Mathematics

AIMS Mathematics, 10(11): 25907–25938.
DOI: 10.3934/math.20251145
Received: 16 August 2025
Revised: 10 October 2025
Accepted: 21 October 2025
Published: 11 November 2025

https://www.aimspress.com/journal/Math

Research article

Stochastic dynamics of solitary waves in the damped mKdV equation: analytical solutions and numerical simulations

Mohra Zayed[1], Hanadi M. AbdelSalam[2], Ibtisam Daqqa[2], S. Abdel-Khalek[3], E A-B Abdel-Salam[4] and Gamal M. Ismail[5,*]

[1] Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia
[2] College of Sciences and Human Studies, Prince Mohammad Bin Fahd University, Khobar 31952, Saudi Arabia
[3] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
[4] Department of Mathematics, Faculty of Science, New Valley University, El-Kharja 72511, Egypt
[5] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia

* **Correspondence:** Email: gismail@iu.edu.sa.

Abstract: In this study, we examined the impact of stochastic factors on the dynamics of nonlinear wave propagation by developing a damped modified Korteweg-de Vries (mKdV) equation incorporating multiplicative noise, mathematically characterized by a Wiener process. Standard deterministic models, although proficient in idealized contexts, frequently fail to accurately depict the complex behaviors elicited by the stochastic fluctuations intrinsic to physical systems. To mitigate this limitation, we utilized a hybrid analytical-numerical approach, employing the modified simple equation method to obtain a spectrum of precise soliton and solitary wave solutions. We ran complementary numerical simulations to find out how different levels of noise affect the time evolution and structural stability of these waveforms. The results showed that solitons keep their structure intact when there is not much noise. Nevertheless, as the noise level grows, the amplitude modulation and potential destabilization become more noticeable. Graphs like density maps and three-dimensional surface plots can be used to see how random changes make waves less predictable. These results demonstrated how crucial it is for nonlinear wave models to include random parts. Future research on complex noise profiles, multi-variable systems, and real-world validation approaches will benefit from this. A stochastic damped mKdV framework, which incorporates multiplicative noise with traditional deterministic or additive-noise analysis, is presented in this article. This paints a fuller picture of the way in which randomness evolves in response to actual system states.

Keywords: solitary solution; exact solution; stochastic differential equations; damped mKdV equation.
Mathematics Subject Classification: 34A34, 34C15, 34C25, 35L65, 37N30

1. Introduction

Numerous physical phenomena, particularly in the fields of fluid dynamics, plasma physics, and optical communications, have been greatly enriched by investigations of solitons and solitary waves in nonlinear systems. A strong theoretical groundwork for soliton theory has been laid by the substantial contributions of Ablowitz and Clarkson [1], Hasegawa and Kodama [2], Lakshmanan and Rajasekar [3], and Wazwaz and Hirota [4,5]. Moreover nonlinear evolution equations characterizing the behavior of solitary waves are shown to be integrable in this paper.

A substantial amount of interest has been shown in the production and propagation of solitons in plasma settings, particularly in dusty plasmas. The dynamics of these systems are determined by the complex interactions that occur between nonlinear, dispersive, and stochastic factors. Among the most significant elements that have an impact on the behavior of waves, ion distribution profiles, dust grain charge fluctuation, and relativistic corrections are among the most essential aspects [6–9]. However, deterministic frameworks usually fail to effectively capture the stochastic properties that are inherent to experimental and natural systems, despite the fact that they have been shown to be effective in idealized circumstances. There is a significant possibility that random fluctuations, which are typical in both laboratory and astrophysical plasmas, can have a significant impact on the development and stability of solitonic structures. This makes purely deterministic models less reliable. The mKdV equation is one of the most important models used in this context. It is a powerful way to describe how nonlinear waves move through dispersive media. In dusty plasmas, this method is commonly used to differentiate dust-acoustic waves from dust-ion-acoustic waves. Researchers such as Asgari et al. [6], Baluku and Helberg [7], and Kalita and Kalita [8] have studied how relativistic dynamics, dust grain charging mechanisms, non-thermal particle distributions, and other factors affect these waves' characteristics. Researchers built on these results by looking at how temperature changes and complex ion populations play a role [10–12].

Solitons are essential for the transport of energy and information in dispersive media; however, realistic environments introduce dissipation and stochastic fluctuations that significantly alter their dynamics. In the domains of optical fiber and plasma, multiplicative white noise, as represented by a Wiener process, affects amplitude, width, and coherence, potentially resulting in broadening, splitting, and degradation [13–20]. Analogous phenomena are evident in stochastic generalizations of integrable models, such as the Gerdjikov-Ivanov, perturbed Triki-Biswas, Schamel, and fractional Fokas–Lenells equations, wherein noise influences the existence, stability, and interaction properties of solitary waves [14,15,17,19]. Complementary analyses of noisy optical concatenation models delineate regimes of robust versus fragile propagation [13,16], while studies of KdV-type and fractional Fokas–Lenells formulations link multiplicative noise to waveform deformation and transitions to incoherent states [18,19]; at a broader dynamical-systems level, advances in symmetry-based reductions for transmission-line models and in attractor/basin geometry reveal how noise and dissipation reconfigure phase-space structure with implications for synchronization and control [20,21]. Inspired by this body of work, we develop and examine a stochastic damped mKdV framework with multiplicative forcing, elucidate its mechanistic differentiation from additive noise, and explore the interplay between randomness and damping in regulating soliton formation, stability, and persistence. By using both analytical methods (like perturbative/collective-coordinate reductions and conservation-law diagnostics adapted to stochastic settings) and high-fidelity numerical simulations, we measure changes in amplitude, width/shape, and coherence caused by noise. We also map parameter regimes that separate robust, metastable, and degraded behavior, and we check continuity with deterministic theory by recovering classical damped-mKdV dynamics in the zero-noise limit [17–19]. The framework improves the classical mKdV model by taking into account recent developments in Schamel, KdV, and fractional Fokas-Lenells models. It also makes optical communication and plasma wave applications more physically accurate. It also talks about predictive and uncertainty-aware strategies in nonlinear wave theory [16,20]. This novel development not only contributes to our comprehension of nonlinear dynamics, but it also paves the way for the investigation of related topics in areas that were previously unexplored. Increasing our capacity to forecast the behavior of a wide variety of physical systems under complex conditions can be accomplished by incorporating these insights into models that are already in existence. In this research, we examine the behavior of solitons in noisy environments by use of analytical and numerical methods. Our purpose of this investigation is to find out how the mKdV equation is impacted by multiplicative white noise. Our undertaking is based on this thorough research. Researching the effects of noise on wave characteristics (amplitude, frequency, and shape) as well as random perturbations on system dynamics and stability (e.g., soliton particles) is our primary objective. Numerical simulations and a modified simple equation technique are employed to comprehend the behavior of solitons in the presence of stochastic events. In light of these results, the theoretical models are better equipped to handle noise and other disruptions in the actual world. Stochastic models are necessary for studying the effects of random perturbations on wave dynamics. When the noise-free damped mKdV equation and other conventional deterministic models fail to provide a sufficient description, this becomes even more crucial. By introducing random variables, we can study many different things, such as how noise affects the stability of solitons and how to generate random solitary waves. The mKdV equation specifies solitons, and we study their behavior in relation to stochastic pressures. This lends credence to the previous discoveries. Coupling analytical methods with numerical experiments, particularly concentrating on the modified simple equation method, allows one to study the noise influence on the soliton system stability and properties. Our purpose of this assessment is to identify the impact of noise on these variables.

In real-world systems, where perfect noise-free conditions are rare, adding random factors to the damped mKdV equation makes it more useful by including the natural presence of noise and randomness. The Wiener process, commonly known as Brownian motion, is a basic stochastic model with continuous, random paths that start from the zero point and have increments that are independent and identically distributed. One of the major things that makes the Wiener process unique is that it is continuous, which means that there are no sudden jumps or breaks. However, its paths are very erratic, and there are no clear tangents at any given time (see Figure 1). The Wiener process is an important tool for modeling random events because it has important features like independent increments, continuous paths, and normally distributed variations. It is used in many fields, such as physics, engineering, and economics.

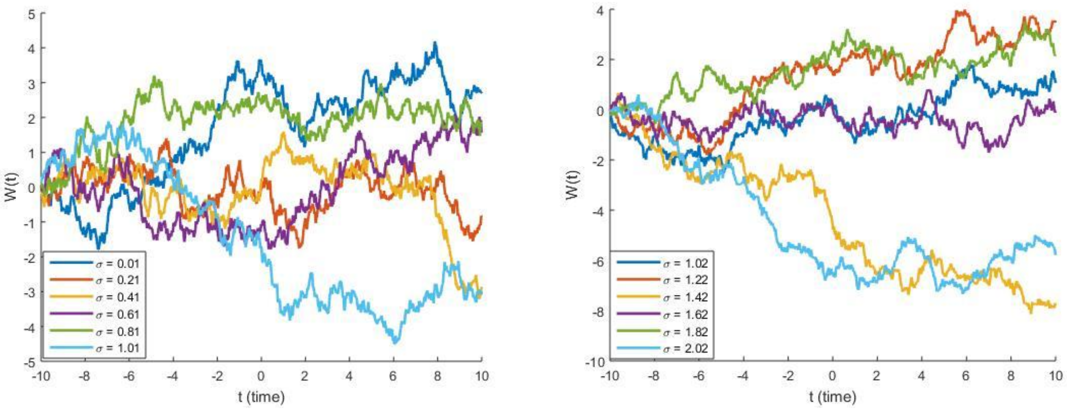


Figure 1. Illustration of the Wiener process trajectories for smooth real-valued random functions over the interval (-10, 10) with varying values of the parameter σ.

We employ a systematic workflow to examine the behavior of nonlinear wave solutions subjected to stochastic influences, with a particular emphasis on the stochastic damped mKdV equation. The study commences with the development of an innovative stochastic model that integrates nonlinear cubic terms, dispersion, damping, and multiplicative space-time white noise in the Itô framework. Then, a wave transformation is used, and the expectation is taken to turn the SPDE into a deterministic nonlinear ordinary differential equation. The modified simple equation method, which is used in a stochastic context, can be used to analytically solve this simplified equation. This gives general forms of soliton and solitary wave solutions. These solutions are divided into four groups based on their shapes: kink-shaped, trigonometric, exponential, and geometric. Each group shows a different behavior when there is no noise. To study how noise affects things, numerical simulations are done at different levels of noise, creating 3D plots, heat maps, and density plots that show how soliton structures change shape and modulate. The solutions are subsequently analyzed within the framework of physical systems, including plasma waves and optical fibers, emphasizing the distinctions between deterministic and stochastic regimes. All mathematical derivations undergo stringent verification, and simulation results are cross-validated for coherence. We end the study by listing the most important results and suggesting what should happen next.

1.1. Problem statement and relevance

The damped mKdV equation is very important for modeling nonlinear dispersive systems where energy loss is important, like plasmas, shallow water waves, and nonlinear optical media. Conventional analyses frequently emphasize deterministic, noise-free conditions, neglecting the inherent randomness introduced by environmental fluctuations, thermal effects, or external forces in actual physical systems. These random factors can significantly change how stable and long-lasting soliton solutions are, which makes purely deterministic models less reliable. To make more realistic descriptions of how solitary waves behave in real life, it is necessary to include both damping and stochastic effects.

1.2. Research objectives

We seek to enhance the damped mKdV framework by incorporating multiplicative stochastic terms to investigate the influence of noise on solitary and soliton dynamics. The goals are to: (i) create a model for the stochastic damped mKdV equation that balances nonlinearity, dispersion, dissipation, and randomness; (ii) use the modified simple equation method to find analytical soliton solutions; (iii) run numerical simulations to see how different noise levels affect wave amplitude, shape, and stability; and (iv) classify and explain solution types in the fields of plasma physics and nonlinear optics. In this way, we connect deterministic theory with real-world noisy environments, giving us information that is useful for communication systems, wave stability, and energy transport.

1.3. Advancement in the field, novelty, and research gap justification

In this study we enhance the examination of nonlinear wave dynamics by extending the damped mKdV equation to include stochastic effects, thereby offering a more accurate representation of physical systems affected by random fluctuations. The research underscores the influence of stochastic perturbations on soliton stability, amplitude, and structure by integrating damping, dispersion, nonlinearity, and multiplicative noise into a cohesive framework-phenomena frequently neglected in deterministic models.

Our innovation consists in applying the modified simple equation method to obtain analytical solutions for the stochastic damped mKdV equation, a methodological approach seldom explored in current literature. Moreover, the integration of analytical derivations and numerical simulations offers a dual perspective, augmenting both theoretical understanding and computational verification. We explicitly address multiplicative stochastic perturbations, unlike conventional approaches that treat solitons under noise-free or purely additive-noise conditions. This better captures state-dependent randomness in real-world environments.

