AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25791–25810.
DOI:10.3934/math.20251141
Received: 23 August 2025
Revised: 02 October 2025
Accepted: 15 October 2025
Published: 10 November 2025

Research article

Power, principal, and isotone $(\odot, \vee)$-higher derivations on MV-algebras

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Abstract: This paper studies three types of $(\odot, \vee)$-higher derivations on MV-algebras: power, principal, and isotone $(\odot, \vee)$-higher derivations. We show that every principal $(\odot, \vee)$-higher derivation is both power and isotone. However, counterexamples demonstrate that no pairwise implications hold beyond the established one. Furthermore, explicit constructions of power $(\odot, \vee)$-higher derivations are provided. Additionally, we characterize isotone $(\odot, \vee)$-higher derivations and show that the fixed point set of any principal $(\odot, \vee)$-higher derivation forms a lattice ideal.

Keywords: MV-algebra; higher derivation; MV-chain; semigroup; fixed point; lattice ideal
Mathematics Subject Classification: 03G20, 06B10, 06D35, 08B26

1. Introduction

Hasse and Schmidt [9] introduced the concept of higher derivations on associative algebras. Subsequent work extended the study of higher derivations to various algebraic structures. Heerema [11, 12] investigated higher derivations on local rings, focusing on their representation of inertial automorphisms, ramification group structures, and convergence properties. Ferrero [8] studied higher derivations of prime and semiprime rings satisfying linear relations. Ribenboim [23] conducted a systematic study of higher derivations on arbitrary rings and modules. Specifically, for a commutative ring R, a higher derivation (or Hasse-Schmidt derivation [7]), on an R-algebra A is a sequence of R-linear maps $D = (d_n)_{n\in\mathbb{N}}$ on A such that $d_0 = \mathrm{Id}_A$ and

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b), \text{ for all } a, b \in A \text{ and } n \geq 1.$$

Further extensions include derivations and higher derivations on lattices [3, 26, 27], incidence algebras [14, 15], triangular algebras [24, 25], and more recently, logical algebras. This line of research has been further expanded to encompass other logical algebraic structures fundamental to fuzzy logic and many-valued reasoning, such as BL-algebras [19], residuated lattices [10], BCK-algebras [6], basic algebras [16], and MV-algebras [1,21,28].

This paper is motivated by the need to bridge the gap between classical derivation theory and its generalizations in MV-algebraic frameworks. The main contributions of this paper are as follows:

1) We introduce and compare three types of $(\odot, \vee)$-higher derivations: power, principal, and isotone. We prove that principal $(\odot, \vee)$-higher derivations imply power and isotone properties (Propositions 3.6 and 3.8), while other pairwise implications fail (Remarks 3.9, 3.10, 3.14; Figure 1).

2) We provide explicit constructions of power $(\odot, \vee)$-higher derivations on two typical MV-chains: the infinite chain C and the finite chain L_m (Theorems 4.1 and 4.2).

3) We characterize a power $(\odot, \vee)$-higher derivation to be isotone (Theorem 3.12).

4) We show that the fixed point set of a principal $(\odot, \vee)$-higher derivation forms a lattice ideal (Proposition 5.2).

2. Preliminaries

An **MV-algebra** [5] is an abelian monoid $(A, \oplus, 0)$ with a unary operation *, which satisfies:

1) $x^{**} = x$;

2) $x \oplus 0^* = 0^*$;

3) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Example 2.1. [5] Let L be the real unit interval $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Define

$$x \oplus y = \min\{1, x + y\} \text{ and } x^* = 1 - x \text{ for any } x, y \in L.$$

Then $(L, \oplus, ^*, 0)$ is an MV-algebra.

For each positive integer $m \geq 2$, the m-element subset of L

$$L_m = \left\{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-2}{m-1}, 1\right\}$$

with the same operations is a subalgebra of MV-algebra L, which is an MV-algebra.

Example 2.2. [4] Let the following two sets of formal symbols be

$$C_0 = \{0, c, 2c, 3c, \cdots\}, \quad C_1 = \{1, c^*, (2c)^*, (3c)^*, \cdots\},$$

where $0c = 0$, $1c = c$, $uc = vc \Leftrightarrow u = v$, $0^* = 1$, $(kc)^* = 1 - kc$ and $(kc)^{**} = ((kc)^*)^* = kc$ for any $u, v, k \in \mathbb{N}$.

Let $+$ (respectively, $-$) be the ordinary sum (respectively, subtraction) between integers. Define the binary operation $\oplus$ on $C = C_0 \cup C_1$ for any $u, v \in \mathbb{N}$:

- $uc \oplus vc = (u+v)c$;
- $(uc)^* \oplus (vc)^* = 1$;
- $(uc)^* \oplus vc = vc \oplus (uc)^* = \begin{cases} 1, & u \leq v, \\ ((u-v)c)^*, & u > v. \end{cases}$

Then, $(C, \oplus, ^*, 0)$ is an MV-algebra.

For every MV-algebra A, denote the constant $1 = 0^*$ and the operation $x \odot y = (x^* \oplus y^*)^*$. Recall that an archimedean MV-algebra [2] is defined to be an MV-algebra satisfying for any $x, y \in A$, $nx \leq y$ for any $n \in \mathbb{N}$ implies $x \odot y = x$.

Let A be an MV-algebra. For any $x, y \in A$, define the natural order of A by $x \leq y$ iff $x^* \oplus y = 1$ iff $x \odot y^* = 0$ [5]. Furthermore, the natural order determines a structure of bounded distributive lattice $(A, \vee, \wedge, 0, 1)$, and

$$x \vee y = (x \odot y^*) \oplus y \text{ and } x \wedge y = x \odot (x^* \oplus y).$$

A linearly ordered MV-algebra is called an MV-chain.

