distribution is a special case of the beta distribution for $\alpha = \frac{\nu}{1-\nu}$ and $\beta = 1$. If $\nu = \frac{1}{2}$, the uniform case arises, and the PDF decreases if $0 < \nu < \frac{1}{2}$, or increases if $\frac{1}{2} < \nu < 1$. The researchers in [176] give other versions of the power function distributions. The power function distribution is mostly applied in fields where data exhibit heavy-tailed, or scale-invariant behavior, making it useful for modeling phenomena where large values occur with non-negligible probability. In economics and finance, it is employed to model wealth distributions, income levels, and stock market fluctuations, while in reliability engineering and survival analysis, it is used to describe lifetimes of components and failure rates under certain stress-strength models. It also applied in hydrology and environmental sciences for modeling extreme events such as flood peaks, or pollutant concentrations, and in physics, biology, and social sciences for systems that follow scaling laws, such as population sizes, word frequencies, and city growth. Its flexibility in representing skewed distributions makes it valuable for risk assessment, resource allocation, and decision-making under uncertainty.

2.18. Unit Gompertz distribution

Suppose Y is a Gompertz random variable with scale parameters α and β. Let $X = \exp(-Y)$. The researchers in [126] showed that the PDF of X is

$$f(x) = \alpha\beta x^{-(\beta+1)} \exp\left[-\alpha\left(x^{-\beta} - 1\right)\right], \tag{2.20}$$

for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters, respectively. The corresponding CDF is given by

$$F(x) = \exp\left[-\alpha\left(x^{-\beta} - 1\right)\right].$$

The kth moment of (2.20) is

$$E\left(X^k\right) = \alpha^{\frac{k}{\beta}} \exp(\alpha)\Gamma\left(1 - \frac{k}{\beta}, \alpha\right),$$

where the moments exists only when $\frac{k}{\beta} < 1$. In particular, the mean and variance are

$$E(X) = \alpha^{\frac{1}{\beta}} \exp(\alpha)\Gamma\left(1 - \frac{1}{\beta}, \alpha\right),$$

and

$$Var(X) = \alpha^{\frac{2}{\beta}} \exp(2\alpha)\,\Gamma\left(1 - \frac{2}{\beta}, \alpha\right) - [E(X)]^2,$$

respectively. Figure 1 in [126] showed that the PDF can be increasing, unimodal, reversed J-shaped, or positively right skewed. This distribution has been applied to model maximum flood level for Susquehanna River in Pennsylvania and measurements of tensile strength of polyester fibers.

2.19. The unit folded normal distribution

Suppose Y is a normal random variable with mean μ and standard deviation σ. Let $X = \left|\tanh(Y)\right|$. The researchers in [104] showed that X has the PDF

$$f(x) = \frac{1}{\sigma\left(1 - x^2\right)}\left[\phi\left(\frac{\text{arctanh}x + \mu}{\sigma}\right) + \phi\left(\frac{\text{arctanh}x - \mu}{\sigma}\right)\right],$$

for $0 < x < 1$, $-\infty < \mu < \infty$ a location parameter, and $\sigma > 0$ a scale parameter. The corresponding CDF is

$$F(x) = \Phi\left(\frac{\text{arctanh}x + \mu}{\sigma}\right) + \Phi\left(\frac{\text{arctanh}x - \mu}{\sigma}\right) - 1,$$

where arctanh$x$ represents the inverse function of tanh$x$. The kth moment is

$$E\left(X^k\right) = \sum_{i=0}^{k}\sum_{l=0}^{\infty}\binom{-k}{i}\binom{-i}{l}(-2)^i\left\{\exp\left[-2(i+l)\mu\right] M_{-\frac{\mu}{\sigma}}\left(-2(i+l)\sigma\right) + \exp\left[2l\mu\right] M_{\frac{\mu}{\sigma}}\left(-2l\sigma\right)\right\},$$

for $k \geq 1$, where $M_a(t) = E\left[\exp(tU)I\{U > a\}\right], t, a \in \mathbb{R}$ is the truncated moment generating function. In reference [104], Figure 1 showed that the PDF possesses a large panel of forms, including increasing, decreasing, bell reversed bathtub, and almost constant shapes. This distribution has been applied to model Better Life Indices data and educational attainment data.

2.20. Unit modified Burr III distribution

Suppose Y is a modified Burr III random variable with shape parameters α, β, and scale parameter γ [14]. Let $X = \frac{Y}{1+Y}$. The researchers in [77] showed that X has the PDF

$$f(x) = \alpha\beta x^{-2}\left(\frac{1-x}{x}\right)^{\beta-1}\left[1+\gamma\left(\frac{1-x}{x}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1},$$

for $0 < x < 1$, where $\alpha > 0$, and $\beta > 0$ are shape parameters, and $\gamma > 0$ is a scale parameter. The corresponding CDF is

$$F(x) = \left[1+\gamma\left(\frac{x}{1-x}\right)^{-\beta}\right].$$

The associated kth moment is

$$E\left(X^k\right) = \alpha\beta\sum_{i=0}^{\infty}(-1)^i(\gamma)^i\binom{\frac{\alpha}{\gamma}+i}{i}B\left(k-\beta(i+1),\beta(i+1)\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \alpha\beta\sum_{i=0}^{\infty}(-1)^i(\gamma)^i\binom{\frac{\alpha}{\gamma}+i}{i}B\left(1-\beta(i+1),\beta(i+1)\right),$$

and

$$Var\left(X\right) = \alpha\beta\sum_{i=0}^{\infty}(-1)^i(\gamma)^i\binom{\frac{\alpha}{\gamma}+i}{i}B\left(2-\beta(i+1),\beta(i+1)\right) - \left[E(X)\right]^2,$$

respectively. In reference [77], Figure 1 showed that the PDF has many different shapes such as bathtub shape, left-skewed (negative skewness), right-skewed (positive skewness). This distribution has been applied to model core samples from petroleum reservoirs and times to infection of kidney dialysis patients.

2.21. Unit log-logistic distribution

Suppose Y is a log-logistic random variable with shape parameter ϕ, and scale parameter α. Let $X = \exp(-Y)$. The researchers in [153] showed that the PDF and CDF of X are

$$f(x) = \frac{\phi}{x\alpha^{\phi}}\left(-\log x\right)^{\phi-1}\left[1+\left(-\frac{1}{\alpha}\log x\right)^{\phi}\right]^{-2},$$

and

$$F(x) = \left[1+\left(-\frac{1}{\alpha}\log x\right)^{\phi}\right]^{-1},$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\phi > 0$ is a shape parameter. In reference [153], Figure 1 showed that the PDF can be monotonically decreasing, unimodal, or bimodal. This distribution has been applied to model proportions of income that is spent on food.

2.22. *Unit Weibull distribution*

Suppose Y is a Weibull random variable with scale parameter α, and shape parameter β. Let $X = \exp(-Y)$. The researchers in [127] showed that the PDF and CDF of X are

$$f(x) = \frac{1}{x}\alpha\beta\,(-\log x)^{\beta-1}\exp\left[-\alpha(-\log x)^{\beta}\right],$$

and

$$F(x) = \exp\left[-\alpha(-\log x)^{\beta}\right],$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The particular case for $\alpha = \beta = 1$ is the standard uniform distribution, $U(0, 1)$. The particular case for $\beta = 1$ is the power function distribution. The particular case for $\beta = 2$ is the unit Rayleigh distribution. The kth moment of X is

$$E\left(X^k\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\alpha^{\frac{n}{\beta}}}\Gamma\left(\frac{n}{\beta}+1\right).$$

When $\beta = 1$, the kth moment is

$$E\left(X^k\right) = \frac{\alpha}{k+\alpha}.$$

The mean and variance in this case are

$$E\,(X) = \frac{\alpha}{1+\alpha},$$

and

$$Var\,(X) = \frac{\alpha}{2+\alpha} - [E(X)]^2,$$

respectively. In reference [127], Figure 1 showed that the PDF can be monotonically increasing, monotonically decreasing, bathtub shaped, or unimodal. In reference [127], Figure 3 showed that skewness can be negative for $\beta \neq 1$. This distribution has been applied to model maximum flood levels for Susquehanna River in Pennsylvania and core samples from petroleum reservoirs.

2.23. *Unit Birnbaum-Saunders distribution*

Let Y be a Birnbaum-Saunders random variable, then by considering the transformation $X = \exp(-Y)$, the unit Birnbaum-Saunders distribution in [125] with the PDF is given by

$$f(x) = \frac{1}{2x\alpha\beta\sqrt{2\pi}}\left[\left(-\frac{\beta}{\log x}\right)^{\frac{1}{2}} + \left(-\frac{\beta}{\log x}\right)^{\frac{3}{2}}\right]\exp\left[\frac{1}{2\alpha^2}\left(\frac{\log x}{\beta} + \frac{\beta}{\log x} + 2\right)\right],$$

for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The corresponding CDF is given by

$$F(x) = 1 - \Phi\left[\frac{1}{\alpha}\left(-\frac{\log x}{\beta}\right)^{\frac{1}{2}} - \left(-\frac{\beta}{\log x}\right)^{\frac{1}{2}}\right],$$

for $0 < x < 1$. The associated kth moment of X is

$$E\left(X^k\right) = \frac{2k\alpha^2\beta + \sqrt{2k\alpha^2\beta+1} + 1}{4k\alpha^2\beta+1} \exp\left(-\frac{\sqrt{2k\alpha^2\beta+1}-1}{\alpha^2}\right),$$

for $k \geq 1$. In particular, the mean and variance are

$$E\left(X\right) = \frac{2\alpha^2\beta + \sqrt{2\alpha^2\beta+1} + 1}{4\alpha^2\beta+1} \exp\left(-\frac{\sqrt{2\alpha^2\beta+1}-1}{\alpha^2}\right),$$

and

$$Var\left(X\right) = \frac{4\alpha^2\beta + \sqrt{4\alpha^2\beta+1} + 1}{8\alpha^2\beta+1} \exp\left(-\frac{\sqrt{4\alpha^2\beta+1}-1}{\alpha^2}\right) - \left[E(X)\right]^2,$$

respectively. In reference [125], Figure 1 reveals that the PDF can be decreasing, or unimodal and, then bathtub shaped. This distribution has been applied to model monthly water capacity data from the Shasta reservoir in California, USA and total milk production in the first cow births.

2.24. *Log-shifted Gompertz distribution*

Let Y be a random variable having a shifted Gompertz (SG) distribution with scale parameter α, and shape parameter $\beta > 0$. By considering the transformation $X = \exp(-Y)$, Jodrá [86] obtained the log-shifted Gompertz (LSG) distribution with the PDF given by

$$f(x) = \beta\left[1 + \alpha\left(1 - x^{\beta}\right)\right] x^{\beta-1} \exp\left(-\alpha x^{\beta}\right), \tag{2.21}$$

for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The associated CDF is given by

$$F(x) = 1 - \left(1 - x^{\beta}\right) \exp\left(-\alpha x^{\beta}\right).$$

It is not difficult to see that Eq (2.21) reduces to the standard power function distribution for $\alpha = 0$, $\beta > 0$ and the uniform distribution for $\alpha = 0, \beta = 1$. The kth moment of X is

$$E\left(X^k\right) = \exp(-\alpha) + \alpha^{-\left(\frac{k}{\beta}+1\right)} \left(\alpha - \frac{k}{\beta}\right) \gamma\left(\frac{k}{\beta} + 1, \alpha\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \exp(-\alpha) + \alpha^{-\left(\frac{1}{\beta}+1\right)} \left(\alpha - \frac{1}{\beta}\right) \gamma\left(\frac{1}{\beta} + 1, \alpha\right),$$

and

$$Var\left(X\right) = \exp(-\alpha) + \alpha^{-\left(\frac{2}{\beta}+1\right)} \left(\alpha - \frac{2}{\beta}\right) \gamma\left(\frac{2}{\beta} + 1, \alpha\right) - \left[E(X)\right]^2,$$

respectively. Jodrá [86, Figure 1] showed that the PDF can be monotonically increasing, monotonically decreasing, or unimodal. This distribution has been applied to model annual percentage of antimicrobial resistant isolates in Portugal and percentages of French speakers in 88 countries.

2.25. Ur Rehman et al.'s unit Fréchet log-logistic distribution

The unit Fréchet log-logistic distribution is a special case of Fréchet log-logistic distribution as found in [179]. The Fréchet log-logistic distribution has its PDF and CDF specified by

$$f(x) = \frac{\beta\theta}{x}\left[\beta \log\left(\frac{\alpha}{x}\right)\right]^{-\theta-1} \exp\left\{-\left[\beta \log\left(\frac{\alpha}{x}\right)\right]^{-\theta}\right\},$$

and

$$F(x) = 1 - \exp\left\{-\left[\beta \log\left(\frac{\alpha}{x}\right)\right]^{-\theta}\right\},$$

respectively, for $0 < x < \alpha$, where $\beta > 0$ is a scale parameter, and $\theta > 0$ is a shape parameter. For $\alpha = 1$, the unit Fréchet log-logistic distribution arises with the PDF given by

$$f(x) = \frac{\beta\theta}{x}\left[\beta \log\left(\frac{1}{x}\right)\right]^{-\theta-1} \exp\left\{-\left[\beta \log\left(\frac{1}{x}\right)\right]^{-\theta}\right\}.$$

The corresponding CDF

$$F(x) = 1 - \exp\left\{-\left[\beta \log\left(\frac{1}{x}\right)\right]^{-\theta}\right\}.$$

The associated kth moment is

$$E\left(X^k\right) = \sum_{i=0}^{\infty} \frac{\alpha^k (-k)^i}{i!} \Gamma\left(1 - \frac{i}{\theta}\right).$$

In particular, the mean and variance are

$$E(X) = \sum_{i=0}^{\infty} \frac{\alpha (-1)^i}{i!} \Gamma\left(1 - \frac{i}{\theta}\right),$$

and

$$Var(X) = \sum_{i=0}^{\infty} \frac{\alpha^2 (-2)^i}{i!} \Gamma\left(1 - \frac{i}{\theta}\right) - [E(X)]^2,$$

respectively. In reference [179], Figures 1 and 2 showed that the PDF can take symmetrical, right skewed, left skewed, L shape, or U shape curves. In reference [179], Figure 4 showed that the distribution can be negatively skewed, positively skewed, or symmetric. Furthermore, the distribution can be leptokurtic, mesokurtic, or platykurtic. This distribution has been applied to model lengths of power failures and death times of patients with cancer of the tongue.

2.26. Unit Gumbel distribution

Suppose Y is a standard Gumbel random variable. Let $X = [1 + \exp(-Y)]^{-\frac{1}{\alpha}}$. Arslan [24] showed that X has the PDF

$$f(x) = \alpha x^{-(\alpha+1)} \exp\left[-\left(x^{-\alpha} - 1\right)\right],$$

for $0 < x < 1$, where $\alpha > 0$ is a shape parameter. The corresponding CDF is

$$F(x) = \exp\left[-\left(x^{-\alpha} - 1\right)\right].$$

The kth moment of the unit Gumbel distribution is given by

$$E\left(X^k\right) = \exp(1) E_{\frac{k}{\alpha}}(1).$$

In particular, the mean and variance are

$$E(X) = \exp(1) E_{\frac{1}{\alpha}}(1),$$

and

$$Var(X) = \exp(1) E_{\frac{2}{\alpha}}(1) - \exp(2)\left[E_{\frac{1}{\alpha}}(1)\right]^2,$$

respectively. Arslan [24, Figure 1] showed that the PDF can take unimodal shapes. Arslan [24, Table 1 and Figure 2] also showed that the distribution is skewed to the right if $\alpha < 1.0295$, skewed to the left if $\alpha > 1.0295$, and symmetric if $\alpha = 1.0295$. This distribution has been applied to model Better Life Indices data and educational attainment data.

2.27. The log-Bilal distribution

Suppose Y is a Bilal random variable with scale parameter θ [6]. Let $X = \exp(-Y)$. The researchers in [17] showed that X has the PDF

$$f(x) = \frac{6}{\theta} x^{\frac{2}{\theta}-1}\left(1 - x^{\frac{1}{\theta}}\right),$$

for $0 < x < 1$, where $\theta > 0$ is a scale parameter. The corresponding CDF is

$$F(x) = 3y^{\frac{2}{\theta}} - 2y^{\frac{3}{\theta}}.$$

The kth moment of X is

$$E\left(X^k\right) = \frac{6}{(k\theta + 2)(k\theta + 3)}.$$

In particular, the mean and variance are

$$E(X) = \frac{6}{(\theta + 2)(\theta + 3)},$$

and

$$Var(X) = \frac{6}{(2\theta + 2)(2\theta + 3)} - \frac{36}{(\theta + 2)^2(\theta + 3)^2},$$

respectively. In reference [17], Figure 1 showed that the PDF can be right skewed, left skewed, or nearly symmetric. This distribution has been applied to model Better Life Indices data and educational attainment data.

2.28. Unit half-logistic geometric distribution

The researchers in [151] proposed a unit half-logistic geometric distribution by using the transformation $X = \exp(-Y)$ on the half logistic-geometric distribution. The unit distribution arising from this has the PDF

$$f(x) = \frac{2\beta}{(\beta + (2-\beta)x)^2},$$

for $0 < x < 1$, where $\beta > 0$ is a scale parameter. The corresponding CDF is

$$F(x) = 1 - \frac{\beta(1-x)}{\beta + (2-\beta)x'}.$$

The kth moment of X is

$$E\left(X^k\right) = \frac{2}{\beta(k+1)}\, {}_2F_1\left(2, k+1; k+2; \frac{\beta-2}{\beta}\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{1}{\beta}\, {}_2F_1\left(2, 2; 3; \frac{\beta-2}{\beta}\right),$$

and

$$Var\left(X\right) = \frac{2}{3\beta}\, {}_2F_1\left(2, 3; 4; \frac{\beta-2}{\beta}\right) - \left[E(X)\right]^2,$$

respectively. In reference [151], Theorem 1 showed that the PDF is a decreasing function if $0 < \beta < 2$, an increasing function if $\beta > 2$, and a constant if $\beta = 2$. In reference [151], Figure 5 showed that the mean is always increasing for β but the variance is increasing if $0 \leq \beta < 2$ and decreasing otherwise. In reference [151], Figure 6 showed that skewness is positive if $0 \leq \beta < 2$ and negative otherwise, and the kurtosis is positive if $0 \leq \beta 0.1$ and negative otherwise. This distribution has been applied to the model cost of a firm's cost management effectiveness.

2.29. The logit slash distribution

Suppose Y is a slash random variable with shape parameter a, scale parameter b, and location parameter c. Let $X = \frac{\exp(Y)}{1+\exp(Y)}$. Korkmaz [98] showed that X has the PDF

$$f(x) = \frac{a}{bx(1-x)} \int_0^1 t^a \phi\left(t \frac{\log \frac{x}{1-x} - c}{b}\right) dt,$$

for $0 < x < 1$, where $a > 0$ is a shape parameter, $b > 0$ is a scale parameter, and $-\infty < c < \infty$ is a location parameter. The corresponding CDF is

$$F(x) = a \int_0^1 t^{a-1} \Phi\left(t \frac{\log \frac{x}{1-x} - c}{b}\right) dt.$$

The kth raw moment of X is

$$E\left(X^k\right) = \frac{a}{\sqrt{2\pi}} \int_0^1 t^{a-1} \int_{-\infty}^{\infty} \left[+\exp\left(-c - \frac{bw}{t}\right)\right]^{-k} \exp\left(-\frac{w^2}{2}\right) dwdt.$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{a}{\sqrt{2\pi}} \int_0^1 t^{a-1} \int_{-\infty}^{\infty} \left[+\exp\left(-c - \frac{bw}{t}\right)\right]^{-} \exp\left(-\frac{w^2}{2}\right) dwdt,$$

and

$$Var\left(X\right) = \frac{a}{\sqrt{2\pi}} \int_0^1 t^{a-1} \int_{-\infty}^{\infty} \left[+\exp\left(-c - \frac{bw}{t}\right)\right]^{-2} \exp\left(-\frac{w^2}{2}\right) dwdt - \left[E(X)\right]^2,$$

respectively. Korkmaz [98, Figure 1] showed that the PDF can be U shaped, bimodal shaped, unimodal shaped, N shaped, decreasing, or increasing. Korkmaz [98, Figure 5] also showed the following. For $c = 0$, skewness is equal to zero. When a increases, the positive (respectively, negative) skewness is seen for negative (respectively, positive) c. Skewness decreases for fixed a and b, when c increases. The negative (respectively, positive) skewness decreases for fixed $c > 0$ (respectively, $c < 0$) and b, while a increases. When $c < 0$, kurtosis decreases. Otherwise, kurtosis increases. When a increases, kurtosis decreases for fixed c. When b increases, skewness first decreases, then increases for fixed a and c. When a increases, skewness first increases, then decreases for fixed b and c. When b increases, kurtosis first increases, then decreases for fixed a and c. When a increases, kurtosis first decreases, then increases for fixed b and c. This distribution has been applied to model Better Life Indices data.

2.30. A new one parameter distribution [36]

Suppose Y is a Lindley random variable and Z is a gamma random variable with shape parameter β independent of Y. Let X denote the limit of $Y \mid 1 - \delta < Y + Z < 1 + \delta$ as δ approaches zero. The researchers in [36] showed that X has the PDF

$$f(x) = \frac{\beta(1+\beta)}{2+\beta}(1+x)(1-x)^{\beta-1},$$

for $0 < x < 1$, where $\beta > 0$ is a shape parameter. The corresponding CDF

$$F(x) = 1 - \frac{(1-x)^{\beta}(2+\beta+x\beta)}{2+\beta}.$$

The kth moment is given by

$$E\left(X^k\right) = \frac{(\beta+2k+2)\Gamma(k+1)\Gamma(\beta+2)}{(\beta+2)\Gamma(\beta+k+2)} = \frac{\beta+2k+2}{(\beta+2)\binom{\beta+k+1}{k}}.$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{\beta+4}{(\beta+2)^2},$$

and

$$Var\left(X\right)=\frac{\beta+6}{\left(\beta+2\right)\binom{\beta+3}{2}}-\left[E(X)\right]^{2},$$

respectively. In reference [36], Figure 6 showed that the PDF can be monotonically increasing, monotonically decreasing, or unimodal. In reference [36], Figure 11 showed that the distribution is skewed toward the left for small β and to the right for moderate to larger values of β. In reference [36], Figure 11 showed that the distribution is platykurtic for moderate β and highly peaked for other values. This distribution has been applied to the model ratio of the actual electricity output to the maximum possible output and proportions of damaged tissues in a patient's blood.

2.31. Unit Lindley distribution [93]

Let Y denote a Lindley random variable. The researchers in [93] proposed a new unit Lindley distribution following the transformation $X=\tanh(Y)$. Its PDF and CDF are

$$f(x)=\frac{\beta^{2}}{\left(\beta+1\right)\left(1-x^{2}\right)}\left[1+\operatorname{arctanh}(x)\right]\exp\left[-\beta\operatorname{arctanh}(x)\right],$$

and

$$F(x)=1-\left[1+\frac{\beta\operatorname{arctanh}(x)}{1+\beta}\right]\exp\left[-\beta\operatorname{arctanh}(x)\right],$$

respectively, for $0<x<1$, where $\beta>0$ is a scale parameter. An alternative form of the CDF can be obtained by applying the logarithmic expression of arctanh(x) as

$$F(x)=1-\left[1+\frac{\beta}{2(1+\beta)}\log\left(\frac{1+x}{1-x}\right)\right]\left(\frac{1-y}{1+y}\right)^{\frac{\beta}{2}},$$

where $0<x<1$ and $\beta>0$. The kth moment is

$$E\left(X^{k}\right)=\frac{\beta^{2}}{1+\beta}\sum_{i=0}^{k}\sum_{l=0}^{\infty}\binom{k}{i}\binom{-i}{l}(-2)^{i}\frac{1+\beta+2(i+l)}{\left[\beta+2(i+l)\right]^{2}}.$$

In particular, the mean and variance are

$$E\left(X\right)=\frac{\beta^{2}}{1+\beta}\sum_{i=0}^{1}\sum_{l=0}^{\infty}\binom{-i}{l}(-2)^{i}\frac{1+\beta+2(i+l)}{\left[\beta+2(i+l)\right]^{2}},$$

and

$$Var\left(X\right)=\frac{\beta^{2}}{1+\beta}\sum_{i=0}^{2}\sum_{l=0}^{\infty}\binom{2}{i}\binom{-i}{l}(-2)^{i}\frac{1+\beta+2(i+l)}{\left[\beta+2(i+l)\right]^{2}}-\left[E(X)\right]^{2},$$

respectively. In reference [93], Figure 2 showed that the PDF is increasing if $0<\beta\leq$, U shaped if $1<\beta\leq 2.207$, inverse N shaped if $2.207<\beta\leq 2.264$, and decreasing if $\beta>2.264$. This distribution has been applied to model Better Life Indices data.

2.32. A new lifetime model with a bounded support

Muhammad [131] proposed a new lifetime probability distribution. The general form of exponential functions provides the basis for their new distribution. Its PDF and CDF are

$$f(x) = \beta \log 2 x^{\beta-1} \exp\left(x^{\beta} \log 2\right), \tag{2.22}$$

and

$$F(x) = \exp\left(x^{\beta} \log 2\right) - 1,$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a shape parameter. The PDF given by (2.22) can be shown as a series of the type

$$f(x) = \beta \sum_{i=0}^{\infty} (i!)^{-1} (\log 2)^{i+1} x^{\beta(i+1)-1}.$$

The kth moment is

$$E\left(X^k\right) = \beta \sum_{i=1}^{\infty} \frac{(\log 2)^{i+1}}{i! \left[\beta(i+1) + k\right]}.$$

However, if $\beta, k \in \mathbb{N}$ and $\alpha = \frac{\beta+k-1}{\beta} \in \mathbb{N}$, then

$$E\left(X^k\right) = 2 \sum_{i=0}^{\alpha-1} \frac{(\alpha-1)!(-1)^{i-\alpha+1}}{(\log 2)^{\alpha-i-1} i!}.$$

Muhammad [131, Figure 1] showed that the PDF can be increasing, or bathtub shaped. Muhammad [131, Figure 3] also demonstrated that the skewness decreases as β increases while the kurtosis is decreasing, then increasing (bathtub) as β increases. This distribution has been applied to model anxiety performance data.

2.33. A new trigonometric distribution with bounded support [1]

The researchers in [1] proposed a trigonometric distribution. The proposed distribution is generated following the transformation $X = \arcsin\left(\frac{\pi}{2} Y\right)$, where the random variable Y is said to follow a truncated exponential distribution $(0, 1)$ with the PDF

$$h(y) = \frac{\beta \exp(-\beta y)}{1 - \exp(-\beta)},$$

for $0 < y < 1$. Then, the PDF and CDF of the new trigonometric distribution are

$$f(x) = \frac{\pi\beta}{2\left[1 - \exp(-\beta)\right]} \cos\left(\frac{\pi x}{2}\right) \exp\left[-\beta \sin\left(\frac{\pi x}{2}\right)\right],$$

and

$$F(x) = \frac{1}{1 - \exp(-\beta)} \left\{1 - \exp\left[-\beta \sin\left(\frac{\pi x}{2}\right)\right]\right\},$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a scale parameter. The kth moment is

$$E\left(X^k\right) = \frac{2^k}{\pi^k\left[1-\exp(-\beta)\right]}\sum_{i=0}^{\infty}\varphi_{k,i}\frac{1}{\beta^{k+2i}}\left[\Gamma(k+2i+1)-\Gamma(k+2i+1,\beta)\right],$$

where, for any fixed r, $\varphi_{r,0} = b_0^r$, and

$$\varphi_{r,k} = (kb_0)^{-1}\sum_{n=1}^{k}\left[n(r+1)-k\right]b_n\varphi_{r,k-n}.$$

The mean and variance of X are

$$E(X) = \frac{2}{\pi\left[1-\exp(-\beta)\right]}\sum_{i=0}^{\infty}\varphi_{1,i}\frac{1}{\beta^{2i+1}}\left[\Gamma(2i+2)-\Gamma(2i+2,\beta)\right],$$

and

$$Var\left(X\right) = \frac{4}{\pi^2\left[1-\exp(-\beta)\right]}\sum_{i=0}^{\infty}\varphi_{2,i}\frac{1}{\beta^{2+2i}}\left[\Gamma(2i+3)-\Gamma(2i+3,\beta)\right]-\left[E(X)\right]^2,$$

respectively. In reference [1], Figure 1 showed that the PDF can be monotonically decreasing. This distribution has been applied to model firms' risk management cost-effectiveness, defined as the total property and casualty premiums and uninsured losses as a percentage of the total asset.