Many researches have looked at deterministic mKdV and damped mKdV equations, but not many have looked at how damping and randomness work together to affect soliton solutions. Current stochastic models predominantly emphasize additive noise or entirely disregard dissipation, resulting in a deficiency in comprehending the interaction between dissipative forces and multiplicative noise in influencing soliton dynamics. This research fills that gap by creating a stochastic damped mKdV framework, categorizing solution types, and testing robustness under noise, which results in a model that is more complete and relevant to physics.

We create a stochastic damped mKdV framework with multiplicative noise that captures state-dependent randomness in real-world environments, in contrast to traditional analyses that focus on deterministic or additive-noise systems. We further adapt the modified simple equation method to the stochastic setting (a methodological step rarely attempted before), enabling us to derive analytical solutions that are then verified by direct numerical simulations. This dual analytical-numerical approach illustrates the impact of stochastic perturbations on soliton stability, amplitude, and structure, effectively connecting deterministic theory with actual noisy physical systems. This study enhances our understanding of soliton dynamics under uncertain conditions and provides innovative insights for practical applications in fields such as fluid dynamics and nonlinear optics. In the future, researchers may concentrate on expanding this framework to encompass more intricate systems, potentially yielding innovative insights into the behavior of solitons within chaotic environments. Researchers may discover patterns and behaviors that contradict current theories by investigating these intricate interactions, facilitating the development of groundbreaking technologies that utilize soliton characteristics. Furthermore, figuring out how these changes affect soliton interactions could help make communication systems and energy transfer methods better.

1.4. Workflow visualization

The workflow of this study commences with the formulation of the damped mKdV equation, integrating multiplicative noise and establishing the requisite parameters, domain, and boundary conditions. We use a wave transformation and an expectation operator to turn the stochastic PDE into a nonlinear ODE. Then, we use the modified simple equation method to find explicit soliton forms. The solutions that come out are grouped into four families: Trigonometric, kink, exponential, and geometric. In addition to the analytical results, numerical simulations are done using appropriate discretization schemes. These simulations show how different noise levels and parameters affect the amplitude, shape, and stability of solitons. The results are shown in 3D plots and density maps, checked against the deterministic limit, and tested for sensitivity. Finally, the results are interpreted in light of their physical ramifications, offering both theoretical understanding and practical significance for nonlinear wave systems subjected to damping and stochastic effects.

This paper is structured as follows. In Section 2, we derive and theorize the stochastic damped mKdV equation, including a thorough mathematical analysis of its most important parts. Section 3 centers on the formulation of soliton and solitary wave solutions, utilizing the modified simple equation method as the principal analytical instrument. In Section 4, numerical simulations are conducted to demonstrate the dynamic behavior of wave structures affected by stochastic perturbations. Last in, Section 5, we give a summary of the major results and talk about possible paths for future research.

2. Mathematical analysis

In this section, we present the governing stochastic damped mKdV equation into explanation as follows:

$$dQ = -\left(AQ^2 Q_x + BQ_{xxx} + CQ\right)dt + \sigma Q dW, \tag{1}$$

here, $Q(x,t)$ is a real-valued function of *x* and *t*, the coefficients A, B, C denote the dispersive, dispersion, and damped coefficients, respectively, σ represents the noise intensity, $W(t)$ denotes the white noise, and $\sigma Q dW$ indicates multiplicative noise in the Itô sense.

This equation captures the core interactions between nonlinearity, dispersion, damping, and randomness. Such interactions are vital in realistic modeling, especially in plasma physics, nonlinear optics, and hydrodynamic systems, where random environmental fluctuations can significantly influence wave dynamics. The damping term simulates the loss of energy, like resistivity in plasma or viscosity in fluids. The stochastic term models random disturbances that can come from a number of things, such as changes in temperature, outside forces, or errors in measurements. This combination causes waves to behave in interesting and complicated ways, especially when looking at solitons and solitary waves, which are important for many nonlinear wave applications. We examine the evolution of solitary wave structures under the influence of deterministic dynamics and stochastic noise through analytical transformations and numerical simulations. These insights are especially important for designing stable communication channels, figuring out how energy moves through plasmas, and predicting how waves will act in noisy places. In the following subsection, we analyze the governing equation using the wave transformation technique.

2.1. Wave transformation of the stochastic damped mKdV equation

The wave equation for stochastic damped mKdV equation (1) is given by considering the following wave transformation:

$$Q(x,t) = q(x,t)e^{[\sigma \mathrm{W(t)} - \sigma^2 t/2]}, \tag{2}$$

where the functions $q(x,t)$ are deterministic. We have, spatial derivatives (the exponential depends on t only):

$$Q_x = q_x e^{\theta},\; Q_{xx} = q_{xx} e^{\theta},\;\; Q_{xxx} = q_{xxx} e^{\theta}. \quad \theta = \sigma W(t) - \sigma^2 t/2. \tag{3a}$$

Itô time increment:

$$Q_t = e^{\theta} q_t + Q\left(\sigma dW + \frac{\sigma^2}{2} - \frac{\sigma^2}{2}\right). \tag{3b}$$

The second term arises from formally differentiating the exponential factor: The contribution $\sigma\, dW$ comes from the stochastic increment of the Wiener process, while the deterministic drift contains two opposing parts, namely $\sigma^2/2$ from the Itô correction and $-\sigma^2/2$ from the explicit $-\sigma^2 t/2$ in the exponent; these two drift terms cancel exactly, so that only the stochastic contribution $\sigma\, dW$ remains. By inserting Eqs (2) and (3) into Eq (1), we have

$$q_t + Aq^2 q_x e^{2[\sigma \mathrm{W(t)} - \sigma^2 t/2]} + B\, q_{xxx} + C\, q = 0. \tag{4}$$

Taking the expectation to both sides on Eq (4) we have:

$$q_t + Aq^2 q_x e^{-\sigma^2 t} E(e^{2\sigma \mathrm{W(t)}}) + B\, q_{xxx} + C\, q = 0. \tag{5}$$

Since W (t) is a Gaussian process, $E(e^{\sigma \mathrm{W(t)}}) = e^{\sigma^2 t/2}$. So, Eq (5) turns into

$$q_t + Aq^2 q_x + B\, q_{xxx} + C\, q = 0. \tag{6}$$

This is used to clearly show the major features of solitary wave structures, which makes it easier to find different solitons that are important to plasma physics. These waveforms are governed by partial differential equations with spatial and temporal variables denoted by x and t, respectively. To convert Eq (6) into a nonlinear ordinary differential equation (NLODE), the following wave transformation is introduced:

$$q(x,t)=q(\xi), \qquad \xi = k\,x+\omega t, \tag{7}$$

here ω represented the wave velocity and k denote the wave number, putting Eq (7) into Eq (6) gives us the following NLODE:

$$\omega q' + A\,k\,q^2 q' + B\,k^3 q''' + C\,q = 0, \qquad ' = \frac{d}{d\xi}, \tag{8}$$

This study presents several novel contributions to the analysis of nonlinear wave equations influenced by stochastic effects, with a particular emphasis on the stochastic damped mKdV equation. We create a new SPDE model that includes nonlinear cubic terms, dispersive and damping effects, and multiplicative space-time white noise in the Itô sense. This makes the model better at representing real-world physical systems like plasmas and optical media that are affected by changes in the environment. A unique methodology is employed that integrates wave transformation with expectation analysis, simplifying the SPDE to a nonlinear ODE, which facilitates the extraction of physically significant wave structures. We also adapt the modified simple equation method, which is usually used for deterministic systems, to the stochastic framework. This enables the researchers to find explicit soliton and solitary wave solutions of the reduced equation, which is a significant methodological gap in the current literature. We use 3D plots, heat maps, and density visualizations to look at how changing the noise intensity affects the amplitude, shape, and stability of soliton structures in great detail through parametric analysis and numerical simulations. The outcomes indicate the existence of specific soliton categories, encompassing kink-type, trigonometric, geometric, and exponential forms, each exhibiting distinct sensitivities to stochastic modulation. Finally, the research connects theoretical advancements and practical implementation by providing insights pertinent to domains such as plasma physics, nonlinear optics, and communication systems, where the interaction among nonlinearity, dispersion, damping, and stochasticity is essential.

3. Soliton and solitary solutions of the deterministic damped mKdV equation

To derive solitary solutions, we employ the modified simple equation technique. By equating orders of q''' with $q^2 q'$, we have $N+3=2N+1$ so we have $N=1$, and we apply the modified simple equation technique. The solution is then expressed in a series form so Eq (8) have the general solution in the formula

$$q(\xi)=\sum_{s=0}^{N} a_s R^s(\xi)+\sum_{s=-1}^{-N} b_{-s} R^s(\xi), \tag{9}$$

where a_s, b_s are later determined constants, and $R(\xi)$ satisfies the ODE

$$R'(\xi)=\rho_0+\rho_1 R(\xi)+\rho_2 R^2(\xi)+\rho_3 R^3(\xi). \tag{10}$$

For N=1, Eq (8) has the following solution form:

$$q(\xi) = a_0 + a_1 R(\xi) + \frac{b_1}{R(\xi)}. \tag{11}$$

Replacing Eq (11) with the assistance of auxiliary ODE (10) into (8), gathering all coefficients of $R^k(\xi)R'^j(\xi)$ (k=0,1,j=0,1,2,3,...,n) and equating by zero, we have an algebraic system. Solving this system with the assistance of software Maple or Mathematica, distinct classes of a_s, b_s, k, and ω are computed depending on ρ's, and by replacing this into Eq (9) various classes of explicit solutions to Eq (1) are derived, as outlined below.

Class 1: When $\rho_3 = 0$, we obtain

$$\text{I}: \quad a_0 = -\rho_1 k\sqrt{\frac{3B}{2A}}, \quad a_1 = 0, \quad b_1 = \rho_0 k\sqrt{\frac{6B}{A}}, \quad \omega = \frac{1}{2}(\rho_0^2 - 4\rho_1\rho_2)Bk^3. \tag{12}$$

Since $A < 0$, putting the values of Eq (12) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_1(x,t) = -\rho_1 k\sqrt{\frac{3B}{2A}} - \rho_0 k\sqrt{\frac{6B}{A}}\left(\frac{2\rho_2}{\rho_1 - \sqrt{4\rho_0\rho_2 - \rho_1^2}\tan\left(\sqrt{4\rho_0\rho_2 - \rho_1^2}\,(\xi + \xi_0)/2\right)}\right), \tag{13}$$
$$\xi = k\,x + (\rho_0^2 - 4\rho_1\rho_2)Bk^3 t/2, \qquad 4\rho_0\rho_2 > \rho_1^2.$$

$$\begin{aligned} \text{II}: \quad & a_0 = \sqrt{\frac{3(2\rho_0\rho_2 Bk^3 + \omega)}{A\,k}}, \; b_1 = 0, \quad a_1 = -\rho_2 k\sqrt{\frac{6B}{A}}, \quad \rho_1 = -\frac{1}{k}\sqrt{\frac{2(2\rho_0\rho_2 Bk^3 + \omega)}{B\,k}}, \\ & D = \frac{1}{k}\sqrt{\frac{2\rho_0\rho_2 Bk^3 + \omega}{B\,k}}\left(\sqrt{2\omega} + 2\rho_0\rho_2 Bk^3\sqrt{2} - (2\rho_0\rho_2 Bk^3 + \omega)\right). \end{aligned} \tag{14}$$

Since $A < 0$, putting the values of Eq (14) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_2(x,t) = \sqrt{\frac{3(2\rho_0\rho_2 Bk^3 + \omega)}{A\,k}} - k\sqrt{\frac{6B}{A}}\left(\rho_1 - \sqrt{4\rho_0\rho_2 - \rho_1^2}\tan\left(\sqrt{4\rho_0\rho_2 - \rho_1^2}\,(\xi + \xi_0)/2\right)\right), \tag{15}$$
$$\xi = k\,x + \omega t, \qquad 4\rho_0\rho_2 > \rho_1^2.$$

$$\begin{aligned} \text{III}: \quad & a_0 = -\sqrt{\frac{3(2\rho_0\rho_2 Bk^3 + \omega)}{A\,k}}, \; b_1 = 0, \quad a_1 = -\rho_2 k\sqrt{\frac{6B}{A}}, \quad \rho_1 = \frac{1}{k}\sqrt{\frac{2(2\rho_0\rho_2 Bk^3 + \omega)}{B\,k}}, \\ & D = -\frac{1}{k}\sqrt{\frac{2\rho_0\rho_2 Bk^3 + \omega}{B\,k}}\left(\sqrt{2\omega} + 2\rho_0\rho_2 Bk^3\sqrt{2} - (2\rho_0\rho_2 Bk^3 + \omega)\right). \end{aligned} \tag{16}$$

Since $A < 0$, putting the values of Eq (16) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_3(x,t)=\sqrt{\frac{3(2\rho_0\rho_2Bk^3+\omega)}{A\,k}}-k\sqrt{\frac{6B}{A}}\left(\rho_1-\sqrt{4\rho_0\rho_2-\rho_1^2}\tan\left(\sqrt{4\rho_0\rho_2-\rho_1^2}\,(\xi+\xi_0)/2\right)\right), \quad \xi=k\,x+\omega t, \qquad 4\rho_0\rho_2>\rho_1^2. \tag{17}$$

$$\text{IV}: \quad a_0=-\rho_1k\sqrt{\frac{3B}{2A}},\ b_1=0,\quad a_1=-\rho_2k\sqrt{\frac{6B}{A}},\ \ \omega=\frac{1}{2}(\rho_1^2-4\rho_0\rho_2)Bk^3. \tag{18}$$