Lemma 2.1. [4,5] *Let A be an MV-algebra and $x, y, z \in A$. Then the following statements hold:*

1) $x \odot y \leq x \wedge y \leq x \leq x \vee y \leq x \oplus y$;

2) If $x \leq y$, then $x \vee z \leq y \vee z$, $x \wedge z \leq y \wedge z$;

3) If $x \leq y$, then $x \oplus z \leq y \oplus z$, $x \odot z \leq y \odot z$;

4) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;

5) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

An element a of MV-algebra A is called idempotent if $a \oplus a = a$. Denote the set of all idempotent elements of A by $\mathbf{B}(A)$. It is known that $\mathbf{B}(A)$ is a subalgebra of the MV-algebra A [5, Corollary 1.5.4].

Lemma 2.2. [5, Theorem 1.5.3] *For every element x in an MV-algebra A, the following conditions are equivalent:*

1) $x \in \mathbf{B}(A)$;

2) $x \oplus x = x$;

3) $x \odot x = x$;

4) $x \odot y = x \wedge y$ for all $y \in A$.

Definition 2.3. [5] A subset I of a lattice L is a lattice ideal if it satisfies:

1) $0 \in I$;

2) $x, y \in I$ imply $x \vee y \in I$;

3) $x \in I$ and $y \leq x$ imply $y \in I$.

By [5, Proposition 1.1.5], a lattice ideal of an MV-algebra $(A, \vee, \wedge)$ is the same as an ideal of the underlying lattice.

Definition 2.4. [28, Definition 3.1] Let A be an MV-algebra. A map $d : A \to A$ is called an $(\odot, \vee)$-derivation on A if it satisfies:

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) \quad \text{for all } x, y \in A. \tag{2.1}$$

Denote the set of all $(\odot, \vee)$-derivations on A by $\mathrm{Der}(A)$. For a finite set $X = \{x_0, x_1, \cdots, x_{m-1}\}$ and a map $d : X \to X$, we shall write d as

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{m-1} \\ d(x_0) & d(x_1) & \cdots & d(x_{m-1}) \end{pmatrix}.$$

If $d \in \mathrm{Der}(L_m)$, then x_i and $d(x_i)$ can be written by $\frac{i}{m-1}$ and $\frac{j_i}{m-1}$, where $0 \le i \le m-1$ and $j_i \le i$. For convenience, we abbreviate d as $j_0 j_1 \cdots j_{m-1} (= [(m-1)d(x_0)][(m-1)d(x_1)] \cdots [(m-1)d(x_{m-1})])$ without ambiguity.

Proposition 2.5. [28, Proposition 3.3] *Let A be an MV-algebra, $x \in A$ and $d \in \mathrm{Der}(A)$. Then for any $n \in \mathbb{N}_+$, the following statements hold:*

1) $d(0) = 0$;

2) $d(x) \le x$.

Lemma 2.6. [28, Corollary 3.12] *Let A be an MV-algebra and $d \in \mathrm{Der}(A)$. Let $u \in A$ be given with $u \le d(1)$ and define an operator d^u on A by*

$$d^u(x) := \begin{cases} u, & \text{if } x = 1; \\ d(x), & \text{otherwise.} \end{cases}$$

Then, d^u is also in $\mathrm{Der}(A)$.

For a given $a \in A$, define the map $d_{a\odot} : A \to A$ by

$$d_{a\odot}(x) := a \odot x \quad \text{for all } x \in A.$$

Then $d_{a\odot}$ is an $(\odot, \vee)$-derivation, called a principal $(\odot, \vee)$-derivation. Recall that the order structure of all $(\odot, \vee)$-derivations on m-element MV-chain L_m $(m \ge 2)$ is given in [28, Theorem 5.6]. Indeed, the lattice $\mathrm{Der}(L_m) = \{d_{(a,b)} \mid (a, b) \in \mathcal{A}(L_m)\}$ is isomorphic to the lattice $\mathcal{A}(L_m)$, where

$$\mathcal{A}(L_m) = \{(x, y) \in L_m \times L_m \mid y \le x\} \backslash \{(0, 0)\},$$

and

$$d_{(a,b)}(x) := \begin{cases} b, & \text{if } x = 1; \\ d_{a\odot}(x) = a \odot x, & \text{otherwise.} \end{cases}$$

Example 2.3. We list all $(\odot, \vee)$-derivations of L_2, L_3, and L_4 in Tables 1–3, respectively:

Table 1. $\mathrm{Der}(L_2)$.

$x \in L_2$	$\mathrm{Id}_{L_2}(x)$	$\mathbf{0}_{L_2}(x)$
0	0	0
1	1	0

Table 2. $\mathrm{Der}(L_3)$.

$x \in L_3$	$\mathrm{Id}_{L_3}(x)$	$d_{\frac{1}{2}\odot}(x)$	$\mathrm{Id}_{L_3}^{\frac{1}{2}}(x)$	$\mathbf{0}_{L_3}(x)$	$\mathrm{Id}_{L_3}^{0}(x)$
0	0	0	0	0	0
$1/2$	$1/2$	0	$1/2$	0	$1/2$
1	1	$1/2$	$1/2$	0	0

Table 3. $\mathrm{Der}(L_4)$.

$x \in L_4$	$\mathrm{Id}_{L_4}(x)$	$d_{\frac{2}{3}\odot}(x)$	$\mathrm{Id}_{L_4}^{\frac{2}{3}}(x)$	$d_{\frac{1}{3}\odot}(x)$	$d_{(\frac{2}{3},\frac{1}{3})}(x)$	$\mathrm{Id}_{L_4}^{\frac{1}{3}}(x)$	$\mathbf{0}_{L_4}(x)$	$d_{(\frac{2}{3},0)}(x)$	$\mathrm{Id}_{L_4}^{0}(x)$
0	0	0	0	0	0	0	0	0	0
$1/3$	$1/3$	0	$1/3$	0	0	$1/3$	0	0	$1/3$
$2/3$	$2/3$	$1/3$	$2/3$	0	$1/3$	$2/3$	0	$1/3$	$2/3$
1	1	$2/3$	$2/3$	$1/3$	$1/3$	$1/3$	0	0	0

Definition 2.7. Let A be an MV-algebra, $k \in \mathbb{N}_+$, and $D_k = (d_n)_{n=0}^k$ be a finite sequence of operators of A such that $d_0 = \mathrm{Id}_A$. Then D_k is called an $(\odot, \vee)$-higher derivation of length k on A if for every $n \leq k$ and $x, y \in A$, we have

$$d_n(x \odot y) = \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). \tag{2.2}$$

Let $\mathrm{HD}_k(A)$ denote the set of all such derivations.