2.34. Unit Johnson SU distribution Gündüz and Korkmaz [73]

A random variable X is said to have the unbounded Johnson distribution system if its PDF is

$$f(y) = \frac{\sigma}{\sqrt{1+y^2}}\phi\left(\mu+\sigma\sinh^{-1}(y)\right),$$

for $-\infty < \mu < \infty$, $\sigma > 0$ and $\sinh^{-1}(w) = \log(w+\sqrt{1+w^2}), w \in \mathbb{R}$ denotes the inverse of the hyperbolic sine function. Thus, following the transformation $X = \frac{\exp(Y)}{1+\exp(Y)}$, Gündüz and Korkmaz [73] proposed a new unit distribution with the PDF and CDF

$$f(x) = \frac{\sigma}{x(1-x)\sqrt{1+\left[\log\left(\frac{x}{1-x}\right)\right]^2}}\phi\left(\mu+\sigma\sinh^{-1}\left(\log\frac{x}{1-x}\right)\right),$$

and

$$F(x) = \Phi\left(\mu+\sigma\sinh^{-1}\left(\log\frac{x}{1-x}\right)\right),$$

respectively, for $0 < x < 1$, where $\sigma > 0$ is a scale parameter, and $-\infty < \mu < \infty$ is a location parameter. The kth moment is given by

$$E\left(X^k\right) = 1+\frac{1}{\sqrt{2\pi}}\sum_{i=1}^{k}(-1)^i\binom{k}{i}\int_{-\infty}^{\infty}\left\{1+\exp\left[\sinh\left(\frac{z-\mu}{\sigma}\right)\right]\right\}^{-i}\exp\left(-\frac{z^2}{2}\right)dz.$$

In particular, the mean and variance are

$$E(X) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{1 + \exp\left[\sinh\left(\frac{z-\mu}{\sigma}\right)\right]\right\}^{-1} \exp\left(-\frac{z^2}{2}\right) dz,$$

and

$$Var(X) = 1 + \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{2} (-1)^i \binom{2}{i} \int_{-\infty}^{\infty} \left\{1 + \exp\left[\sinh\left(\frac{z-\mu}{\sigma}\right)\right]\right\}^{-i} \exp\left(-\frac{z^2}{2}\right) dz - [E(X)]^2,$$

respectively. Gündüz and Korkmaz [73, Figure 1] showed that the PDF can be W shaped, U shaped, unimodal shaped, N shaped, inverse N shaped, decreasing, or increasing. Gündüz and Korkmaz [73, Figure 3] showed that the distribution can be left skewed, right skewed, and symmetrical. For $\mu = 0$, the skewness of distribution is equal to zero. For fixed μ, when σ increases, the skewness goes to zero, and the kurtosis goes to 3. This distribution has been applied to model burr measurements on iron sheets and recovery rates of COVID-19 in Turkey.

2.35. *Chesneau's unit power-log distribution*

Chesneau's [42] unit power-log distribution is specified by the PDF and CDF

$$f(x) = \frac{1}{\log(1+\beta)} \frac{x^\beta - 1}{\log(x)},$$

and

$$F(x) = \frac{1}{\log(1+\beta)} \left\{\mathrm{Ei}\left[(1+\beta)\log(x)\right] - \mathrm{Li}(x)\right\},$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a shape parameter. The kth moment of the unit power-log distribution is given by

$$E\left(X^k\right) = \frac{1}{\log(1+\beta)} \log\left(1 + \frac{\beta}{1+k}\right).$$

In particular, the mean and variance are

$$E(X) = \frac{1}{\log(1+\beta)} \log\left(1 + \frac{\beta}{2}\right),$$

and

$$Var(X) = \frac{1}{\log(1+\beta)} \log\left(1 + \frac{\beta}{3}\right) - \frac{1}{(\log(1+\beta))^2} \left[\log\left(1 + \frac{\beta}{2}\right)\right]^2,$$

respectively.

2.36. Log-extended exponential-geometric distribution

Following the transformation $(X = \exp(-Y))$ of the extended exponential-geometric distribution in [10], Jodrá and Jiménez-Gamero [87] proposed a log-extended exponential-geometric distribution. The PDF of the proposed distribution is

$$f(x) = \frac{\alpha(1+\beta)x^{\alpha-1}}{(1+\beta x^{\alpha})^2},$$

for $0 < x < 1$, where $\alpha > 0$ is a shape parameter, and $\beta > -1$ is a scale parameter. The corresponding CDF is given by

$$F(x) = \frac{(1+\beta)x^{\alpha}}{1+\beta x^{\alpha}}.$$

The kth moment is

$$E\left(X^k\right) = 1 - \frac{k(1+\beta)}{\alpha}\Phi\left(-\beta, 1, 1+\frac{k}{\alpha}\right).$$

The mean and variance are

$$E(X) = 1 - \frac{1+\beta}{\alpha}\Phi\left(-\beta, 1, 1+\frac{1}{\alpha}\right),$$

and

$$Var(X) = 1 - \frac{2(1+\beta)}{\alpha}\Phi\left(-\beta, 1, 1+\frac{2}{\alpha}\right) - [E(X)]^2,$$

respectively. Convergence of the Lerch transcendent function is achieved for any real number $\nu > 0$, provided z and λ are any complex numbers with either $|z| < 1$, or $|z| = 1$ and $\lambda > 1$. Jodrá and Jiménez-Gamero [87, Figure 1] showed that the PDF can be monotonically increasing, monotonically decreasing, or unimodal. This distribution has been applied to model cost effectiveness to management's philosophy of controlling a company's exposure to various property and casualty losses after adjusting for company effects such as size and industry type.

2.37. Log-Lindley distribution

Suppose Y is a Lindley random variable. Let $X = \exp(-Y)$. The researchers in [68] showed that X has the PDF

$$f(x) = \frac{\sigma^2}{1+\lambda\sigma}(\lambda - \log x)x^{\sigma-1},$$

for $0 < x < 1$, where $\lambda \geq 0$ is a scale parameter, and $\sigma > 0$ is a shape parameter. The corresponding CDF is

$$F(x) = \frac{x^{\sigma}\left[1+\sigma(\lambda - \log x)\right]}{1+\lambda\sigma}.$$

The kth moment is

$$E\left(X^k\right) = \frac{\sigma}{1+\lambda\sigma}\frac{1+\lambda(\sigma+k)}{(\sigma+k)^2}.$$

Specifically, the mean and variance are

$$E(X) = \frac{\sigma^2}{1+\lambda\sigma}\frac{1+\lambda(1+\sigma)}{(1+\sigma)^2},$$

and

$$Var(X) = \left(\frac{\sigma}{1+\lambda\sigma}\right)^2\left\{\frac{(1+\lambda\sigma)\left[1+\lambda(2+\sigma)\right]}{(2+\sigma)^2} - \frac{\sigma^2\left[1+\lambda(1+\sigma)\right]^2}{(1+\sigma)^4}\right\},$$

respectively. In reference [68], Figure 1 showed that the PDF is U shaped for $\sigma < 1$. Also, the larger is the value of λ, the thicker is the tail of the PDF. This distribution has been applied to model cost effectiveness to management's philosophy of controlling a company's exposure to various property and casualty losses, after adjusting for company effects such as size and industry type.

2.38. A new three-parameter distribution with bounded domain

Muhammad [132] proposed a new three parameter distribution with bounded support through some algebraic manipulation of the inverse of the exponential-Pareto distribution. The PDF and CDF of the proposed three parameter distribution with bounded support are

$$f(x) = \alpha\beta x^{-1}\left[1-\beta\log(x)\right]^{-(\alpha+1)},$$

and

$$F(x) = \left[1-\beta\log(x)\right]^{-\alpha},$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a scale parameter, and $\alpha > 0$ is a shape parameter. The $k$the moment is

$$E\left(X^k\right) = \alpha\left(\frac{k}{\beta}\right)^{\frac{\alpha+1}{2}}\exp\left(\frac{k}{2\beta}\right)W_{-\frac{\alpha+1}{2},-\frac{\alpha}{2}}\left(\frac{k}{\beta}\right).$$

In particular, the mean and variance are

$$E(X) = \alpha\left(\frac{1}{\beta}\right)^{\frac{\alpha+1}{2}}\exp\left(\frac{1}{2\beta}\right)W_{-\frac{\alpha+1}{2},-\frac{\alpha}{2}}\left(\frac{1}{\beta}\right),$$

and

$$Var(X) = \alpha\left(\frac{2}{\beta}\right)^{\frac{\alpha+1}{2}}\exp\left(\frac{1}{\beta}\right)W_{-\frac{\alpha+1}{2},-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right) - \left[E(X)\right]^2,$$

respectively. Muhammad [132, Figure 1] showed that the PDF can be monotonically increasing, monotonically decreasing, or bathtub shaped. Muhammad [132, Figure 3] showed that skewness decreases as α and β increase, while the kurtosis decreases, then increases as β and α increase. This distribution has been applied to model lifetimes of devices and quarterly capital expenditures and appropriations for USA manufacturing firms.

2.39. Korkmaz and Chesneau's unit Burr XII distribution

Suppose Y is a Burr XII random variable with shape parameters α, β. Let $X = \exp(-Y)$. Korkmaz and Chesneau [102] showed that X has the PDF

$$f(x) = \alpha\beta x^{-1}(-\log x)^{\beta-1}\left[1 + (-\log x)^{\beta}\right]^{-\alpha-1},$$

for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$ are shape parameters. The corresponding CDF is

$$F(x) = \left[1 + (-\log x)^{\beta}\right]^{-\alpha}.$$

Korkmaz and Chesneau [102, Figure 1] showed that the PDF can be U shaped, increasing, decreasing, inverse N shaped, or unimodal. This distribution has been applied to model recovery rate of CD34+ cells after peripheral blood stem cell transplants.

2.40. Unit Burr XII distribution [152]

The researchers in [152] proposed another unit Burr XII distribution by reparameterizing of the quantile function associated with Korkmaz and Chesneau [102]'s unit Burr XII distribution. Its PDF and CDF are

$$f(x) = \frac{\log \tau^{-c} \log^{c-1} x^{-1}}{x \log\left(1 + \log^{c} q^{-1}\right)} \left(1 + \log^{c} x^{-1}\right)^{\frac{\log \tau}{\log\left(1+\log^{c} q^{-1}\right)} - 1},$$

and

$$F(x) = \left(1 + \log^{c} x^{-1}\right)^{\frac{\log \tau}{\log\left(1+\log^{c} q^{-1}\right)}},$$

respectively, for $0 < x < 1$, where $c > 0$ is a shape parameter, $0 < \tau < 1$ is a scale parameter, $q = Q_X(\tau)$, $Q_X(\tau) = \exp\left[-\left(\tau^{-\frac{1}{d}} - 1\right)^{\frac{1}{c}}\right]$, and d is one of the shape parameters of $G_Y(y; c, d)$, as defined in [152]. In reference [152], Figure 1 showed that the PDF can be decreasing, increasing, reverse J shaped, U shaped, reverse tilde shaped (decreasing, increasing, and, then decreasing), non skewed, or skewed left. This distribution has been applied to model dropout rates in Brazilian undergraduate courses.

2.41. A new distribution based on the arccosine function [46]

A new unit distribution based on the arccosine function in [46] has its PDF and CDF specified by

$$f(x) = \alpha x^{\alpha-1} \arccos\left(x^{\alpha}\right),$$

and

$$F(x) = x^{\alpha} \arccos\left(x^{\alpha}\right) + 1 - \sqrt{1 - x^{2\alpha}},$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a shape parameter. The kth moment is

$$E\left(X^{k}\right) = \frac{\alpha^{2} \sqrt{\pi}\, \Gamma\left(\frac{k}{2\alpha} + 1\right)}{(\alpha + k)^{2}\, \Gamma\left(\frac{k}{2\alpha} + \frac{1}{2}\right)},$$

for $k \geq 1$. In particular, the mean and variance are

$$E(X) = \frac{\alpha^2 \sqrt{\pi}\Gamma\left(\frac{r}{2\alpha}+1\right)}{(\alpha+1)^2\Gamma\left(\frac{1}{2\alpha}+\frac{1}{2}\right)},$$

and

$$Var(X) = \frac{\alpha^2 \sqrt{\pi}\Gamma\left(\frac{1}{alpha}+1\right)}{(\alpha+1)^2\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)} - [E(X)]^2,$$

respectively. In reference [46], Figure 1 showed that the PDF can be decreasing, or left skewed. In reference [46], Table 1 showed that, when $\alpha > 1$, the variance decreases as α increases, and when $\alpha < 1$, the variance increases as α increases. This distribution has been applied to model times to infection of kidney dialysis patients in months and failure times of an air conditioning system of an airplane.

2.42. Unit Chen distribution [101]

Suppose Y is a Chen random variable. The researchers in [101] employed the inverted transformation $X = \exp(-Y)$ to propose a unit Chen distribution. Its PDF and CDF are

$$f(x) = \frac{\alpha\beta}{x}(-\log x)^{\beta-1}\exp\left[(-\log x)^{\beta}\right]\exp\left[\alpha\left\{1-\exp\left[(-\log x)^{\beta}\right]\right\}\right],$$

and

$$F(x) = \exp\left[\alpha\left\{1-\exp\left[(-\log x)^{\beta}\right]\right\}\right],$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The kth moment of X is

$$E\left(X^k\right) = \int_0^1 \exp\left\{-k\left[\log\left(1-\frac{\log t}{\alpha}\right)\right]^{\frac{1}{\beta}}\right\}dt.$$

In particular, the mean and variance are

$$E(X) = \int_0^1 \exp\left\{-\left[\log\left(1-\frac{\log t}{\alpha}\right)\right]^{\frac{1}{\beta}}\right\}dt,$$

and

$$Var(X) = \int_0^1 \exp\left\{-2\left[\log\left(1-\frac{\log t}{\alpha}\right)\right]^{\frac{1}{\beta}}\right\}dt - [E(X)]^2,$$

respectively. In reference Korkmaz et al. 2022, Figure 1 showed that the PDF can be U shaped, unimodal, or right skewed. In reference [101], Figure 2 showed that the mean increases as α increases regardless of β, while for $\alpha < 0.4$, the mean increases with β, but for $\alpha > 0.4$, the mean decreases with β. This distribution has been applied to model failure times of mechanical components and recovery rate of CD34+ cells after peripheral blood stem cell transplants.

2.43. One parameter unit Lindley distribution [124]

The researchers in [124] proposed a one parameter unit Lindley distribution using the transformation $X = \frac{Y}{1+Y}$, where Y follows a one parameter Lindley distribution. The PDF and CDF of this new random variable are

$$f(x) = \frac{\beta^2}{1+\beta}(1-x)^{-3} \exp\left(-\frac{\beta x}{1-x}\right),$$

and

$$F(x) = 1 - \left[1 - \frac{\beta x}{(1+\beta)(x-1)}\right] \exp\left(-\frac{\beta x}{1-x}\right),$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{\beta^2}{1+\beta} \int_0^1 t^k (1-t)^{-3} \exp\left(-\frac{\beta t}{1-t}\right) dt.$$

In particular, the mean and variance are

$$E(X) = \frac{1}{1+\beta},$$

and

$$Var\,(X) = \frac{1}{1+\beta}\left[\beta^2 \exp(\beta)\mathrm{Ei}(\beta) - \beta + 1\right] - [E(X)]^2,$$

respectively. In reference [124], Figure 1 showed that the PDF is unimodal when $\beta < 3$ and monotonically decreasing otherwise. In reference [124], Figure 2 showed that the mean decreases and skewness increases with the increase in β, whereas the kurtosis initially decreases, then increases with β. This distribution has been applied to model access of people in households with inadequate water supply and sewage in cities of Brazil.

The researchers in [123] proposed another version of the one parameter unit Lindley distribution using the transformation $X = \frac{1}{1+Y}$. For this version, the PDF and CDF are

$$f(x) = \frac{\theta^2}{x^3}(1+\theta) \exp\left(-\theta \frac{1-x}{x}\right),$$

and

$$F(x) = \frac{\theta + x}{x(1+\theta)} \exp\left(-\theta \frac{1-x}{x}\right),$$

respectively, for $0 < x < 1$, where $\theta > 0$ is a scale parameter. For this version, the kth moment of X is

$$E\left(X^k\right) = \theta^2(1+\theta) \int_0^1 t^{k-3} \exp\left(-\theta \frac{1-t}{t}\right) dt.$$

In particular, the mean and variance are

$$E(X) = \frac{\theta}{1+\theta},$$

and

$$Var\,(X) = \frac{\theta^2 \exp(\theta) Ei(\theta)}{1+\theta} - [E(X)]^2\,,$$

respectively. In reference [123], Figure 1 showed that the PDF is unimodal for all values of β. This distribution has been applied to model cost effectiveness with the management philosophy of controlling a company's exposure to various property losses and accidents, taking into account company characteristics such as size and type of industry.

2.44. Unit Nadarajah-Haghighi distribution [165]

Suppose Y is a Nadarajah-Haghighi random variable with shape parameter α and scale parameter λ [134]. Let $X = \exp(-Y)$. The researchers in [165] showed that X has the PDF

$$f(x) = \frac{\alpha\lambda}{x}\,(1-\lambda\log x)^{\alpha-1}\exp\left[1-(1-\lambda\log x)^{\alpha}\right],$$

for $0 < x < 1$, where $\alpha > 0$ is a shape parameter, and $\lambda > 0$ is a scale parameter. The corresponding CDF is

$$F(x) = \exp\left[1-(1-\lambda\log x)^{\alpha}\right].$$

The kth moment of X is

$$E\left(X^k\right) = \alpha\lambda\sum_{i=1}^{\infty}(-\lambda)^i\binom{\alpha-1}{i}\int_0^1 t^{k-1}(\log t)^i\exp\left[1-(1-\lambda\log t)^{\alpha}\right]dt.$$

In particular, the mean and variance are

$$E\,(X) = \alpha\lambda\sum_{i=1}^{\infty}(-\lambda)^i\binom{\alpha-1}{i}\int_0^1 (\log t)^i\exp\left[1-(1-\lambda\log t)^{\alpha}\right]dt,$$

and

$$Var\,(X) = \alpha\lambda\sum_{i=1}^{\infty}(-\lambda)^i\binom{\alpha-1}{i}\int_0^1 t(\log t)^i\exp\left[1-(1-\lambda\log t)^{\alpha}\right]dt - [E(X)]^2\,,$$

respectively. In reference [165], Figure 1 showed that the PDF can be decreasing, or increasing. This distribution has been applied to model daily rainfall at a location in Florida and anxiety performance data.

2.45. Truncated exponentiated-exponential distribution

The truncated exponentiated-exponential distribution in Ribeiro-Reis [154] has the PDF

$$f(x) = \frac{\lambda\alpha}{K} \exp(-\lambda x) \left[1 - \exp(-\lambda x)\right]^{\alpha-1},$$

for $0 < x \leq 1$, where $\lambda > 0$ is a scale parameter, $\alpha > 0$ is a shape parameter, and $K = [1 - \exp(-\lambda)]^{\alpha}$. The corresponding CDF is

$$F(x) = \frac{1}{K} \left[1 - \exp(-\lambda x)\right]^{\alpha}.$$

The kth moment is

$$E\left(X^k\right) = \frac{\lambda\alpha}{K} \sum_{i=0}^{\infty} \frac{(-1)^i}{(\lambda + i\lambda)^{k+1}} \binom{\alpha-1}{k} \gamma(k+1, \lambda + k\lambda).$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{\lambda\alpha(\alpha-1)}{K} \sum_{i=0}^{\infty} \frac{(-1)^i}{(\lambda + i\lambda)^2} \gamma(2, \lambda + k\lambda),$$

and

$$Var\left(X\right) = \frac{\lambda\alpha(\alpha-1)(\alpha-2)}{2K} \sum_{i=0}^{\infty} \frac{(-1)^i}{(\lambda + i\lambda)^3} \gamma(3, \lambda + k\lambda) - \left[E(X)\right]^2,$$

respectively.

2.46. Continuous Bernoulli distribution

The continuous Bernoulli distribution has the PDF

$$f(x) = \begin{cases} C(\beta)\beta^x(1-\beta)^{1-x}, & x \in (0,1), \\ 0, & x \notin (0,1), \end{cases}$$

where $0 < \beta < 1$ is a scale parameter and $C(\beta)$ is defined by

$$C(\beta) = \begin{cases} 2, & \beta = \frac{1}{2}, \\ \dfrac{2\tanh^{-1}(1-2\beta)}{1-2\beta}, & \beta \in (0,1)\backslash\{\frac{1}{2}\}, \end{cases}$$

where $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ is the inverse hyperbolic tangent. The corresponding CDF is

$$F(x) = \begin{cases} 0, & x \leq 0, \\ x, & \beta = \frac{1}{2}, \quad \text{and} \quad x \in (0,1), \\ \dfrac{\beta^x(1-\beta)^{1-x} + \beta - 1}{2\beta - 1}, & \beta \in (0,1)\backslash\{\frac{1}{2}\}, \quad \text{and} \quad x \in (0,1), \\ 1 & x \geq 1. \end{cases}$$

The mean and variance of a continuous Bernoulli random variable are

$$E(X) = \begin{cases} \dfrac{1}{2}, & \beta = \frac{1}{2}, \\ \dfrac{\beta}{2\beta - 1} + \dfrac{1}{2\tanh^{-1}(1-2\beta)}, & \beta \in (0,1)\backslash\{\frac{1}{2}\}, \end{cases}$$

and

$$Var(X) = \begin{cases} \dfrac{1}{12}, & \beta = \frac{1}{2}, \\ \dfrac{(1-\beta)\beta}{(1-2\beta)^2} + \dfrac{1}{\left[2\tanh^{-1}(1-2\beta)\right]^2}, & \beta \in (0,1)\backslash\{\frac{1}{2}\}, \end{cases}$$

respectively. This distribution is especially useful in machine learning and computer vision. A prominent application is in variational autoencoders (VAEs) for modeling normalized pixel intensities, where standard Bernoulli, or Gaussian assumptions fail to capture the true data distribution. It is also relevant in probabilistic modeling of proportions, uncertainty quantification, and Bayesian inference for bounded data. More broadly, it provides a mathematically consistent way to handle continuous relaxations of binary variables, which supports tasks like differentiable optimization, stochastic gradient estimation, and generative modeling where gradients are crucial.

2.47. Altun and Cordeiro's unit improved second-degree Lindley distribution

Suppose Y is an improved second-degree Lindley random variable [97]. Let $X = \frac{Y}{Y+1}$. Altun and Cordeiro [16] showed that the PDF of X is

$$f(x) = \frac{\lambda^3 (1-x)^{-2}}{\lambda^2 + 2\lambda + 2} \left(1 + \frac{x}{1-x}\right)^2 \exp\left(-\frac{x\lambda}{1-x}\right),$$

for $0 < x < 1$, where $\lambda > 0$ is a scale parameter. The corresponding CDF of X is

$$F(x) = 1 - \left\{1 + \frac{\lambda^2 \frac{x^2}{(1-x)^2} + 2\left(\lambda^2 + \lambda\right) \frac{x}{1-x}}{\lambda^2 + 2\lambda + 2}\right\} \exp\left(-\frac{x\lambda}{1-x}\right).$$

The kth moment of X is

$$E\left(X^k\right) = \frac{\lambda^3}{\lambda^2 + 2\lambda + 2} \int_0^1 t^k (1-t)^{-2} \left(1 + \frac{t}{1-t}\right)^2 \exp\left(-\frac{t\lambda}{1-t}\right) dt.$$

The mean and variance are

$$E(X) = \frac{\lambda + 2}{\lambda^2 + 2\lambda + 2},$$

and

$$Var(X) = \frac{2}{\lambda + 2\lambda + 2} - \frac{(\lambda + 2)^2}{\left(\lambda^2 + 2\lambda + 2\right)^2},$$

respectively. Altun and Cordeiro [16, Figure 1] showed that the PDF can be extremely left skewed, or right skewed. This distribution has been applied to model monthly water capacity from the Shasta reservoir in California, USA.

2.48. Krishna et al.'s unit Teissier distribution

Suppose Y is a Teissier random variable with scale parameter β [178]. Let $X = \exp(-Y)$. The researchers in [108] showed that X has the PDF and CDF

$$f(x) = \beta\left(x^{-\beta} - 1\right)x^{-(\beta+1)}\exp\left(-x^{-\beta} + 1\right),$$

and

$$F(x) = x^{-\beta}\exp\left(-x^{-\beta} + 1\right),$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a shape parameter. The kth moment is

$$E\left(X^k\right) = e\left[\Gamma\left(-\frac{k}{\beta} + 2, 1\right) - \Gamma\left(-\frac{k}{\beta} + 1, 1\right)\right].$$

In particular, the mean and variance are

$$E\left(X\right) = e\left[\Gamma\left(-\frac{1}{\beta} + 2, 1\right) - \Gamma\left(-\frac{1}{\beta} + 1, 1\right)\right],$$

and

$$Var\left(X\right) = e\left[\Gamma\left(-\frac{2}{\beta} + 2, 1\right) - \Gamma\left(-\frac{2}{\beta} + 1, 1\right)\right] - \left[E(X)\right]^2,$$

respectively. In reference [108], Figure 1 showed that the PDF can be monotonically decreasing, or unimodal. This distribution has been applied to model maximum flood levels for the Susquehanna River in Pennsylvania and times between failures of secondary reactor pumps.

2.49. Unit inverse Gaussian distribution [64]

Suppose Y is an inverse Gaussian random variable with scale parameter λ, μ. Following the transformation $X = \exp(-Y)$, the researchers in [64] proposed a unit inverse Gaussian distribution. Its PDF and CDF are

$$f(x) = \sqrt{\frac{\lambda}{2\pi}}\frac{1}{x(-\log x)^{\frac{3}{2}}}\exp\left[\frac{\lambda}{2\mu^2\log x}(\log x + \mu)^2\right],$$

and

$$F(x) = 1 - \left[\Phi\left(\sqrt{\frac{\lambda}{-\log x}}\left(\frac{-\log x}{\mu} - 1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{-\log x}}\left(\frac{-\log x}{\mu} + 1\right)\right)\right],$$

respectively, for $0 < x < 1$, where $\lambda > 0$ and $\mu > 0$ are scale parameters. The kth moment of X is

$$E\left(X^k\right) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2k\mu^2}{\lambda}}\right)\right],$$

for $k \geq 1$. The mean and variance are

$$E(X) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2}{\lambda}}\right)\right],$$

and

$$Var(X) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{4\mu^2}{\lambda}}\right)\right] - [E(X)]^2,$$

respectively. In reference [64], Figure 1 showed that the PDF can be unimodal, decreasing, or decreasing-increasing-decreasing shape. In reference [64], Figure 3 showed that the skewness can be negative. This distribution has been applied to model Municipal Human Development Indices data.

2.50. *Unit generalized log Burr XII distribution [33]*

By using both the Pearson differential equation,

$$\frac{d \log f(y)}{dy} = \frac{\sum_{i=0}^{m} a_i y^m}{\sum_{i=0}^{n} b_i y^n}, \; m, n > 0,$$

and variable transformation $x = \exp(-y)$, [33] proposed the unit generalized log Burr XII distribution with the PDF

$$f(x) = \frac{2\alpha\beta}{\lambda x}\left(-\frac{\log x}{\lambda}\right)^{(2\beta-1)}\left[1 + \left(-\frac{\log x}{\lambda}\right)^{2\beta}\right]^{-(\alpha+1)},$$

for $0 < x < 1$, where $\alpha > 0$, and $\beta > 0$ are shape parameters, and $\lambda > 0$ is a scale parameter. The corresponding CDF is

$$F(x) = \left[1 + \left(-\frac{\log x}{\lambda}\right)^{2\beta}\right]^{-\alpha}.$$

Remarkably, the link between the gamma and exponential random variables may also be used to construct the unit generalized log Burr XII distribution. That is, if $Y_1 \exp(1)$ and $Y_2 \sim gamma(\alpha, 1)$ are independent random variables, then

$$Y_1 = \left(-\frac{\log X}{\lambda}\right)^{2\beta} Y_2,$$

and we can see that

$$X = \exp\left[-\lambda\left(\frac{Y_1}{Y_2}\right)^{\frac{1}{2\beta}}\right],$$

follows the unit generalized log Burr XII distribution. The kth moment of X is

$$E\left(X^k\right) = \alpha \sum_{i=0}^{\infty} \frac{(-k\lambda)^i}{i!} B\left(1 + \frac{i}{2\beta}, \alpha - \frac{i}{2\beta}\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \alpha \sum_{i=0}^{\infty} \frac{(-\lambda)^i}{i!} B\left(1 + \frac{i}{2\beta}, \alpha - \frac{i}{2\beta}\right),$$

and

$$Var\left(X\right) = \alpha \sum_{i=0}^{\infty} \frac{(-2\lambda)^i}{i!} B\left(1 + \frac{i}{2\beta}, \alpha - \frac{i}{2\beta}\right) - \left[E(X)\right]^2,$$

respectively. In reference [33], Figure 1 showed that the PDF can be decreasing, unimodal, U shaped, or N shaped. This distribution has been applied to model monthly water capacity at the Shasta reservoir, California, USA and proportions of total milk production in the first cow births.