Since $A<0$, putting the values of Eq (18) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_4(x,t)=-\rho_1k\sqrt{\frac{3B}{2A}}-k\sqrt{\frac{6B}{A}}\left(\rho_1-\sqrt{4\rho_0\rho_2-\rho_1^2}\tan\left(\sqrt{4\rho_0\rho_2-\rho_1^2}\,(\xi+\xi_0)/2\right)\right), \quad \xi=k\,x+(\rho_1^2-4\rho_0\rho_2)Bk^3t/2, \qquad 4\rho_0\rho_2>\rho_1^2. \tag{19}$$

$$\text{V}: \quad a_0=-\rho_1k\sqrt{\frac{3B}{2A}},\ a_1=-\rho_2k\sqrt{\frac{6B}{A}},\ b_1=-\rho_0k\sqrt{\frac{6B}{A}},\ \ \omega=\frac{1}{2}(\rho_1^2+8\rho_0\rho_2)Bk^3. \tag{20}$$

Since $A<0$, putting the values of Eq (20) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_5(x,t)=-\rho_1k\sqrt{\frac{3B}{2A}}-k\sqrt{\frac{6B}{A}}\left(\rho_1-\sqrt{4\rho_0\rho_2-\rho_1^2}\tan\left(\sqrt{4\rho_0\rho_2-\rho_1^2}\,(\xi+\xi_0)/2\right)\right) -\rho_0k\sqrt{\frac{6B}{A}}\left(\frac{\rho_2}{\rho_1-\sqrt{4\rho_0\rho_2-\rho_1^2}\tan\left(\sqrt{4\rho_0\rho_2-\rho_1^2}\,(\xi+\xi_0)/2\right)}\right), \quad \xi=k\,x+(\rho_1^2+8\rho_0\rho_2)Bk^3t/2, \qquad 4\rho_0\rho_2>\rho_1^2. \tag{21}$$

Class 2: When $\rho_0=\rho_3=0,$ we obtain

$$\text{I}: \quad a_0=-\rho_1k\sqrt{\frac{3B}{2A}},\qquad b_1=0,\quad a_1=-\rho_2k\sqrt{\frac{6B}{A}},\quad \omega=\frac{1}{2}\rho_1^2Bk^3. \tag{22}$$

Since $A<0$, putting the values of Eq (22) in Eq (11), then we get the following soliton solution for Eq (6) in exponential form as:

$$q_6(x,t)=\rho_1k\sqrt{\frac{3B}{2A}}\left(\frac{\rho_2e^{\rho_1(\xi+\xi_0)}+1}{\rho_2e^{\rho_1(\xi+\xi_0)}-1}\right),\qquad \xi=k\,x+\frac{1}{2}\rho_1^2Bk^3t.\quad \rho_1>0. \tag{23}$$

$$q_7(x,t)=\rho_1 k\sqrt{\frac{3B}{2A}}\left(\frac{\rho_2 e^{\rho_1(\xi+\xi_0)}-1}{\rho_2 e^{\rho_1(\xi+\xi_0)}+1}\right),\qquad \xi=k\,x+\frac{1}{2}\rho_1^2 Bk^3 t.\quad \rho_1<0. \tag{24}$$

$$\text{II}:\qquad a_0=\rho_1 k\sqrt{\frac{3B}{2A}},\qquad b_1=0,\quad a_1=\rho_2 k\sqrt{\frac{6B}{A}},\quad \omega=\frac{1}{2}\rho_1^2 Bk^3. \tag{25}$$

Since $A<0$, putting the values of Eq (25) in Eq (11), then we get the following soliton solution for Eq (6) in exponential form as:

$$q_8(x,t)=-\rho_1 k\sqrt{\frac{3B}{2A}}\left(\frac{\rho_2 e^{\rho_1(\xi+\xi_0)}+1}{\rho_2 e^{\rho_1(\xi+\xi_0)}-1}\right),\qquad \xi=k\,x+\frac{1}{2}\rho_1^2 Bk^3 t.\quad \rho_1>0. \tag{26}$$

$$q_9(x,t)=-\rho_1 k\sqrt{\frac{3B}{2A}}\left(\frac{\rho_2 e^{\rho_1(\xi+\xi_0)}-1}{\rho_2 e^{\rho_1(\xi+\xi_0)}+1}\right),\qquad \xi=k\,x+\frac{1}{2}\rho_1^2 Bk^3 t.\quad \rho_1<0. \tag{27}$$

Class 3: When $\rho_1=\rho_3=0,$ we obtain

$$\text{I}:\qquad a_0=0,\qquad a_1=0,\quad b_1=\rho_0 k\sqrt{\frac{6B}{A}},\quad \omega=-2\rho_0\rho_2 Bk^3. \tag{28}$$

Since $A<0$, putting the values of Eq (28) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_8(x,t)=k\sqrt{\frac{6B}{A}}\left(\sqrt{\rho_0\rho_2}\cot\left(\sqrt{\rho_0\rho_2}(\xi+\xi_0)\right)\right),\quad \xi=k\,x-2\rho_0\rho_2 Bk^3 t,\quad \rho_0\rho_2>0. \tag{29}$$

$$q_9(x,t)=-k\sqrt{\frac{6B}{A}}\left(\sqrt{-\rho_0\rho_2}\coth\left(\sqrt{-\rho_0\rho_2}(\xi+\xi_0)\right)\right),\quad \xi=k\,x-2\rho_0\rho_2 Bk^3 t,\quad \rho_0\rho_2<0. \tag{30}$$

$$\text{II}:\qquad a_0=0,\qquad b_1=0,\quad a_1=-\rho_0 k\sqrt{\frac{6B}{A}},\quad \omega=-2\rho_0\rho_2 Bk^3. \tag{31}$$

Since $A<0$, putting the values of Eq (31) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_{10}(x,t)=-k\sqrt{\frac{6B}{A}}\left(\sqrt{\rho_0\rho_2}\tan\left(\sqrt{\rho_0\rho_2}(\xi+\xi_0)\right)\right),\quad \xi=k\,x-2\rho_0\rho_2 Bk^3 t,\quad \rho_0\rho_2>0. \tag{32}$$

$$q_{11}(x,t)=-k\sqrt{\frac{6B}{A}}\left(\sqrt{-\rho_0\rho_2}\tanh\left(\sqrt{-\rho_0\rho_2}(\xi+\xi_0)\right)\right),\quad \xi=k\,x-2\rho_0\rho_2 Bk^3 t,\quad \rho_0\rho_2<0. \tag{33}$$

$$\text{III}: \quad a_0 = 0, \quad b_1 = -\rho_0 k \sqrt{\frac{6B}{A}}, \quad a_1 = -\rho_2 k \sqrt{\frac{6B}{A}}, \quad \omega = 4\rho_0\rho_2 B k^3. \tag{34}$$

Since $A < 0$, putting the values of Eq (34) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_{12}(x,t) = -2k\sqrt{\frac{6B}{A}}\left(\sqrt{\rho_0\rho_2}\csc\left(2\sqrt{\rho_0\rho_2}(\xi+\xi_0)\right)\right), \quad \xi = kx + 4\rho_0\rho_2 Bk^3 t, \quad \rho_0\rho_2 > 0. \tag{35}$$

$$q_{13}(x,t) = 2k\sqrt{\frac{6B}{A}}\left(\sqrt{-\rho_0\rho_2}\operatorname{csch}\left(2\sqrt{-\rho_0\rho_2}(\xi+\xi_0)\right)\right), \quad \xi = kx + 4\rho_0\rho_2 Bk^3 t, \quad \rho_0\rho_2 < 0. \tag{36}$$

$$\text{IV}: \quad a_0 = 0, \quad b_1 = \rho_0 k \sqrt{\frac{6B}{A}}, \quad a_1 = \rho_2 k \sqrt{\frac{6B}{A}}, \quad \omega = 4\rho_0\rho_2 B k^3. \tag{37}$$

Since $A < 0$, putting the values of Eq (34) in Eq (11), then we get the following soliton solution for Eq (6) in trigonometric form as:

$$q_{14}(x,t) = 2k\sqrt{\frac{6B}{A}}\left(\sqrt{\rho_0\rho_2}\csc\left(2\sqrt{\rho_0\rho_2}(\xi+\xi_0)\right)\right), \quad \xi = kx + 4\rho_0\rho_2 Bk^3 t, \quad \rho_0\rho_2 > 0. \tag{38}$$

$$q_{15}(x,t) = -2k\sqrt{\frac{6B}{A}}\left(\sqrt{-\rho_0\rho_2}\operatorname{csch}\left(2\sqrt{-\rho_0\rho_2}(\xi+\xi_0)\right)\right), \quad \xi = kx + 4\rho_0\rho_2 Bk^3 t, \quad \rho_0\rho_2 < 0. \tag{39}$$

We have checked all of the equations in this study for mathematical correctness and internal consistency. At every step, the derivations (from the original stochastic damped mKdV equation to the wave transformation, expectation operations, and reduction to nonlinear ordinary differential equations) have been carefully checked. The use of the modified simple equation method and the explicit solution forms that come from it have been checked using symbolic computation and algebraic simplification. Any differences found during the derivation process were fixed to ensure that the analytical results fit with the stochastic framework that governs them. This thorough verification backs the accuracy of the analytical and numerical results shown in this work. Our approach offers a more thorough and physically plausible framework than current methods in the literature that address deterministic or simplified stochastic wave equations.

Conventional analyses of the mKdV equation frequently overlook damping and stochastic effects, concentrating instead on integrable systems that can be resolved using methods like the inverse scattering transform or Hirota's technique. These methods are very useful, but they work only in perfect, noise-free situations. Some recent stochastic models have included additive noise, but they do not always show the amplitude-dependent changes that happen in real systems. We, on the other hand, use multiplicative white noise in the Itô sense, which is a better way to show how environmental randomness affects wave amplitude. We want to give a more accurate picture of how waves behave in uncertain conditions by using this method. This study may yield novel insights into the propagation and interaction of waves across diverse media, thereby advancing our comprehension of intricate dynamical systems.

This study also presents a novel deterministic reduction method to the stochastic PDE literature: Wave transformation with expectation analysis. It lets you look at how nonlinear waves act in a way that is easy to understand. The most important new idea is that the modified simple equation method can now be used in a stochastic context. This method has only been used with deterministic equations in the past. The classification of soliton and solitary wave solutions subjected to stochastic influence (bolstered by a comprehensive parametric analysis and numerical simulations) enhances the comprehension of wave stability and dynamics in the presence of noise. As a result, we not only improve the theoretical framework for stochastic nonlinear equations, but it also fill a methodological gap by using deterministic solution methods in real-world stochastic situations. This new way of looking at things gives us new ideas about how complex systems behave when they are affected by random changes. Researchers can make more reliable predictive tools that take into account uncertainty in real-world situations by bridging the gap between deterministic and stochastic models. These advancements could result in substantial enhancements across multiple disciplines, including finance, engineering, and environmental science, where comprehending the interaction between deterministic and stochastic components is essential. Consequently, the outcomes of this research may enable practitioners to make more informed decisions grounded in enhanced models that more accurately represent the intricacies of their specific fields.

3.1. Verification in comparison to current techniques

The dependability of the suggested analytical and numerical framework for the stochastic damped mKdV equation can be validated through comparison with established deterministic and stochastic methodologies in the literature. As the noise intensity approaches zero, the obtained solutions converge to those of the classical damped mKdV equation, which are well-established. This reduction is an initial consistency verification. Furthermore, the analytical soliton profiles obtained from the modified simple equation method can be juxtaposed with solutions generated by alternative techniques, such as the inverse scattering transform or the tanh-coth expansion method, when suitable, to validate their structural integrity and coherence. These comparisons not only make the analytical solutions more believable, but they also give us hints about how strong different mathematical methods are when studying nonlinear wave equations. This method could also lead to new discoveries about how solitons behave in different situations, which would add to our overall understanding of the dynamics at play.

In terms of numbers, simulations can be compared to current deterministic soliton propagation schemes to ensure that the current method reproduces known waveforms and dynamics when there are no random perturbations. When using stochastic forcing, it is possible to check for stability and convergence by comparing the results with those of well-known stochastic numerical integrators, like the Euler-Maruyama or Milstein methods. The proposed framework gains credibility by being in line with classical deterministic outcomes and established stochastic simulation methods. This shows how innovative it is in capturing the combined effects of damping and multiplicative noise on soliton dynamics. This alignment confirms the framework's strength and makes it possible to study more complex systems where noise is important. Researchers are still working on these models, and they could be useful in many areas, such as fluid dynamics and financial modeling. They could help us understand how stability and randomness work together.