Example 2.4. 1) The identity and zero $(\odot, \vee)$-higher derivation of length $k \in \mathbb{N}_+$ on A are respectively defined by

$$D_k^{\mathrm{Id}_A} = (d_n)_{n=0}^k, \quad \text{where } d_n = \mathrm{Id}_A \text{ for } 0 \leq n \leq k,$$
$$D_k^{\mathbf{0}_A} = (d_n)_{n=0}^k, \quad \text{where } d_0 = \mathrm{Id}_A \text{ and } d_n = \mathbf{0}_A \text{ for } 1 \leq n \leq k.$$

2) We list all $(\odot, \vee)$-higher derivations of length 2 on L_3 in Table 4.

Table 4. $\mathrm{HD}_2(L_3) = \{(d_0, d_1, d_2) \mid d_0 = \mathrm{Id}_{L_3}$, and d_1, d_2 satisfy Eq (2.2).

	$[2d(0)][2d(\frac{1}{2})][2d(1)]$																				
d_0	012																				
d_1	000					001					010					011					012
d_2	000	001	010	011	012	000	001	010	011	012	000	001	010	011	012	000	001	010	011	012	012

Proposition 2.8. *Let A be an MV-algebra, $k \in \mathbb{N}_+$, and $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$. Then the following statements hold for all $x, y \in A$:*

1) $d_n(0) = 0$, for any $n \leq k$;

2) $d_i(x) \odot d_{n-i}(x^*) = 0$, for any $n \leq k$ and $i \leq n$. In particular, $d_n(x) \leq x$;

3) $d_i(x) \odot d_{n-i}(1) \leq d_n(x)$, for any $n \leq k$ and $i \leq n$. In particular, $x \odot d_n(1) \leq d_n(x)$;

4) If $d_1 = \mathrm{Id}_A$, then $d_n = \mathrm{Id}_A$ for all $2 \leq n \leq k$.

Proof. (1) We prove $d_n(0) = 0$ for any $n \leq k$ by induction on n. First, if $n = 0$, then $d_0(0) = 0$.

Assume $d_l(0) = 0$ for all $0 < l < n$. Putting $x = y = 0$ in Eq (2.2), we get

$$d_n(0) = d_n(0 \odot 0) = \bigvee_{i=0}^{n} (d_i(0) \odot d_{n-i}(0)) = (0 \odot d_n(0)) \vee (d_n(0) \odot 0) \vee \bigvee_{i=1}^{n-1} (d_i(0) \odot d_{n-i}(0)) = 0,$$

since $d_i(0) \odot d_{n-i}(0) = 0$ for all $1 \leq i \leq n-1$. Hence, $d_n(0) = 0$ for any $n \leq k$.

(2) Since $x \odot x^* = 0$, it follows that for any $n \leq k$,

$$0 = d_n(0) = d_n(x \odot x^*) = \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(x^*))$$

by (1) and Eq. (2.2). Thus $d_i(x) \odot d_{n-i}(x^*) = 0$, for all $i \leq n$. In particular, we have $d_n(x) \odot x^* = 0$, which implies $d_n(x) \leq x$.

(3) Since $x \odot 1 = x$, it follows from Eq (2.2) that for any $n \leq k$,

$$d_n(x) = d_n(x \odot 1) = \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(1)).$$

Thus $d_i(x) \odot d_{n-i}(1) \leq d_n(x)$ for all $i \leq n$. In particular, when $i = 0$, we get $x \odot d_n(1) \leq d_n(x)$.

(4) Assume $d_1 = \mathrm{Id}_A$; we have $d_1(x) = x$ for any $x \in A$.

We prove the conclusion by induction; when $n = 2$, we get

$$\begin{aligned} d_2(x) &= d_2(x \odot 1) \\ &= (x \odot d_2(1)) \vee (d_2(x) \odot 1) \vee (d_1(x) \odot d_1(1)) && \text{(Eq. (2.2))} \\ &= (x \odot d_2(1)) \vee d_2(x) \vee x && (d_1 = \mathrm{Id}_A) \\ &= x. && \text{(Proposition 2.8 (2) and (3))} \end{aligned}$$

Now, assume $d_l = \mathrm{Id}_A$ for all $l \leq n$. It follows that

$$\begin{aligned} d_{n+1}(x) &= d_{n+1}(x \odot 1) \\ &= \bigvee_{i=0}^{n+1} (d_i(x) \odot d_{n+1-i}(1)) && \text{(Eq. (2.2))} \end{aligned}$$

$$= (x \odot d_{n+1}(1)) \vee (d_{n+1}(x) \odot 1) \vee \bigvee_{i=1}^{n} (d_i(x) \odot d_{n+1-i}(1))$$
$$= (x \odot d_{n+1}(1)) \vee d_{n+1}(x) \vee x \quad \text{(Assumptions)}$$
$$= x. \quad \text{(Proposition 2.8 (2) and (3))}$$

Hence, $d_n = \mathrm{Id}_A$ for all $n \geq 2$. □

Notation summary: To enhance readability and avoid potential confusion in heavy notation, we provide the following summary of frequently used symbols and their meanings; see Table 5.

Table 5. Summary of notations used throughout the paper.