2.51. Unit generalized half-normal distribution [99]

Suppose Y ia a generalized half-normal random variable with shape parameter β and scale parameter α [49]. Let $X = \exp(-Y)$. Korkmaz [99] showed that X has the PDF

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{\beta}{x\left[-\log(x)\right]} \left(-\frac{\log(x)}{\alpha}\right)^{\beta} \exp\left\{-\frac{1}{2}\left[-\frac{\log(x)}{\alpha}\right]^{2\beta}\right\},$$

for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The corresponding CDF is

$$F(x) = 2\Phi\left(-\left(-\frac{\log(x)}{\alpha}\right)^{\beta}\right).$$

The kth moment of X is

$$E\left(X^k\right) = \sum_{i=0}^{\infty} \frac{(-k)^i}{k!} \alpha^i \sqrt{\frac{2^{\frac{i}{\beta}}}{\pi}} \Gamma\left(\frac{i+\beta}{2\beta}\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{i=0}^{\infty} \frac{(}{-} 1)^i \alpha^i \sqrt{\frac{2^{\frac{i}{\beta}}}{\pi}} \Gamma\left(\frac{i+\beta}{2\beta}\right),$$

and

$$Var\left(X\right) = \frac{1}{2} \sum_{i=0}^{\infty} (-2)^i \alpha^i \sqrt{\frac{2^{\frac{i}{\beta}}}{\pi}} \Gamma\left(\frac{i+\beta}{2\beta}\right) - \left[E(X)\right]^2,$$

respectively. Korkmaz [99, Figure 1] showed that the PDF can be decreasing, increasing, unimodal, U shaped, or N shaped. This distribution has been applied to model failure times of mechanical components.

2.52. A modified power function distribution [138]

The modified power function distribution in [138] has the PDF

$$f(x) = \gamma\delta(1-x)^{\delta-1}\left[1-(1-\gamma)(1-x)^{\delta}\right]^{-2},$$

for $0 < x < 1$, where $\delta > 0$ is a shape parameter, and $\gamma > 0$ is a scale parameter. The corresponding CDF is

$$F(x) = 1 - \frac{\gamma(1-x)^{\delta}}{1-(1-\gamma)(1-x)^{\delta}}.$$

The kth moment of the modified power function is

$$E\left(X^k\right) = \gamma\delta\sum_{i=0}^{\infty}(i+1)(1-\gamma)^i B\left(k+1, \delta(i+1)\right),$$

for $k \geq 1$. In particular, the mean and variance are

$$E\left(X\right) = \gamma\delta\sum_{i=0}^{\infty}(i+1)(1-\gamma)^i B\left(2, \delta(i+1)\right),$$

and

$$Var\left(X\right) = \gamma\delta\sum_{i=0}^{\infty}(i+1)(1-\gamma)^i B\left(3, \delta(i+1)\right) - \left[E(X)\right]^2,$$

respectively. In reference [138], Figure 1 showed that the PDF can be constant monotonically increasing, monotonically decreasing, unimodal, or bathtub shaped. In reference [138], Figure 2 showed that skewness is a decreasing function of γ for fixed δ and kurtosis is bathtub shaped with respect to γ for fixed δ. This distribution has been applied to model anxiety performance data and evaporation data.

2.53. Altun and Hamedani [18]'s unit distribution

Suppose Y is a Lindley random variable with parameter β. Let $X = \exp(-Y)$. Altun and Hamedani [18] showed that X has the PDF

$$f(x) = \frac{\beta^2}{1+\beta}\left[1+\frac{\beta}{2}\log(x)^2\right]x^{\beta-1},$$

for $0 < x < 1$, where $\beta > 0$ is a shape parameter. The corresponding CDF is

$$F(x) = x^{\beta}(\beta+1)^{-1}\left[1+\beta-\beta\log(x)+\frac{\beta^2\log(x)^2}{2}\right].$$

The kth moment is

$$E\left(X^k\right) = \frac{\beta^2\left[\beta^2+(2k+1)\right]\beta+k^2}{(\beta+1)(\beta+k)^3}.$$

In particular, the mean and variance are

$$E(X) = \frac{\beta^2\left(\beta^2+3\right)\beta+1}{(\beta+1)(\beta+k)^3},$$

and

$$Var(X) = \frac{\beta^2\left(\beta^2+5\right)\beta+k^2}{(\beta+1)(\beta+2)^3} - [E(X)]^2,$$

respectively. Altun and Hamedani [18, Figure 1] showed that the PDF can be increasing, decreasing, or increasing-decreasing-increasing. Altun and Hamedani [18, Figure 3] also showed that skewness, and kurtosis can correspond to the distribution being symmetric, left skewed, or right skewed. This distribution has been applied to model for estimating unit capacity factors.

2.54. A log exponential-power distribution [100]

Suppose Y is an exponential power random variable with scale parameter α and shape parameter β Let $X = \exp(-Y)$. The researchers in [100] showed that X has the PDF

$$f(x) = \frac{\alpha\beta}{x}\exp\left[\alpha(-\log x)^{\beta}\right](-\log x)^{\beta-1}\exp\left\{1-\exp\left[\alpha(-\log x)^{\beta}\right]\right\},$$

for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The corresponding CDF is

$$F(x) = \exp\left\{1-\exp\left[\alpha(-\log x)^{\beta}\right]\right\}.$$

The associated kth moment is

$$E\left(X^k\right) = 1+\exp(1)+\exp(1)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha i)^j}{i!j!}k^{-\beta j}\Gamma(\beta j+1).$$

In particular, the mean and variance are

$$E(X) = 1+\exp(1)+\exp(1)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha i)^j}{i!j!}\Gamma(\beta j+1),$$

and

$$Var(X) = 1+\exp(1)+\exp(1)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha i)^j}{i!j!}2^{-\beta j}\Gamma(\beta j+1) - [E(X)]^2,$$

respectively. In reference [100], Figure 1 showed that the PDF can be U shaped, increasing, decreasing, or unimodal. This distribution has been applied to model flood levels for the Susquehanna River, Pennsylvania and Better Life Indices.

2.55. Log-gamma distribution

Let Y denote a Stacy random variable [172]. Making the transformation $\left(\frac{Y}{a}\right)^p = -\nu \log X$, $\frac{d}{p} = r$, Consul and Jain [48] proposed the log gamma distribution with the PDF

$$f(x) = \frac{\nu^r x^{\nu-1} \left(-\log x\right)^{r-1}}{\Gamma(r)},$$

for $0 < x < 1$, where $\nu, r \geq 1$ are shape parameters. The corresponding CDF is

$$F(x) = \int_0^y \frac{\nu^r x^{\nu-1} \left(-\log x\right)^{r-1}}{\Gamma(r)} dx.$$

The mean and variance of the log-gamma distribution are

$$E(X) = \left(\frac{\nu}{\nu+1}\right)^r,$$

and

$$Var(X) = \left(\frac{\nu}{\nu+2}\right)^r - \left(\frac{\nu}{\nu+1}\right)^{2r},$$

which indicate that the variance falls off as ν rises for a given value of r. Consul and Jain [48] showed that skewness first decreases with the increase in the value of r and, then begins to increase. This distribution is particularly useful in survival analysis and reliability engineering for modeling failure times and lifetimes of systems since it can capture long-tail behaviors better than simpler models. In finance and insurance, it is applied to model claim sizes, risk measures, and asset returns that exhibit asymmetry and heavy tails. The distribution also appears in Bayesian statistics, especially as priors, or in transformations of gamma-distributed parameters, and in environmental studies for modeling extreme events such as rainfall, or flood data. Additionally, its flexibility in handling skewness makes it relevant in medical statistics and biology for representing positively skewed measurements such as reaction times, incubation periods, and growth processes.

2.56. An arcsecant hyperbolic normal distribution [103]

If Y is a Gaussian random variable $X = \mathrm{sech} Y$ is a hyperbolic secant random variable [103]. So, $\mathrm{sech} y = \frac{2}{\exp(y)+\exp(-y)} = \frac{2\exp(y)}{\exp(2y)+1} \in (0, 1)$ for $y \in R$. The PDF and CDF of X are

$$f(x) = \frac{1}{\sigma x \sqrt{1-x^2}} \left[\phi\left(\frac{\mathrm{arcsech} x + \mu}{\sigma}\right) + \phi\left(\frac{\mathrm{arcsech} x - \mu}{\sigma}\right)\right],$$

and

$$F(x) = 2 - \Phi\left(\frac{\mathrm{arcsech} x + \mu}{\sigma}\right) - \Phi\left(\frac{\mathrm{arcsech} x - \mu}{\sigma}\right),$$

respectively, $0 < x < 1$, where $-\infty < \mu < \infty$ is a location parameter, $\mu > 0$ is a scale parameter, and $\mathrm{arcsech} z = \log\left[\frac{1}{z}\left(1 + \sqrt{1-z^2}\right)\right] > 0$, $0 < z < 1$ is the inverse hyperbolic secant function. The kth moment of X is

$$E\left(X^k\right) = \frac{1}{\sigma} \int_0^1 \frac{x^{k-1}}{\sqrt{1-x^2}} \left[\phi\left(\frac{\mathrm{arcsech} x + \mu}{\sigma}\right) + \phi\left(\frac{\mathrm{arcsech} x - \mu}{\sigma}\right)\right] dx,$$

for $k \geq 1$. In reference [103], Figure 1 showed that the PDF can be J shaped, reversed J shaped, U shaped, or bell shaped. In reference [103], Figure 2 showed that skewness can be negative and positive, and the kurtosis can be either very small, or very large. This distribution has been applied to model failure times of mechanical components and Better Life Indices.

2.57. A bounded truncated Cauchy power exponential distribution [135]

A random variable X is said to have the bounded truncated Cauchy power distribution [135] if its PDF is

$$f(x) = \frac{4\alpha\lambda x^{\lambda-1}\left(1 - x^{\lambda}\right)^{\alpha}}{\pi\left[1 + (1 - x^{\lambda})^{2\alpha}\right]},$$

for $0 < x < 1$, where $\alpha > 0$ and $\lambda > 0$ are shape parameters. The corresponding CDF is

$$F(x) = 1 - \frac{4}{\pi}\arctan\left[\left(1 - x^{\lambda}\right)^{\alpha}\right].$$

The kth moment is

$$E\left(X^k\right) = \frac{4\alpha}{\pi}\sum_{i=0}^{\infty}(-1)^i B\left(\frac{k}{\lambda} + 1, \alpha(1 + 2i)\right).$$

In particular, the mean and variance are

$$E(X) = \frac{4\alpha}{\pi}\sum_{i=0}^{\infty}(-1)^i B\left(\frac{1}{\lambda} + 1, \alpha(1 + 2i)\right),$$

and

$$Var(X) = \frac{4\alpha}{\pi}\sum_{i=0}^{\infty}(-1)^i B\left(\frac{2}{\lambda} + 1, \alpha(1 + 2i)\right) - \left[E(X)\right]^2,$$

respectively. In reference [135], Figure 1 showed that the PDF can be symmetric, bathtub shaped, left skewed, or right skewed. In reference [135], Figure 2 showed that skewness approaches zero as α and λ increase. In reference [135], Figure 3 showed that the distribution displays a platykurtic shape when α and λ are equal. This distribution has been applied to model body fat percentages measured in five regions: Android, arms, gynoids, legs, and trunk.

2.58. Kumaraswamy distribution

The generalized double-bounded distribution in Kumaraswamy [110] has the PDF

$$f(x) = \alpha\beta x^{\alpha-1}\left(1 - x^{\alpha}\right)^{\beta-1},$$

for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$ are shape parameters. The corresponding CDF is

$$F(x) = 1 - \left(1 - x^{\alpha}\right)^{\beta}.$$

The kth moment is

$$E\left(X^k\right) = \beta B\left(1 + \frac{k}{\alpha}, \beta\right).$$

In particular, the mean and variance are

$$E(X) = \beta B\left(1 + \frac{1}{\alpha}, \beta\right),$$

and

$$Var(X) = \beta B\left(1 + \frac{2}{\alpha}, \beta\right) - [E(X)]^2,$$

respectively. The PDF can be monotonically increasing, monotonically decreasing, bathtub shaped, or unimodal. This distribution is particularly useful in reliability analysis, survival studies, and hydrology, where lifetimes, failure rates, or proportions naturally fall within the unit interval. In environmental sciences, it is applied to model rainfall proportions, pollutant concentrations, and other normalized measurements. In finance and economics, the distribution is employed for modeling rates of return, credit risk, and income proportions. Additionally, it is applied in engineering for quality control and software reliability, in medical sciences for dose-response and recovery proportions, and in Bayesian statistics as a prior for parameters constrained to [0, 1], making it a versatile tool across diverse scientific and applied domains.

2.59. Unit Vasicek distribution

The unit Vasicek distribution in Vasicek [182] has the PDF

$$f(x) = \sqrt{\frac{1-b}{b}} \exp\left\{\frac{1}{2}\left[\Phi^{-1}(x) - \left(\frac{\sqrt{1-b}\Phi^{-1}(x) - \Phi^{-1}(a)}{\sqrt{b}}\right)^2\right]\right\},$$

for $0 < x < 1$, where $0 < a < 1$ and $0 < b > 1$ are scale parameters. The associated CDF is

$$F(x) = \Phi\left(\frac{\sqrt{1-b}\Phi^{-1}(x) - \Phi^{-1}(a)}{\sqrt{b}}\right),$$

for $0 < x < 1$. The mean and variance are

$$E(X) = a,$$

and

$$Var(X) = \Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a); b\right) - a^2,$$

respectively, where $\Phi_2(\cdot, \cdot; \theta)$ denotes the bivariate standard normal CDF with b as the correlation parameter. This distribution is widely used by banks, regulators, and financial institutions to estimate portfolio default rates, loss distributions, and economic capital requirements, as it captures the correlation between defaults in large loan portfolios. Beyond credit risk, it is applied in stress testing, risk-based pricing of loans, and regulatory frameworks such as Basel II/III, where it underpins the Internal Ratings-Based (IRB) approach for calculating capital adequacy. Its ability to model correlated default events makes it especially relevant for assessing systemic risk, portfolio diversification effects, and the stability of the financial system.

2.60. Unit extended Weibull families of distributions [72]

Let Y denote an extended Weibull random variable [74]. By employing the transformations $X = \exp(-Y)$ and $X = 1 - \exp(-Y)$, The researchers in [72] proposed two new unit general families of distributions: Unit extended Weibull (UEW) and complemented unit extended Weibull (CUEW) distributions. Their PDFs and CDFs are

$$f(x) = \frac{\alpha}{x} h\left(-\log(x); \boldsymbol{\xi}\right) \exp\left[-\alpha H\left(-\log(x); \boldsymbol{\xi}\right)\right],$$

and

$$F(x) = \exp\left[-\alpha H\left(-\log(x); \boldsymbol{\xi}\right)\right],$$

as well as

$$f(x) = f(1-x) = \frac{\alpha}{1-x} h\left(-\log(1-x); \boldsymbol{\xi}\right) \exp\left[-\alpha H\left(-\log(1-x); \boldsymbol{\xi}\right)\right],$$

and

$$F(x) = 1 - F(1-x) = 1 - \exp\left[-\alpha H\left(-\log(1-x); \boldsymbol{\xi}\right)\right],$$

where $0 < x < 1$, $\alpha > 0$ is a scale parameter, $H\left(\cdot; \boldsymbol{\xi}\right)$ is a non-negative monotonically increasing function, and $h\left(\cdot; \boldsymbol{\xi}\right)$ denotes the derivative of $H\left(\cdot; \boldsymbol{\xi}\right)$. The kth moment of unit extended Weibull distribution is

$$E\left(X^k\right) = E\left[\exp(-kY)\right] = M_Y(-k),$$

for $k \geq 1$, where

$$M_Y(t) = \int_0^\infty \exp(ty) f(y) dy,$$

denotes the moment generating function of Y. The kth moment of the complemented unit extended Weibull (CUEW) distribution is

$$E\left(X^k\right) = E\left\{\left[1 - \exp(-Y)\right]^k\right\} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_Y(-j).$$

Based on these general classes, or families of distributions, the researchers in [72] also gave instances of unit distributions that emerge as unique models. These include

- The unit Gompertz distribution with the PDF and CDF given by

$$f_{UGo}(x) = \frac{\beta \log 2}{\mu^{-\beta} - 1} x^{-(\beta+1)} 2^{\frac{x^{-\beta}-1}{1-\mu^{-\beta}}},$$

  and

$$F_{UGo}(x) = 2^{\frac{x^{-\beta}-1}{1-\mu^{-\beta}}},$$

  respectively, for $0 < x < 1$, where $\beta > 0$ is a shape parameter, and $0 < \mu < 1$ is a scale parameter.

- The unit Lomax distribution with the PDF and CDF given by

$$f_{UL}(x) = \frac{\log 2}{\beta x}\left[\log\left(1-\beta^{-1}\log\mu\right)\right]^{-1}\left(1-\beta^{-1}\log x\right)^{-\frac{\log 2}{\log\left(1-\beta^{-1}\log\mu\right)}-1},$$

and

$$F_{UL}(x) = \left(1-\beta^{-1}\log x\right)^{-\frac{\log 2}{\log\left(1-\beta^{-1}\log\mu\right)}-1},$$

respectively, for $0 < x < 1$, where $\beta > 0, 0 < \mu < 1$ are scale parameters.

- The complementary unit Gompertz distribution with the PDF and CDF given by

$$f_{CUGo}(x) = \frac{\beta\log(2)}{(1-\mu)^{-\beta}-1}(1-x)^{-(\beta+1)}\,2^{\frac{(1-x)^{-\beta}-1}{1-(1-\mu)^{-\beta}}},$$

and

$$F_{CUGo}(x) = 1-2^{\frac{(1-x)^{-\beta}-1}{1-(1-\mu)^{-\beta}}},$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a shape parameter, and $0 < \mu < 1$ is a scale parameter.

- The complementary unit Lomax distribution with the PDF and CDF given by

$$\begin{aligned} f_{CUL}(x) &= \frac{\log(2)}{\beta(1-x)}\left\{\log\left[1-\beta^{-1}\log(1-\mu)\right]\right\}^{-1} \\ &\quad\cdot\left[1-\beta^{-1}\log(1-x)\right]^{-\frac{\log 2}{\log\left[1-\beta^{-1}\log(1-\mu)\right]}-1}, \end{aligned}$$

and

$$F_{CUL}(x) = 1-\left[1-\beta^{-1}\log(1-x)\right]^{-\frac{\log 2}{\log\left[1-\beta^{-1}\log(1-\mu)\right]}},$$

respectively, for $0 < x < 1$, where $\beta > 0$, $0 < \mu < 1$ are scale parameters.

These distributions have been applied to model proportions of people aged 15 years old, or more who can read, or write a simple note.

2.61. A mixture power function and logarithmic distribution [3]

The unit distribution based on mixture of power function and continuous logarithmic distribution termed "power logarithmic" (PL) distribution in [3] has the PDF

$$f(x) = \frac{(\alpha+1)^2}{\beta+\delta+\alpha\beta}x^{\alpha}\left[\beta-\delta\log(x)\right], \tag{2.23}$$

for $0 < x < 1$, where $\alpha > 0, \beta > 0, \delta > 0$ are scale parameters. The corresponding CDF is

$$F(x) = \frac{x^{\alpha+1}\left\{\delta+(\alpha+1)\left[\beta-\delta\log(x)\right]\right\}}{\beta+\delta+\alpha\beta}.$$

It is possible to define the PDF provided by (2.23) as a two-component mixture as

$$f(x) = pf_p(x) + (1-p)f_L(x),$$

where $f_p(x) = (\alpha+1)x^{\alpha}$ for $0 \leq x \leq 1$ is the PDF of the power function distribution, $f_L(x) = (\alpha+1)^2 x^{\alpha} \log\left(\frac{1}{x}\right)$ for $0 \leq x \leq 1$ is the PDF of logarithmic distribution and $p = \frac{(\alpha+1)\beta}{\delta+(\alpha+1)\beta}$. The moment generating function and the kth moment of the power logarithmic distribution are

$$M(t) = \frac{(\alpha+1)^2}{\beta+\delta+\alpha\beta} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{(1+k+\alpha)\beta+\delta}{(1+k+\alpha)^2},$$

and

$$E\left(X^k\right) = \frac{(\alpha+1)^2}{\beta+\delta+\alpha\beta} \frac{(1+k+\alpha)\beta+\delta}{(1+k+\alpha)^2},$$

respectively. In particular, the mean and variance are

$$E(X) = \frac{(\alpha+1)^2}{(\alpha+2)^2} \frac{(\alpha+2)\beta+\delta}{(\alpha+1)\beta+\delta},$$

and

$$Var(X) = \frac{(\alpha+1)^2}{\beta+\delta+\alpha\beta} \left\{ \frac{(\alpha+3)\beta+\delta}{(\alpha+3)^2} - \frac{(\alpha+1)^2 \left[(\alpha+2)\beta+\delta\right]^2}{(\alpha+2)^4(\beta+\delta+\alpha\beta)} \right\},$$

respectively. In reference [3], Figure 2 showed that PDF can be unimodal, increasing, or decreasing. In reference [3], Table 2 showed that for fixed β and δ, skewness decreases as α increases, while kurtosis first decreases and, then increases as α increases. Skewness, and kurtosis decrease when β increases for fixed α and δ. Also, for fixed α and β, skewness increases as δ increases while the kurtosis increases when δ increases. This distribution has been applied to model proportion of income spent on food in a city and proportions of total milk production in the first cow births.

2.62. Chesneau's logarithmic weighted power distribution

The logarithmic weighted power distribution in [42] has its PDF and CDF specified by

$$f(x) = \alpha x^{\alpha-1} \left[1 - \lambda - \lambda\alpha \log(x)\right],$$

and

$$F(x) = x^{\alpha} \left[1 - \lambda\alpha \log(x)\right],$$

respectively, for $0 < x < 1$, where $0 \leq \lambda \leq 1$ is a scale parameter, and $\alpha > 0$ is a shape parameter. The kth moment is

$$E\left(X^k\right) = \frac{\alpha}{(k+\alpha)^2} \left[k(1-\lambda) + \alpha\right].$$

In particular, the mean and variance are

$$E(X) = \frac{\alpha}{(1+\alpha)^2}\left[(1-\lambda)+\alpha\right],$$

and

$$Var(X) = \frac{\alpha}{(2+\alpha)^2}\left[2(1-\lambda)+\alpha\right] - \frac{\alpha^2}{(1+\alpha)^4}(1-\lambda+\alpha)^2,$$

respectively.

2.63. Grassia's transformed gamma distribution

Grassia [71] proposed distributions based on the gamma random variable by using the logarithmic transformation $X = \exp(-Y)$ and the complementary version $X = 1-\exp(-Y)$, where Y is a gamma random variable with shape parameter α and scale parameter β. The probability distribution arising from this transformation has the PDF

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\beta-1}\left[\log\left(\frac{1}{x}\right)\right]^{\alpha-1}, \tag{2.24}$$

for $0 < x < 1$, where $\alpha > 0, \beta > 0$ are shape parameters. The PDF of the complementary version of this distribution is

$$f(x') = \frac{\beta^\alpha}{\Gamma(\alpha)}(1-x')^{\beta-1}\left[\log\left(\frac{1}{1-x'}\right)\right]^{\alpha-1},$$

for $0 < x' < 1$. Furthermore, the mean, variance, and the kth moment of (2.24) are

$$E(X) = \left(\frac{\beta}{\beta+1}\right)^\alpha,$$

$$Var(X) = \left(\frac{\beta}{\beta+2}\right)^\alpha - \left(\frac{\beta}{\beta+1}\right)^{2\alpha},$$

and

$$E\left(X^k\right) = \left(\frac{\beta}{\beta+k}\right)^\alpha,$$

respectively. Grassia [71] gave a full description of the behavior of the PDF: If $\alpha < 1$ and $\beta < 1$, then the PDF has a U shape, is asymptotic at both $x = 0$ and $x = 1$ and has a minimum at $x = \exp\left(-\frac{\beta-1}{\alpha-1}\right)$; if $\alpha < 1$ and $\beta = 1$, then the PDF has a distorted J shape tangential to $x = 0$ at the origin and then a proper J shape becoming asymptotic to $x = 1$; if $\alpha < 1$ and $\beta > 1$, then the PDF a proper J shape starting at 0 and then increasing steeply to become asymptotic at $x = 1$; if $\alpha = 1$ and $\beta < 1$, then the PDF has an inverted J shape and is asymptotic at $x = 0$; if $\alpha = 1$ and $\beta = 1$, then the PDF is a rectangular shaped; if $\alpha = 1$ and $\beta > 1$, then the PDF starts from the origin and then intercepts $x = 1$, taking a finite value; if $\alpha > 1$ and $\beta \leq 1$, then the PDF has a distorted reversed J shape and is asymptotic at $x = 0$; if $\alpha > 1$ and $\beta > 1$, then the PDF starts from 0, increases to reach a maximum at $x = \exp\left(-\frac{\beta-1}{\alpha-1}\right)$, and then decreases.

2.64. Hahn's rectangular beta distribution

Let Y denote a beta random variable with shape parameters $\alpha > 0$ and $\beta > 0$. The rectangular beta distribution in Hahn [75] can be specified by the PDF

$$f(x) = \theta + (1-\theta) f_Y(x;\alpha,\beta),$$

for $0 < x < 1$, where $0 \leq \theta \leq 1$ is a mixture parameter. From the rectangular beta distribution, we obtain the uniform and beta distributions by setting $\theta = 1$ and $\theta = 0$, respectively. The mean and variance associated with the rectangular beta distribution are

$$E(X) = \frac{\theta}{2} + (1-\theta)\frac{\alpha}{\alpha+\beta},$$

and

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}(1-\theta)\{1-\theta[1+(\alpha+\beta)]\} + \frac{\theta}{12}(4-3\theta),$$

respectively. Hahn [75, Figure 2] showed that the variance has a minimum of $\frac{1}{36}$ when $\theta = 1$ and thereafter increases as a function of θ. They also showed that the mean becomes more moderated as uncertainty grows and θ declines. As θ tends to 0, the mean tends toward $\frac{1}{2}$ given the lack of certainty. This distribution has been applied to model a real-world electronic module development project.