3.2. Limitations of the methods

Our integrated analytical–numerical framework provides a thorough comprehension of stochastic soliton dynamics; however, it is essential to recognize certain methodological constraints. The revised simple equation method provides an effective analytical instrument for obtaining closed-form soliton and solitary wave solutions. This method, on the other hand, depends on deterministic reductions of the stochastic damped mKdV equation through wave transformations and expectation operations. Consequently, it reflects the system's average behavior instead of the complete range of stochastic variations. As a result, some temporary or localized random effects, especially those caused by high-intensity or non-Gaussian noise, may not be fully captured in the analytical solutions. This limitation indicates the necessity for supplementary numerical methods capable of encompassing the entire dynamics of the system. Researchers can gain a deeper understanding of the underlying phenomena by combining analytical and numerical methods. This will lead to better predictions and a better understanding of how complex wave systems behave.

To address this limitation, numerical simulations are utilized to directly model the stochastic partial differential equation through stochastic integration methods, including the Euler–Maruyama scheme. These simulations confirm the analytical findings and provide insights into the temporal evolution, amplitude modulation, and possible destabilization of soliton structures under different noise levels. The dual-method approach guarantees mathematical feasibility and physical authenticity: The analytical solutions provide accurate soliton configurations, while the numerical experiments validate their resilience and stability in stochastic contexts. This all-encompassing framework improves our understanding of how solitons behave and opens new avenues for research on complex systems that are affected by random changes. We can better predict how solitons will act in the real world, like in optical fibers and fluid dynamics, by combining theoretical and computational points of view.

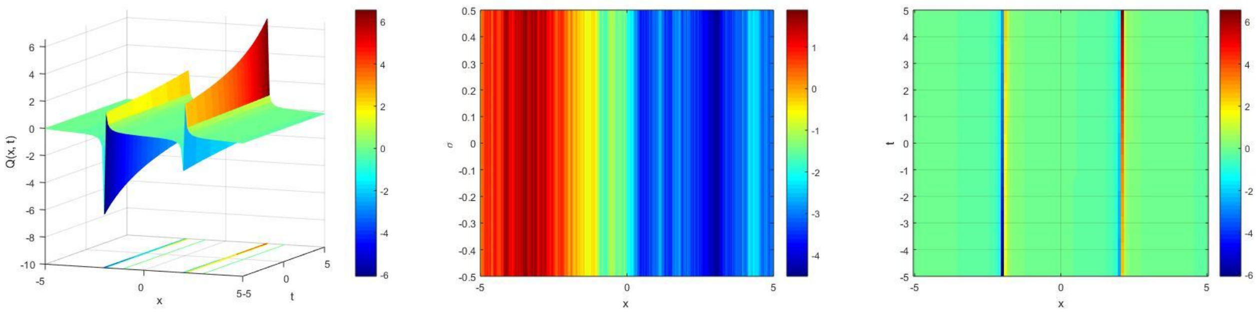
Although these two things work well together, there are still some issues that need to be addressed. This analysis is limited to one-dimensional formulations defined by multiplicative white noise. Conversely, higher-dimensional extensions may exhibit fundamentally different behaviors, such as noise-induced pattern formation or wave collapse, requiring more advanced numerical and analytical techniques. These complexities may yield novel insights into the comprehension of dynamical systems and their reactions to stochastic perturbations. Subsequent investigations ought to concentrate on examining these higher-dimensional contexts to facilitate a more profound understanding of their fundamental mechanisms. This study could show how noise affects different dimensions, which could lead to new uses in, for example, physics, biology, and engineering. Researchers can better predict and control how these complex systems behave by combining advanced computational methods and theoretical frameworks. Empirical investigations in plasma systems, optical fibers, or fluid channels would be instrumental in validating the theoretical predictions delineated herein. Researchers ought to examine alternative stochastic processes, such as colored, Lévy, or non-Markovian noise, to assess the universality of the proposed methodology. This investigation may provide significant insights into the resilience of the methodology under diverse conditions and applications. Additionally, looking at these processes side by side could help us find any rules that govern how they act in complicated systems. These studies may reveal novel insights into the dynamics of intricate systems. Researchers will gain a more profound comprehension of the fundamental mechanisms driving these phenomena by broadening the spectrum of analyzed stochastic processes. To sum up, the combination of analytical and numerical methods makes it easier to study stochastic nonlinear wave systems in a balanced and reliable way. However, more work needs to be done to make it more useful, test its predictions in the lab, and improve its ability to capture the full complexity of real-world stochastic phenomena. This endeavor will augment our comprehension of these systems and facilitate novel applications across diverse domains, including physics and engineering. By accepting a wider variety of stochastic processes, researchers can find new dynamics and interactions that are not fully understood yet.

4. Simulation results of the behavior of wave patterns

In this section, we examine how random noise (specifically white noise) affects the solitary and soliton solutions of the stochastic damped mKdV Eq (1). The main focus of the analysis is on how random changes affect the system's dynamic behavior, especially when compared to its deterministic counterpart. This comparison is crucial for figuring out how the model changes over time and stays stable over the long term. Different types of graphs illustrate the system's behavior under varying levels of noise (σ) to highlight these effects. The results of the simulation present us with useful information about how stochastic modulation affects the behavior of waves. As the noise level rises, the solitary and soliton structures start to show significant changes in amplitude, which makes them look different from their deterministic profiles. These solutions keep their general shape at lower noise levels, but as σ grows, their structural modulation becomes more pronounced. Heat maps that show the link between noise level and the Wiener process make this trend obvious. Higher noise levels cause more noticeable changes. Furthermore, density plots of how amplitude changes over time and space show that higher noise levels cause waves to behave in strange ways, which shows how stochasticity can make things less stable. These results show how important it is to include stochastic effects in nonlinear wave models if you want to accurately predict how a system will behave in real-world, noisy situations. Moreover, the results suggest that the soliton and solitary stay fairly stable when there is not much noise, keeping their shape and other features. However, as the noise level rises, the soliton's modulation becomes stronger, causing it to deviate significantly from the deterministic solution. We expect this kind of change from nonlinear systems, where even small changes can have a big effect on how the system works over time. The Wiener process records these random changes, and the soliton's time-dependent modulation shows how the noise affects it. These changes become more obvious as the noise level (σ) goes up, which shows that the soliton's stability has changed. The time evolution of the soliton shows that when there is more noise, the soliton's amplitude changes more, which could cause it to break down. The choice of parameters, like those related to the system's characteristics, is also vital for determining the solution and its sensitivity to noise. Changing these parameters can make the soliton and solitary more stable when there is noise, which shows how complicated the relationship is between solitons (solitary) and noise.

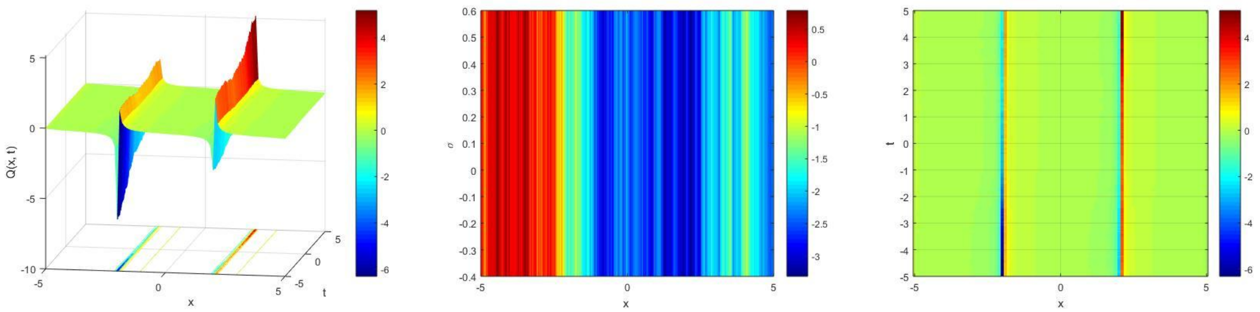
The findings underscore the essential influence of noise intensity on the behavior and stability of solitary and soliton wave structures. This dynamic interaction highlights the imperative for ongoing research into the long-term stability of these waveforms amidst diverse stochastic conditions and the examination of alternative stochastic processes. The graphs in this study provide more information about how levels of noise affect the model's dynamic evolution. Furthermore, comprehending these interactions may facilitate the development of sophisticated applications in domains such as telecommunications and materials science, where meticulous regulation of wave behavior is crucial. Future studies may concentrate on creating predictive models that integrate these stochastic factors to improve the dependability of soliton-based technologies.

The system behaves in a purely deterministic way when the noise intensity is set to $\sigma=0$. As seen in Figure 2, the corresponding solitary solution is a flat, periodic trigonometric structure that is described by Eq (13). Figure 3 shows a non-deterministic, non-flat solitary solution when the noise level rises to $\sigma=0.1$. This shows that stochastic modulation is starting to happen. Figure 4 shows that raising σ to 0.5 makes the non-flat solitary solution even more pronounced. The parameters are $\rho_0=1.5;$ $\rho_1=1.5;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.01;$ $A=1;$ and $\xi_0=2$. This change shows how sensitive the system is to noise, which means that even small changes can cause big changes in behavior. As the parameters continue to change, the solitary solution's complexity might uncover new dynamics that necessitate a more thorough examination of the underlying mechanisms involved.



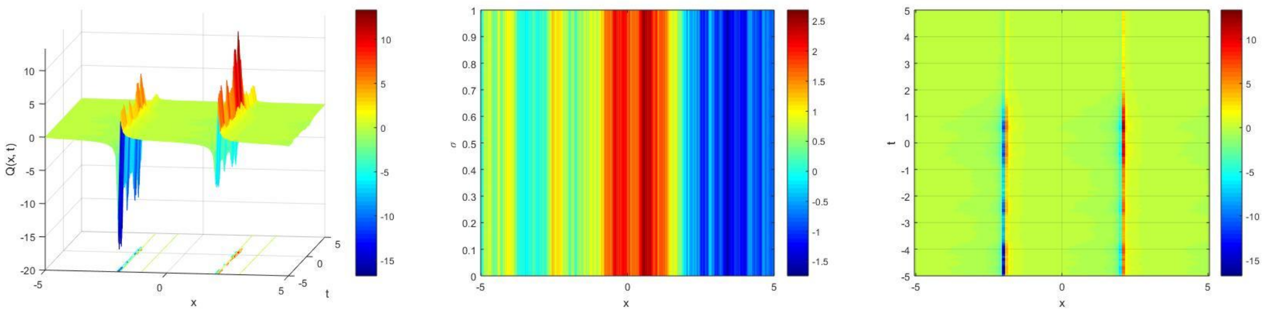
(a) (b) (c)

Figure 2. Periodic trigonometric behavior given by Eq (13), (2a) is the 3D plotting, (2b) is the heat map, (2c) is the density plot with $\sigma=0$; $\rho_0=1.5;$ $\rho_1=1.5;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.01;$ $A=1;$ and $\xi_0=2$.



(a) (b) (c)

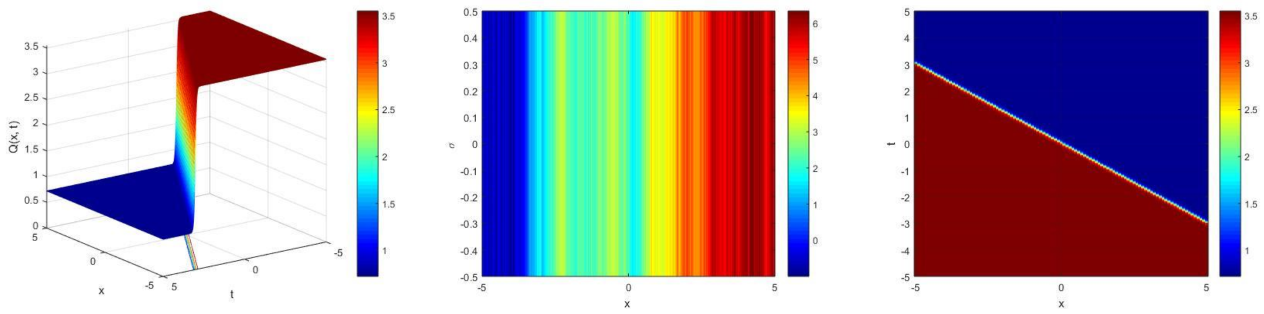
Figure 3. Stochastic periodic trigonometric behavior, (3a) is the 3D plotting, (3b) is the heat map, (3c) is the density plot with $\sigma=0.1$; $\rho_0=1.5;$ $\rho_1=1.5;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.01;$ $A=1;$ and $\xi_0=2$.