Notation	Meaning
A	An MV-algebra
$\mathbb{N}_+$	Set of positive integers
d	A single $(\odot, \vee)$-derivation on A
d^n	The n-fold composition of d with itself ($d \circ d \circ \cdots \circ d$, n times)
d_n	The n-th component of an $(\odot, \vee)$-higher derivation sequence $D_k = (d_0, d_1, \ldots, d_k)$
$d_{a\odot}$	The principal $(\odot, \vee)$-derivation defined by $d_{a\odot}(x) = a \odot x$
$d_{(a,b)}$	An $(\odot, \vee)$-derivation on L_m with parameters a, b (see Lemma 2.6)
d^u	The $(\odot, \vee)$-derivation obtained by modifying d at $x = 1$: $d^u(x) = \begin{cases} u, & \text{if } x = 1 \\ d(x), & \text{otherwise} \end{cases}$ (for $u \leq d(1)$)
$d^b_{a\odot}$	The $(\odot, \vee)$-derivation defined as $d^b_{a\odot}(x) = \begin{cases} b, & \text{if } x = 1 \\ a \odot x, & \text{otherwise} \end{cases}$ (for $b \leq a$)
D_k	An $(\odot, \vee)$-higher derivation of length k, i.e., a sequence $(d_0, d_1, \ldots, d_k)$
$D_{\{k,a\}}$	A principal $(\odot, \vee)$-higher derivation of length k generated by a
$\mathrm{Der}(A)$	The set of all $(\odot, \vee)$-derivations on A
$\mathrm{HD}_k(A)$	The set of all $(\odot, \vee)$-higher derivations of length k on A
$\mathbf{B}(A)$	The set of all idempotent elements of A
$\mathrm{Fix}_{D_k}(A)$	The set of all fixed points of the $(\odot, \vee)$-higher derivation D_k

3. Power, principal, and isotone $(\odot, \vee)$-higher derivations on MV-algebras

3.1. Definitions of the three type $(\odot, \vee)$-higher derivations

Let us get started with the definition of a power $(\odot, \vee)$-higher derivation.

Definition 3.1. Let A be an MV-algebra and $k \in \mathbb{N}_+$. An $(\odot, \vee)$-higher derivation $D_k = (d_n)_{n=0}^k$ of length k is power if $d_n = d_1^n$ for any $1 \leq n \leq k$. Here, d_1 is called the generator of D_k.

Remark 3.2. 1) Clearly, for any $k \in \mathbb{N}_+$, the identity and zero $(\odot, \vee)$-higher derivation of length k (cf. Example 2.4), $D_k^{\mathrm{Id}_A}$ and $D_k^{\mathbf{0}_A}$ are power.

2) We will construct explicitly some power $(\odot, \vee)$-higher derivations on MV-chains C and L_m in Theorems 4.1 and 4.2. However, by the counterexample in Remark 4.3, we will see that not all $(\odot, \vee)$-higher derivations on an MV-algebra are power.

Let A be an MV-algebra, $k \in \mathbb{N}_+$, and $D_k = (d_n)_{n=0}^k$ be an operator sequence of A. Recall that D_k is decreasing on n if $n_1, n_2 \in \mathbb{N}_+$ and $n_1 \le n_2 \le k$, then $d_{n_2}(x) \le d_{n_1}(x)$ for any $x \in A$.

Proposition 3.3. *Let A be an MV-algebra and $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$ be power. Then the following statements hold for all $x, y \in A$:*

1) D_k is decreasing on n;

2) $d_n(x) \odot d_n(y) \le d_n(x \odot y)$, for all $n \le k$;

3) $(d_n(x))^m \le d_n(x^m)$, for all $m \ge 1$ and $n \le k$;

4) If there exists $m \le k$ such that $d_m(x) = x$, then $d_m(y) = y$ for any $y \le x$.

Proof. 1) Since $d_1 \in \mathrm{Der}(A)$, we know $d_1(x) \le x$ for all $x \in A$ by Proposition 2.5 (2). Definition 3.1 of power $(\odot, \vee)$-higher derivation implies $d_n(x) = d_1(d_{n-1}(x))$ for any $1 \le n \le k$, so $d_n(x) \le d_{n-1}(x)$ for any $1 \le n \le k$ and $x \in A$. Hence, D_k is decreasing on n.

2) Taking a certain $n \le k$. By Proposition 3.3 (1), it is easy to see that for all $i \le n$,

$$d_n(x) \le d_i(x), d_n(y) \le d_{n-i}(y).$$

Hence, by Lemma 2.1 (3) for each $i \le n$,

$$d_n(x) \odot d_n(y) \le d_i(x) \odot d_{n-i}(y).$$

Finally, we get $d_n(x) \odot d_n(y) \le \bigvee_{i=0}^n (d_i(x) \odot d_{n-i}(y)) = d_n(x \odot y)$ by Lemma 2.1 (1) and Eq. (2.2). Due to the arbitrariness of n selection, $d_n(x) \odot d_n(y) \le d_n(x \odot y)$ for all $n \le k$.

3) For every $n \le k$, we prove it by induction on m. First, if $m = 1$, it is clear that $(d_n(x))^1 = d_n(x) = d_n(x^1)$. Now, assume that $(d_n(x))^m \le d_n(x^m)$; we need to show the case of $m+1$. By Lemma 2.1 (3), we have

$$(d_n(x))^{m+1} = (d_n(x))^m \odot d_n(x) \le d_n(x^m) \odot d_n(x).$$

According to Proposition 3.3 (2), we know $d_n(x^m) \odot d_n(x) \le d_n(x^m \odot x) = d_n(x^{m+1})$. Hence, $(d_n(x))^{m+1} \le d_n(x^{m+1})$ for all $m \ge 1$ and $n \le k$.