2.65. A trapezoidal beta distribution [60]

Let Y denote a beta random variable with shape parameters $\alpha > 0$ and $\beta > 0$. The trapezoidal beta distribution in [60] can be specified by the PDF

$$f(x) = a + (b-a)x + \left(1 - \frac{a+b}{2}\right) f_Y(x;\alpha,\beta), \tag{2.25}$$

for $0 < x < 1$, where $0 \leq a, b \leq 2, 0 \leq a+b \leq 2$ are scale parameters. An alternative parameterization of the trapezoidal beta distribution can be obtained by taking into consideration three different beta distributions. For this, the PDF in (2.25) can be modified as

$$\begin{aligned} f(x) &= \omega_1 f_1(x) + \omega_2 f_2(x) + \omega_3 f_3(x) \\ &= \frac{a}{2}(2-2x) + \frac{b}{2}(2x) + \left(1 - \frac{a+b}{2}\right) f_Y(x;\alpha,\beta), \end{aligned}$$

where $f_1(x) = f_Y(x;1,2) = 2-2x$, $f_2(x) = f_Y(x;2,1) = 2x$ and $f_3(x) = f_Y(x;\alpha,\beta)$. Furthermore $\omega_1 = \frac{a}{2}, \omega_2 = \frac{b}{2}, \omega_3 = \left(1 - \frac{a+b}{2}\right)$ are the mixture weights adjusted so that $\omega_1 + \omega_2 + \omega_3 = 1$, this implies that $\leq \omega_1, \omega_2, \omega_3 \leq 1$. The associated kth moment of the trapezoidal beta distribution is

$$E\left(X^k\right) = \frac{a}{k+1} + \frac{b-a}{k+2} + \left(1 - \frac{a+b}{2}\right)\left(\prod_{i=0}^{k-1} \frac{\alpha+i}{\alpha+\beta+i}\right).$$

In particular, the mean and variance of the trapezoidal beta distribution are

$$E(X) = \frac{a+2b}{6} + \left(1 - \frac{a+b}{2}\right)\frac{\alpha}{\alpha+\beta},$$

and

$$\begin{aligned} Var(X) &= \frac{3a+9b-(a+2b)^2}{36} + \left(\frac{\alpha}{\alpha+\beta}\right)\left(1-\frac{a+b}{2}\right) \\ &\cdot \left[\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha(2-a-b)}{2(\alpha+\beta)} - \frac{a+2b}{3}\right], \end{aligned}$$

respectively. The parameters a and b can be intuitively interpreted as lifting the left and right tails, respectively. This distribution has been applied to model average scores of a university selection test for school establishments in the Metropolitan Region of Chile.

2.66. Unit ratio-extended Weibull family of distributions [144]

Following the transformation $X = \frac{Y}{1+Y}$, where Y is a random variable based on the extended Weibull class of distributions, the researchers in [144] proposed the unit ratio extended Weibull class of distributions, The PDF and CDF of X are

$$f(x) = \alpha\,(1-x)^{-2}\, h\left(\frac{x}{1-x};\boldsymbol{\xi}\right)\exp\left[-\alpha H\left(\frac{x}{1-x};\boldsymbol{\xi}\right)\right], \tag{2.26}$$

and

$$F(x) = 1 - \exp\left[-\alpha H\left(\frac{x}{1-x};\boldsymbol{\xi}\right)\right], \tag{2.27}$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, $H(x;\boldsymbol{\xi})$ is a non-negative function that increases monotonically and is dependent on the parameter vector $\boldsymbol{\xi}$, and $h(x;\boldsymbol{\xi}) = H'(x;\boldsymbol{\xi})$. Based on (2.26) and (2.27), the researchers in [144] proposed five special cases of this distribution. They included the unit ratio-Gompertz distribution, the unit ratio-Burr XII distribution, the unit ratio-Lomax distribution, the unit ratio-Weibull distribution, and the unit ratio-Rayleigh distribution. The mathematical expressions for these special cases are well documented in [144]. This distribution has been applied to model educational attainment data.

2.67. A log-cosine-power unit distribution [136]

The log-cosine-power distribution in [136] has the PDF

$$f(x) = -\frac{\alpha\lambda}{\log\left[\cos(\lambda)\right]} x^{\alpha-1}\tan\left(\lambda x^{\alpha}\right), \tag{2.28}$$

for $0 < x < 1$, where $\alpha > 0$ is a shape parameter, and $0 < \lambda < \frac{\pi}{2}$ is a scale parameter. The associated CDF is

$$F(x) = \frac{1}{\log\left[\cos(\lambda)\right]}\log\left[\cos\left(\lambda x^{\alpha}\right)\right].$$

To aid for the derivation of the statistical properties of the PDF, the authors expanded (2.28) as

$$f(x) = -\frac{\alpha\lambda}{\log\left[\cos(\lambda)\right]} x^{\alpha-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2^{2j} \left(2^{2j}-1\right) B_{2j}}{(2j!)} \left(\lambda x^{\alpha}\right)^{2j-1} = \sum_{j=1}^{\infty} \omega_j x^{2\alpha j-1},$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where B_{2j} are Bernoulli numbers,

$$\tan(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2^{2j} \left(2^{2j}-1\right) B_{2j}}{(2j!)},$$

and

$$\omega_j = \frac{\alpha\lambda}{\log\left[\cos(\lambda)\right]} x^{\alpha-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2^{2j} \left(2^{2j}-1\right) B_{2j}}{(2j!)} \lambda^{2j-1}.$$

Thus, the kth moment is

$$E\left(X^k\right) = \sum_{j=1}^{\infty} \frac{\omega_j}{2\alpha j + k}.$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{j=1}^{\infty} \frac{\omega_j}{2\alpha j + 1},$$

and

$$Var\left(X\right) = \sum_{j=1}^{\infty} \frac{\omega_j}{2\alpha j + 2} - \left[E(X)\right]^2,$$

respectively. In reference [136], Figure 1 showed that the PDF can be J shaped, reversed J shaped, or bathtub shaped. In reference [136], Figure 2 showed that the distribution can be left skewed, or right skewed. This distribution has been applied to model proportion of uninsured English speaking adults in the USA and failure times of Kevlar49/epoxy strands data.

2.68. *Unit exponentiated Lomax distribution [58]*

Suppose Y is an exponentiated Lomax random variable with shape parameters δ, η, and scale parameter λ. Let $X = \exp(-Y)$. The researchers in [58] showed that X has the PDF and CDF

$$f(x) = \frac{\lambda\delta\eta}{x} \left(1 - \lambda\log(x)\right)^{-\delta-1} \left\{1 - \left[1 - \lambda\log(x)\right]^{-\delta}\right\}^{\eta-1},$$

and

$$F(x) = 1 - \left\{1 - \left[1 - \lambda\log(x)\right]^{-\delta}\right\}^{\eta},$$

respectively, for $0 < x < 1$, where $\delta, \eta > 0$ are shape parameters, and $\lambda > 0$ is a scale parameter. The kth moment is

$$E\left(X^k\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} k^j \delta\eta}{j!\lambda^j} \binom{\eta - 1}{i} B\left(1 + j, \delta(i+1) - j\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} \delta\eta}{j!\lambda^j} \binom{\eta - 1}{i} B\left(1 + j, \delta(i+1) - j\right),$$

and

$$Var\left(X\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} 2^j \delta\eta}{j!\lambda^j} \binom{\eta - 1}{i} B\left(1 + j, \delta(i+1) - j\right) - \left[E(X)\right]^2,$$

respectively. In reference [58], Figure 1 showed that the PDF can be symmetric, U shaped, right skewed, J shaped, or normal tapered. Skewness values in [58, Table 1] showed that the distribution can be left skewed, or right skewed. Kurtosis values in [58, Table 1] showed that the distribution leptokurtic, or platykurtic. This distribution has been applied to model data with responses of naive mock jurors, proportion of income spent on food in a city in the USA, and mortality rate of COVID-19.

2.69. Unit Johnson-t distribution [122]

Suppose Y is a Student's t random variable with ν degrees of freedom. Let $X = \left[1 + \exp\left(-\frac{Y-\alpha}{\theta}\right)\right]^{-1}$. The researchers in [122] showed that X has the PDF and CDF

$$f(x) = \frac{\theta \nu^{\frac{\nu}{2}} B\left(\frac{1}{2}, \frac{\nu}{2}\right)^{-1}}{x(1-x)} \left\{\nu + \left[\alpha + \theta\, l(x)\right]^2\right\}^{\frac{-\nu-1}{2}},$$

and

$$F(x) = \frac{1}{2}\left\{1 + \mathrm{sign}\left[\alpha + \theta\, l(x)\right]\left[1 - I_{m(\alpha+\theta h(x))}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right]\right\},$$

respectively, for $0 < x < 1$, where $\nu > 0$ denotes the degree of freedom, $-\infty < \alpha < \infty$, $\theta > 0$ are scale parameters, $l(x) = \log\left(\frac{x}{1-x}\right)$ and $m(z) = \frac{z}{\nu+z^2}$. This distribution has been applied to model body fat percentage of individuals assisted in a public hospital in Brazil and recovery rates of COVID-19 in the USA.

2.70. Unit half-normal distribution

Suppose Y is a half-normal random variable with scale parameter a. Let $X = \frac{Y}{1+Y}$. The researchers in [28] showed that X has the PDF and CDF

$$f(x) = \frac{2}{a(1-x)^2} \phi\left(\frac{x}{a(1-x)}\right),$$

and

$$F(x) = 2\Phi\left(\frac{x}{a(1-x)}\right) - 1,$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The associated kth moment is

$$E\left(X^k\right) = a^k E\left[\frac{X^k}{(1+aZ)^k}\right],$$

for $k \geq 1$, where Z denotes a half-normal random variable. In particular, the mean and variance are

$$E\left(X\right) = aE\left[\frac{X}{1+aZ}\right],$$

and

$$Var\left(X\right) = a^2 E\left[\frac{X^2}{(1+aZ)^2}\right] - \left[E(X)\right]^2,$$

respectively. In reference [28], Figure 1 showed that the PDF can be unimodal, or asymmetric (left and right skewed). This distribution has been applied to model a database extracted from an image of Foulum (Denmark).

2.71. A power Johnson SB distribution [38]

The power Johnson SB distribution in [38] has the PDF and CDF

$$f(x) = \delta m\left(\gamma + \delta Q(x); \alpha\right) \phi\left(\gamma + \delta Q(x)\right) \left|\frac{dQ(x)}{dx}\right|,$$

and

$$F(x) = \left[\Phi\left(\gamma + \delta\, Q(x)\right)\right]^{\alpha},$$

respectively, for $0 < x < 1$, where $m(x) = \alpha^2\Phi(x)$, $-\infty < \gamma < \infty$, $\delta > 0$ are scale parameters, $\alpha > 0$ is a shape parameter, and $Q(x) = G^{-1}(x)$ represents the convenient quantile of a valid CDF $G(\cdot)$. The kth moment is

$$E\left(X^k\right) = \sum_{i=0}^{k} \binom{k}{i} \gamma^i \delta^{k-i} \mu_{k-i},$$

for $k \geq 1$, where

$$\mu_k = \alpha \int_0^1 \left[G\left(\Phi^{-1}(\mu)\right)\right]^k \mu^{\alpha-1} d\mu.$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{i=0}^{1} \gamma^i \delta^{1-i} \mu_{1-i},$$

and

$$Var\,(X) = \sum_{i=0}^{2} \binom{2}{i} \gamma^i \delta^{2-i} \mu_{2-i} - [E(X)]^2\,,$$

respectively. In reference [38], Figure 1 showed that PDF can be left skewed, right skewed, or unimodal. This distribution has been applied to model colorectal cancer incidence and mortality rates by race/ethnicity in the USA.

2.72. An arcsecant hyperbolic Weibull distribution [105]

Following the transformation $X = \sec\text{h}(Y)$, where Y is a Weibull random variable with shape parameter α and scale parameter θ, The researchers in [105] proposed the arcsecant hyperbolic Weibull distribution with the PDF and CDF given by

$$f(x) = \frac{\alpha\theta}{x\sqrt{1-x^2}} \left[\text{arcsinh}(x)\right]^{\theta-1} \exp\left\{-\alpha\ \left[\text{arcsinh}(x)\right]^{\theta}\right\},$$

and

$$F(x) = \exp\left\{-\alpha\ \left[\text{arcsinh}(x)\right]^{\theta}\right\},$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, $\theta > 0$ is a shape parameter, and $\text{arcsinh}(x) = \log\left[\left(1 + \frac{\sqrt{1-x^2}}{x}\right)\right] \in (0, \infty)$. The associated kth moment is

$$E\left(X^k\right) = \sum_{i,j=0}^{\infty} 2^k \binom{-k}{i} \frac{(-1)^j}{j!} (k+2i)^j \alpha^{-\frac{j}{\theta}} \Gamma\left(\frac{j}{\theta}+1\right).$$

In particular, the mean and variance are

$$E\,(X) = 2 \sum_{i,j=0}^{\infty} \binom{-1}{i} \frac{(-1)^j}{j!} (1+2i)^j \alpha^{-\frac{j}{\theta}} \Gamma\left(\frac{j}{\theta}+1\right),$$

and

$$Var\,(X) = 4 \sum_{i,j=0}^{\infty} \binom{-2}{i} \frac{(-2)^j}{j!} (1+i)^j \alpha^{-\frac{j}{\theta}} \Gamma\left(\frac{j}{\theta}+1\right) - [E(X)]^2\,,$$

respectively. In reference [105], Figures 1 and 2 showed that the PDF can be U shaped, increasing, decreasing, unimodal, or N shaped. In reference [105], Table 1 showed that skewness can be negative, or positive. This distribution has been applied to model educational attainment data.

2.73. Unit Rayleigh distribution [30]

Suppose Y is a Rayleigh random variable with scale parameter β. Let $X = \exp(-Y)$. The researchers in [30] showed that X has the PDF and CDF

$$f(x) = -\frac{2\beta}{x} \log(x) \exp\left\{-\beta \left[\log(x)\right]^2\right\},$$

and

$$F(x) = \exp\left\{-\beta\left[\log(x)\right]^2\right\},$$

respectively, for $0 < x < 1$, where $\beta > 0$ is a scale parameter. The kth moment is

$$E\left(X^k\right) = 1 - \exp\left(\frac{k^2}{4\beta}\right)\frac{k}{\sqrt{\beta}}\frac{\sqrt{\pi}}{2}\operatorname{erfc}\left(\frac{k}{2\sqrt{\beta}}\right).$$

In particular, the mean and variance are

$$E\left(X\right) = 1 - \exp\left(\frac{1}{4\beta}\right)\frac{1}{\sqrt{\beta}}\frac{\sqrt{\pi}}{2}\operatorname{erfc}\left(\frac{1}{2\sqrt{\beta}}\right),$$

and

$$Var\left(X\right) = 1 - \exp\left(\frac{1}{\beta}\right)\frac{k}{\sqrt{\beta}}\frac{\sqrt{\pi}}{2}\operatorname{erfc}\left(\frac{1}{\sqrt{\beta}}\right) - \left[E(X)\right]^2,$$

respectively. In reference [30], Figure 1 showed that the PDF can be decreasing, increasing, increasing-decreasing, skewed abruptly to the right, moderate right skewed, or decreasing-increasing. Skewness values in [30, Table 1] suggest that the distribution can be negatively, or positively skewed. Kurtosis values in [30, Table 1] are lower than 3, nearly equal to 3, or greater than 3. This distribution has been applied to model times to infection of kidney dialysis patients in months, failure times of the air conditioning system of an airplane and maximum flood levels of a particular river in Pennsylvania.

2.74. Mazucheli and Alves [121]'s unit Gumbel distribution

Suppose Y is a Gumbel random variable and let $X = \frac{1}{1+\exp(-Y)}$. Mazucheli and Alves [121] showed that X has the PDF and CDF

$$f(x) = \frac{\theta}{x(1-x)}\exp\left\{-\alpha - \theta\log\left(\frac{x}{1-x}\right) - \exp\left[-\alpha - \theta\log\left(\frac{x}{1-x}\right)\right]\right\},$$

and

$$F(x) = \exp\left[-\exp(-\alpha)\left(\frac{1-x}{x}\right)^{\theta}\right],$$

respectively, for $0 < x < 1$, where $\theta > 0$ is a shape parameter, and $-\infty < \alpha < \infty$ is a scale parameter.

2.75. Unit Gumbel type-II distribution [164]

Suppose Y is a Gumbel type-II random variable with shape parameter μ and scale parameter ν. Let $X = \frac{Y}{1+Y}$. The researchers in [164] showed that X has the PDF and CDF

$$f(x) = \mu\nu\frac{x^{-\mu-1}}{(1-x)^{1-\mu}}\exp\left[-\nu\left(\frac{x}{1-x}\right)^{-\mu}\right],$$

and

$$F(x)=\exp\left[-\nu\left(\frac{x}{1-x}\right)^{-\mu}\right],$$

respectively, for $0<x<1$, where $\mu>0$ is a shape parameter, and $\nu>0$ is a scale parameter. The kth moment of the unit Gumbel type-II distribution is

$$E\left(X^{k}\right)=\sum_{i=0}^{\infty}(-1)^{i}\binom{k+i-1}{i}\frac{\Gamma\left(1+\frac{i}{\mu}\right)}{\nu^{\frac{i}{\mu}}},$$

provided that $\frac{k}{\mu}>-1$. In particular, the mean and variance are

$$E(X)=\sum_{i=0}^{\infty}(-1)^{i}\frac{\Gamma\left(1+\frac{i}{\mu}\right)}{\nu^{\frac{i}{\mu}}},$$

and

$$Var(X)=\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\frac{\Gamma\left(1+\frac{i}{\mu}\right)}{\nu^{\frac{i}{\mu}}}-[E(X)]^{2},$$

provided that $\frac{1}{\mu}>-1$. In reference [164], Figure 1 showed that the PDF can be right skewed, reversed J shaped, symmetrical, or U formed. In reference [164], Table 1 showed that, for fixed μ, mean, variance, skewness, and kurtosis decrease with increasing ν. This distribution has been applied to model core specimens from cross-sections of petroleum wells and flood level observations for the Susquehanna River at Harrisburg, Pennsylvania.

2.76. A composite quantile probability distribution [156]

The composite quantile probability distributions proposed by [156] have the PDFs

$$f_1(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left[-\log(x)\right]^{\phi\alpha-1}\exp\left\{-\beta\left[-\log(x)\right]^{\phi}\right\}\frac{\phi}{x},$$

and

$$f_2(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left[-\log(1-x)\right]^{\phi\alpha-1}\exp\left\{-\beta\left[-\log(1-x)\right]^{\phi}\right\}\frac{\phi}{1-x},$$

for $0<x<1$, where $\theta>0,\beta>0$ scale parameters, and $\alpha>0,\phi>0$ are shape parameters. It is important to note that if $\phi=1$, then $f_1(\cdot)$ and $f_2(\cdot)$ reduce to transformed gamma distributions in Grassia [71]. This suggests that composite quantile probability distributions encompass the Grassia's distribution as special cases. The associated kth moments of these distributions are

$$E\left(X^{k}\right)=\sum_{i=0}^{\infty}\frac{\left(-k\beta^{-\frac{1}{\phi}}\right)^{i}}{i!}\frac{\Gamma\left(\alpha+\frac{i}{\phi}\right)}{\Gamma\left(\alpha\right)},$$

and

$$E\left(X^k\right)=\sum_{i=0}^{k}\binom{k}{i}(-1)^i\sum_{j=0}^{\infty}\frac{\left(-i\beta^{-\frac{1}{\phi}}\right)^j}{j!}\frac{\Gamma\left(\frac{\phi\alpha+j}{\phi}\right)}{\Gamma\left(\alpha\right)},$$

respectively. The associated means and variances of these distributions are

$$E\left(X\right)=\sum_{i=0}^{\infty}\frac{\left(-\beta^{-\frac{1}{\phi}}\right)^i}{i!}\frac{\Gamma\left(\alpha+\frac{i}{\phi}\right)}{\Gamma\left(\alpha\right)},\qquad Var\left(X\right)=\sum_{i=0}^{\infty}\frac{\left(-2\beta^{-\frac{1}{\phi}}\right)^i}{i!}\frac{\Gamma\left(\alpha+\frac{i}{\phi}\right)}{\Gamma\left(\alpha\right)}-\left[E(X)\right]^2,$$

and

$$E\left(X\right)=\sum_{i=0}^{1}(-1)^i\sum_{j=0}^{\infty}\frac{\left(-i\beta^{-\frac{1}{\phi}}\right)^j}{j!}\frac{\Gamma\left(\frac{\phi\alpha+j}{\phi}\right)}{\Gamma\left(\alpha\right)},\qquad Var\left(X\right)=\sum_{i=0}^{2}\binom{2}{i}(-1)^i\sum_{j=0}^{\infty}\frac{\left(-i\beta^{-\frac{1}{\phi}}\right)^j}{j!}\frac{\Gamma\left(\frac{\phi\alpha+j}{\phi}\right)}{\Gamma\left(\alpha\right)}-\left[E(X)\right]^2,$$

respectively. This distribution has been applied to model proportion of poverty in Peru.

2.77. Unit Maxwell-Boltzmann distribution [34]

Suppose Y is a Maxwell-Boltzmann random variable with scale parameter θ. Let $X=\exp(-Y)$. The researchers in [34] showed that X has the PDF and CDF

$$f(x)=\frac{\sqrt{\frac{2}{\pi}}\log^2\left(\frac{1}{x}\right)\exp\left[-\frac{\log^2\left(\frac{1}{x}\right)}{2\theta^2}\right]}{\theta^3 x},$$

and

$$F(x)=1-\frac{\sqrt{\log^2\left(\frac{1}{x}\right)}\operatorname{erf}\left(\frac{\sqrt{\log^2\left(\frac{1}{x}\right)}}{\sqrt{2}\theta}\right)}{\log\left(\frac{1}{x}\right)}+\frac{\sqrt{\frac{2}{\pi}}\log\left(\frac{1}{x}\right)\exp\left[-\frac{\log^2\left(\frac{1}{x}\right)}{2\theta^2}\right]}{\theta},$$

respectively, for $0<x<1$, where $\theta>0$ is a scale parameter. The associated kth moment is

$$E\left(X^k\right)=\exp\left(\frac{\theta^2k^2}{2}\right)\left(\theta^2k^2+1\right)\operatorname{erfc}\left(\frac{\theta k}{\sqrt{2}}\right)-\sqrt{\frac{2}{\pi}}\theta k.$$

In particular, the mean and variance are

$$E\left(X\right)=\exp\left(\frac{\theta^2}{2}\right)\left(\theta^2+1\right)\operatorname{erfc}\left(\frac{\theta}{\sqrt{2}}\right)-\sqrt{\frac{2}{\pi}}\theta,$$

and

$$Var\left(X\right)=\exp\left(2\theta^2\right)\left(4\theta^2+1\right)\operatorname{erfc}\left(\sqrt{2}\theta\right)-2\sqrt{\frac{2}{\pi}}\theta-\left[E(X)\right]^2,$$

respectively. In reference [34], Figures 1 and 2 showed that the PDF can be unimodal. This distribution has been applied to model concentration of air pollutant CO in Alberta, Canada, air quality monitoring of the annual average concentration of a pollutant and concentration of sulfate in Calgary, Canada.

2.78. A generalized biparabolic distribution [62]

The standardized generalized biparabolic distribution proposed by [62] has the PDF

$$f(x) = C(m) \cdot \begin{cases} \left(\frac{x}{\theta}\right)^{2m} - 2\left(\frac{x}{\theta}\right)^{m}, & \text{if } 0 < x \leq \theta, \\ \left(\frac{1-x}{1-\theta}\right)^{2m} - 2\left(\frac{1-x}{1-\theta}\right)^{m}, & \text{if } \theta < x < 1, \end{cases} \tag{2.29}$$

where $m > 0$ is a shape parameter, $0 < \theta < 1$ is a cutoff point, and $C(m) = \frac{(2m+1)(m+1)}{-3m-1}$. The corresponding CDF is

$$F(x) = \begin{cases} C(m)\theta \frac{\left(\frac{x}{\theta}\right)^{2m+1}}{2m+1} - \frac{2\left(\frac{x}{\theta}\right)^{m+1}}{m+1}, & \text{if } 0 < x \leq \theta, \\ 1 + C(m)(\theta - 1)\frac{\left(\frac{1-x}{1-\theta}\right)^{2m+1}}{2m+1} - \frac{2\left(\frac{1-x}{1-\theta}\right)^{m+1}}{m+1}, & \text{if } \theta < x < 1. \end{cases}$$

By denormalizing the PDF (2.29) using the change of variable approach discussed by [62], the unstandardized version of biparabolic distribution known as the generalized biparabolic distribution has the PDF

$$f(x) = C(m) \cdot \begin{cases} \left(\frac{x-a}{\theta'-a}\right)^{2m} - 2\left(\frac{x-a}{\theta'-a}\right)^{m}, & \text{if } a < x \leq \theta', \\ \left(\frac{b-x}{b-\theta'}\right)^{2m} - 2\left(\frac{b-x}{b-\theta'}\right)^{m}, & \text{if } \theta' < x < b, \end{cases}$$

where $C(m) = \frac{(2m+1)(m+1)}{(-3m-1)(b-a)}$.

2.79. Okorie and Afuecheta's unit upper truncated Weibull distribution

Following the transformation $X = \frac{Y}{\alpha}$, where Y is an upper truncated Weibull random variable, the unit upper truncated Weibull distribution in Okorie and Afuecheta [137] has the PDF

$$f(x) = \frac{\lambda\beta}{1 - \exp(-\lambda)} x^{\beta-1} \exp\left(-\lambda x^{\beta}\right),$$

for $0 \leq x \leq 1$, where $\lambda > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The CDF associated is

$$F(x) = \frac{1 - \exp\left(-\lambda x^{\beta}\right)}{1 - \exp(-\lambda)}.$$

The kth moment is

$$E\left(X^{k}\right) = \frac{\gamma\left(\frac{k}{\beta} + 1, \lambda\right)}{\lambda^{\frac{k}{\beta}}\left[1 - \exp(-\lambda)\right]},$$

for $k \geq 1$. In particular, the mean and variance are

$$E(X) = \frac{\gamma\left(\frac{1}{\beta}+1,\lambda\right)}{\lambda^{\frac{1}{\beta}}\left[1-\exp(-\lambda)\right]},$$

and

$$Var(X) = \frac{\gamma\left(\frac{2}{\beta}+1,\lambda\right)\left[1-\exp(-\lambda)\right]-\gamma\left(\frac{1}{\beta}+1,\lambda\right)^2}{\lambda^{\frac{2}{\beta}}\left[1-\exp(-\lambda)\right]^2},$$

respectively. Okorie and Afuecheta [137, Figure 1] showed that the PDF can be monotonic increasing, monotonic decreasing, or unimodal. Okorie and Afuecheta [137, Figure 2] showed that the distribution can be symmetric, or asymmetric while its tail can be only platykurtic. This distribution has been applied to model Better Life Indices data and recovery rate of CD34+ cells after peripheral blood stem cell transplants.

2.80. Chesneau's variable-power parametric distributions

Using the idea of variable-power parametric (VPP) functions, the researchers in [44] proposed a set of eight unique CDFs. These include

- VPP CDF of the first kind with the functional form denoted as

$$F(x) = x^{x^{\frac{[-\log(x)]^b}{a}}},$$

  for $0 < x < 1$, where $a \geq 0$ and $b > -1$ are shape parameters.
- VPP CDF of the second kind with the functional form denoted as

$$F(x) = x^{a+bx+cx\log(x)},$$

  for $0 < x < 1$, where $a \geq I\{c \neq 0\} - b$, $b \leq 0$, $a > 0$ and $0 \leq c \leq 1$ are shape parameters.
- VPP CDF of the third kind with the functional form denoted as

$$F(x) = (1-x)^{a(1-x)^b} - (1-x),$$

  for $0 < x < 1$, where $0 \leq a \leq 1$ and $b \geq 1$ are shape parameters.
- VPP CDF of the fourth kind with the functional form denoted as

$$F(x) = \frac{1}{a}\left\{[1+ax+bx\log(x)]^x - x^x\right\},$$

  for $0 < x < 1$, where $a \leq 0$ and $a \geq 1-b$ are scale parameters.
- VPP CDF of the fifth kind with the functional form denoted as

$$F(x) = (1+a)\left[1+\frac{a+b\log(x)+cx\log(x)}{x}\right]^{-\frac{1}{x^d}},$$

  for $0 < x < 1$, where $a \geq 0$, $b \geq 0$, $c \leq 0$ are scale parameters such that $a-b-c \neq 0$ and $d \geq 0$ is a shape parameter.

- VPP CDF of the sixth kind with the functional form denoted as

$$F(x) = \left[1 + \frac{a + b\log(x) + cx\log(x)}{x}\right]^{1-\frac{1}{x^d}},$$

for $0 < x < 1$, where $a \geq 0$, $b \geq 0$, $c \leq 0$ are scale parameters such that $a - b - c \neq 0$ and $d > 0$ is a shape parameter.

- VPP CDF of the seventh kind with the functional form denoted as

$$F(x) = x^{x^{-a\left(1-x^b\right)}},$$

for $0 < x < 1$, where $a \geq 0$ and $b \geq 0$ are shape parameters.

- VPP CDF of the eight kind with the functional form denoted as

$$F(x) = 1 - (1 - x)\frac{x^a}{[-\log(x)]^b},$$

for $0 < x < 1$, where $a \geq -1$ and $b \geq 0$ are shape parameters such that $1 + a + b \neq 0$.