(a) (b) (c)

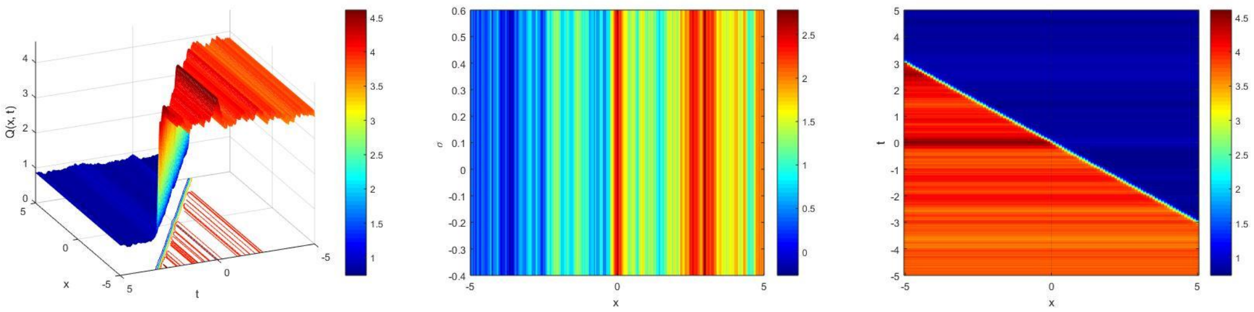
Figure 4. Stochastic periodic trigonometric behavior, (4a) is the 3D plotting, (4b) is the heat map, (4c) is the density plot with σ=0.5; $\rho_0=1.5$; $\rho_1=1.5$; $\rho_2=1.5$; $k=0.3$; $B=0.01$; $A=1$; and $\xi_0=2$.

In the same way, Figure 5 shows the deterministic solitary solution for σ=0, which is the same as the flat kink-shaped profile shown in Eq (15). Figure 6 shows that adding weak noise (σ=0.1) makes the solitary structure slightly deformed and non-deterministic. Figure 7 shows that adding stronger stochastic forcing (σ=0.5) makes the configuration clearly kink-shaped and not flat. This case uses the following values: $\rho_0=1.5$; $\omega=0.5$; $\rho_2=1.5$; $k=0.3$; $B=0.06$; $A=10$; and $\xi_0=-0.02$. These results indicate that the stability of the kink-shaped profile is influenced by the noise level, underscoring the nuanced interplay between deterministic and stochastic elements in the system's dynamics. More research on these parameters might help us understand how solitary structures act under different conditions.



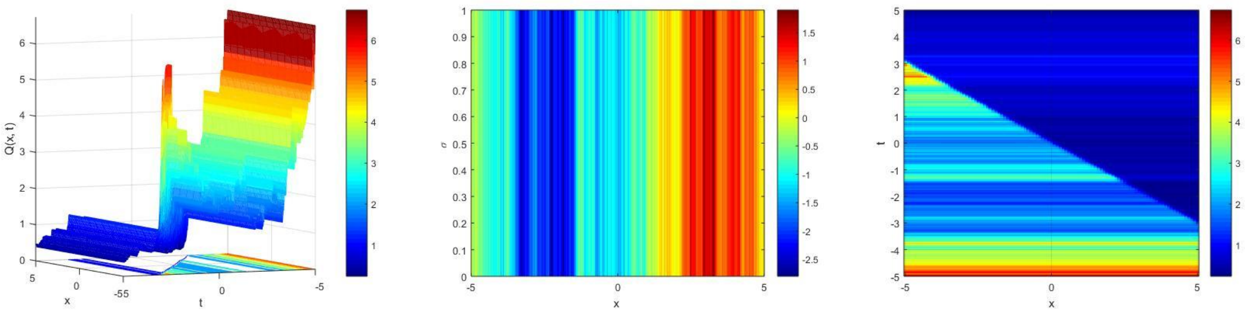
(a) (b) (c)

Figure 5. Kink-shaped type behavior given by Eq (15), (5a) is the 3D plotting, (5b) is the heat map, (5c) is the density plot with σ=0; $\rho_0=1.5$; $\omega=0.5$; $\rho_2=1.5$; $k=0.3$; $B=0.06$; $A=10$; and $\xi_0=-0.02$.



(a) (b) (c)

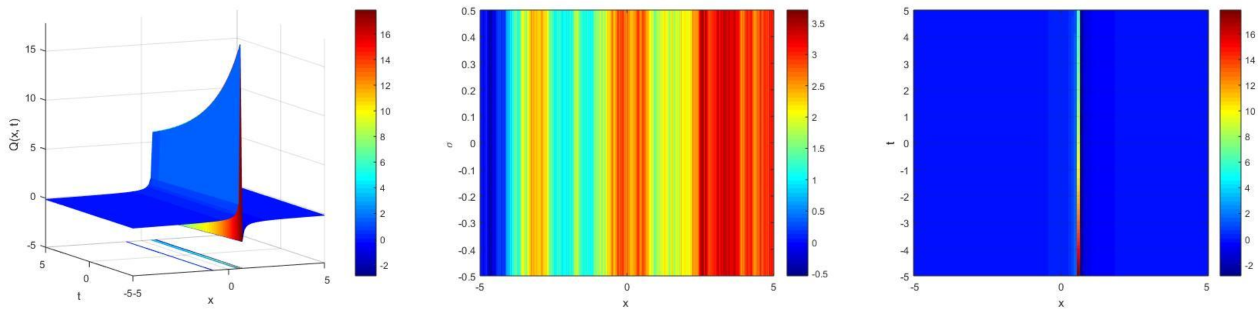
Figure 6. Stochastic kink-shaped type behavior given by Eq (15), (6a) is the 3D plotting, (6b) is the heat map, (6c) is the density plot with σ=0.1; $\rho_0=1.5;$ $\omega=0.5;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.06;$ $A=10;$ and $\xi_0=-0.02$.



(a) (b) (c)

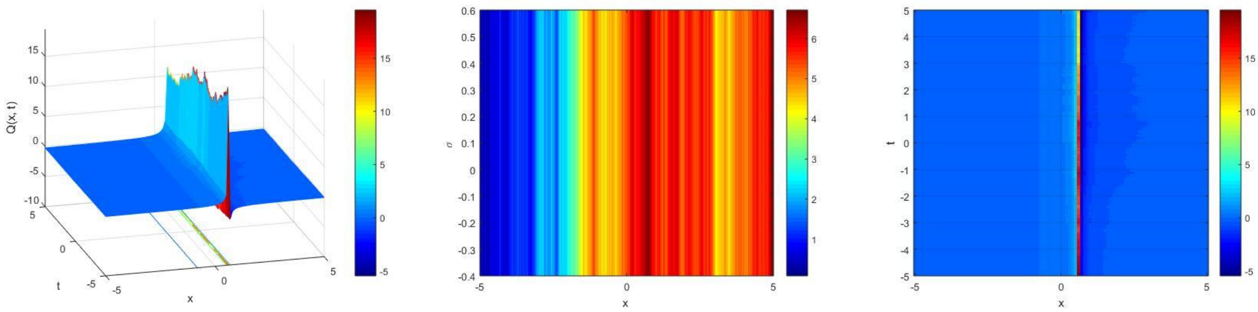
Figure 7. Stochastic kink-shaped type behavior given by Eq (15), (7a) is the 3D plotting, (7b) is the heat map, (7c) is the density plot with σ=0.5; $\rho_0=1.5;$ $\omega=0.5;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.06;$ $A=10;$ and $\xi_0=-0.02$.

A comparable trend is evident in the geometric function-type solutions. Figure 8 shows the deterministic solitary solution for σ=0. This is a flat waveform that can be described by Eq (19). Figure 9 shows that adding moderate noise (σ=0.1) creates a non-deterministic, non-flat solitary pattern. Figure 10 shows that adding high noise (σ=0.5) creates a more irregular geometric function-shaped solution. The values for this situation are $\rho_0=0.5;$ $\rho_1=1.5;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.06;$ $A=10;$ and $\xi_0=-2$. The findings demonstrate that an augmentation in stochastic intensity modifies both the amplitude and the morphology of soliton and solitary formations over time. This illustrates that noise significantly affects the duration, shape modifications, and stability of nonlinear waveforms in stochastic contexts. The diversity in solitary-wave solutions illustrates the considerable influence of stochastic forces on system dynamics.



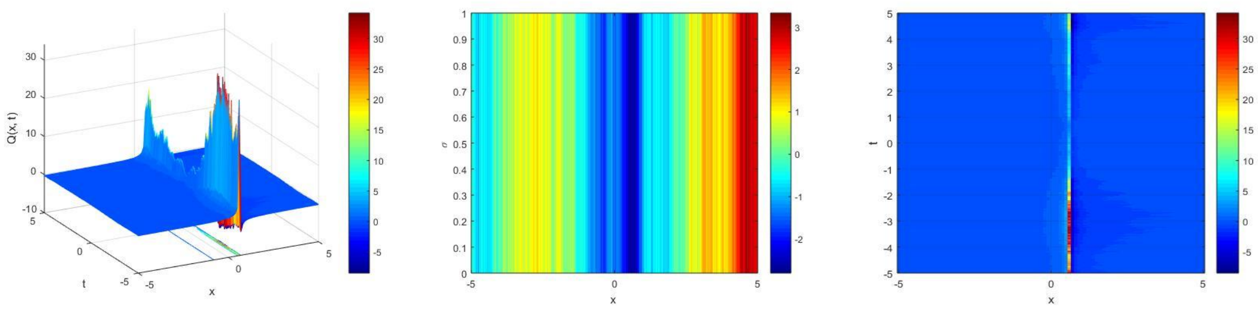
(a) (b) (c)

Figure 8. Geometric function shaped type behavior given by Eq (19), (8a) is the 3D plotting, (8b) is the heat map, (8c) is the density plot with σ=0; *ρ_0=0.5; ρ_1=1.5; ρ_2=1.5; k=0.3; B=0.06; A=10;* and *ξ_0=-2*.



(a) (b) (c)

Figure 9. Stochastic geometric function shaped type behavior given by Eq (19), (9a) is the 3D plotting, (9b) is the heat map, (9c) is the density plot with σ=0.1; *ρ_0=0.5; ρ_1=1.5; ρ_2=1.5; k=0.3; B=0.06; A=10;* and *ξ_0=-2.*

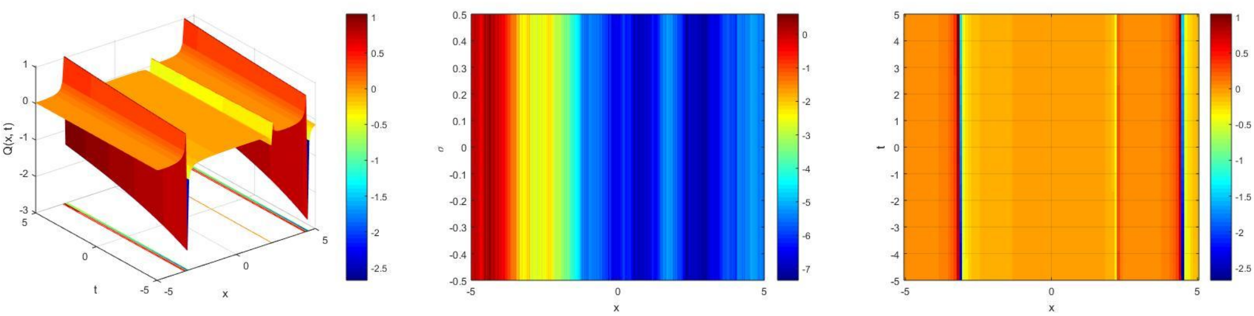


(a) (b) (c)

Figure 10. Stochastic geometric function shaped type behavior given by Eq (19), (10a) is the 3D plotting, (10b) is the heat map, (10c) is the density plot with σ=0.5; *ρ_0=0.5; ρ_1=1.5; ρ_2=1.5; k=0.3; B=0.06; A=10;* and *ξ_0=-2*.

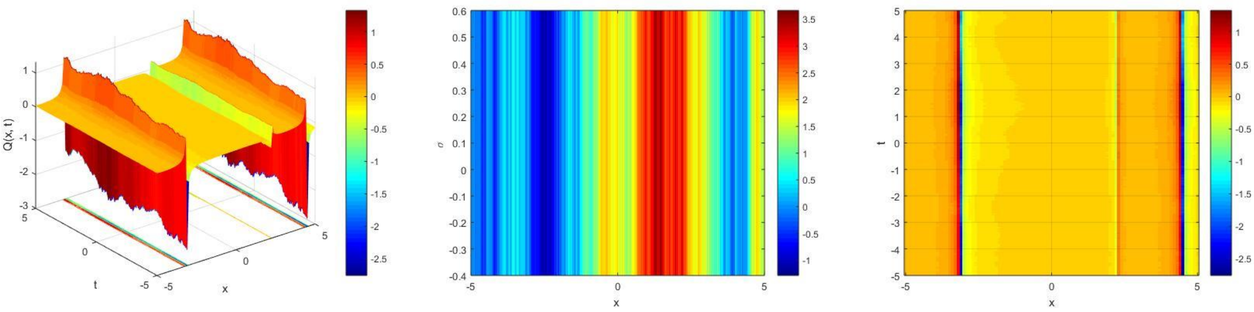
Deviations from the deterministic baseline, imperceptible at lower noise levels, become increasingly apparent as noise levels escalate, revealing nonlinear interactions. The interplay between stochastic disturbances and the properties of long-term wave evolution, including dispersion, nonlinearity, and damping, requires comprehensive examination in light of this behavior. In the future researchers should concentrate on employing uncertainty quantification and inference tools to accurately assess these implications. Calculating response statistics like amplitude, width, energy, and positional means and variances, along with stochastic stability metrics like Lyapunov exponents and coherence measures, represents several intriguing possibilities. Methods like generalized polynomial chaos, Fokker-Planck, or moment-closure analyses can also be used to investigate transition probabilities. Statistical methods that work together, like ensemble and particle filters for data assimilation, Gaussian-process surrogate modeling, hierarchical Bayesian calibration of σ and other parameters, and efficient sensitivity assessments, are made possible. Environmental scientists, engineers, and economists will all gain from enhanced decision-making due to the refined predictive frameworks that more effectively manage the intrinsic uncertainty in real systems. These methods can yield more robust designs and risk-aware tactics for managing complex systems under uncertainty by explicitly integrating stochastic factors in modeling and control.