4) Assume $d_m(x) = x$ for $m \le k$. Since $y \le x$, it follows that

$$\begin{aligned}
d_m(y) &= d_m(x \wedge y)\\
&= d_m(x \odot (x^* \oplus y))\\
&= (d_m(x) \odot (x^* \oplus y)) \vee (x \odot d_m(x^* \oplus y)) \vee \bigvee_{i=1}^{m-1} (d_i(x) \odot d_{m-i}(x^* \oplus y)) && \text{(Eq. (2.2))}\\
&= (x \odot (x^* \oplus y)) \vee (x \odot d_m(x^* \oplus y)) \vee \bigvee_{i=1}^{m-1} (d_i(x) \odot d_{m-i}(x^* \oplus y)) && (d_m(x) = x)
\end{aligned}$$

$$= x \odot (x^* \oplus y) \qquad \text{(Proposition 3.3 (1))}$$
$$= x \wedge y$$
$$= y. \qquad \square$$

We next introduce the principal and isotone $(\odot, \vee)$-higher derivations.

Definition 3.4. Let A be an MV-algebra, $a \in A$, and $k \in \mathbb{N}_+$. Define $D_{\{k,a\}} = (d_{a^n\odot})_{n=0}^k$ by $d_{a^n\odot}(x) = a^n \odot x$, for all $n \leq k$ and $x \in A$. Let $x, y \in A$. Then,

$$d_{a^n\odot}(x \odot y) = a^n \odot x \odot y = \bigvee_{i=0}^{n} (a^i \odot x \odot a^{n-i} \odot y) = \bigvee_{i=0}^{n} (d_{a^i\odot}(x) \odot d_{a^{n-i}\odot}(y)),$$

which implies that $D_{\{k,a\}} = (d_{a^n\odot})_{n=0}^k \in \mathrm{HD}_k(A)$. We call $D_{\{k,a\}}$ a principal $(\odot, \vee)$-higher derivation of length k on A.

Definition 3.5. An $(\odot, \vee)$-higher derivation $D_k = (d_n)_{n=0}^k$ of length k on A is isotone if, for any $n \leq k$, d_n is isotone. That is, $x \leq y$ implies $d_n(x) \leq d_n(y)$ for all $n \leq k$ and $x, y \in A$.

3.2. The relationship between principal and power $(\odot, \vee)$*-higher derivations*

We have mentioned in Remark 3.2 that $D_k^{\mathrm{Id}_A}$ and $D_k^{\mathbf{0}_A}$ are power. Clearly, both $D_k^{\mathrm{Id}_A}$ and $D_k^{\mathbf{0}_A}$ are also principal. We note that a principal $(\odot, \vee)$-higher derivation is power in general.

Proposition 3.6. *Let A be an MV-algebra. Then all principal* $(\odot, \vee)$*-higher derivations on A are power.*

Proof. Without loss of generality, assume $a \in A$, and $D_{\{k,a\}} = (d_{a^n\odot})_{n=0}^k$ is an arbitrary principal $(\odot, \vee)$-higher derivation of length k on A.

For any $n \leq k$ and $x \in A$, we have $d_{a^n\odot}(x) = a^n \odot x$ by the definition of principal $(\odot, \vee)$-higher derivations.

Also, $(d_{a\odot}^n)(x) = d_{a\odot}^{n-1}(d_{a\odot}(x)) = d_{a\odot}^{n-1}(a \odot x) = a^n \odot x$ for any $1 \leq n \leq k$ and $x \in A$.

Hence, $d_{a^n\odot} = d_{a\odot}^n$, which indicates the generator of $D_{\{k,a\}}$ is $d_{a\odot}$. Thus $D_{\{k,a\}}$ is power. $\square$

Remark 3.7. Conversely, a power $(\odot, \vee)$-higher derivation $D_k = (d_n)_{n=0}^k$ of length k on A may not be principal, and a counterexample that is more general than might be anticipated will be constructed in Remark 3.9. In fact, we note here in advance that the interested reader will get a clear relationship picture among principal, power, and isotone $(\odot, \vee)$-higher derivations, as shown in Figure 1 after Remark 3.14.

3.3. The relationship between principal and isotone $(\odot, \vee)$*-higher derivations*

Propositions 3.6 and 3.8 establish that a principal $(\odot, \vee)$-higher derivation is both power and isotone.

Proposition 3.8. *Let A be an MV-algebra. Then any principal* $(\odot, \vee)$*-higher derivation is isotone.*

Proof. Without loss of generality, assume $a \in A$, and $D_{\{k,a\}} = (d_{a^n\odot})_{n=0}^k$ is an arbitrary principal $(\odot, \vee)$-higher derivation of length k on A.

Let $x \leq y$. Then $a^n \odot x \leq a^n \odot y$ for any $n \leq k$ by Lemma 2.1 (3) and so $d_{a^n\odot}(x) \leq d_{a^n\odot}(y)$ for any $n \leq k$. Thus $D_{\{k,a\}}$ is isotone. $\square$

However, an isotone $(\odot, \vee)$-higher derivation is not always principal. Example 3.1 below shows that there are isotone $(\odot, \vee)$-higher derivations on an MV-algebra A other than the principal ones.

Example 3.1. Let $D_2 = (d_n)_{n=0}^2$ with $d_0 = 012, d_1 = 001, d_2 = 011$ be an $(\odot, \vee)$-higher derivation on L_3 in Table 4. Then D_2 is isotone, while D_2 is not principal, since $d_2(1) = \frac{1}{2} = \frac{1}{2} \odot 1$, but $d_2(\frac{1}{2}) = \frac{1}{2} \neq 0 = \frac{1}{2} \odot \frac{1}{2}$.

Remark 3.9. Here we construct a promised counterexample to show that a power isotone $(\odot, \vee)$-higher derivation is not principal. On L_4, let $d = 0122$ be Example 3.1 in [28] (as $\mathrm{Id}_{L_4}^{\frac{2}{3}}$ in Table 3) and $d_n = d^n = d$. It is clear that $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(L_4)$ is power and isotone, but $d_2(1) = \frac{2}{3} \neq a^2 = a^2 \odot 1$ for any $a \in L_4$, hence it is not principal.

Theorem 3.12 will tell us that if $D_k = (d_n)_{n=0}^k$ is a power $(\odot, \vee)$-higher derivation on an MV-algebra A with $d_1(1) \in \mathbf{B}(A)$, then D_k is isotone iff D_k is principal.