2.81. *Bakouch et al.'s unit exponential distribution*

By employing the epsilon function $\varepsilon_{\lambda,a}(x) = \left(\frac{a+x}{a-x}\right)^{\frac{\lambda a}{2}}$, $-a < x < a$ as documented in [53], the researchers in [27] proposed unit exponential distribution with the PDF and CDF as

$$f(x) = \frac{2\alpha\beta}{1 - x^2}\left(\frac{1 + x}{1 - x}\right)^{\beta}\overline{F}(x), \tag{2.30}$$

and

$$F(x) = \begin{cases} 1 - \exp\left\{\alpha\left[1 - \left(\frac{1 + x}{1 - x}\right)^{\beta}\right]\right\}, & \text{for } 0 \leq x < 1, \\ 1, & \text{for } x = 1, \end{cases}$$

respectively, where α is a scale parameter, β is a shape parameter, and $\overline{F}(x) = 1 - F(x)$. The kth moment associated with (2.30) is

$$E\left(X^k\right) = \frac{2k\alpha^{\frac{1}{\beta}}\exp(\alpha)}{\beta}\sum_{i=0}^{k-1}\sum_{j=0}^{\infty}\binom{k-1}{i}\binom{-(k+1)}{j}(-1)^i\alpha^{\frac{i+j+1}{\beta}}\Gamma\left(-\frac{i+j+1}{\beta}, \alpha\right).$$

The mean and variance are

$$E(X) = \frac{2\alpha^{\frac{1}{\beta}}\exp(\alpha)}{\beta}\sum_{j=0}^{\infty}\binom{-(k+1)}{j}\alpha^{\frac{j+1}{\beta}}\Gamma\left(-\frac{j+1}{\beta}, \alpha\right),$$

and

$$Var\left(X\right) = \frac{4\alpha^{\frac{1}{\beta}}\exp(\alpha)}{\beta}\sum_{i=0}^{1}\sum_{j=0}^{\infty}\binom{-3}{j}(-1)^{i}\alpha^{\frac{i+j+1}{\beta}}\Gamma\left(-\frac{i+j+1}{\beta},\alpha\right)-\left[E(X)\right]^{2},$$

respectively. In reference [27], Figure 2 showed that the PDF can be monotonically decreasing, or unimodal. This distribution has been applied to model soil moisture content and permanent wilting points.

2.82. Unit Zeghdoudi distribution [32]

Suppose Y is a Zeghdoudi random variable with scale parameter α [129]. Let $X = \frac{Y}{1+Y}$. The researchers in [32] showed that X has the PDF and CDF

$$f(x) = 1\exp\left(-\frac{\alpha x}{1-x}\right)\left[1+\frac{\alpha x(\alpha+2-2x)}{(\alpha+2)(1-x)^{2}}\right],$$

and

$$F(x) = \frac{\alpha^{3}x}{(1-x)^{4}(\alpha+2)}\exp\left(-\frac{\alpha x}{1-x}\right),$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter. The associated kth moment is

$$E\left(X^{k}\right) = \sum_{j=0}^{\infty}(-1)^{j}\binom{k+j-2}{j}\frac{\Gamma(k+j+2)}{(\alpha+2)\alpha^{k}+j-1}.$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{j=0}^{\infty}(-1)^{j}\binom{j-1}{j}\frac{\Gamma(j+3)}{(\alpha+2)\alpha+j-1},$$

and

$$Var\left(X\right) = \sum_{j=0}^{\infty}(-1)^{j}\binom{j}{j}\frac{\Gamma(j+4)}{(\alpha+2)\alpha^{2}+j-1}-\left[E(X)\right]^{2},$$

respectively. In reference [32], Figures 1 and 2 showed that the distribution can exhibit positive skewness, negative skewness, and unimodal characteristics. In reference [32], Table 1 showed that, as α grows, skewness climbs steadily and slowly, exhibiting a little asymmetry towards larger values. Kurtosis shows consistent tail behavior with modest range-wide expansion, remaining quite steady. This distribution has been applied to model COVID-19 related recovery in Spain, amount of water the California Shasta Reservoir could hold each month, and insecticidal and biochemical effects of Jatropha oil on cotton leaf worm.

2.83. Korkmaz and Korkmaz's unit distribution

Let Y denote a log-log random variable in Pham [145]. Let $X = \exp(-Y)$. The researchers in [106] showed that the PDF and CDF of X are

$$f(x) = a \log b x^{-1} - \log x)^{a-1} b^{(-\log x)^a} \exp\left[1 - b^{(-\log x)^a}\right],$$

and

$$F(x) = 1 - \exp\left[1 - b^{(-\log x)^a}\right],$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 1$ is a scale parameter. In reference [106], Figure 1 showed that the PDF can be skewed, bathtub shaped, inverse N shaped, decreasing, increasing, or unimodal. This distribution has been applied to model educational attainment data.

2.84. A unit distribution [79]

Let Y denote an inverse exponentiated Pareto random variable in [65]. Let $X = \frac{1}{1+Y}$. The researchers in [79] showed that the PDF and CDF of X are

$$f(x) = ab(1-x)^{b-1}\left[1-(1-x)^b\right]^{a-1},$$

and

$$F(x) = \left[1-(1-x)^b\right]^a,$$

respectively, for $0 < x < 1$, where $a > 0$ and $b > 0$ are both shape parameters. The kth moment of X is

$$E\left(X^k\right) = ab\sum_{i=0}^{k}(-1)^i\binom{a-1}{i}B\left(k+1, b(i+1)\right).$$

In particular, the mean and variance are

$$E\left(X\right) = ab\sum_{i=0}^{1}(-1)^i\binom{a-1}{i}B\left(2, b(i+1)\right),$$

and

$$Var\left(X\right) = ab\sum_{i=0}^{2}(-1)^i\binom{a-1}{i}B\left(3, b(i+1)\right) - [E(X)]^2,$$

respectively. In reference [79], Figure 1 showed that the PDF can be reversed J shaped, left skewed, right skewed, or unimodal. In reference [79], Table 1 showed that variance decreases when $a1$ increases and b is kept constant. In reference [79], Figure 3 showed that skewness ranges from about −2 to 1, showing a wide variety of options. Furthermore, kurtosis can have both small and large values, enabling the distribution to display all three kurtosis types: Leptokurtic, mesokurtic, and platykurtic. This distribution has been applied to model COVID-19 death rates were recorded in England and rock samples taken from a petroleum reservoir.

2.85. A unit distribution [91]

Let Y denote a zero-centered Gumbel random variable with scale parameter b. Let $X = \log\left(Y^{-a} - 1\right)$ for $a > 0$. The researchers in [91] showed that the PDF and CDF of X are

$$f(x) = \frac{abx^{ab-1}}{\left(1 - x^a\right)^{b+1}} \exp\left[-\frac{x^{ab}}{\left(1 - x^a\right)^b}\right],$$

and

$$F(x) = 1 - \exp\left[-\frac{x^{ab}}{\left(1 - x^a\right)^b}\right],$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. In reference [91], Theorem 1 showed that the PDF can be monotonically decreasing, bathtub shaped, unimodal on the left, or unimodal on the right. This distribution has been applied to model percentage of service usage time of end users.

2.86. A unit distribution [169]

Let Y denote a zero-centered Laplace random variable with scale parameter b. Let $X = \log\left(Y^{-a} - 1\right)$ for $a > 0$. The researchers in [169] showed that the PDF and CDF of X are

$$f(x) = \begin{cases} \dfrac{abx^{ab-1}}{2\left(1 - x^a\right)^{b+1}}, & \text{if } x \leq 2^{-\frac{1}{a}}, \\[2ex] \dfrac{ab\left(1 - x^a\right)^{b-1}}{2x^{ab+1}}, & \text{if } x > 2^{-\frac{1}{a}}, \end{cases}$$

and

$$F(x) = \begin{cases} \dfrac{x^{ab}}{2\left(1 - x^a\right)^{b}}, & \text{if } x \leq 2^{-\frac{1}{a}}, \\[2ex] 1 - \dfrac{\left(1 - x^a\right)^{b}}{2x^{ab}}, & \text{if } x > 2^{-\frac{1}{a}}, \end{cases}$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{b}{2}\left[B_{\frac{1}{2}}\left(\frac{k}{a} + b, -b\right) + B_{\frac{1}{2}}\left(b, \frac{k}{a} - b\right)\right].$$

In particular, the mean and variance are

$$E(X) = \frac{b}{2}\left[B_{\frac{1}{2}}\left(\frac{1}{a} + b, -b\right) + B_{\frac{1}{2}}\left(b, \frac{1}{a} - b\right)\right],$$

and

$$Var(X) = \frac{b}{2}\left[B_{\frac{1}{2}}\left(\frac{2}{a}+b,-b\right)+B_{\frac{1}{2}}\left(b,\frac{2}{a}-b\right)\right]-[E(X)]^2,$$

respectively. In reference [169], Theorem 1 showed that the PDF is decreasing if $a(1+2b) < 1$ and $b > 1$, increasing if $a(1-2b) > 1$ and $ab > 1$; left tailed and right vanishing if $a(1+2b) > 1$, $ab \le 1$ and $b > 1$; right tailed and left vanishing if $a(1-2b) \le 1$, $ab > 1$ and $b \le 1$; both sides tailed if $ab \le 1$ and $b \le 1$; and both sides vanishing if $ab > 1$ and $b > 1$. This distribution has been applied to model broadband usage in rural counties in the United States, historical data on the melting rate of the South Greenland ice sheet and log-returns of daily changes in natural gas prices.

2.87. A unit distribution [31]

Let Y denote a gamma/Gompertz random variable with shape parameter a, and scale parameters b, c. Let $X = -\log Y$. The researchers in [31] showed that the PDF and CDF of X are

$$f(x) = \frac{abc^a x^{ab-1}}{\left[1+(c-1)x^b\right]^{a+1}},$$

and

$$F(x) = \frac{c^a}{\left[c-1+x^{-b}\right]^a},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$, $c > 0$ are scale parameters. The kth moment of X is

$$E\left(X^k\right) = \begin{cases} \dfrac{abc^a}{k+ab}\,{}_2F_1\left(a+1,a+\dfrac{k}{b};a+\dfrac{k}{b}+1;1-c\right), & \text{if } c \ne 1, \\[2ex] \dfrac{ab}{k+ab}, & \text{if } c = 1. \end{cases}$$

In particular, the mean and variance are

$$E(X) = \begin{cases} \dfrac{abc^a}{1+ab}\,{}_2F_1\left(a+1,a+\dfrac{1}{b};a+\dfrac{1}{b}+1;1-c\right), & \text{if } c \ne 1, \\[2ex] \dfrac{ab}{1+ab}, & \text{if } c = 1, \end{cases}$$

and

$$Var(X) = \begin{cases} \dfrac{abc^a}{2+ab}\,{}_2F_1\left(a+1,a+\dfrac{2}{b};a+\dfrac{2}{b}+1;1-c\right)-[E(X)]^2, & \text{if } c \ne 1, \\[2ex] \dfrac{ab}{2+ab}-[E(X)]^2, & \text{if } c = 1, \end{cases}$$

respectively. In reference [31], Proposition 1 showed that the PDF is increasing if $ab \geq 1$, $\frac{ab-1}{c-1} < 0$, and $\frac{ab-1}{c-1} > b+1$; decreasing if $ab < 1$, $\frac{ab-1}{c-1} < 0$ and $\frac{ab-1}{c-1} > b+1$; increasing-decreasing if $ab \geq 1$ and $0 \leq \frac{ab-1}{c-1} \leq b+1$; decreasing-increasing if $ab < 1$ and $0 \leq \frac{ab-1}{c-1} \leq b+1$. In reference [31], Table 1 showed that the distribution can be left and right skewed. Furthermore, kurtosis values can be lower, nearly equal, or greater than 3. This distribution has been applied to model individuals having HIV+.

2.88. *A unit distribution [47]*

The researchers in [47] introduced a unit distribution with the PDF and CDF specified by

$$f(x) = \frac{2}{ax}\left(\frac{\log x}{a}\right)^2 \phi\left(\frac{\log x}{a}\right),$$

and

$$F(x) = 2\Phi\left(\frac{\log x}{a}\right) - \frac{2\log x}{a}\phi\left(\frac{\log x}{a}\right),$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = 2^{\frac{a^2k^2}{2}}\left[ak\phi(ak) + \Phi(-ak) - 2ak\phi(ak) + a^2k^2 - a^2k^2\Phi(ak)\right].$$

In particular, the mean and variance are

$$E(X) = 2\exp\left(\frac{a^2}{2}\right)\left[\left(1+a^2\right)\Phi(-a) - a\phi(a)\right],$$

and

$$Var(X) = 2\exp\left(2a^2\right)\left[\left(1+a^2\right)\Phi(-2a) - 2a\phi(2a)\right] - [E(X)]^2,$$

respectively. The parameter a controls the skewness, and kurtosis of the distribution. This distribution has been applied to model dynamics of the Chilean inflation in the post-military dictatorship period and relative humidity of the air in a northern Chilean city.

2.89. *Unit power-logarithmic distribution*

The researchers in [43] introduced a unit power-logarithmic distribution with the PDF and CDF specified by

$$f(x) = \frac{ax^a\log x + 1 - x^a}{ax(\log x)^2},$$

and

$$F(x) = 1 - \frac{a\log x + 1 - x^a}{a\log x},$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = 1 - \frac{k}{a}\log\left(1 + \frac{a}{k}\right).$$

In particular, the mean and variance are

$$E(X) = 1 - \frac{1}{a}\log\left(1 + a\right),$$

and

$$Var(X) = 1 - \frac{2}{a}\log\left(1 + \frac{a}{2}\right) - \left[E(X)\right]^2,$$

respectively. In reference [43], Figure 2 showed that the PDF is decreasing for the small values of a, or U shaped. In reference [43], Figure 7 showed that skewness is a strictly decreasing function with respect to a and can be negative, or positive, meaning that the distribution can be left, or right skewed, respectively. Also, kurtosis is a non-monotonic function with a V shape. It can be inferior, equal, or superior to 3, meaning that the distribution may be platykurtic, mesokurtic, or leptokurtic, respectively.

2.90. Unit Muth distribution

Let Y denote a Muth random variable with shape parameter a and scale parameter b. Let $X = \exp(-Y)$. The researchers in [120] showed that the PDF and CDF of X are

$$f(x) = \frac{\exp\left(\frac{1}{a}\right)}{b}\left(x^{-\frac{a}{b}} - a\right)x^{-\frac{a}{b}-1}\exp\left(-\frac{1}{a}x^{-\frac{a}{b}}\right),$$

and

$$F(x) = \exp\left(\frac{1}{a}\right)x^{-\frac{a}{b}}\exp\left(-\frac{1}{a}x^{-\frac{a}{b}}\right),$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = 1 - bka^{-\frac{bk}{a}}\exp\left(\frac{1}{a}\right)\Gamma\left(1 - \frac{bk}{a}, \frac{1}{a}\right).$$

In particular, the mean and variance are

$$E(X) = 1 - ba^{-\frac{b}{a}}\exp\left(\frac{1}{a}\right)\Gamma\left(1 - \frac{b}{a}, \frac{1}{a}\right),$$

and

$$Var(X) = 1 - 2ba^{-\frac{2b}{a}}\exp\left(\frac{1}{a}\right)\Gamma\left(1 - \frac{2b}{a}, \frac{1}{a}\right) - \left[E(X)\right]^2,$$

respectively. In reference [120], Proposition 2.1 showed that the PDF is unimodal if $0 < a \leq 1$ and $b > 0$. In reference [120], Figure 2 showed that mean is a decreasing function of b and variance is unimodal with respect to b when a is fixed. In reference [120], Figure 3 showed that skewness is an increasing function of b and kurtosis is bathtub shaped with respect to b when a is fixed. This distribution has been applied to model SC 16 and P3 algorithms for estimating unit capacity factors and times between failures of secondary reactor pumps.

2.91. Cauchy-logistic unit distribution

Let Y denote a Cauchy random variable with scale parameter b. Let $X = \log\left(Y^{-a} - 1\right)$. The researchers in [174] showed that the PDF and CDF of X are

$$f(x) = \frac{ab}{\pi x\left(1 - x^{a}\right)\left[b^{2} + \log^{2}\left(x^{-a} - 1\right)\right]},$$

and

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\arctan\left[\frac{\log\left(x^{-a} - 1\right)}{b}\right],$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter. The kth moment of X is

$$E\left(X^{k}\right) = \frac{1}{2} + \frac{bk}{a\pi}\sum_{i=0}^{\infty}\binom{-\frac{k}{a} - 1}{i}\left[w_{i+\frac{k}{a}}(b) - w_{i+1}(b)\right],$$

where

$$w_{i}(b) = \frac{\mathrm{Ci}(ib)\sin(ib) + \cos(ib)\left[\frac{\pi}{2} - \mathrm{Si}(ib)\right]}{i}.$$

In particular, the mean and variance are

$$E(X) = \frac{1}{2} + \frac{b}{a\pi}\sum_{i=0}^{\infty}\binom{-\frac{1}{a} - 1}{i}\left[w_{i+\frac{1}{a}}(b) - w_{i+1}(b)\right],$$

and

$$Var(X) = \frac{1}{2} + \frac{2b}{a\pi}\sum_{i=0}^{\infty}\binom{-\frac{2}{a} - 1}{i}\left[w_{i+\frac{2}{a}}(b) - w_{i+1}(b)\right] - \left[E(X)\right]^{2},$$

respectively. This distribution has been applied to model percentage of use of certain types of antibiotics before and during the SarS CoV virus pandemic and percentage of time spent using the service by end users and is part of the training data set relating to network and telecommunication traffic in India.

2.92. Unit Xgamma distribution

Let Y denote an Xgamma random variable with scale parameter b [163]. Let $X = \frac{Y}{1+Y}$. The researchers in [78] showed that the PDF and CDF of X are

$$f(x) = \frac{b^{2}}{1+b}\frac{1}{(1-x)^{2}}\left[1 + \frac{b}{2}\left(\frac{x}{1-x}\right)^{2}\right]\exp\left(-b\frac{x}{1-x}\right),$$

and

$$F(x) = 1 - \frac{1}{1+b}\left[1 + b + b\frac{x}{1-x} + \frac{b^{2}}{2}\left(\frac{x}{1-x}\right)^{2}\right]\exp\left(-b\frac{x}{1-x}\right),$$

respectively, for $0 < x < 1$, where $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{b\exp(b)}{1+b}\sum_{i=0}^{\infty}(-1)i\binom{k}{i}b^i\Gamma(1-i,b) + \frac{b^3\exp(b)}{2(1+b)}\sum_{i=0}^{\infty}(-1)i\binom{k+1}{i}b^i\Gamma(2-i,b).$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{b\exp(b)}{1+b}\sum_{i=0}^{\infty}(-1)i\binom{1}{i}b^i\Gamma(1-i,b) + \frac{b^3\exp(b)}{2(1+b)}\sum_{i=0}^{\infty}(-1)i\binom{2}{i}b^i\Gamma(2-i,b),$$

and

$$Var\left(X\right) = \frac{b\exp(b)}{1+b}\sum_{i=0}^{\infty}(-1)i\binom{2}{i}b^i\Gamma(1-i,b) + \frac{b^3\exp(b)}{2(1+b)}\sum_{i=0}^{\infty}(-1)i\binom{3}{i}b^i\Gamma(2-i,b) - \left[E(X)\right]^2,$$

respectively. In reference [78], Figure 1 showed that the PDF can be monotonically decreasing, or unimodal. In reference [78], Table 1 showed that the distribution can be negatively skewed for $b \leq 1$, or positively skewed for $b > 1$. Furthermore, the distribution is leptokurtic $b \leq 0.5$, or $b > 5$, mesokurtic for $b = 5$, and l is platykurtic for $0.5 < b < 5$. This distribution has been applied to model water capacity month-wise from the Shasta reservoir in California.

2.93. Unit exponential Pareto distribution

Let Y denote an exponential Pareto random variable with shape parameter a and scale parameters b, c [12]. Let $X = \frac{Y}{1-Y}$. The researchers in [76] showed that the PDF and CDF of X are

$$f(x) = \frac{ac}{b}\left[\frac{x}{b(1-x)}\right]^{a-1}\exp\left\{-c\left[\frac{x}{b(1-x)}\right]^a\right\},$$

and

$$F(x) = 1 - \exp\left\{-c\left[\frac{x}{b(1-x)}\right]^a\right\},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$, $c > 0$ are scale parameters. The kth moment of X is

$$E\left(X^k\right) = \frac{c^{\frac{2}{a}}}{b^2}\sum_{i=0}^{\infty}\binom{-k-2}{i}\frac{c^{\frac{i+2}{a}}}{b^{i+2}}\Gamma\left(1-\frac{i}{a}\right).$$

In particular, the mean and variance are

$$E(X) = \frac{c^{\frac{2}{a}}}{b^2}\sum_{i=0}^{\infty}\binom{-3}{i}\frac{c^{\frac{i+2}{a}}}{b^{i+2}}\Gamma\left(1-\frac{i}{a}\right),$$

and

$$Var(X) = \frac{c^{\frac{2}{a}}}{b^2}\sum_{i=0}^{\infty}\binom{-4}{i}\frac{c^{\frac{i+2}{a}}}{b^{i+2}}\Gamma\left(1-\frac{i}{a}\right) - \left[E(X)\right]^2,$$

respectively. In reference [76], Figure 1 showed that the PDF can be decreasing, upside down with unimodal, skewed to the left, or symmetric. This distribution has been applied to model recovery rate of COVID-19 in Turkey, proportions of total milk production in the first cow births and failure times of components.

2.94. Chesneau's unit gamma distribution

Let Y denote a gamma random variable with shape parameter a and scale parameter b. Let $X = \frac{1}{1+Y}$. The researchers in [45] showed that the PDF and CDF of X are

$$f(x) = \frac{b^a \exp(b)}{\Gamma(a)} x^{-a-1} (1-x)^{b-1} \exp\left(-\frac{b}{x}\right),$$

and

$$F(x) = 1 - \frac{1}{\Gamma(a)} \gamma\left(a, \frac{b}{x} - b\right),$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{b^k \exp(b)}{\Gamma(a)} \sum_{i=0}^{\infty} (-1)^i \binom{a-1}{i} b^i \Gamma\left(a-i-k, b\right).$$

In particular, the mean and variance are

$$E(X) = \frac{b \exp(b)}{\Gamma(a)} \sum_{i=0}^{\infty} (-1)^i \binom{a-1}{i} b^i \Gamma\left(a-i-1, b\right),$$

and

$$Var(X) = \frac{b^2 \exp(b)}{\Gamma(a)} \sum_{i=0}^{\infty} (-1)^i \binom{a-1}{i} b^i \Gamma\left(a-i-2, b\right) - \left[E(X)\right]^2,$$

respectively. In reference [45], Figure 2 showed that the PDF can exhibit a variety of unimodal shapes, including left skewed, nearly symmetric and right skewed unimodal. In reference [45], Table 1 showed that skewness can be negative, or positive. Furthermore, kurtosis can be lower, or greater than 3. This distribution has been applied to model proportions of income spent on food.

2.95. Unit gamma distribution [59]

Let Y denote a power Burr X random variable with shape parameters b, c and scale parameter a [180]. Let $X = \exp(-Y)$. The researchers in [59] showed that the PDF and CDF of X are

$$f(x) = 2a^2 bcx^{-1} (-\log x)^{2b-1} \left[1 - \exp\left\{-\left[a(-\log x)^b\right]^2\right\}\right]^{c-1},$$

and

$$F(x) = 1 - \left[1 - \exp\left\{-\left[a(-\log x)^b\right]^2\right\}\right]^c,$$

respectively, for $0 < x < 1$, where $b > 0$, $c > 0$ are shape parameters, and $a > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = c \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} k^i}{i!(j+1)^{1+\frac{i}{2b}}} a^{-\frac{i}{b}} \binom{c-1}{j} \Gamma\left(1 + \frac{i}{2b}\right).$$

In particular, the mean and variance are

$$E(X) = c \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!(j+1)^{1+\frac{i}{2b}}} a^{-\frac{i}{b}} \binom{c-1}{j} \Gamma\left(1 + \frac{i}{2b}\right),$$

and

$$Var(X) = c \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} 2^i}{i!(j+1)^{1+\frac{i}{2b}}} a^{-\frac{i}{b}} \binom{c-1}{j} \Gamma\left(1 + \frac{i}{2b}\right) - [E(X)]^2,$$

respectively. In reference [58], Figure 1 showed that the PDF can be symmetric, U shaped, right skewed, J shaped, or normal tapered. In reference [58], Figure 3 showed that skewness, and kurtosis increase as b and c increase. Furthermore, as a increases, skewness decreases and kurtosis increases. This distribution has been applied to model proportions of income spent on food, COVID-19 data, and data with responses of naive mock jurors.

2.96. Unit Ishita distribution

Let Y denote an Ishita random variable with shape parameter a [167]. Let $X = \exp(-Y)$. The researchers in [35] showed that the PDF and CDF of X are

$$f(x) = \frac{a^3 x^{a-1} \left[a + \log^2 x\right]}{a^3 + 2},$$

and

$$F(x) = \frac{x^a \left[a^3 + a^2 \log^2 x - 2a \log x + 2\right]}{a^3 + 2},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{a^3 \left[a(a+k)^2 + 2\right]}{(a^3 + 2)(a+k)^3}.$$

In particular, the mean and variance are

$$E(X) = \frac{a^3 \left[a(a+1)^2 + 2\right]}{(a^3 + 2)(a+1)^3},$$

and

$$Var(X) = \frac{a^3 \left[a(a+2)^2 + 2\right]}{(a^3 + 2)(a+2)^3} - [E(X)]^2,$$

respectively. In reference [35], Figure 1 showed that the PDF can be monotonically increasing, monotonically decreasing, or unimodal.

2.97. Unit extended exponential distribution

Let Y denote an extended exponential random variable with shape parameter a and scale parameter b [67]. Let $X = \exp(-Y)$. The researchers in [150] showed that the PDF and CDF of X are

$$f(x) = \frac{a^2 (1 - b \log x)\, x^{a-1}}{a+b},$$

and

$$F(x) = \left(1 - \frac{ab \log x}{a+b}\right) x^a,$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{a^2 (k+a+b)}{(a+b)\,(a+k)^2}.$$

In particular, the mean and variance are

$$E(X) = \frac{a^2 (1+a+b)}{(a+b)\,(a+1)^2},$$

and

$$Var(X) = \frac{a^2 (2+a+b)}{(a+b)\,(a+2)^2} - [E(X)]^2,$$

respectively. In reference [150], Figure 3 showed that the PDF can be decreasing, left skewed, right skewed, or unimodal. This distribution has been applied to model tensile strength observations of polyester fibers and computing times of P3 algorithms.

2.98. Unit distribution based on a half-logistic map distribution

Let Y denote a quasi Lindley random variable. Let X by proportional to $\left[\log\left(1 - Y^2\right)\right]^{-1}$. The researchers in [173] showed that the PDF and CDF of X are

$$f(x) = \frac{2ax}{b+1}\left(1 - x^2\right)^{a-1}\left[b - a \log\left(1 - x^2\right)\right],$$

and

$$F(x) = 1 - \frac{\left(1 - x^2\right)^a \left[b + 1 - a \log\left(1 - x^2\right)\right]}{b+1},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{\Gamma\left(\frac{k}{2}+1\right)\Gamma(a+1)\left[b + a\psi\left(\frac{k}{2}+a+1\right) - a\psi(a)\right]}{(b+1)\Gamma\left(\frac{k}{2}+a+1\right)}.$$

In particular, the mean and variance are

$$E(X)=\frac{\sqrt{\pi}\Gamma(a+1)\left[b+a\psi\left(\frac{3}{2}+a\right)-a\psi(a)\right]}{2(b+1)\Gamma\left(\frac{3}{2}+a\right)},$$

and

$$Var(X)=\frac{\Gamma(a+1)\left(b+2-\frac{1}{a+1}\right)}{(b+1)\Gamma\left(a+2\right)}-\left[E(X)\right]^2,$$

respectively. In reference [173], Theorem 1 showed that the PDF is unimodal if $a > 1$ and is monotonically increasing if $0 < a \leq 1$. In reference [173], Theorem 3 showed that the distribution can be positively asymmetric if either $b > \xi(a)$ and $a_1 < a < a_2$, or $b > 0$ and $a > a_2$, where a_1 is an asymptotic point and a_2 is a positive zero of $\xi(a)=\frac{2a3^a\log\left(\frac{3}{4}\right)}{2\cdot 3^a-4^a}$. The distribution can be negatively asymmetric if either $a \leq a_1$, or $0 < b < \xi(a)$ and $a_1 < a < a_2$. This distribution has been applied to model ratings of organizations involved in organizing polls, antibiotic usage before and during the COVID-19 pandemic, and percentages of time spent by end-users using telecommunications services.