The following figures show how stochastic perturbations affect the behavior and shape of solitary wave solutions when the noise level changes. Figure 11 shows the deterministic solitary solution for $\sigma=0$, which gives the flat solitary waveform described by Eq (21). This waveform has a profile that looks like a periodic geometric function. Figure 12 shows a non-deterministic, non-flat solitary solution as the stochastic intensity rises to $\sigma=0.1$. This shows that stochastic modulation is starting. Figure 13 shows a clearly non-flat periodic geometric function-type solution for a stronger noise level of $\sigma=0.5$. The chosen parameters are $\rho_0=0.5;$ $\rho_1=1.05;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.006;$ $A=10;$ and $\xi_0=-0.2$. This change implies a more profound relationship between the noise and the fundamental dynamics of the system, resulting in more complex and diverse solution landscapes. As the stochasticity increases, the likelihood of chaotic behavior may arise, necessitating a more comprehensive examination of the stability and attributes of these solutions.



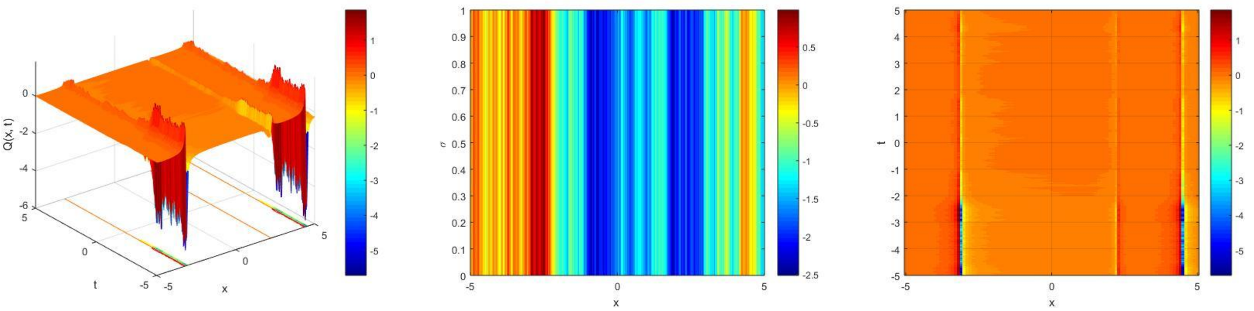
(a) (b) (c)

Figure 11. Periodic shaped type behavior given by Eq (21), (11a) is the 3D plotting, (11b) is the heat map, (11c) is the density plot with $\sigma=0;$ $\rho_0=0.5;$ $\rho_1=1.05;$ $\rho_2=1.5;$ $k=0.3;$ $B=0.006;$ $A=10;$ and $\xi_0=-0.2$.



(a) (b) (c)

Figure 12. Stochastic periodic shaped type behavior given by Eq (21), (12a) is the 3D plotting, (12b) is the heat map, (12c) is the density plot with σ=0.1; $\rho_0=0.5$; $\rho_1=1.05$; $\rho_2=1.5$; $k=0.3$; $B=0.006$; $A=10$; and $\xi_0=-0.2$.



(a) (b) (c)

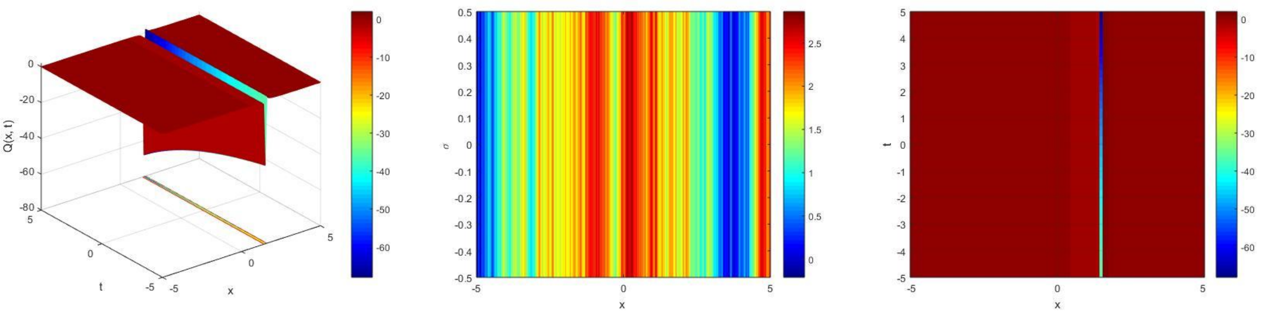
Figure 13. Stochastic periodic shaped type behavior given by Eq (21), (13a) is the 3D plotting, (13b) is the heat map, (13c) is the density plot with σ=0.5; $\rho_0=0.5$; $\rho_1=1.05$; $\rho_2=1.5$; $k=0.3$; $B=0.006$; $A=10$; and $\xi_0=-0.2$.

Figure 14 shows the deterministic solitary solution for σ=0, which is the flat solitary profile described by Eq (23) and shows exponential-type behavior. Figure 15 illustrates a non-flat solitary waveform upon the introduction of a weak stochastic perturbation (σ=0.1). As seen in Figure 16, raising σ to 0.5 results in a non-deterministic, non-flat exponential-type solitary solution with the parameters $\rho_1=1.5$; $\rho_2=0.9$; $k=0.06$; $B=0.06$; $A=10$ and $\xi_0=-0.02$. These plots clearly show how complicated the dynamics are behind the single solution as the parameter σ changes. The pictures show how small changes in σ can affect how the system works, which shows how important it is to choose the right parameters when making theoretical models.

Figure 17 also shows the deterministic solitary solution for σ=0, which is given by Eq (24) and makes a flat kink-shaped waveform. The addition of randomness with σ=0.1 (Figure 18) makes the solitary structure not flat. The higher noise level of σ=0.5 (Figure 19) makes the configuration more pronounced, with the parameters $\rho_1=-1.5$; $\rho_2=0.1$; $k=6$; $B=0.06$; $A=10$; and $\xi_0=0$. Figure 20 shows the three-dimensional backward plots that go with this. They show the behavior of the exponential function defined by Eq (25). Plot (20a) is for σ=0, plot (20b) is for σ=0.1, and plot (20c) is for σ=0.5, with the same parameter choices. Figure 21 shows the backward three-dimensional plots that go with Eq (26), which shows the kink-type behavior. Plot (21a) shows σ=0, plot (21b) shows σ=0.1, and plot (21c) shows σ=0.5. The parameter values are the same as those used in Figure 17. These plots show how the system changes over time as it moves through different states of σ. They also show how the defined parameters affect the kink-type behavior in a complex way. This picture gives us important information about the underlying mechanisms that cause the observed events.

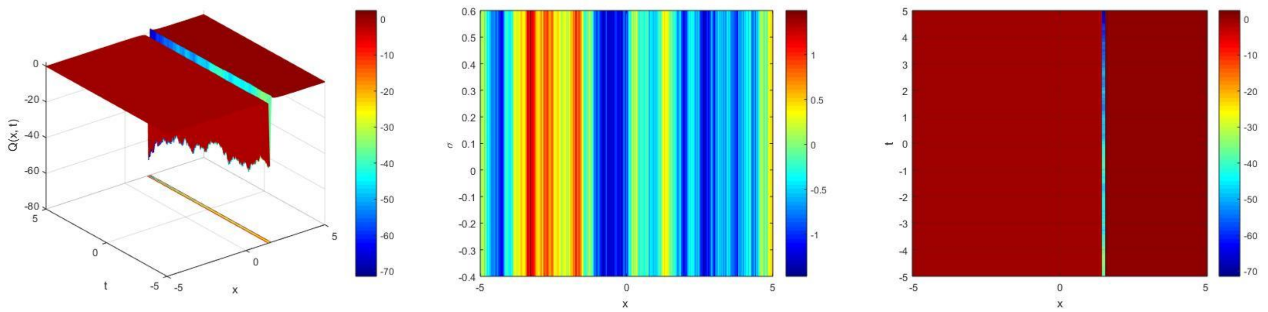
Last, Figure 22 shows the deterministic solitary solution for σ=0, which gives us the flat solitary waveform described by Eq (38). This waveform has a geometric function-type profile. Figure 23 shows a non-flat solitary solution when the noise level is raised to σ=0.1. Figure 24 shows a non-flat periodic geometric function-type waveform when the noise level is raised to σ=0.5. We chose the following values for this case: $\rho_0=0.1$; $\rho_2=0.01$; $k=1.1$; $B=0.2$; $A=0.1$; and $\xi_0=-0.1$. These results show that solitary waveforms are very sensitive to changes in noise intensity. This means that even small changes in the parameters can have a big effect on the waveform's properties. Further research into how these parameters interact may show that these systems work more complexly.

Every single one of these plots collectively illustrates the degree to which the stochastic intensity (σ) influences the manner in which structures change in single solutions. As the value of π increases, the solutions undergo a transformation from deterministic, smooth, and flat shapes to increasingly complex patterns that are not flat. It is clear from this graph that the system is extremely susceptible to arbitrary alterations. Considering these changes, it is difficult to provide an explanation for how deterministic and non-deterministic processes interact with one another. They suggest that a closer examination of the parameter space could reveal more intriguing patterns. In the future, scientists may try to figure out how to measure these changes, figure out how stable they are, and find out what role they play in real-life systems where random effects are important for moving waves and energy around.



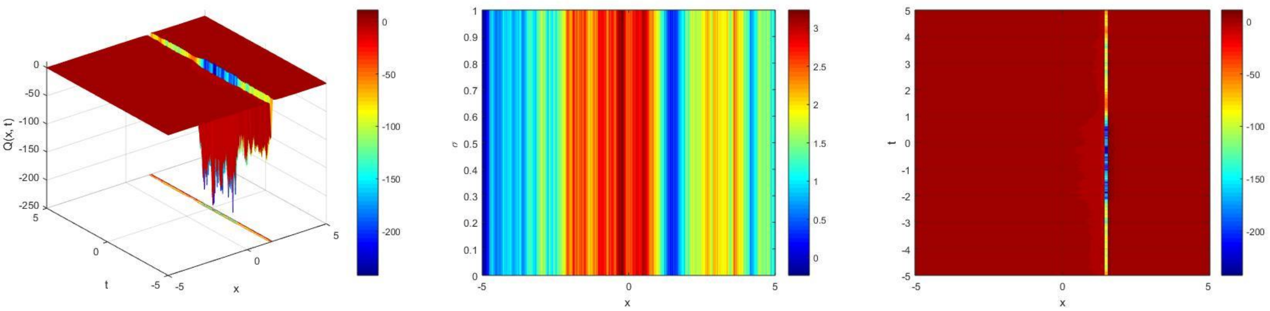
(a) (b) (c)

Figure 14. Exponential function behavior given by Eq (23), (14a) is the 3D plotting, (14b) is the heat map, (14c) is the density plot with σ=0; $\rho_1=1.5$; $\rho 2=0.9$; $k=0.06$; $B=0.06$; $A=10$; and $\xi_0=-0.02$.



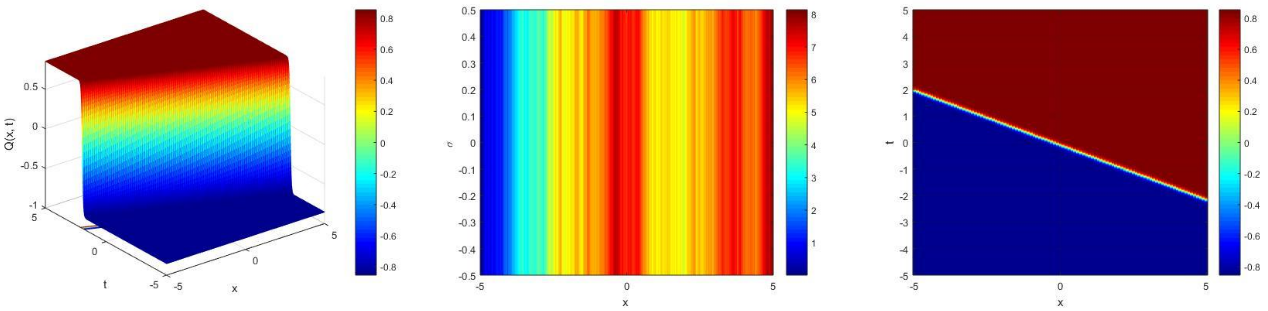
(a) (b) (c)

Figure 15. Stochastic exponential function behavior given by Eq (23), (15a) is the 3D plotting, (15b) is the heat map, (15c) is the density plot with σ=0.1; $\rho_1=1.5$; $\rho_2=0.9$; $k=0.06$; $B=0.06$; $A=10$; and $\xi_0=-0.02$.