3.4. The relationship between power and isotone $(\odot, \vee)$-higher derivations

Remarks 3.10 and 3.14 illustrate that the power and isotone cannot imply each other for $(\odot, \vee)$-higher derivations.

Remark 3.10. A power $(\odot, \vee)$-higher derivation may not be isotone. For example, let $D_2 = (d_n)_{n=0}^2$ with $d_0 = 012, d_1 = 010$, and $d_2 = 010$ on L_3 as in Table 4; we can verify that $d_2 = d_1^2$. Then D_2 is a power $(\odot, \vee)$-higher derivation but is not isotone. Indeed, $\frac{1}{2} \leq 1$ but $d_2(\frac{1}{2}) = \frac{1}{2} \geq 0 = d_2(1)$.

In what condition will a power $(\odot, \vee)$-higher derivation be isotone? Proposition 3.11 gives a sufficient condition.

Proposition 3.11. *Let $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$ be power. If for some m, $d_m(1) = 1$, then D_k is the identity $(\odot, \vee)$-higher derivation, and thus it is isotone.*

Proof. By Proposition 3.3 (4), we know that $d_m(1) = 1$ implies $d_m(x) = x$ for all $x \in A$. By Proposition 3.3 (1), for all $i \leq m$, $d_m(x) \leq d_i(x)$, it implies $d_i(x) = x$. Furthermore, $d_i = \mathrm{Id}_A$ for $m + 1 \leq i \leq k$ by Proposition 2.8 (4). Thus we conclude that D_k is the identity $(\odot, \vee)$-higher derivation, so is isotone naturally. □

Moreover, we give 5 equivalent characterizations of a power $(\odot, \vee)$-higher derivation to be isotone.

Theorem 3.12. *Let $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$ be power and $d_1(1) \in \mathbf{B}(A)$. The following statements are equivalent for all $x, y \in A$ and $n \leq k$:*

1) D_k is isotone;

2) $d_n(x) \leq d_n(1)$;

3) $d_n(x) = d_n(1) \odot x = (d_1(1))^n \odot x$;

4) $d_n(x \wedge y) = d_n(x) \wedge d_n(y)$;

5) $d_n(x \vee y) = d_n(x) \vee d_n(y)$;

6) $d_n(x \odot y) = d_n(x) \odot d_n(y)$.

We need Lemma 3.13 to prepare for the proof of Theorem 3.12.

Lemma 3.13. *Let $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$ be power. If $d_m(1) \in \mathbf{B}(A)$ for some $m \in \mathbb{N}_+$, then $d_m(d_m(1)) = d_m(1)$.*

Proof. Let $d_m(1) \in \mathbf{B}(A)$. Then $d_m(d_m(1)) \le d_m(1)$ by Proposition 3.3 (1). Also, $d_m(1) \in \mathbf{B}(A)$ implies that $d_m(1) \odot d_m(1) = d_m(1)$, and so,

$$\begin{aligned} d_m(d_m(1)) &= d_m(1 \odot d_m(1)) \\ &= (d_m(1) \odot d_m(1)) \vee (1 \odot d_m(d_m(1))) \vee \bigvee_{i=1}^{m-1} (d_i(1) \odot d_{m-i}(1)) \\ &= d_m(1) \vee d_m(d_m(1)) \vee \bigvee_{i=1}^{m-1} (d_i(1) \odot d_{m-i}(1)), \end{aligned}$$

then we get that $d_m(1) \le d_m(d_m(1))$. Hence, $d_m(d_m(1)) = d_m(1)$. □

We next prove Theorem 3.12 by demonstrating the following cyclical implication chains: (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1), (1) ⇒ (2) ⇒ (3) ⇒ (5) ⇒ (1), and (2) ⇒ (3) ⇒ (6) ⇒ (2).

Proof of Theorem 3.12. By Lemma 3.13, $d_1(1) \in \mathbf{B}(A)$ implies $d_n(1) = d_1(1) \in \mathbf{B}(A)$ for any $1 \le n \le k$.

(1) ⇒ (2): By the definition of isotone $(\odot, \vee)$-higher derivations.

(2) ⇒ (3): Assume $d_n(x) \le d_n(1)$. Proposition 3.3 (1) induces $d_n(x) \le x$; it follows that

$$\begin{aligned} d_n(x) &= d_n(1) \wedge d_n(x) && (d_n(x) \le d_n(1)) \\ &= d_n(1) \odot d_n(x) && \text{(Lemma 2.2 (4))} \\ &\le d_n(1) \odot x. && \text{(Lemma 2.1 (3))} \end{aligned}$$

Proposition 2.8 (3) gives $d_n(1) \odot x \le d_n(x)$. Thus, $d_n(x) = d_n(1) \odot x$. Furthermore, $d_n(1) = d_1(1) = (d_1(1))^n$ by Lemma 2.2 (3).

(3) ⇒ (4): Assume $d_n(x) = d_n(1) \odot x$. By Lemma 2.1 (4), we have

$$d_n(x \wedge y) = d_n(1) \odot (x \wedge y) = (d_n(1) \odot x) \wedge (d_n(1) \odot y) = d_n(x) \wedge d_n(y).$$

(4) ⇒ (1): If $x \le y$, then

$$d_n(x) = d_n(x \wedge y) = d_n(x) \wedge d_n(y).$$

Hence, $d_n(x) \le d_n(y)$.

(3) ⇒ (5): Assume $d_n(x) = d_n(1) \odot x$. By Lemma 2.1 (5), we have

$$d_n(x \vee y) = d_n(1) \odot (x \vee y) = (d_n(1) \odot x) \vee (d_n(1) \odot y) = d_n(x) \vee d_n(y).$$

(5) ⇒ (1): If $x \le y$, then

$$d_n(x) \le d_n(x) \vee d_n(y) = d_n(x \vee y) = d_n(y).$$

Thus, $d_n(x) \leq d_n(y)$.