2.99. *Unit Mirra distribution*

Let Y denote a Mirra random variable [162]. Let $X=\frac{1}{1+Y}$. The researchers in [13] showed that the PDF and CDF of X are

$$f(x)=\frac{b^3\left[(a+2)x^2+a-2ax\right]}{2x^4\left(a+b^2\right)}\exp\left[-\frac{b(1-x)}{x}\right],$$

and

$$F(x)=\left\{1+\frac{ab}{a+b^2}\left[\frac{1-x}{x}+\frac{b(1-x)^2}{2x^2}\right]\right\}\exp\left[-\frac{b(1-x)}{x}\right],$$

respectively, for $0 < x < 1$, where both $a > 0$ and $b > 0$ are scale parameters. The kth moment of X is

$$E\left(X^k\right)=\frac{b^2\left\{a+\exp(b)\left[(a+2)bE_k(b)-a(2b+k-2)E_{k-1}(b)\right]\right\}}{2\left(a+b^2\right)}.$$

In particular, the mean and variance are

$$E(X)=\frac{b^2\left\{a+\exp(b)\left[(a+2)bE_1(b)-a(2b-1)E_0(b)\right]\right\}}{2\left(a+b^2\right)},$$

and

$$Var(X)=\frac{b^2\left\{a+\exp(b)\left[(a+2)bE_2(b)-2abE_1(b)\right]\right\}}{2\left(a+b^2\right)}-\left[E(X)\right]^2,$$

respectively. In reference [13], Figures 2 and 3 showed that the PDF can be positively skewed, increasing, or increasing-decreasing. This distribution has been applied to model times of kidney dialysis patients to infection measured in months and trade share values.

2.100. Unit exponentiated half logistic distribution

Let Y denote an exponentiated half logistic random variable [92]. Let $X = \exp(-Y)$. The researchers in [81] showed that the PDF and CDF of X are

$$f(x) = \frac{2abx^{a-1}}{(1+x^a)^2}\left(\frac{1-x^a}{1+x^a}\right)^{b-1},$$

and

$$F(x) = 1 - \left(\frac{1-x^a}{1+x^a}\right)^{b},$$

respectively, for $0 < x < 1$, where both $a > 0$ and $b > 0$ are shape parameters. The kth moment of X is

$$E\left(X^k\right) = 2b\sum_{i=0}^{\infty}(-1)^i\binom{b+i}{i}B\left(\frac{k}{a}+i+1, b\right).$$

In particular, the mean and variance are

$$E(X) = 2b\sum_{i=0}^{\infty}(-1)^i\binom{b+i}{i}B\left(\frac{1}{a}+i+1, b\right),$$

and

$$Var(X) = 2b\sum_{i=0}^{\infty}(-1)^i\binom{b+i}{i}B\left(\frac{2}{a}+i+1, b\right) - [E(X)]^2,$$

respectively. In reference [81], Figure 1 showed that the PDF can be right skewed, left skewed, reverse J shaped, U shaped, or asymmetric. In reference [81], Table 1 showed that when a increases for a fixed b, the first four moments, variance, and skewness decrease, while kurtosis increases. When b increases for a fixed a, the first four moments and kurtosis decrease, while variance, and skewness increase. This distribution has been applied to model trade share values, measurements of polyester fibers' tensile strength, daily new deaths, daily cumulative cases, and daily cumulative deaths.

2.101. Unit exponentiated Fréchet distribution

Let Y denote an exponentiated Fréchet random variable with shape parameters a, b and scale parameter c. Let $X = \frac{Y}{1+Y}$. The researchers in [9] showed that the PDF and CDF of X are

$$f(x) = \frac{abc^b}{x^2}\left(\frac{1-x}{x}\right)^{b-1}\exp\left[-c^b\left(\frac{1-x}{x}\right)^b\right]\left\{1-\exp\left[-c^b\left(\frac{1-x}{x}\right)^b\right]\right\}^{a-1},$$

and

$$F(x) = 1 - \left\{1-\exp\left[-c^b\left(\frac{1-x}{x}\right)^b\right]\right\}^{a},$$

respectively, for $0 < x < 1$, where both $a > 0$ and $b > 0$ are shape parameters. The kth moment of X is

$$E\left(X^k\right) = a \sum_{i,j=0}^{\infty} \binom{a-1}{i}\binom{-k}{j} \frac{(-1)^i c^{-j}}{(i+1)^{\frac{j}{b}+1}} \Gamma\left(\frac{j}{b}+1\right).$$

In particular, the mean and variance are

$$E(X) = a \sum_{i,j=0}^{\infty} \binom{a-1}{i}\binom{-1}{j} \frac{(-1)^i c^{-j}}{(i+1)^{\frac{j}{b}+1}} \Gamma\left(\frac{j}{b}+1\right),$$

and

$$Var(X) = a \sum_{i,j=0}^{\infty} \binom{a-1}{i}\binom{-2}{j} \frac{(-1)^i c^{-j}}{(i+1)^{\frac{j}{b}+1}} \Gamma\left(\frac{j}{b}+1\right) - [E(X)]^2,$$

respectively. In reference [9], Figure 1 showed that the PDF can be increasing, decreasing, left skewed, right skewed, or approximately symmetric. In reference [9], Table 1 showed that skewness can be positive, or negative. This distribution has been applied to model insurance loss data and risk management.

2.102. Modified unit half-normal distribution

Suppose Y is a half-normal random variable with scale parameter a. Let $X = \frac{1}{1+Y}$. The researchers in [19] showed that X has the PDF and CDF

$$f(x) = \frac{2}{ax^2}\phi\left(\frac{1-x}{ax}\right),$$

and

$$F(x) = 2\Phi\left(\frac{x-1}{ax}\right),$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The associated kth moment is

$$E\left(X^k\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\exp\left(-\frac{t^2}{2}\right)}{(1+at)^k} dt.$$

In particular, the mean and variance are

$$E(X) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\exp\left(-\frac{t^2}{2}\right)}{1+at} dt,$$

and

$$Var(X) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\exp\left(-\frac{t^2}{2}\right)}{(1+at)^2} dt - [E(X)]^2,$$

respectively. In reference [19], Proposition 2 showed that the PDF is unimodal at $x = \frac{\sqrt{1+8a^2}-1}{4a^2}$. This distribution has been applied to model shape perimeter of rocks from an oil reservoir and proportions formed by COVID information taken from a Chilean database.

2.103. Unit Haq distribution

Let Y denote a Haq random variable with scale parameter a [11]. Let $X = \exp(-Y)$. Alzahrani and Almohaimeed [21] showed that the PDF and CDF of X are

$$f(x) = \frac{a^2}{(1+a)^2}\left(2 + a + \frac{a\log^2 x}{2}\right)x^{a-1},$$

and

$$F(x) = \frac{1}{(1+a)^2}\left[(1+a)^2 - a\log x + \frac{a^2\log^2 x}{2}\right]x^a,$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{a^2\left[a + (2+a)(k+a)^2\right]}{(1+a)^2(k+a)^3}.$$

In particular, the mean and variance are

$$E(X) = \frac{a^2\left[a + (2+a)(1+a)^2\right]}{(1+a)^5},$$

and

$$Var(X) = \frac{a^2\left[a + (2+a)^3\right]}{(1+a)^2(2+a)^3} - [E(X)]^2,$$

respectively. Alzahrani and Almohaimeed [21, Figure 1] showed that the PDF can be monotonically increasing, or monotonically decreasing. Alzahrani and Almohaimeed [21, Table 1] showed that mean increases and variance decreases with respect to a. This distribution has been applied to model distinct algorithms, P3 and SC16, used for estimating unit capacity factors.

2.104. Unit Burr-Hatke distribution

Let Y denote a Burr-Hatke random variable with scale parameter a [115]. Let $X = \exp(-Y)$. Saglam and Karakaya [158] showed that the PDF and CDF of X are

$$f(x) = \frac{x^{a-1}\left[1 - a(\log x - 1)\right]}{(1-\log x)^2},$$

and

$$F(x) = \frac{x^a}{1-\log x},$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = 1 - k\exp(a+k)E_1(a+k).$$

In particular, the mean and variance are

$$E(X) = 1 - k \exp(a+1) E_1(a+1),$$

and

$$Var(X) = 1 - k \exp(a+2) E_1(a+2) - [E(X)]^2,$$

respectively. Saglam and Karakaya [158, Figure 1] showed that the PDF can be monotonically increasing, or bathtub shaped. Saglam and Karakaya [158, Table 1] showed that mean increases as a increases. Furthermore, variance, skewness, and kurtosis decrease as a increases. The distribution is skewed to the right for small a and skewed to the left for large a.

2.105. Unit Garima distribution

Let Y denote a Garima random variable with scale parameter a [166]. Let $X = \exp(-Y)$. Ayuyuen and Bodhisuwan [25] showed that the PDF and CDF of X are

$$f(x) = \frac{a(a+x)}{(2+a)x^3} \exp\left[-a\left(\frac{1}{x} - 1\right)\right],$$

and

$$F(x) = \left[1 - \frac{a}{2+a}\left(\frac{1}{x} - 1\right)\right] \exp\left[-a\left(\frac{1}{x} - 1\right)\right],$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. This distribution has been applied to model failure times of Kevlar 49/epoxy strands test and measurements on petroleum rock samples.

2.106. Unit power skew normal distribution

Let Y denote a power skew normal random variable [118] with the PDF and CDF specified by

$$f(y) = [\Phi_{SN}(y; a)]^b,$$

and

$$F(y) = b\phi_{SN}(y)\left[\Phi_{SN}(y; a)\right]^{b-1},$$

respectively, for $-\infty < y < \infty$, where $b > 0$ is a shape parameter. Furthermore, $\phi_{SN}(\cdot; a)$ and $\Phi_{SN}(\cdot; a)$ denote the PDF and CDF of a skew normal random variable with shape parameter a [26]. Let $X = \exp[-\exp(Y)]$. [117] showed that the PDF and CDF of X are

$$f(x) = -\frac{b}{dx \log x}\phi_{SN}\left(\frac{\log(-\log x) - c}{d}; a\right)\left[\Phi_{SN}\left(\frac{\log(-\log x) - c}{d}; a\right)\right]^{b-1},$$

and

$$F(x) = \left[\Phi_{SN}\left(\frac{\log(-\log x) - c}{d}; a\right)\right]^b,$$

respectively, for $0 < x < 1$, where $-\infty < a < \infty$, $b > 0$ are shape parameters, c is a location parameter, and d is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = b \int_0^1 t^{b-1} \exp\left\{-k \exp\left[\Phi_{SN}^{-1}(t;a)\right]\right\} dt.$$

In particular, the mean and variance are

$$E\,(X) = b \int_0^1 t^{b-1} \exp\left\{-\exp\left[\Phi_{SN}^{-1}(t;a)\right]\right\} dt,$$

and

$$Var\,(X) = b \int_0^1 t^{b-1} \exp\left\{-2 \exp\left[\Phi_{SN}^{-1}(t;a)\right]\right\} dt - \left[E(X)\right]^2,$$

respectively. This distribution has been applied to model percentages of teachers of the fundamental level of the municipalities of Brazil and food/income taxa.

2.107. Unit generalized Rayleigh distribution

Let Y denote a generalized Rayleigh random variable with shape parameter a and scale parameter b [111]. Let $X = \exp(-Y)$. The researchers in [85] showed that the PDF and CDF of X are

$$f(x) = -\frac{2ab^2 \log x}{x} \exp\left[-(b \log x)^2\right] \left\{1 - \exp\left[-(b \log x)^2\right]\right\}^a,$$

and

$$F(x) = 1 - \left\{1 - \exp\left[-(b \log x)^2\right]\right\}^{a-1},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. In reference [85], Figure 1 showed that the PDF can be monotonically increasing, monotonically decreasing, or unimodal.

2.108. Unit Fav-Jerry distribution

Suppose Y is a Fav-Jerry random variable with shape parameter a [55]. Let $X = \exp\left(-\frac{Y}{b}\right)$. Karakaya and Sağlam [94] showed that the PDF and CDF of X are

$$f(x) = \frac{ab\left(2 - a^3 b \log x\right) x^{ab-1}}{a^2 + 1},$$

and

$$F(x) = \frac{\left(a^2 + 2 - a^3 b \log x\right) x^{ab}}{a^2 + 2},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{2abk + 2a^2b^2 + a^4b^2}{(a^2+2)\,(k+ab)^2}.$$

In particular, the mean and variance are

$$E(X) = \frac{2ab + 2a^2b^2 + a^4b^2}{(a^2+2)\,(1+ab)^2},$$

and

$$Var(X) = \frac{4ab + 2a^2b^2 + a^4b^2}{(a^2+2)\,(2+ab)^2} - [E(X)]^2,$$

respectively. In reference [94], Figure 1 showed that the PDF can be decreasing, increasing, unimodal, or decreasing-increasing. In reference [94], Table 1 showed that, as a and b increase, mean increases while variance, skewness, and kurtosis decrease. For small a and b, the distribution is skewed to the right. For large a and b, the distribution is skewed to the left. This distribution has been applied to model proportions of total milk production in the first cow births from the Carnauba farm in Brazil and Better Life Indices based on self-reported health data from 2015.

2.109. Unit power half-normal distribution

Suppose Y is a power half-normal random variable with shape parameter a and scale parameter b [66]. Let $X = \frac{1}{1+Y}$. The researchers in [159] showed that the PDF and CDF of X are

$$f(x) = \frac{2a}{bx^2}\phi\left(\frac{1-x}{bx}\right)\left[2\Phi\left(\frac{1-x}{bx}\right) - 1\right]^{a-1},$$

and

$$F(x) = 1 - \left[2\Phi\left(\frac{1-x}{bx}\right) - 1\right]^{a},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = a\int_0^1 t^{-a-1}\left[1 + b\Phi^{-1}\left(\frac{t+1}{2}\right)\right]^{-k} dt.$$

In particular, the mean and variance are

$$E\left(X\right) = a\int_0^1 t^{-a-1}\left[1 + b\Phi^{-1}\left(\frac{t+1}{2}\right)\right]^{-1} dt,$$

and

$$Var\left(X\right) = a\int_0^1 t^{-a-1}\left[1 + b\Phi^{-1}\left(\frac{t+1}{2}\right)\right]^{-2} dt - [E(X)]^2,$$

respectively. In reference [159], Figure 1 showed that the PDF can be unimodal. In reference [159], Table 1 showed that skewness is negative for small a and b and negative for large a and b. This distribution has been applied to model patient health outcomes and survival rates under a specific treatment.

2.110. Unit Monsef distribution

Suppose Y is a Monsef random variable with scale parameter a [4]. Let $X = \frac{Y}{1+Y}$. The researchers in [5] showed that the PDF and CDF of X are

$$f(x) = \frac{a^3}{(1-x)^4 \left(a^2+2a+2\right)} \exp\left(-\frac{ax}{1-x}\right),$$

and

$$F(x) = 1 - \frac{\left(2+2x^2-4x-2ax+2a+a^2\right)}{(1-x)^2 \left(a^2+2a+2\right)} \exp\left(-\frac{ax}{1-x}\right),$$

respectively, for $0 < x < 1$, where $a > 0$ is a scale parameter. The kth moment of X is

$$f(x) = \frac{a^3}{a^2+2a+2} \int_0^1 \frac{t^k}{(1-x)^4} \exp\left(-\frac{ax}{1-x}\right) dt.$$

In particular, the mean and variance are

$$E(X) = \frac{2+a}{2+2a+a^2},$$

and

$$Var(X) = \frac{2}{2+2a+a^2} - \left[E(X)\right]^2,$$

respectively. In reference [5], Figure 1 showed that the PDF can be monotonically decreasing, or unimodal. This distribution has been applied to model educational attainment for OECD countries.

2.111. Unit bimodal Birnbaum-Saunders distribution

Suppose Y is a bimodal Birnbaum-Saunders random variable with scale parameters a, b and location parameter c [139]. Let $X = \exp(-Y)$. The researchers in [119] showed that the PDF and CDF of X are

$$f(x) = \frac{1}{4ab\Phi(-c)x} \left[\left(-\frac{b}{\log x}\right)^{\frac{1}{2}} + \left(-\frac{b}{\log x}\right)^{\frac{3}{2}}\right] \phi\left(\left|t(x)\right| + c\right),$$

and

$$F(x) = \begin{cases} \dfrac{1-\Phi\left(t(x)+c\right)}{2\left[1-\Phi(c)\right]}, & \text{if } x \le \exp(-b), \\ \\ 1 - \dfrac{\Phi\left(t(x)+c\right)}{2\left[1-\Phi(c)\right]}, & \text{if } x > \exp(-b), \end{cases}$$

respectively, for $0 < x < 1$, where $t(x) = \frac{1}{a}\left(\sqrt{-\frac{\log x}{b}} - \sqrt{-\frac{b}{\log x}}\right)$, $a > 0$, $b > 0$ are scale parameters, and $-\infty < c > \infty$ is a location parameter. The mean and variance of X are

$$E(X) = \frac{1}{2\Phi(-c)} \left\{ \exp(-b) \left[1 - \Phi(c) \right] + \int_0^1 \Phi \left(t(x) + c \right) dx \right\},$$

and

$$Var(X) = \frac{\exp\left(-\frac{c^2}{2}\right)}{4\sqrt{2\pi}\sqrt{b}\Phi(-c)} \int_0^\infty x^{-\frac{3}{2}} (x+b) \exp\left\{ \frac{(1+4b)x^2 + 2bx\left[c\left|t(x)\right| - 1\right] + b^2}{2bx} \right\} dx - [E(X)]^2,$$

respectively. This distribution has been applied to model biomedical measurements of 100 female athletes and 102 male athletes competing in different sports and clinical marker of periodontal diseases.

2.112. Unit Omega distribution

Suppose Y is an Omega random variable with shape parameters a, b and location parameter c [54]. Setting $c = 1$ gives a unit distribution as shown in Prataviera and Cordeiro [146]. Its PDF and CDF are

$$f(x) = \frac{2abx^{b-1}}{1 - x^{2b}} \left(\frac{1 + x^b}{1 - x^b} \right)^{-a},$$

and

$$F(x) = 1 - \left(\frac{1 + x^b}{1 - x^b} \right)^{-a},$$

respectively, for $0 < x < 1$, where $a > 0$ and $b > 0$ are shape parameters. The kth moment of X is

$$E\left(X^k\right) = 2aB\left(\frac{k}{b} + 1, a + 1\right) {}_1F_2\left(a + 1; \frac{k}{b} + 1, \frac{k}{b} + a + 1; -1\right).$$

In particular, the mean and variance are

$$E(X) = 2aB\left(\frac{1}{b} + 1, a + 1\right) {}_1F_2\left(a + 1; \frac{1}{b} + 1, \frac{1}{b} + a + 1; -1\right),$$

and

$$Var(X) = 2aB\left(\frac{2}{b} + 1, a + 1\right) {}_1F_2\left(a + 1; \frac{2}{b} + 1, \frac{2}{b} + a + 1; -1\right) - [E(X)]^2,$$

respectively. Prataviera and Cordeiro [146, Figure 1] showed that the PDF can be asymmetric left, symmetric, asymmetric right, or U shaped. This distribution has been applied to model annual percentages of antimicrobial resistant isolates in Portugal in 2012.

2.113. Unit log-normal distribution

Suppose Y is a log-normal random variable with location parameter a and scale parameter b. Let $X = \frac{Y}{1+Y}$. Ribeiro-Reis [155] showed that the PDF and CDF of X are

$$f(x) = \frac{1}{\sqrt{2\pi} b x(1-x)} \exp\left\{-\frac{1}{2b^2}\left[\log\frac{x}{1-x} - a\right]^2\right\},$$

and

$$F(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\log\frac{x}{1-x} - a}{\sqrt{2} b}\right),$$

respectively, for $0 < x < 1$, where $-\infty < a < \infty$ is a location parameter, and $b > 0$ is a scale parameter. Ribeiro-Reis [155, Figure 1] showed that the PDF can be symmetric, right symmetric, left symmetric, U shaped, or M shaped. This distribution has been applied to model Firjan health indices of 853 municipalities in the State of Minas Gerais, Brazil and proportions of households with per capita household income below the extreme poverty line.

2.114. Unit power Lindley distribution

Suppose Y is a power Lindley random variable with shape parameter a and scale parameter b [63]. Let $X = \frac{Y}{1+Y}$. The researchers in [96] showed that the PDF and CDF of X are

$$f(x) = \frac{ab^2}{1+b}\frac{x^{a-1}}{(1-x)^{a+1}}\left[1 + \left(\frac{x}{1-x}\right)^a\right]\exp\left[-b\left(\frac{x}{1-x}\right)^a\right],$$

and

$$F(x) = 1 - \left[1 + \frac{1}{1+b}\left(\frac{x}{1-x}\right)^a\right]\exp\left[-b\left(\frac{x}{1-x}\right)^a\right],$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. In reference [96], Figure 1 showed that the PDF can be monotonically decreasing, unimodal, or bathtub shaped. In reference [96], Table 1 showed that the distribution is platykurtic. This distribution has been applied to model measures on burrs.

2.115. Unit gamma Lindley distribution

Suppose Y is a gamma Lindley random variable with shape parameter a and scale parameter b [7]. Let $X = \frac{Y}{1+Y}$. Karakaya and Saglam [95] showed that the PDF and CDF of X are

$$f(x) = \frac{ab^2\left[1 + (a+b)(1-x)\right] x^{a-1}(b - bx + x)^{-a}}{(1+b)(bx - x - b)^2},$$

and

$$F(x) = \frac{\left(\frac{x}{1-x}\right)^a}{\left(b + \frac{x}{1-x}\right)^a} + \frac{ab\left(\frac{x}{1-x}\right)^a}{(1+b)\left(b + \frac{x}{1-x}\right)^{a+1}},$$

respectively, for $0 < x < 1$, where $a > 0$ is a shape parameter, and $b > 0$ is a scale parameter. Karakaya and Saglam [95, Figure 1] showed that the PDF can be increasing, decreasing, or unimodal. This distribution has been applied to model cost-effectiveness in firms' risk management and educational attainment for OECD countries.

2.116. Hassana and Alharbi's unit inverse exponentiated Weibull distribution

Suppose Y is an inverse exponentiated Weibull random variable with scale parameter ε and shape parameters δ, ϕ [112]. Let $X = \exp(-Y)$. Hassan and Alharbi [80] showed that the PDF and CDF of X are

$$f(x) = \delta\varepsilon\phi x^{-1}(-\log x)^{-\delta-1}\exp\left[-\varepsilon(-\log x)^{-\delta}\right]\left\{1-\exp\left[-\varepsilon(-\log x)^{-\delta}\right]\right\}^{\phi-1},$$

and

$$F(x) = \left\{1-\exp\left[-\varepsilon(-\log x)^{-\delta}\right]\right\}^{\phi},$$

respectively, for $0 < x < 1$, where $\varepsilon > 0$ is a scale parameter and $\delta > 0$, $\phi > 0$ are shape parameters. The kth moment of X is

$$E\left(X^k\right) = \sum_{i,j=0}^{\infty}\binom{\phi-1}{j}\frac{(-1)^{j+i}k^i\varepsilon\phi}{i!\left[\varepsilon(j+1)\right]^{1-\frac{i}{\delta}}}\Gamma\left(1-\frac{i}{\delta}\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{i,j=0}^{\infty}\binom{\phi-1}{j}\frac{(-1)^{j+i}\varepsilon\phi}{i!\left[\varepsilon(j+1)\right]^{1-\frac{i}{\delta}}}\Gamma\left(1-\frac{i}{\delta}\right),$$

and

$$Var\left(X\right) = \sum_{i,j=0}^{\infty}\binom{\phi-1}{j}\frac{(-1)^{j+i}2^i\varepsilon\phi}{i!\left[\varepsilon(j+1)\right]^{1-\frac{i}{\delta}}}\Gamma\left(1-\frac{i}{\delta}\right)-\left[E(X)\right]^2,$$

respectively. In reference [80], Figure 1 showed that the PDF can be left skewed, right skewed, asymmetric, or unimodal. In reference [80], Table 1 showed that the distribution can be leptokurtic, or platykurtic. This distribution has been applied to model total milk production in the first cow births and remission times of a random sample of bladder cancer patients.

2.117. A two–parameter unit probability model distribution [83]

The researchers in [83] proposed a two–parameter unit distribution with the PDF and CDF given by

$$f(x) = (1-x)^{\alpha-1}(1+x)^{\beta-1}\left[x(\alpha+\beta)+\alpha-\beta\right],$$

and

$$F(x) = 1-(1-x)^{\alpha}(1+x)^{\beta},$$

respectively, for $0 < x < 1$, where $\alpha \geq \beta$, $\alpha \in \mathbb{R}^+$, and $\beta \in \mathbb{R}$ are shape parameters. The kth moment is

$$E\left(X^k\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{\beta-1}{j}(-1)^i\left[2\alpha(k+i+j)+3\alpha+\beta\right]}{(k+i+j+1)(k+i+j+2)}.$$

In particular, the mean and variance are

$$E\left(X\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{\beta-1}{j}(-1)^i\left[2\alpha(1+i+j)+3\alpha+\beta\right]}{(i+j+2)(i+j+3)},$$

and

$$Var\left(X\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{\beta-1}{j}(-1)^i\left[2\alpha(2+i+j)+3\alpha+\beta\right]}{(i+j+3)(i+j+4)} - \left[E(X)\right]^2,$$

respectively. This distribution has been applied to model COVID-19 data, glass fiber strengths, total milk production from the first cow births and unit capacity factors through a comparative study of two distinct algorithms.

2.118. Unit Weibull probability distribution [160]

Suppose Y is a Weibull random variable with scale parameter α and shape parameter β. Let $X = \frac{1}{1+Y}$. The researchers in [160] showed that the PDF and CDF are

$$f(x) = \alpha\beta\left(\frac{1-x}{x}\right)^{\beta-1} x^{-2}\exp\left[-\alpha\left(\frac{1-x}{x}\right)^{\beta}\right],$$

and

$$F(x) = \exp\left[-\alpha\left(\frac{1-x}{x}\right)^{\beta}\right],$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter. The kth moment of X is

$$E\left(X^k\right) = \alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{i,j}B\left(i+k-\beta j-\beta, \beta j+1\right),$$

where $\Delta_{i,j} = \frac{(-1)^{i+j}}{j!}\binom{\beta-1}{i}\alpha^j$. In particular, the mean and variance are

$$E\left(X\right) = \alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{i,j}B\left(i+1-\beta j-\beta, \beta j+1\right),$$

and

$$Var\left(X\right) = \alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{i,j}B\left(i+2-\beta j-\beta, \beta j+1\right) - \left[E(X)\right]^2,$$

respectively. In reference [160], Figure 1 showed that the PDF can be bathtub shaped, right skewed, or unimodal. This distribution has been applied to model educational attainment data and effectiveness of a firm's risk management in terms of cost.