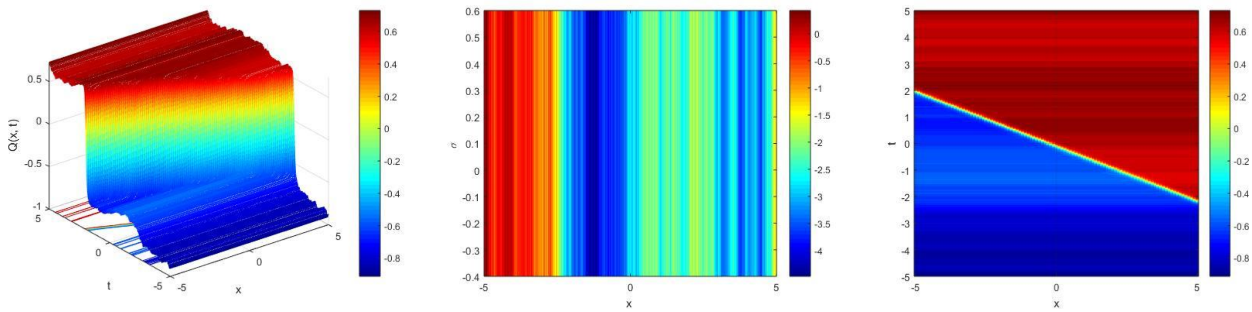
(a) (b) (c)

Figure 16. Stochastic exponential function behavior given by Eq (23), (16a) is the 3D plotting, (16b) is the heat map, (16c) is the density plot with σ=0.5; $\rho_1=1.5$; $\rho_2=0.9$; $k=0.06$; $B=0.06$; $A=10$; and $\xi_0=-0.02$.



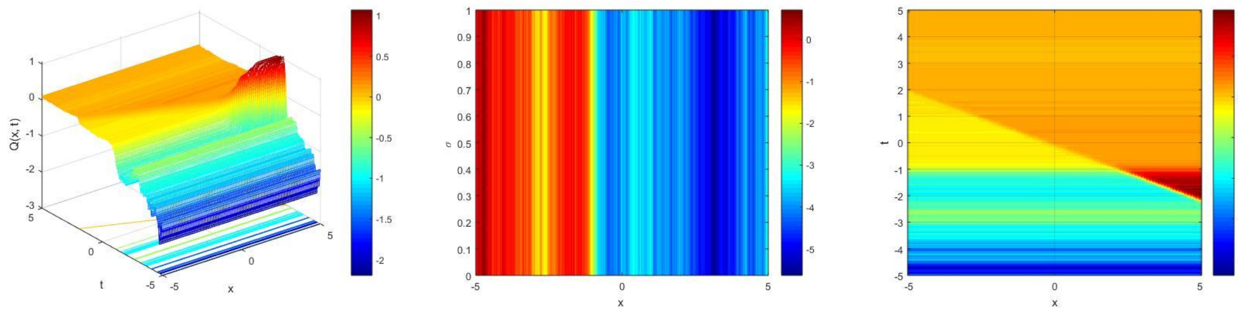
(a) (b) (c)

Figure 17. Kink-shaped type behavior given by Eq (24), (17a) is the 3D plotting, (17b) is the heat map, (17c) is the density plot with σ=0; $\rho_1=-1.5$; $\rho_2=0.1$; $k=6$; $B=0.06$; $A=10$; and $\xi_0=0$.



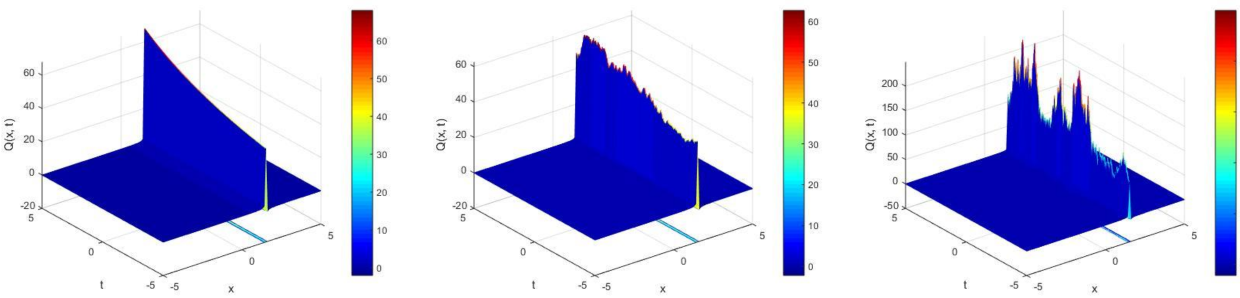
(a) (b) (c)

Figure 18. Stochastic kink-shaped type behavior given by Eq (24), (18a) is the 3D plotting, (18b) is the heat map, (18c) is the density plot with σ=0.1; $\rho_1=-1.5$; $\rho_2=0.1$; $k=6$; $B=0.06$; $A=10$; and $\xi_0=0$.



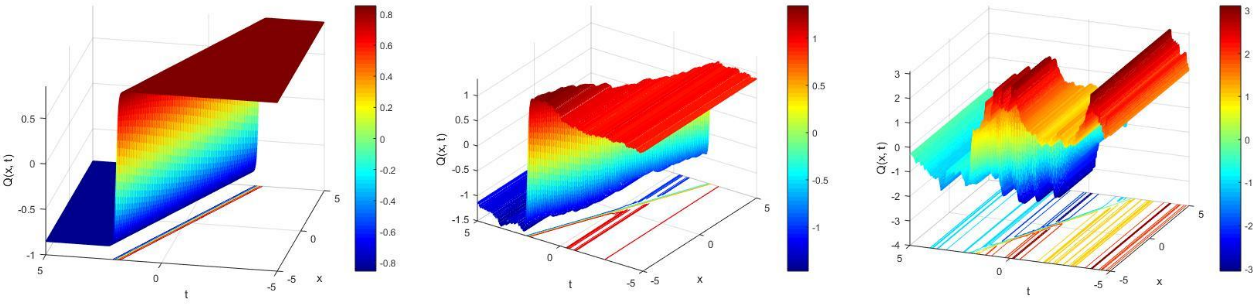
(a) (b) (c)

Figure 19. Stochastic kink-shaped type behavior given by Eq (24), (19a) is the 3D plotting, (19b) is the heat map, (19c) is the density plot with σ=0.5; $\rho_1=-1.5$; $\rho_2=0.1$; $k=6$; $B=0.06$; $A=10$; and $\xi_0=0$.



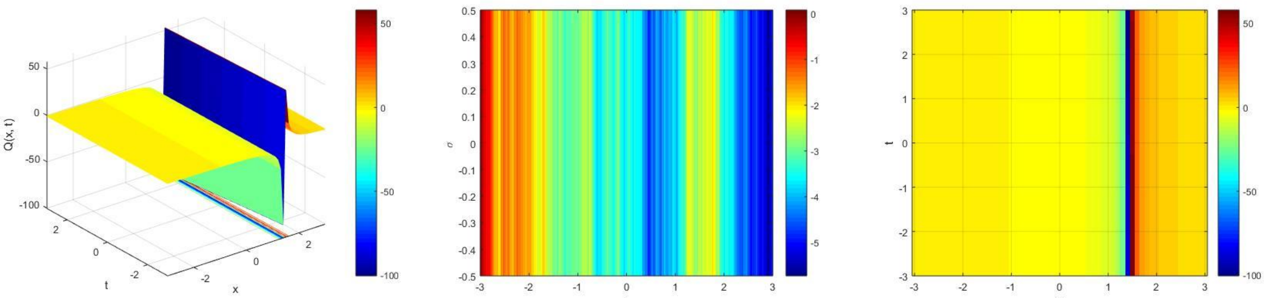
(a) (b) (c)

Figure 20. Backward 3D plotting exponential function behavior given by Eq (25), (20a) when σ=0, (20b) when σ=0.1, (20c) when σ=0.5 with $\rho_1=1.5$; $\rho_2=0.9$; $k=0.06$; $B=0.06$; $A=10$; and $\xi_0=-0.02$.



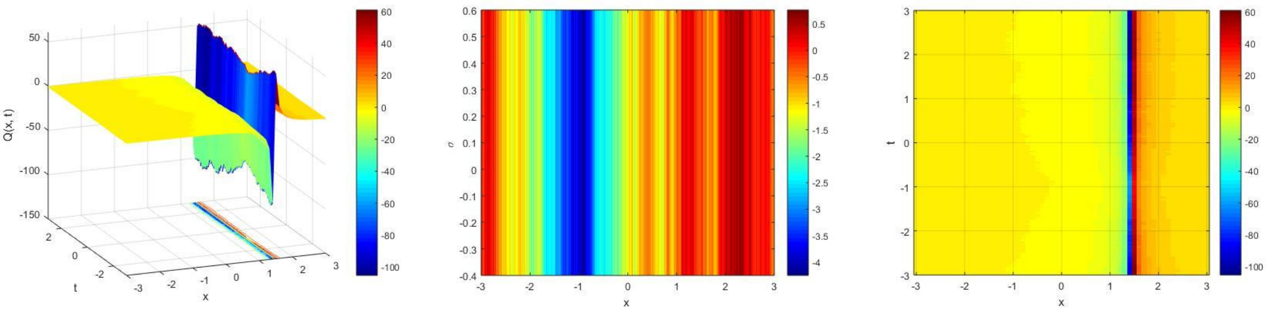
(a) (b) (c)

Figure 21. Backward 3D plotting kink-shaped type behavior given by Eq (26), (21a) when σ=0, (21b) when σ=0.1, (21c) when σ=0.5 with $\rho_1=-1.5;\ \rho_2=0.1;\ k=6;\ B=0.06;\ A=10;$ and $\xi_0=0$.



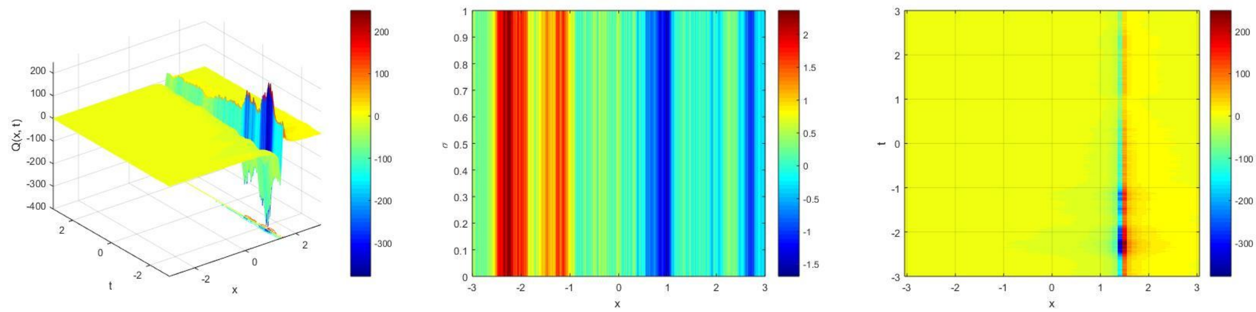
(a) (b) (c)

Figure 22. Geometric function shaped type behavior given by Eq (38), (22a) is the 3D plotting, (22b) is the heat map, (22c) is the density plot with σ=0; $\rho_0=0.1;\ \rho_2=0.01;\ k=1.1;\ B=0.2;\ A=0.1;$ and $\xi_0=-0.1$.



(a) (b) (c)

Figure 23. Stochastic geometric function shaped type behavior given by Eq (38), (23a) is the 3D plotting, (23b) is the heat map, (23c) is the density plot with σ=0.1; $\rho_0=0.1;\ \rho_2=0.01;\ k=1.1;\ B=0.2;\ A=0.1;$ and $\xi_0=-0.1$.



(a) (b) (c)

Figure 24. Stochastic geometric function shaped type behavior given by Eq (38), (24a) is the 3D plotting, (24b) is the heat map, (24c) is the density plot with σ=0.5; $\rho_0=0.1$; $\rho_2=0.01$; $k=1.1$; $B=0.2$; $A=0.1$; and $\xi_0=-0.1$.