(3) ⇒ (6): Since $d_n(1) \in \mathbf{B}(A)$, we know $d_n(1) \odot d_n(1) = d_n(1)$, it follows that

$$\begin{aligned} d_n(x \odot y) &= d_n(1) \odot (x \odot y) \\ &= d_n(1) \odot d_n(1) \odot (x \odot y) \\ &= (d_n(1) \odot x) \odot (d_n(1) \odot y) \\ &= d_n(x) \odot d_n(y). \end{aligned}$$

(6) ⇒ (2): Since $d_n(1) \in \mathbf{B}(A)$, it follows that

$$d_n(x) = d_n(x \odot 1) = d_n(x) \odot d_n(1) = d_n(x) \wedge d_n(1)$$

by Lemma 2.2 (4). Hence, $d_n(x) \leq d_n(1)$. □

The equivalences established in Theorem 3.12 reveal that for a power $(\odot, \vee)$-higher derivation with idempotent generator value $d_1(1)$, the isotone property is equivalent to the derivation behaving as a "localization" or "restriction" to the principal ideal generated by $d_1(1)$. Specifically, condition (3) $d_n(x) = (d_1(1))^n \odot x$ shows that each d_n acts as multiplication by the idempotent element $(d_1(1))^n$, which naturally preserves order. Conditions (4)–(6) further demonstrate that such derivations respect the lattice operations and the MV-algebraic product in a well-behaved manner. This provides a clear structural characterization of when power derivations preserve the natural order of the MV-algebra.

Remark 3.14. Note that $D_2 = (d_n)_{n=0}^2$ with $d_0 = 012, d_1 = 001, d_2 = 011$ on L_3, as in Table 4, is an example of isotone $(\odot, \vee)$-higher derivations but not power since $d_2 \neq d_1 \circ d_1$. Now, we have finished the following Figure 1.

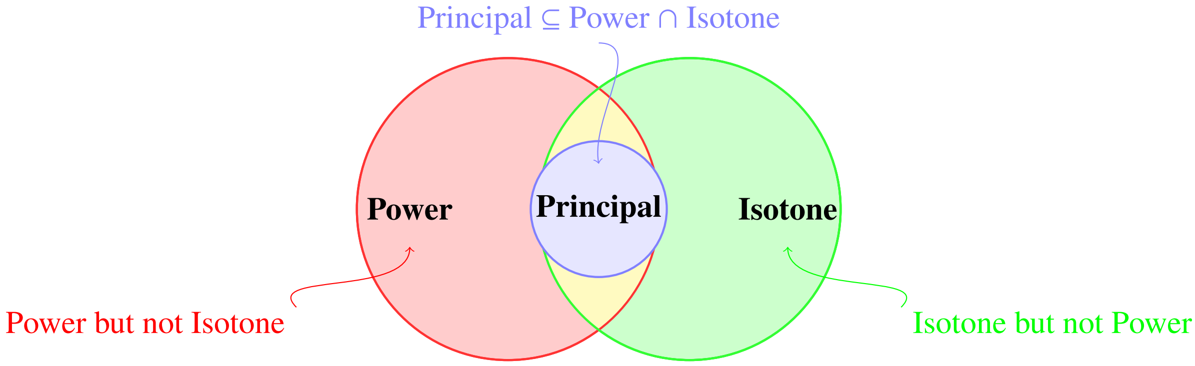

Figure 1. Interrelations among the three $(\odot, \vee)$-higher derivations.

4. Power $(\odot, \vee)$-higher derivations on two typical MV-chains

In this section, we construct explicitly power $(\odot, \vee)$-higher derivations on two typical MV-chains C and L_m.

Theorem 4.1. *Let C be the infinite MV-algebra in Example 2.2 and $k \in \mathbb{N}_+$. Let $a, b \in C$ with $b \leq a$. Define*

$$d_1(x) = \begin{cases} b \odot x, & \text{if } x \in C_1; \\ a \odot x, & \text{if } x \in C_0, \end{cases} \quad \text{and} \quad d_n(x) = d_1^n(x) = \begin{cases} b^n \odot x, & \text{if } x \in C_1; \\ a^n \odot x, & \text{if } x \in C_0. \end{cases}$$

for any $n \leq k$. Then $D_k = (d_n)_{n=0}^k$ is a power $(\odot, \vee)$-higher derivation on C.

Proof. Let $x, y \in C$. It suffices to prove that $D_k \in \mathrm{HD}_k(C)$ since $d_n = d_1^n$ for any $n \leq k$. Consider the following four cases:

Case 1: $x, y \in C_0$. Then,

$$\begin{aligned} d_n(x \odot y) &= a^n \odot (x \odot y) \\ &= \bigvee_{i=0}^{n} (a^i \odot x) \odot (a^{n-i} \odot y) \\ &= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). \end{aligned}$$

Case 2: $x, y \in C_1$. Then

$$\begin{aligned} d_n(x \odot y) &= b^n \odot (x \odot y) \\ &= \bigvee_{i=0}^{n} (b^i \odot x) \odot (b^{n-i} \odot y) \\ &= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). \end{aligned}$$

Case 3: $x \in C_1, y \in C_0$; let $x = (uc)^*, y = vc$, where $u, v \in \mathbb{N}_+$.
If $u \geq v$, then $x \odot y = 0$, and

$$\begin{aligned} d_n(x \odot y) &= d(0) = 0 = \bigvee_{i=0}^{n} 0 \\ &= \bigvee_{i=0}^{n} b^i \odot a^{n-i} \odot (x \odot y) \\ &= \bigvee_{i=0}^{n} (b^i \odot x) \odot (a^{n-i} \odot y) \\ &= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). \end{aligned}$$

If $u < v$, then

$$\begin{aligned} d_n(x \odot y) &= a^n \odot (x \odot y) \\ &= b^0 \odot x \odot a^n \odot y \\ &= \bigvee_{i=0}^{n} (b^i \odot x) \odot (a^{n-i} \odot y) \qquad (b \leq a) \end{aligned}$$

$$= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)).$$

Case 4: $x \in C_0, y \in C_1$. Symmetry of Case 3, we can show that $d_n(x \odot y) = \bigvee_{i=0}^{n}(d_i(x) \odot d_{n-i}(y))$.