2.119. Unit exponentiated Weibull probability distribution Sarhan and Sobh [161]

Suppose Y is an exponentiated Weibull random variable with scale parameter α and shape parameters β, γ [130]. Let $X = \exp(-Y)$. Sarhan and Sobh [161] showed that the PDF and CDF of X are

$$f(x) = \frac{\alpha\beta\gamma}{x}\left(\log\frac{1}{x}\right)^{\beta-1}\exp\left[-\alpha\left(\log\frac{1}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\alpha\left(\log\frac{1}{x}\right)^{\beta}\right]\right\}^{\gamma-1},$$

and

$$F(x) = 1-\left\{1-\exp\left[-\alpha\left(\log\frac{1}{x}\right)^{\beta}\right]\right\}^{\gamma},$$

respectively, for $0 < x < 1$, where $\alpha > 0$ is a scale parameter, and $\beta > 0$, $\gamma > 0$ are shape parameters. The kth moment of X is

$$E\left(X^k\right) = \begin{cases} \displaystyle\sum_{i=0}^{\gamma-1}\frac{\gamma(-1)^i\Gamma(\gamma)}{\Gamma(\gamma-i)(i+1)!}\mu_k'\left((i+1)\alpha,\beta\right), & \gamma \in N^+, \\ \displaystyle\sum_{i=0}^{\infty}\frac{\gamma(-1)^i(\gamma-1)(\gamma-2)\cdots(\gamma-i)}{(i+1)!}\mu_k'\left((i+1)\alpha,\beta\right), & \gamma / \in N^+, \end{cases}$$

where

$$\mu_k'\left((i+1)\alpha,\beta\right) = \sum_{j=0}^{\infty}\frac{(-1)^j k^j}{j!\alpha^{\frac{j}{\beta}}}(i+1)^{-\frac{j}{\beta}}\Gamma\left(\frac{j}{\beta}+1\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \begin{cases} \displaystyle\sum_{i=0}^{\gamma-1}\frac{\gamma(-1)^i\Gamma(\gamma)}{\Gamma(\gamma-i)(i+1)!}\mu_1'\left((i+1)\alpha,\beta\right), & \gamma \in N^+, \\ \displaystyle\sum_{i=0}^{\infty}\frac{\gamma(-1)^i(\gamma-1)(\gamma-2)\cdots(\gamma-i)}{(i+1)!}\mu_1'\left((i+1)\alpha,\beta\right), & \gamma / \in N^+, \end{cases}$$

and

$$Var\left(X\right) = \begin{cases} \displaystyle\sum_{i=0}^{\gamma-1}\frac{\gamma(-1)^i\Gamma(\gamma)}{\Gamma(\gamma-i)(i+1)!}\mu_2'\left((i+1)\alpha,\beta\right) - \left[E(X)\right]^2, & \gamma \in N^+, \\ \displaystyle\sum_{i=0}^{\infty}\frac{\gamma(-1)^i(\gamma-1)(\gamma-2)\cdots(\gamma-i)}{(i+1)!}\mu_2'\left((i+1)\alpha,\beta\right) - \left[E(X)\right]^2, & \gamma / \in N^+, \end{cases}$$

respectively. Sarhan and Sobh [161, Figure 1] showed that the PDF can be monotonically increasing, monotonically decreasing, bathtub shaped, or unimodal. Sarhan and Sobh [161, Figure 2] showed that skewness is always positive and initially decreasing, then increasing. Furthermore, kurtosis is positive and increasing. This distribution has been applied to model breaking stress of carbon fibers, breaking stress of carbon fibers, and failure times of the air conditioning system of an airplane.

2.120. Unit log symmetric distribution [183]

Let Z denote a random variable symmetric around 0 with the PDF and CDF denoted by $f_Z(\cdot)$ and $F_Z(\cdot)$, respectively. The researchers in [183] proposed a unit log symmetric distribution with the PDF and CDF

$$f(x) = \frac{1}{\sigma x(1-x)} f_Z\left(\log\left[\left(\frac{x}{\eta(1-x)}\right)^{\frac{1}{\sigma}}\right]\right),$$

and

$$F(x) = F_Z\left(\log\left[\left(\frac{x}{\eta(1-x)}\right)^{\frac{1}{\sigma}}\right]\right),$$

respectively, for $0 < x < 1$, where $\sigma > 0$ is a shape parameter, and $\eta > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \frac{1}{2^k} E\left[\left(\sum_{i=0}^{\sigma} a_i(\log(\eta) + \sigma Z\right)^k\right], \quad a_i = E_i(1)\frac{1}{i!}.$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{1}{2} E\left[\sum_{i=0}^{\sigma} a_i(\log(\eta) + \sigma Z\right],$$

and

$$Var\left(X\right) = \frac{1}{4} E\left[\left(\sum_{i=0}^{\sigma} a_i(\log(\eta) + \sigma Z\right)^2\right] - \left[E(X)\right]^2,$$

respectively. In reference [183], Figures 1 and 2 showed that the PDF can be unimodal, decreasing-increasing-decreasing bimodal, or trimodal. This distribution has been applied to model internet access data.

2.121. Vila and Quintino's unit asymmetric distribution

Suppose (Y_1, Y_2) have the bivariate extreme distribution with Fr´echet margins [57, p. 14], with scale parameter σ, shape parameter α and correlation coefficient ρ. Let $X = \frac{Y_1}{Y_1+Y_2}$. Vila and Quintino [184] showed that the PDF and CDF of X are

$$f(x) = \frac{\alpha}{\sigma^2} s^{\alpha-1}(s+1)^2 \left\{\frac{2\left(\frac{s^\alpha}{\sigma^\alpha}+1\right)^2 - \rho\left[\left(\frac{s^\alpha}{\sigma^\alpha}+1\right)^2\right]}{\left[\left(\frac{s^\alpha}{\sigma^\alpha}+1\right)^2 - \rho\frac{s^\alpha}{\sigma^\alpha}\right]^2} - \frac{1}{\left(\frac{s^\alpha}{\sigma^\alpha}+1\right)^2}\right\},$$

and

$$F(x) = \frac{\frac{s^\alpha}{\sigma^\alpha}}{\frac{s^\alpha}{\sigma^\alpha}+1} \frac{\left(\frac{s^\alpha}{\sigma^\alpha}+1\right)^2 - \rho}{\left(\frac{s^\alpha}{\sigma^\alpha}+1\right)^2 - \rho\frac{s^\alpha}{\sigma^\alpha}}, \tag{2.31}$$

respectively, for $0 < x < 1$, where $s = \frac{x}{1-x}$, $\sigma > 0$ is a scale parameter, $\alpha > 0$ is a shape parameter, and $0 < \rho < 1$ is correlation coefficient. The kth moment of X can be approximated as

$$E\left(X^k\right) \approx \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)^k + \frac{k\mu_1^{k-1}}{2\left(\mu_1 + \mu_2\right)^{k+2}} \left\{\frac{\mu_2}{\mu_1}\left[(k-1)\mu_2 - 2\mu_1\right] Var\left(Y_1\right) + 2\left(\mu_1 - k\mu_2\right) Cov\left(Y_1, Y_2\right) + (k+1)\mu_1 Var\left(Y_2\right)\right\},$$

where

$$\mu_i = \sigma\Gamma\left(1 - \frac{1}{\alpha}\right), \quad \alpha > 1,$$

$$Var\left(Y_i\right) = \sigma\left[\Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma^2\left(1 - \frac{1}{\alpha}\right)\right], \quad \alpha > 2$$

and

$$\left|Cov\left(Y_1, Y_2\right)\right| \le \sigma^2\left[\Gamma\left(1 - \frac{2}{\alpha}\right) - \Gamma^2\left(1 - \frac{1}{\alpha}\right)\right], \quad \alpha > 2.$$

In particular, the mean and variance can be approximated as

$$E(X) \approx \frac{\mu_1}{\mu_1 + \mu_2} - \frac{1}{\left(\mu_1 + \mu_2\right)^3}\left\{\mu_2 Var\left(Y_1\right) + \left(\mu_2 - \mu_1\right) Cov\left(Y_1, Y_2\right) - \mu_1 Var\left(Y_2\right)\right\},$$

and

$$\begin{aligned} Var(X) \approx\ & \frac{\mu_1}{\left(\mu_1 + \mu_2\right)^4}\left\{\frac{\mu_2}{\mu_1}\left(\mu_2 - 2\mu_1\right) Var\left(Y_1\right) + 2\left(\mu_1 - 2\mu_2\right) Cov\left(Y_1, Y_2\right) + 3\mu_1 Var\left(Y_2\right)\right\} \\ & - \frac{1}{\left(\mu_1 + \mu_2\right)^6}\left\{\mu_2 Var\left(Y_1\right) + \left(\mu_2 - \mu_1\right) Cov\left(Y_1, Y_2\right) - \mu_1 Var\left(Y_2\right)\right\}^2 \\ & + \frac{2\mu_1}{\left(\mu_1 + \mu_2\right)^4}\left\{\mu_2 Var\left(Y_1\right) + \left(\mu_2 - \mu_1\right) Cov\left(Y_1, Y_2\right) - \mu_1 Var\left(Y_2\right)\right\}, \end{aligned}$$

respectively. Vila and Quintino [184, Figures 3 and 4] showed that the PDF can be unimodal, or bathtub shaped. This distribution has been applied to model medium pass completion proportions in UEFA Champions League and expenditures and income data from Italy.

2.122. Unit bimodal distribution [185]

Suppose $\left(Y_1, Y_2\right)$ are correlated Birnbaum-Saunders random variables. Let $X = \frac{Y_2}{Y_1 + Y_2}$. The researchers in [185] showed that the PDF and CDF of X are

$$f(x) = \frac{\exp\left\{\frac{1}{1-\rho^2}\left(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2}\right)\right\}}{4\pi\alpha_1\alpha_2\sqrt{1-\rho^2}} \frac{(s+1)^2}{s} \exp\left\{-\frac{\rho}{\alpha_1\alpha_2\left(1-\rho^2\right)}\left[\sqrt{\frac{\beta_2 s}{\beta_1}} + \sqrt{\frac{\beta_1}{\beta_2 s}}\right]\right\}$$

$$\cdot \left\{ \left(\sqrt{\frac{\beta_2 s}{\beta_1}} + \sqrt{\frac{\beta_1}{\beta_2 s}} \right) K_0 \left(\frac{\sqrt{u_\rho v_\rho}}{1-\rho^2} \right) + \left[\sqrt{\frac{\frac{v_\rho}{u_\rho} s}{\beta_1 \beta_2}} + \sqrt{\frac{\beta_1 \beta_2}{\frac{v_\rho}{u_\rho} s}} \right] K_1 \left(\frac{\sqrt{u_\rho v_\rho}}{1-\rho^2} \right) \right\},$$

and

$$F(x) = \frac{1}{2} - \frac{1}{2} \int_{-\infty}^{\infty} \text{erf} \left\{ \frac{\left(\frac{\alpha_2}{\alpha_1} \sqrt{\frac{s\beta_2}{\beta_1}} - \rho \right)}{\sqrt{2(1-\rho^2)}} \omega + \frac{\sqrt{2} \left(s\beta_2 - \beta_1 \right)}{\alpha_1 \sqrt{s\beta_1 \beta_2 \left(1-\rho^2\right)}} \frac{1}{\alpha_2 \omega + \sqrt{\left(\alpha_2 \omega\right)^2 + 4}} \right\} \phi(\omega) d\omega,$$

respectively, for $0 < x < 1$, where $u_\rho \equiv \frac{s}{\alpha_1^2 \beta_1} + \frac{1}{\alpha_2^2 \beta_2} - \frac{2\rho}{\alpha_1 \alpha_2} \sqrt{\frac{s}{\beta_1 \beta_2}}$, $v_\rho \equiv \frac{\beta_1}{\alpha_1^2 s} + \frac{\beta_2}{\alpha_2^2} - \frac{2\rho}{\alpha_1 \alpha_2} \sqrt{\frac{\beta_1 \beta_2}{s}}$, $\alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0$ are scale parameters, and $-1 < \rho < 1$ is a correlation coefficient. The kth moment of X is

$$E\left(X^k\right) = \frac{1}{2} + \frac{k}{2} \int_0^1 t^{k-1} \int_{-\infty}^{\infty} \text{erf} \left\{ \frac{\left(\frac{\alpha_2}{\alpha_1} \sqrt{\frac{s\beta_2}{\beta_1}} - \rho \right)}{\sqrt{2(1-\rho^2)}} \omega + \frac{\sqrt{2} \left(s\beta_2 - \beta_1 \right)}{\alpha_1 \sqrt{s\beta_1 \beta_2 \left(1-\rho^2\right)}} \frac{1}{\alpha_2 \omega + \sqrt{\left(\alpha_2 \omega\right)^2 + 4}} \right\} \phi(\omega) d\omega dt.$$

In particular, the mean and variance are

$$E\left(X\right) = \frac{1}{2} + \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \text{erf} \left\{ \frac{\left(\frac{\alpha_2}{\alpha_1} \sqrt{\frac{s\beta_2}{\beta_1}} - \rho \right)}{\sqrt{2(1-\rho^2)}} \omega + \frac{\sqrt{2} \left(s\beta_2 - \beta_1 \right)}{\alpha_1 \sqrt{s\beta_1 \beta_2 \left(1-\rho^2\right)}} \frac{1}{\alpha_2 \omega + \sqrt{\left(\alpha_2 \omega\right)^2 + 4}} \right\} \phi(\omega) d\omega dt,$$

and

$$Var\left(X\right) = \frac{1}{2} + \int_0^1 t \int_{-\infty}^{\infty} \text{erf} \left\{ \frac{\left(\frac{\alpha_2}{\alpha_1} \sqrt{\frac{s\beta_2}{\beta_1}} - \rho \right)}{\sqrt{2(1-\rho^2)}} \omega + \frac{\sqrt{2} \left(s\beta_2 - \beta_1 \right)}{\alpha_1 \sqrt{s\beta_1 \beta_2 \left(1-\rho^2\right)}} \frac{1}{\alpha_2 \omega + \sqrt{\left(\alpha_2 \omega\right)^2 + 4}} \right\} \phi(\omega) d\omega dt - \left[E(X)\right]^2,$$

respectively. In reference [185], Figures 1–4 showed that the PDF can be bimodal. This distribution has been applied to model data on income and consumption from Italy and percentage of the body fat complement of athletes.

2.123. Unit power generalized Weibull distribution [56]

Suppose Y is a power generalized Weibull random variable with shape parameters α, β, and scale parameter λ [52]. Let $X = \exp(-Y)$. The researchers in [56] showed that the PDF and CDF are X are

$$f(x) = \frac{\alpha\beta\lambda}{x} (-\log x)^{\alpha-1} \left[1 + \lambda(-\log x)^{\alpha}\right]^{\beta-1} \exp\left\{1 - \left[1 + \lambda(-\log x)^{\alpha}\right]^{\beta}\right\},$$

and

$$F(x) = \exp\left\{1 - \left[1 + \lambda(-\log x)^{\alpha}\right]^{\beta}\right\},$$

respectively, for $0 < x < 1$, $\alpha > 0, \beta > 0$ are shape parameters, and $\lambda > 0$ is a scale parameter. The kth moment of X is

$$E\left(X^k\right) = \alpha\beta e\left(\frac{\lambda}{k^\alpha}\right)^{k+1}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\binom{(i+1)\beta-1}{j}\Gamma\left(\alpha(j+1)\right).$$

In particular, the mean and variance are

$$E\left(X\right) = \alpha\beta\lambda^2 e\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\binom{(i+1)\beta-1}{j}\Gamma\left(\alpha(j+1)\right),$$

and

$$Var\left(X\right) = \alpha\beta e\frac{\lambda^3}{8^\alpha}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\binom{(i+1)\beta-1}{j}\Gamma\left(\alpha(j+1)\right) - [E(X)]^2,$$

respectively. In reference [56], Figure 1 showed that the PDF can be constant, bathtub shaped, unimodal, J shaped (increasing), and inverted J shaped (decreasing). This distribution has been applied to model COVID-19 mortality rates in the Kingdom of Saudi Arabia, failure times of an airplane's air-cooling system, burr measurements on iron sheets, and susceptibility indices for irradiated and unirradiated peppermint packages.

3. Application with real data sets

In this section, we use real-life data sets to illustrate the application of several models reviewed in Section 2. These data sets have been used by different authors and were obtained from various sources. To ensure consistency, we prioritized data sets that have been widely used and recognized for their relevance to unit probability distributions. These include:

- Tensile strength of polyester fibers data: This dataset consists of 30 measurements of tensile strength of polyester fibers, as reported in Quesenberry and Hales [148].
- Capacity factor datasets: Capacity factor is defined as the ratio of actual electricity output to the maximum possible output from a power unit, and lies within the unit interval $[0, 1]$. It is widely used as a reliability metric for evaluating the efficiency of energy generation systems. Different algorithms have been proposed to estimate this metric under varying assumptions. Two datasets of estimated capacity factors generated using two algorithms discussed in [39]: The SC16 algorithm and the P3 algorithm are considered. Accordingly, we refer to these datasets as
  - Capacity factor dataset 1 (SC16): Estimated using the SC16 algorithm, this dataset contains 23 values of the capacity factor derived from probabilistic production costing models.
  - Capacity factor dataset 2 (P3): Estimated using the P3 algorithm, this dataset includes 22 values and is closely related in structure and purpose to the SC16 dataset, but utilizes a different estimation scheme.
- Core specimens of petroleum wells data: This dataset contains 48 observations obtained from 12 core samples taken from petroleum reservoirs, which were sampled across 4 cross-sections, as reported in [149].

- The capacity factor datasets: This data includes the capacity factor, which is the ratio of actual electricity output to the maximum possible output generated using various algorithms.
- Soil fertility influence data: This data includes measurement of phosphorus concentration in the leaves of 128 plants, as reported in Fonseca and França [61].
- Failure times data: This data consists of the failure times of 20 mechanical components, as reported in [133].
- Secondary reactor pumps data: This data consists of times between failures of secondary reactor pumps, as reported in Suprawhardana and Prayoto [175].
- Anxiety performance data: The data come from a study on anxiety conducted with a group of 166 'normal' women in Townsville, Queensland, Australia, as reported by [138].
- Daily pan evaporation data: The data pertains to daily pan evaporation, measured in hundredths of inches, recorded in September 2016 at the San Joaquin Drainage 05 Friant Government Camp, California, USA.

Similarly, in choosing models for illustration, we prioritize those that are widely recognized and commonly used among practitioners, particularly those that are derived from popular classical distributions. These include: The unit upper truncated Weibull (UUTW) distribution in [137]; the unit Gumbel type II (UG-II) distribution in [164]; the unit modified power function (UMP) distribution in [138]; the unit Teissier (UT) distribution in [108]; the unit Lindley (UL) distribution in [123]; the log-extended exponential-geometric (LEEG) distribution in [87]; Lindley conditional Lindley+Gamma (LCG): A new one parameter distribution in [36]; the one parameter unit Gumbel (OPUG) distribution in [24]; the unit Gompertz (UGz) distribution in [126]; and the Kumaraswamy distribution, the power function (PF) distribution and beta distribution.

To illustrate the application of these selected models, we fit each of the chosen models to every data set listed, see Tables 1–9 and Figures 1–9. This enables us to systematically compare the performance of each model across real-life scenarios. By fitting every model to every dataset, we can assess how well each model captures the underlying patterns and behaviors within different types of data. This comparison enables us to identify the most effective models for specific datasets and provides valuable insights into their practical applications in the context of unit probability distributions.

Table 1. Anxiety performance data.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-493.8465	-487.6226	-493.7729	-485.6226	-491.3202	0.2895	< 0.0001
UG-II	-557.8554	-551.6315	-557.7818	-549.6315	-555.3291	0.33227	< 0.0001
MPF	-514.9746	-508.7506	-514.9010	-506.7506	-512.4482	0.28291	< 0.0001
UT	-467.9559	-464.8439	-467.9315	-463.8439	-466.6927	0.2621	< 0.0001
UL	756.4667	759.5787	756.4911	760.5787	757.7299	0.4311	< 0.0001
LEEG	-515.2125	-508.9885	-515.1389	-506.9885	-512.6862	1	< 0.0001
LCG	-431.5715	-428.4595	-431.5471	-427.4595	-430.3083	0.4139	< 0.0001
OPUG	-456.0906	-452.9787	-456.0663	-451.9787	-454.8275	0.6707	< 0.0001
UGz	-368.0573	-361.8334	-367.9837	-359.8334	-365.5310	0.2700	< 0.0001
Kumaraswamy	-484.6297	-478.4057	-484.5560	-476.4057	-482.1033	0.2879	< 0.0001
Beta	-474.8960	-468.6720	-474.8224	-466.6720	-472.3697	0.2989	< 0.0001
PF	-373.4542	-370.3422	-373.4298	-369.3422	-372.1910	0.2708	< 0.0001

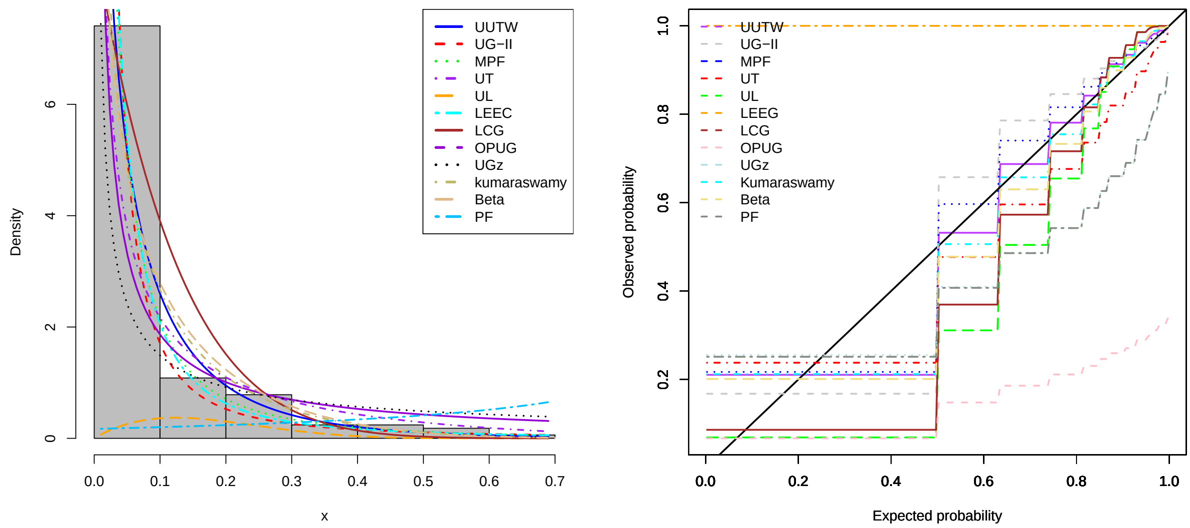

Figure 1. Plots of the fitted PDFs and P-P plot for anxiety performance data.

Table 2. Core specimens of petroleum wells data.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-101.1826	-97.4402	-100.9159	-95.4402	-99.7683	0.1648	0.1473
UG-II	-	-	-	-	-	-	–
MPF	-98.6255	-94.8831	-98.35881	-92.8831	-97.2112	0.1464	0.2549
UT	-84.2570	-82.3858	-84.1700	-81.3858	-83.5498	0.2643	0.00245
UL	108.0842	109.9554	108.1711	110.9554	108.7913	0.3210	0.0001
LEEG	-101.3644	-97.6220	-101.0977	-95.6220	-99.9501	1	< 0.0001
LCG	-60.6019	-58.7307	-60.5149	-57.7307	-59.8948	0.3477	< 0.0001
OPUG	-40.6256	-38.7544	-40.5386	-37.7544	-39.9185	0.7677	< 0.0001
UGz	-108.8036	-105.0612	-108.5369	-103.0612	-107.3894	0.1013	0.7086
Kumaraswamy	-100.6970	-96.9546	-100.4303	-94.9546	-99.2827	0.1679	0.1335
Beta	-106.7326	-102.9902	-106.4659	-100.9902	-105.3183	0.1597	0.1727
PF	-10.2067	-8.3355	-10.1197	-7.3355	-9.4995	0.4296	< 0.0001

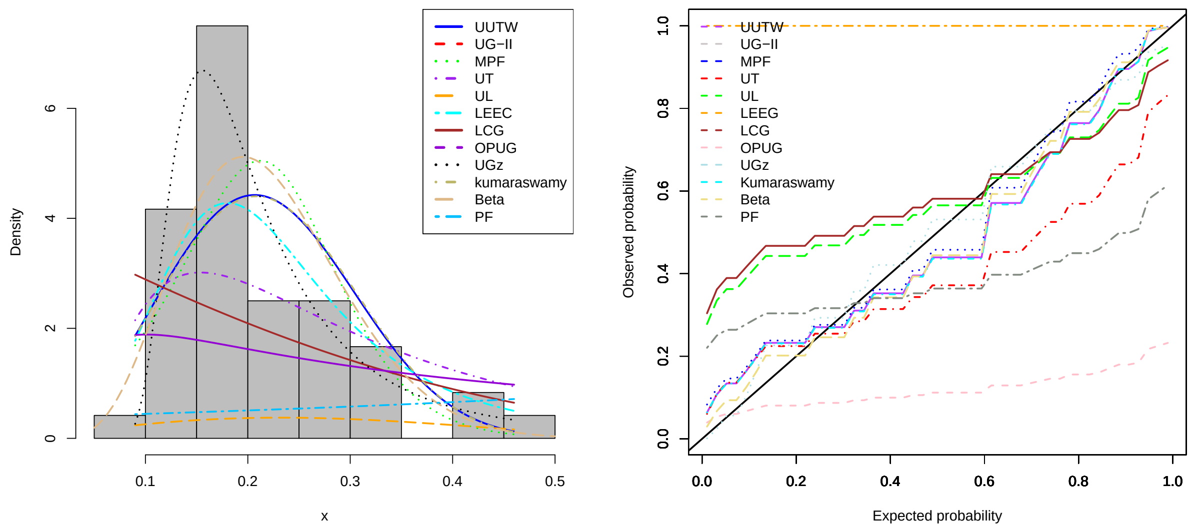

Figure 2. Plots of the fitted PDFs and P-P plot for core specimens of petroleum wells data.

Table 3. Daily pan evaporation data.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-76.5694	-73.7670	-76.1249	-71.7670	-75.6729	0.1999	0.182
UG-II	-	-	-	-	-	-	–
MPF	-76.9754	-74.1730	-76.5310	-72.1730	-76.0789	0.2069	0.1533
UT	-48.9215	-47.5203	-48.7786	-46.5203	-48.4732	0.3768	0.0004
UL	63.42959	64.8308	63.5725	65.8308	63.8778	0.3725	0.0005
LEEG	-63.3737	-60.5713	-62.9292	-58.5713	-62.4772	1	< 0.0001
LCG	-27.6520	-26.2508	-27.5092	-25.2508	-27.2038	0.3974	0.0002
OPUG	-20.2517	-18.8505	-20.1088	-17.8505	-19.8034	0.8099	< 0.0001
UGz	-57.4261	-54.6237	-56.9817	-52.6237	-56.5296	0.2355	0.0717
Kumaraswamy	-76.0951	-73.2927	-75.6506	-71.2927	-75.1985	0.2030	0.1685
Beta	-77.6190	-74.8166	-77.1746	-72.8166	-76.7225	0.1897	0.2308
PF	-0.7948	0.6064	-0.6519	1.6064	-0.3465	0.5009	< 0.0001

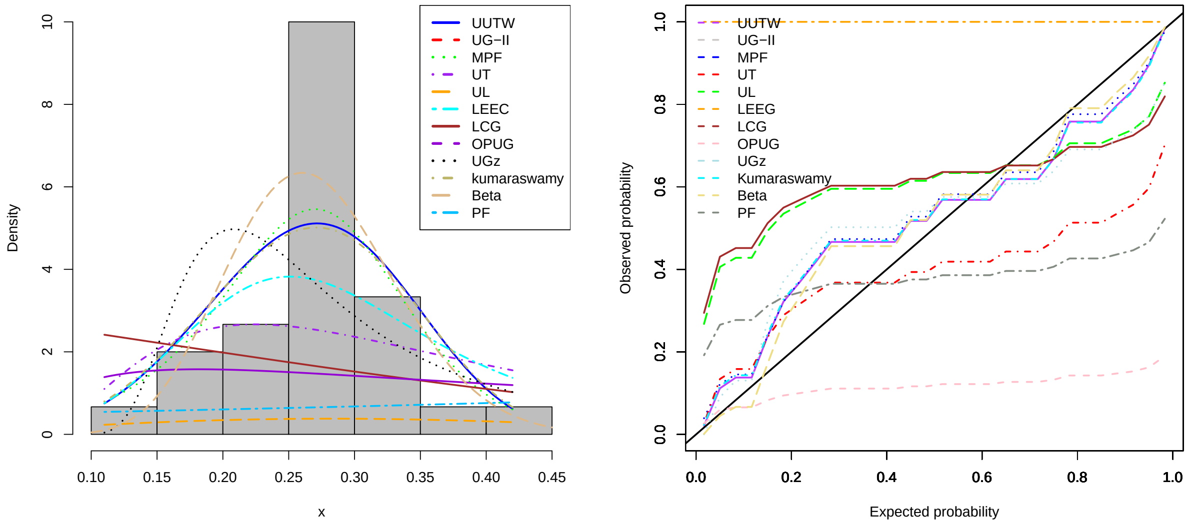

Figure 3. Plots of the fitted PDFs and P-P plot for daily pan evaporation data.