The solutions obtained in this study correspond to solitary and soliton waveforms that emerge in physical systems described by the stochastic damped mKdV equation, which encapsulates the equilibrium among nonlinearity, dispersion, damping, and stochastic disturbances. In terms of physics, these solutions are like localized wave packets or traveling waves that keep their shape over time and space but can be changed by random changes in the environment. In a deterministic environment (when the noise intensity σ=0), the soliton and solitary solutions exhibit varying forms (trigonometric, kink-shaped, exponential, or geometric) influenced by the interaction of dispersion and nonlinearity. Many physical systems, like nonlinear ion-acoustic waves in plasma, surface waves in shallow water, or optical pulses in nonlinear fibers, have these kinds of structures. Each type of solution corresponds to a different physical regime. For example, kink-type solutions show how two stable states can change (like shocks or domain walls), while trigonometric and exponential solutions show how localized pulses can be periodic or quickly fade away. When stochasticity is added (σ>0), the system acts like random changes that happen in real life, like thermal noise, turbulence, or electromagnetic disturbances. This model uses multiplicative white noise, which means that the random effects change based on the wave's amplitude. This technique causes the system's behavior to depend on its state. Such behavior causes changes in the amplitude, phase, and structure of the original wave profiles. The stochastic solutions show how energy spreads out and is moved around when there is noise, which is important for predicting wave instability, pattern change, or wave packet breakdown. Simulations show that as noise increases louder, solitons and solitary waves become less regular, with changes in height, width, and speed. However, they keep some of their localized nature at moderate noise levels. Similar changes occur in plasma turbulence, randomly disturbed shallow water flows, and optical systems with amplifier noise. An examination of these solutions from a scientific perspective sheds light on the dynamics of energy transfer, signal integrity, and wave stability within complex physical systems. This research also demonstrates the development of nonlinear wave structures that are influenced by dissipative forces and random disturbances.

4.1. Simulation results

We simulate the stochastic damped mKdV

$$dQ = -\left(AQ^2Q_x + BQ_{xxx} + CQ\right)dt + \sigma Q dW\,, \tag{40}$$

on a domain $x \in [0, L]$. The deterministic part is treated using the modified simple equation method, while the stochastic forcing is advanced with the Euler–Maruyama scheme in the Itô sense with $\Delta W_n \sim N(0,\Delta t)$. The linear dispersive–damping operator is handled exactly through an integrating-factor approach, and the nonlinear term $-AQ^2Q_x$ is evaluated in physical space with 3/2-rule dealiasing. Time-stepping uses Δt chosen by stability tests; the grid uses N equispaced points with $\Delta x = L/N$. Initial conditions are taken as a localized pulse (or a superposition of pulses), and periodic boundary conditions are imposed. For each noise intensity σ, we perform M independent realizations to compute ensemble-averaged fields and visualize the results via three-dimensional surfaces, spatiotemporal density maps, and heat plots. Convergence is assessed by halving Δt and doubling N, as well as by confirming that the $\sigma \to 0$ limit reproduces the deterministic damped mKdV soliton.

In these present simulations, the temporal discretization parameter Δt was determined on the basis of stability analysis and numerical testing. The stochastic forcing enters through the transformation

$$Q(x,t) = \psi(x,t)e^{\sigma W(t) - \sigma^2 t/2}, \tag{41}$$

where $W(t)$ denotes a Wiener process. For the Euler–Maruyama discretization of the associated stochastic term, mean-square stability of the linear test problem

$$dX = \lambda X_t\, dt + \mu X\, dW\,, \tag{42}$$

requires

$$|1 + \lambda \Delta t|^2 + |\mu|^2 \Delta t < 1, \tag{43}$$

which, in this multiplicative-noise setting, constrains $\Delta t \ll 1/\sigma^2$. For the nonlinear convective term, stability is governed by a CFL-type condition,

$$\Delta t \le C\ \frac{\Delta x}{\max|\mathrm{Q}|}, \qquad C < 1. \tag{44}$$

Since the dispersive–damping operator is integrated exactly, it imposes no further restriction. Thus, the admissible time step is given by

$$\Delta t < \min\left(\frac{C\Delta x}{\max|\mathrm{Q}|}, \frac{1}{\sigma^2}\right). \tag{45}$$

In practice, an initial trial value of Δt is reduced until computed solutions remain bounded for the simulation interval without artificial growth. Convergence is confirmed by halving Δt and ensuring that ensemble-averaged quantities exhibit no significant variation. The final choice of Δt therefore represents the largest value consistent with both the analytical stability conditions and empirical convergence tests, ensuring numerical reliability and efficiency.

If we define the stochastic multiplier

$$M(t)=e^{\sigma W(t)-\sigma^2 t/2}. \tag{46}$$

Then $Q=\psi M$. Since $W(t) \sim N(0,t)$. Over one step Δt with $\Delta W_n \sim N(0,\Delta t)$,

$$M_{n+1}=M_n e^{\sigma \Delta W_n-\sigma^2 \Delta t/2}. \tag{47}$$

Thus,

$$E[M_{n+1}^2 \mid M_n]=M_n^2 e^{\sigma^2 \Delta t}=M_n^2\left(1+\sigma^2 \Delta t+O(\Delta t^2)\right). \tag{48}$$

Therefore, per step, the second moment of Q is multiplied by $\exp(\sigma^2\Delta t)$. If the numerical update for p has mean-square amplification factor $G(\Delta t)$ (from the deterministic/nonlinear discretization), the overall mean-square amplification for Q is

$$A(\Delta t)=|G(\Delta t)|^2 e^{\sigma^2 \Delta t}. \tag{49}$$

Mean-square stability requires $A(\Delta t)<1$. For small Δt,

$$|G(\Delta t)|^2(1+\sigma^2 \Delta t+O(\Delta t^2))<1 \Rightarrow \sigma^2 \Delta t \leq 1-|G(0)|^2. \tag{50}$$

Since the integrating-factor treats the linear part exactly (so $|G(0)|=1$) and the residual nonlinear step is near-neutral for sufficiently small Δt, the dominant constraint is

$$\sigma^2 \Delta t << 1 \Leftrightarrow \Delta t << \frac{1}{\sigma^2}. \tag{51}$$

The exponential factor M amplifies the second moment by $\exp(\sigma^2\Delta t)$ at each step, and to prevent uncontrolled mean-square growth (and thus maintain stability of the ensemble statistics), the time step Δt must be chosen such that $\sigma^2\Delta t$ remains small; this leads to the practical stability restriction $\Delta t < \min\left(C\,\Delta x / \max|Q|, 1/\sigma^2\right)$, which combines the stochastic constraint with the CFL-type condition imposed by the nonlinear convective term, while convergence is subsequently verified empirically by halving Δt and confirming invariance of ensemble-averaged quantities.

4.2. Physical interpretation

The results show how damping and random perturbations work together to change the behavior of solitary waves. Without noise, the damped mKdV solitons keep their classical shape, but their amplitude slowly decreases because of dissipative effects. When stochastic forcing is added, the soliton structure becomes sensitive to the level of noise. Low levels of multiplicative noise mostly cause small changes around the deterministic profile, while higher levels of noise can cause the soliton core to lose amplitude, spread out, or change shape. In systems like plasma waves, optical fibers, and shallow water dynamics, where random changes in the environment interact with built-in dissipation to determine wave stability and energy transport, these behaviors have physical meaning. Therefore, the study emphasizes the fragile equilibrium among nonlinearity, dispersion, damping, and randomness in influencing the long-term dynamics of nonlinear wave phenomena. Understanding these interactions is critical to forecasting the behavior of waves in various applications, such as telecommunications and environmental modeling.

The results underscore the practical importance of formulating strategies to stabilize and regulate solitary waves in the presence of stochastic influences, in addition to providing theoretical insights. Progress in this area can help make communication systems more reliable, make better predictions about how the environment works, and lead to more technological breakthroughs. For instance, better management of soliton dynamics could make data transmission more efficient, and better stochastic modeling could make predictions about changes in the ocean and climate more accurate. It is necessary for the fields of physics, engineering, and computer science to collaborate in order to address the issues that are brought about by noise-driven wave phenomena. In a range of domains, such as weather forecasting and the harvesting of renewable energy, researchers can develop complicated models that make use of nonlinear wave phenomena. This is made possible by the combining of approaches from different fields. This all-encompassing strategy has the potential to provide solutions to significant global problems that are not only long-lasting but also scalable, and it has the potential to assist us in increasing our understanding of theory.

5. Conclusions

In this study, we formulated and examined a stochastic damped mKdV equation featuring multiplicative noise, characterized by a Wiener process, to explore the effects of stochastic perturbations on nonlinear wave propagation. By integrating the modified simple equation method with supplementary numerical simulations, we successfully identified numerous analytical soliton and solitary wave solutions and examined their responses to varying noise levels. The results showed that noise can change the stability and dynamics of soliton solutions in a big way. This research paves the way for additional exploration into the ramifications of stochastic effects in diverse physical systems where these equations are relevant.

It has been demonstrated through these discoveries that soliton structures are stable and continue to maintain their configuration despite the presence of weak stochastic influences. When the noise level increases, however, solitons exhibit amplitude modulation in a clear and obvious manner and, in some instances, become less stable. The complex interplay between dispersion, damping, and randomness is demonstrated by numerical simulations, which also confirm these predictions. Two ways to show how multiplicative noise reduces coherence in waveforms' temporal behavior are density maps and three-dimensional surface plots. Plasmas, optical cables, and shallow sea waves are only a few examples of real-world physical systems whose behaviors cannot be accurately predicted using deterministic methods. Deterministic techniques cannot accurately represent the behaviors exhibited by these systems. The objective of this research is to develop a model that effectively addresses state-dependent unpredictability, thereby bridging a gap in the current body of knowledge. In this study, we demonstrate substantial procedural modifications through the deployment of a modified simple equations method inside a stochastic context. This technique enhances the understanding of how systems adapt to their environments.

This study not only contributes to a better understanding of the theoretical framework, but it also sheds light on the practical consequences that noise has on the dynamics of soliton systems. Understanding how random changes affect the stability and amplitude of solitons can help improve the design of plasma devices, wave-based communication systems, and optical transmission channels. The integration of environmental changes facilitates the creation of predictive models that are fundamentally more reliable and robust, suitable for engineering and experimental applications. Furthermore, the changes could have a huge impact on how many various technologies are made and work in the future, which would make systems more reliable and useful in the end. Potential new uses for solitons in fields like materials science and telecommunications are being revealed as we learn more about their context-dependent behavior.

5.1. Suggestions for future work

Additional research into higher-dimensional model extensions, experimental validation, and the incorporation of various noise processes, such as colored noise and Lévy noise, could improve this paradigm. Machine learning can enhance data-driven modeling of stochastic nonlinear systems by predicting system performance and estimating parameters. Making predictions could be one way to achieve this. This study lays the groundwork for the integration of deterministic and stochastic perspectives in soliton theory, enhancing our understanding of the equilibrium between order and chaos in nonlinear wave dynamics and fostering further developments in both theory and practice.

In the future, researchers could build on this study by looking at other stochastic processes, like Lévy noise or colored noise, to see how they affect soliton dynamics in their own way. Another promising avenue is to employ probabilistic and stability-theoretic approaches to investigate the long-term stability and potential decay or persistence of solitons subjected to continuous stochastic influence. There is also the possibility that the creation of stochastic models in higher dimensions, such as two or three dimensions, can make it simpler to explain spatial behaviors that are more complicated. Control and optimization techniques, such as adaptive tuning and feedback loops, are both potential methods that could be applied in order to achieve the goal of stabilizing solitons. This is the option of inspecting the utilization of either of these techniques. In addition, the use of machine learning could be of assistance in the identification of patterns, the forecasting of the reaction of a system, and the development of data-driven modeling in situations that are generated at random.

There is a possibility that the numerical and theoretical results could be improved by using physical system validation studies that contain optical fibers, shallow water waves, or plasma settings. Based on the comparison of these random effects with other nonlinear stochastic models, such as the KdV, sine-Gordon, or nonlinear Schrodinger equations, it is possible to notice the usefulness of these random effects in the field of wave dynamics. For the purpose of performing this comparative research, it is possible that the fundamental mechanics of complex systems will be better understood, and that new insight into the dynamics of these systems will be disclosed. We may uncover novel applications for these linkages in a variety of fields, such as telecommunications and fluid dynamics, as a result of our research. These kinds of interdisciplinary approaches could help create stronger predictive models, which would make technologies that depend on wave dynamics work better and more reliably. Moreover, the incorporation of these findings with experimental data will guarantee that theoretical progress is anchored in practical reality, which will improve awareness of both the phenomena and their applications.

Author contributions

All authors participated equally to the idea, approach, analysis, and writing of this publication. Every author has seen and sanctioned the final version of the manuscript and consents to be responsible for all facets of the work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors hereby declare that artificial intelligence tools were used in this research only as supportive instruments for tasks such as language refinement, idea organization, and data summarization. All intellectual contributions, critical analysis, and final interpretations are solely our own. The use of AI has been properly documented to maintain transparency and uphold academic integrity.

Acknowledgments

Mohra Zayed extends her appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through Large Research Project under grant number RGP2/419/45.

Conflict of interest

The authors declare that they have no conflicts of interest.

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