Table 4. Failure times data.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-48.8456	-46.8542	-48.1397	-44.8542	-48.4569	0.1999	0.182
UG-II	-	-	-	-	-	-	–
MPF	-47.8879	-45.8964	-47.1820	-43.8964	-47.4991	0.2069	0.1533
UT	-31.4800	-30.4843	-31.2578	-29.4843	-31.2856	0.3768	0.0004
UL	51.5714	52.5671	51.7936	53.5671	51.7657	0.3725	0.0005
LEEG	-56.4383	-54.4469	-55.7324	-52.4469	-56.0496	1	< 0.0001
LCG	-43.3055	-42.3097	-43.0832	-41.3097	-43.1111	0.3974	0.0002
OPUG	-28.2670	-27.2713	-28.0448	-26.2713	-28.0726	0.8099	< 0.0001
UGz	-12.6512	-10.6597	-11.9453	-8.6597	-12.2624	0.2355	0.0717
Kumaraswamy	-47.2968	-45.3054	-46.5910	-43.3054	-46.9081	0.2030	0.1685
Beta	-51.7626	-49.7711	-51.0567	-47.7711	-51.3739	0.1897	0.2308
PF	-15.1164	-14.1207	-14.8942	-13.1207	-14.9220	0.5009	< 0.0001

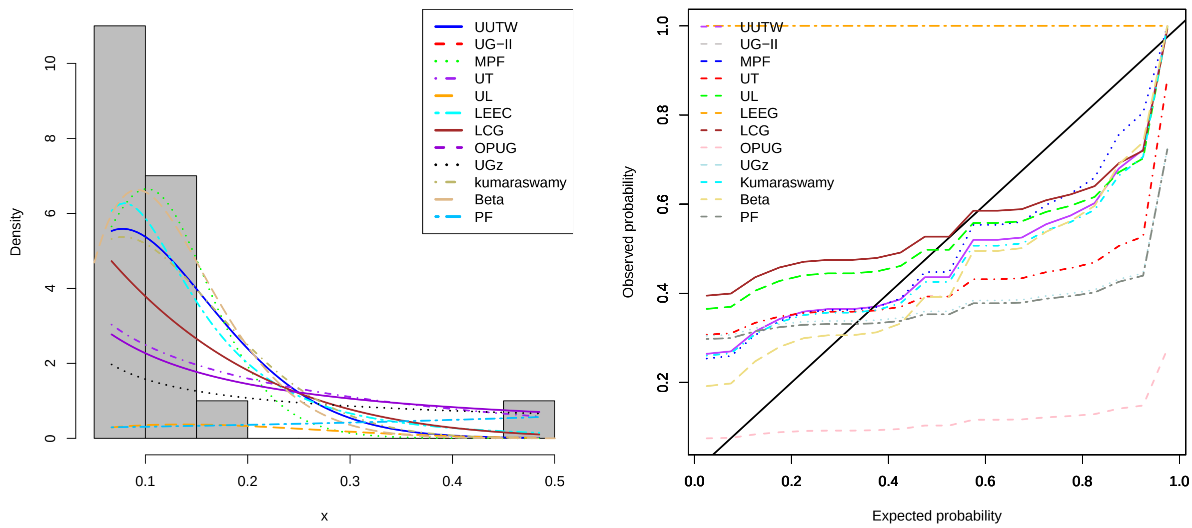

Figure 4. Plots of the fitted PDFs and P-P plot for failure times data.

Table 5. Phosphorus concentration data.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-384.2896	-378.5855	-384.1936	-376.5855	-381.9720	0.1027	0.1346
UG-II	-	-	-	-	-	-	-
MPF	-374.6277	-368.9237	-374.5317	-366.9237	-372.3101	0.1140	0.0720
UT	-286.2761	-283.4241	-286.2444	-282.4241	-285.1173	0.3094	< 0.0001
UL	282.9622	285.8143	282.9940	286.8143	284.1210	0.30552	< 0.0001
LEEG	-352.9109	-347.2068	-352.8149	-345.2068	-350.5933	1	< 0.0001
LCG	-263.2934	-260.4414	-263.2617	-259.4414	-262.1346	0.3199	< 0.0001
OPUG	-164.3754	-161.5233	-164.3436	-160.5233	-163.2166	0.8039	< 0.0001
UGz	-76.9956	-71.2916	-76.8996	-69.2916	-74.6780	0.4696	< 0.0001
Kumaraswamy	-384.0737	-378.3696	-383.9777	-376.3696	-381.7561	0.0954	0.1947
Beta	-390.1785	-384.4745	-390.0825	-382.4745	-387.8609	0.0971	0.1792
PF	-81.4684	-78.6164	-81.4367	-77.6164	-80.3096	0.47005	< 0.0001

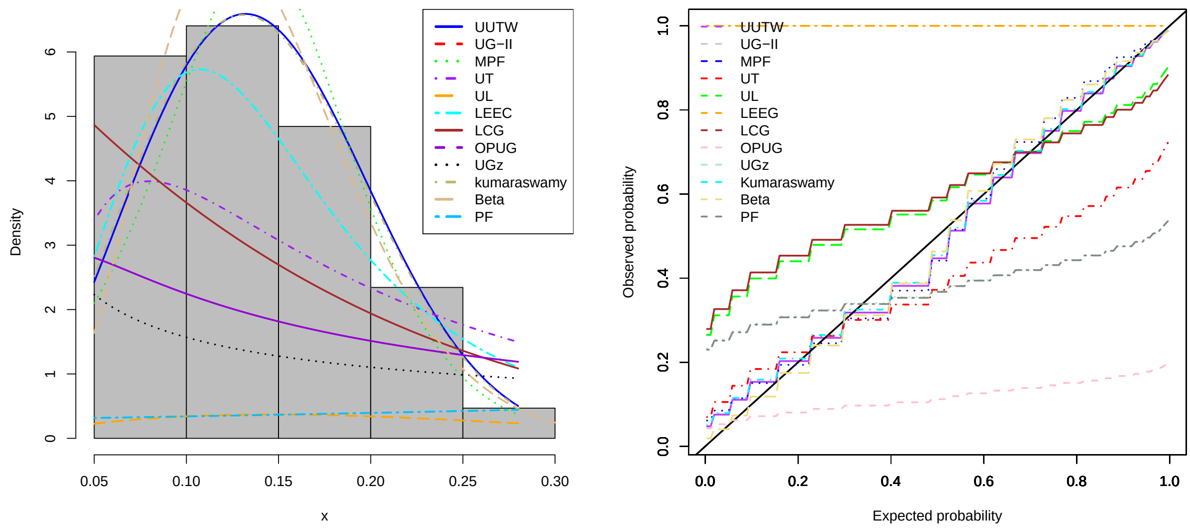

Figure 5. Plots of the fitted PDFs and P-P plot for Phosphorus concentration data.

Table 6. Tensile strength data.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-2.8938	-0.0914	-2.4493	1.9086	-1.9973	0.0578	0.9998
UG-II	-	-	-	-	-	-	-
MPF	-2.9222	-0.1198	-2.4777	1.8802	-2.0256	0.0515	1
UT	-2.7375	-1.3363	-2.5947	-0.3363	-2.2893	0.0807	0.9808
UL	138.9697	140.3709	139.1126	141.3709	139.4180	0.2721	0.0188
LEEG	-3.0504	-0.2480	-2.6059	1.7519	-2.1538	1	< 0.0001
LCG	-3.8932	-2.4920	-3.7504	-1.4920	-3.4450	0.0960	0.9203
OPUG	5.8950	-4.4938	-5.7521	-3.4938	-5.4467	0.6443	< 0.0001
UGz	-3.8976	-1.0952	-3.4532	0.9048	-3.0011	0.0733	0.9932
Kumaraswamy	-2.6221	0.1803	-2.1776	2.1803	-1.7256	0.0650	0.9987
Beta	-2.6101	0.1923	-2.1657	2.1923	-1.7136	0.0669	0.9979
PF	-1.4495	-0.0483	-1.3067	0.9517	-1.0013	0.1374	0.5755

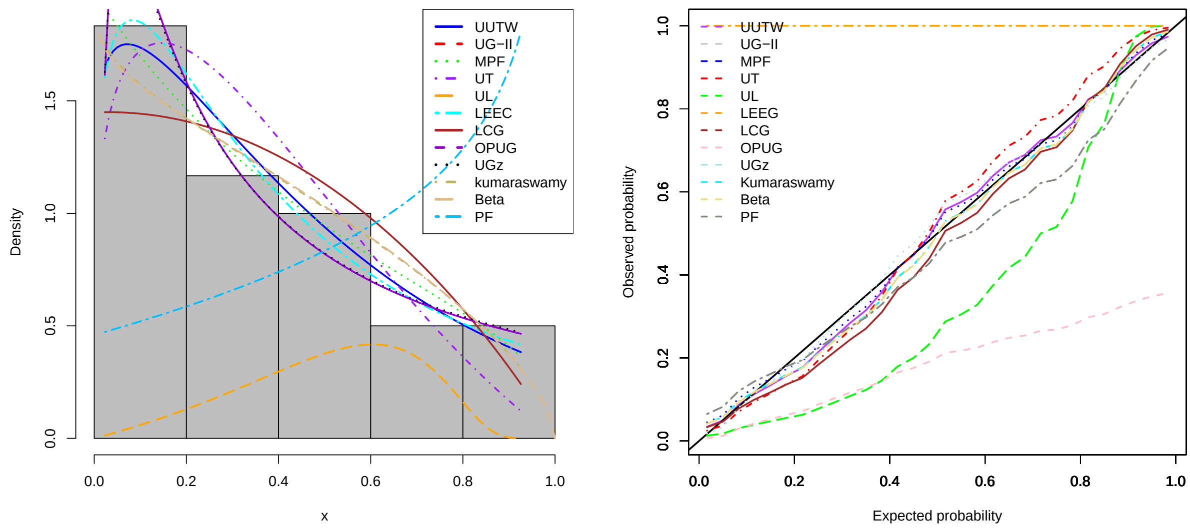

Figure 6. Plots of the fitted PDFs and P-P plot for tensile strength data.

Table 7. Capacity factor data I.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-15.9192	-13.6482	-15.3192	-11.6482	-15.3480	0.1550	0.6384
UG-II	-19.2780	-17.0070	-18.6780	-15.0070	-18.7069	0.1139	0.9266
MPF	-14.5776	-12.3066	-13.9776	-10.3066	-14.0064	0.1484	0.6919
UT	-11.1913	-10.0558	-11.0009	-9.0558	-10.9058	0.1878	0.392
UL	140.0437	141.1792	140.2341	142.1792	140.3292	0.4755	< 0.0001
LEEG	-16.3738	-14.1029	-15.7738	-12.1029	-15.8027	1	< 0.0001
LCG	-4.4904	-3.3549	-4.2999	-2.3549	-4.2048	0.3640	0.0045
OPUG	-20.6198	-19.4843	-20.4293	-18.4843	-20.3342	0.6466	< 0.0001
UGz	-18.6728	-16.4018	-18.0728	-14.4018	-18.1017	0.1336	0.8065
Kumaraswamy	-15.3416	-13.0706	-14.7416	-11.0706	-14.7704	0.1790	0.4529
Beta	-16.9690	-15.8335	-16.7785	-14.8335	-16.6834	0.1836	0.4202
PF	-15.2149	-12.9439	-14.6149	-10.9439	-14.6438	0.1893	0.3817

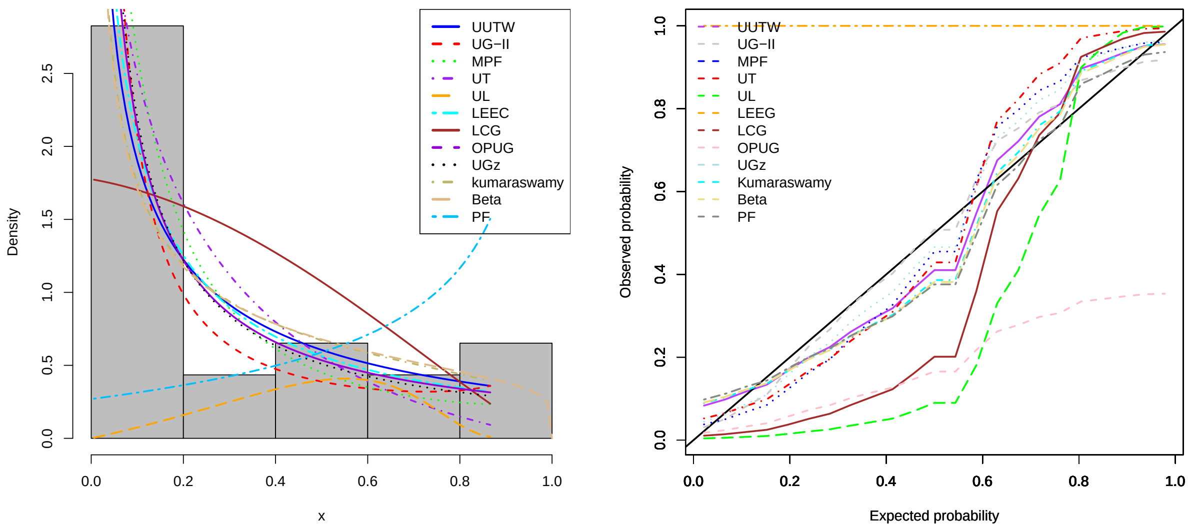

Figure 7. Plots of the fitted PDFs and P-P plot for capacity factor data I.

Table 8. Capacity factor data II.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-10.2649	-8.0828	-9.6333	-6.0828	-9.7508	0.1748	0.5122
UG-II	-	-	-	-	-	-	
MPF	-10.2269	-8.0448	-9.5954	-6.0448	-9.7129	0.1439	0.7523
UT	-6.5524	-5.4614	-6.3524	-4.4614	-6.2954	0.1910	0.3982
UL	125.8079	126.8989	126.0079	127.8989	126.0649	0.4600	0.0002
LEEG	-10.7873	-8.6052	-10.1557	-6.6052	-10.2733	1	< 0.0001
LCG	-3.2717	-2.1807	-3.0717	-1.1807	-3.0147	0.3513	0.0088
OPUG	-15.0104	-13.9194	-14.8104	-12.9194	-14.7534	0.6472	< 0.0001
UGz	-13.1320	-10.9500	-12.5005	-8.9500	-12.6180	0.1370	0.8036
Kumaraswamy	-9.6872	-7.5051	-9.0557	-5.5051	-9.1732	0.1963	0.365
Beta	-9.5639	-7.3818	-8.9323	-5.3818	-9.0498	0.2002	0.3413
PF	-11.1706	-10.0795	-10.9706	-9.0795	-10.9136	0.2047	0.3151

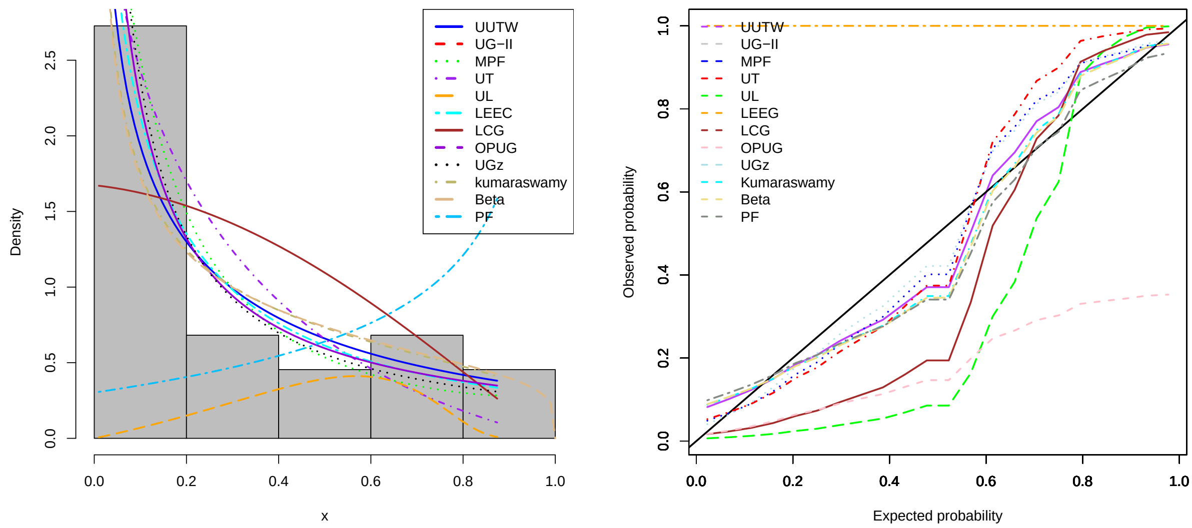

Figure 8. Plots of the fitted PDFs and P-P plot for capacity factor data II.

Table 9. Times between failures of secondary reactor pumps.

	AIC	BIC	AICc	CAIC	HQC	K-S	P-value
UUTW	-37.3035	-35.0325	-36.7035	-33.0325	-36.7323	0.1183	0.8671
UG-II	-	-	-	-	-	-	
MPF	-39.1096	-36.8387	-38.5096	-34.8387	-38.5385	0.0975	0.9659
UT	–38.5716	-37.4361	-38.3811	-36.4361	-38.2860	0.1430	0.6826
UL	104.3503	105.4858	104.5408	106.4858	104.6359	0.3274	0.0107
LEEG	-38.9377	-36.6667	-38.3377	-34.6667	-38.3665	1	< 0.0001
LCG	-33.4175	-32.2820	-33.2270	-31.2820	-33.1320	0.2633	0.0676
OPUG	-38.0862	-36.9507	-37.8957	-35.9507	-37.8006	0.6718	< 0.0001
UGz	-26.5726	-24.3017	-25.9726	-22.3017	-26.0015	0.2217	0.1788
Kumaraswamy	-36.6592	-34.3883	-36.0592	-32.3883	-36.0881	0.1393	0.7123
Beta	-36.0571	-33.7861	-35.4571	-31.7861	-35.4859	0.1541	0.5918
PF	-29.0615	-27.9260	-28.8710	-26.9260	-28.7759	0.2248	0.1674

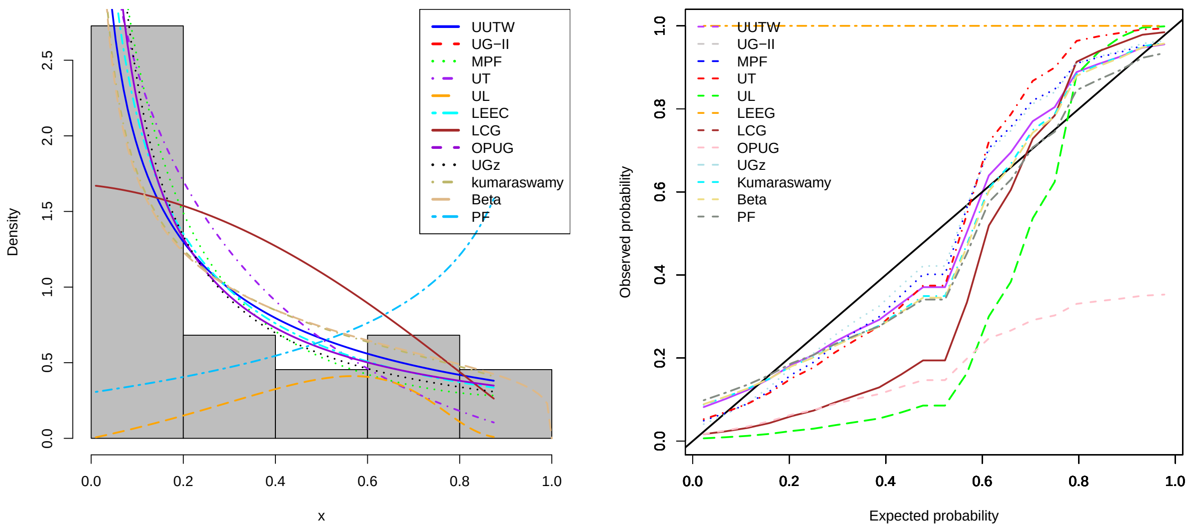

Figure 9. Plots of the fitted PDFs and P-P plot for times between failures of secondary reactor pumps.

The empirical analysis conducted in this review serves to assess the practical performance of twelve selected unit continuous distributions across nine real-world datasets. These datasets span a range of areas, including economics, engineering, and biological sciences with each characterized by variables naturally bounded within the unit interval [0,1], such as proportions and probabilities. The method of maximum likelihood was used for parameter estimation. The values of the five selection criteria, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (AICc), Consistent Akaike Information Criterion (CAIC), and Hanna-Quinn Information Criterion (HQC) are provided in Tables 1–9.

Overall, the results demonstrate that the selected unit distributions exhibit a high degree of flexibility in modeling bounded data, with several models yielding good fits depending on the nature and characteristics of the dataset. For instance, the Beta and Kumaraswamy distributions continue to serve as robust baseline models due to their tractable forms and interpretability. However, in many cases, the unit upper truncated Weibull, the unit Gumbel type II distribution, the log-extended exponential-geometric (LEEG) distribution, etc. outperformed traditional models by better capturing skewness, and kurtosis, or by providing more flexible tail behavior. In particular,

- For anxiety performance data: According to the selection criteria values in Table 1, the best fitting model is the unit Gumbel type II distribution (UG-II), the second best fitting model is the log-extended exponential-geometric (LEEG) distribution, and the third best fitting model is the modified power function distribution (MPF). The worst fit is given by the UL.
- For the core specimens of petroleum wells data: According to the selection criteria values in Table 2, the best fitting model is the unit Gompertz distribution (UGz), the second best fitting model is the beta distribution, and the third best fitting model is the log-extended exponential-geometric (LEEG) distribution. The worst fit is given by the UL.
- For the daily pan evaporation data: According to the selection criteria values in Table 3, the best

fitting model is the beta distribution, the second best fitting model is the unit modified power function distribution (MPF), and the third best fitting model is the unit upper truncated Weibull distribution. The worst fit is given by the UL.

- For the failure times data: According to the selection criteria values in Table 4, the best fitting model is the log-extended exponential-geometric (LEEG) distribution, the second best fitting model is the beta distribution, and the third best fitting model is the unit modified power function distribution (MPF). The worst fit is given by the UL.
- Soil fertility influence (Phosphorus concentration) data: According to the selection criteria values in Table 5, the best fitting model is the beta distribution, the second best fitting model is the Kumaraswamy distribution, and the third best fitting model is the unit upper truncated Weibull (UUTW) distribution. The worst fit is given by the UL.
- Tensile strength of polyester fibers data: According to the selection criteria values in Table 6, the best fitting model is the unit Gompertz distribution (UGz), the second best fitting model is Lindley conditional+Gamma distribution (LCG) and the third best fitting model is the LEEG distribution. The worst fit is given by the UL.
- The capacity factor datasets I: According to the selection criteria values in Table 7, the best fitting model is the one parameter unit Gumbel distribution (OPUG), the second best fitting model is the unit Gumbel type II distribution (UG-II), and the third best fitting model is the unit Gompertz distribution (UGz). The worst fit is given by the UL.
- The capacity factor datasets II: According to the selection criteria values in Table 8, the best fitting model is the one parameter unit Gumbel distribution (OPUG), the second best fitting model is the unit Gompertz distribution (UGz), and the third best fitting model is the power function distribution (PF). The worst fit is given by the UL.
- The secondary reactor pumps data: According to the selection criteria values in Table 9, the best fitting model is the unit modified power function distribution (MPF), the second best fitting model is the log-extended exponential-geometric (LEEG) distribution, and the third best fitting model is the UT. The worst fit is given by the UL.

It is important to highlight that no single distribution emerged as the universally best-performing model across all datasets. Rather, the performance of each distribution is largely data dependent, with different models performing in different contexts. For example, distributions such as the unit log-logistic and unit Gumbel perform well in datasets with heavy tails, while others like the power function and unit exponential variants are more suited to symmetric, or lightly skewed datasets. This variation highlights a key finding of this review, the empirical features of the data, and the theoretical properties of the distributions. In conclusion, the analysis supports the usefulness and versatility of unit continuous distributions, particularly their contemporary extensions. It does, however, also make it apparent that there is no one-size-fits-all model. Rather, this compendium's strength is its breadth, providing researchers with a wealth of tools from which they can choose the best model based on actual data.

4. Conclusions

In this review, we have provided a detailed and comprehensive overview of over one-hundred-unit continuous probability distributions, encompassing classical models. These distributions play a vital role in modeling proportions, rates, and other bounded data that arise naturally in many scientific and engineering fields. In particular, we fit twelve of these distributions to nine diverse datasets. The empirical findings demonstrate the high flexibility and fitting capability of these models, confirming their value in applied statistics. Notably, the analysis show that while all selected models perform well under certain conditions, there is no universally best-performing distribution across all datasets. This highlights the importance of context-specific model selection and the need to understand the underlying structure of the data being analyzed. We also highlight the continued evolution of unit distributions, with ongoing developments introducing greater flexibility, improved tail behavior, and better alignment with complex data patterns. By documenting these advancements, we hope this work will serve as a foundational resource for researchers and practitioners seeking to model bounded data accurately and effectively.

Future research directions include: (1) Developing multivariate extensions of unit distributions; (2) proposing copula-based constructions for modeling joint dependencies; and (3) exploring Bayesian estimation methods tailored to the unit setting. The wide range of distributions reviewed here offers a strong foundation for such future developments, and we hope this compendium stimulates further innovation and practical application in this growing area of statistics. The wealth of unit distributions reviewed here provides a strong basis for such future explorations.

Author contributions

All authors contributed equally to the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors have no conflicts of interest.

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Appendix

The expressions in Section 2 include various special functions, including the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt;$$

the upper incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} \exp(-t) dt;$$

the lower incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t) dt;$$

the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt;$$

the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt;$$

the sine function defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt;$$

the cosine function defined by

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt;$$

the standard normal probability density function defined by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right);$$

the standard normal cumulative distribution function defined by

$$\Phi(x) = \int_{-\infty}^x \phi(t) dt;$$

the beta function defined by

$$B(a,b) = \int_0^\infty t^{a-1}(1-t)^{b-1} dt;$$

the incomplete beta function defined by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt;$$

the incomplete beta function ratio defined by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt;$$

the exponential integral defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{\exp(t)}{t} dt;$$

the logarithmic integral defined by

$$\mathrm{Li}(x) = \int_0^x \frac{1}{\log(t)} dt;$$

the generalized exponential integral defined by

$$E_a(b) = \int_1^\infty t^{-a} \exp(-bt) dt;$$

the Whittaker function defined by

$$W_{a,b}(x) = \frac{\exp\left(-\frac{x}{2}\right) x^a}{\Gamma\left(\frac{1}{2} - a + b\right)} \int_0^\infty t^{b-a-\frac{1}{2}} \left(1 + \frac{t}{x}\right)^{a+b-\frac{1}{2}} \exp(-t) dt;$$

the Lerch function defined by

$$\Phi(a,b) = \sum_{i=0}^{\infty} \frac{x^i}{(i+b)^a};$$

the modified Bessel function of the second kind defined by

$$K_\nu(x) = \begin{cases} \frac{\pi \csc(\pi\nu)}{2} \left[I_{-\nu}(x) - I_\nu(x)\right], & \text{if } \nu \notin \mathbb{Z}, \\ \lim_{\mu \to \nu} K_\mu(x), & \text{if } \nu \in \mathbb{Z}; \end{cases}$$

the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)k!} \left(\frac{x}{2}\right)^{2k+\nu};$$

the confluent hypergeometric function defined by

$${}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial; the hypergeometric function defined by

$${}_1F_2(a;b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!};$$

the Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!};$$

the hypergeometric function defined by

$${}_2F_2(a,b;c,d;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{x^k}{k!};$$

the hypergeometric function defined by

$${}_3F_2(a,b,c;d,e;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{x^k}{k!};$$

the generalized hypergeometric function defined by

$$ {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{x^k}{k!}; $$

Humbert's confluent hypergeometric function of the first kind defined by

$$ \Psi_1\left(a,b,c;x,y\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{i+j}(b)_i}{(c)_{i+j}}\frac{x^i}{i!}\frac{y^j}{j!}; $$

Humbert's confluent hypergeometric function of the second kind defined by

$$ \Psi_2\left(a,b,c;x,y\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{i+j}}{(b)_i(c)_j}\frac{x^i}{i!}\frac{y^j}{j!}. $$

The properties of these functions can be found in Abramowitz and Stegun [8], Srivastava and Karlsson [170], Prudnikov et al. [147] and Gradshteyn and Ryzhik [70].