Summarizing the above arguments, D_k is a power $(\odot, \vee)$-higher derivation on C. □

Theorem 4.1 constructs some power $(\odot, \vee)$-higher derivation on a typical non-archimedean infinite MV-chain C. Next, we consider the construction of an archimedean finite MV-chain, which gives a method to construct some $(\odot, \vee)$-higher derivations on a general MV-algebra in Corollary 4.4.

Theorem 4.2. *Let $d \in \mathrm{Der}(L_m)$ and $k \in \mathbb{N}_+$. Then $D_k = (d_n)_{n=0}^{k} \in \mathrm{HD}_k(L_m)$, where $d_n = d^n$ for any $n \leq k$.*

Proof. Recall that

$$\mathrm{Der}(L_m) = \{d_{(a,b)} \mid (a, b) \in \mathcal{A}(L_m)\},$$

where

$$\mathcal{A}(L_m) = \{(a, b) \in L_m \times L_m \mid b \leq a\}\backslash\{(0, 0)\}.$$

Assume $(a, b) \in \mathcal{A}(L_m)$ and $d = d_{(a,b)} \in \mathrm{Der}(L_m)$. In the case of $n = 0$, we know $d_0 = \mathrm{Id}_{L_m}$, then $d_0(x \odot y) = x \odot y = d_0(x) \odot d_0(y)$, so Eq. (2.2) holds.

Hence, it is enough to verify the cases $1 \leq n \leq k$. Since $d_n = d^n$, we can get d_n with

$$d_n(x) = \begin{cases} a^{n-1} \odot b, & \text{if } x = 1; \\ a^n \odot x, & \text{otherwise.} \end{cases} \tag{4.1}$$

There are only three cases:

Case 1. If $x = 1$ and $y = 1$, then we have

$$\begin{aligned}
d_n(x \odot y) &= a^{n-1} \odot b && \text{(Eq (4.1))}\\
&= \left(a^{n-1} \odot b\right) \vee \left(a^{n-1} \odot b\right) \vee \bigvee_{i=1}^{n-1} \left(a^{n-2} \odot b^2\right) && (b \leq a)\\
&= \left(a^{n-1} \odot b \odot y\right) \vee \left(x \odot a^{n-1} \odot b\right) \vee \bigvee_{i=1}^{n-1} \left(a^{i-1} \odot b \odot a^{n-i-1} \odot b\right) && (x = y = 1)\\
&= (d_n(x) \odot d_0(y)) \vee (d_0(x) \odot d_n(y)) \vee \bigvee_{i=1}^{n-1} (d_i(x) \odot d_{n-i}(y)) && \text{(Eq (4.1))}\\
&= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). && \text{(right side of Eq (2.2))}
\end{aligned}$$

Case 2. If $x = 1$ or $y = 1$ (but not both), say $x \neq 1$ and $y = 1$, then

$$\begin{aligned}
d_n(x \odot y) &= d_n(x) = a^n \odot x && \text{(Eq (4.1))}\\
&= (a^n \odot x) \vee \left(a^{n-1} \odot b \odot x\right) \vee \bigvee_{i=1}^{n-1} \left(a^{n-1} \odot b \odot x\right) && (b \leq a)
\end{aligned}$$

$$= (a^n \odot x \odot y) \vee \left(x \odot a^{n-1} \odot b\right) \vee \bigvee_{i=1}^{n-1} \left(a^i \odot x \odot a^{n-i-1} \odot b\right) \qquad (y = 1)$$

$$= (d_n(x) \odot d_0(y) \vee (d_0(x) \odot d_n(y) \vee \bigvee_{i=1}^{n-1} (d_i(x) \odot d_{n-i}(y)) \qquad \text{(Eq (4.1))}$$

$$= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). \qquad \text{(right side of Eq (2.2))}$$

Case 3. If $x < 1, y < 1$, then

$$d_n(x \odot y) = a^n \odot x \odot y \qquad \text{(Eq (4.1))}$$

$$= (a^n \odot x \odot y) \vee (x \odot a^n \odot y) \vee \bigvee_{i=0}^{n-1} \left(a^i \odot x \odot a^{n-i} \odot y\right)$$

$$= (d_n(x) \odot d_0(y) \vee (d_0(x) \odot d_n(y) \vee \bigvee_{i=1}^{n-1} (d_i(x) \odot d_{n-i}(y)) \qquad \text{(Eq (4.1))}$$

$$= \bigvee_{i=0}^{n} (d_i(x) \odot d_{n-i}(y)). \qquad \text{(right side of Eq (2.2))}$$

Thus, we conclude that $D_k \in \mathrm{HD}_k(L_m)$. □

Remark 4.3. The converse of Theorem 4.2 is not true. That is, not all $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(L_m)$ can be expressed as $d_n = d^n$ for some $d \in \mathrm{Der}(L_m)$. For example, consider $D_2 = (d_n)_{n=0}^2$ with $d_0 = 012, d_1 = 000, d_2 = 001$ on L_3 as in Table 4, $D_2 \in \mathrm{HD}_2(L_3)$ and $d_1 \in \mathrm{Der}(L_3)$, but $d_2 = 001 \neq 000 = d_1 \circ d_1$.

By observing the proof of Theorem 4.2, we find that the proof is only related to whether the values of x, y are 1 or less than 1. Hence, we wonder if the conclusion holds on any MV-algebra? We will show below that the conclusion does hold for general MV-algebras.

Let $d_{a\odot}$ be a principal $(\odot, \vee)$-derivation on A and $b \leq a$. It is known that $d_{a\odot}^b$ is an $(\odot, \vee)$-derivation on A by Lemma 2.6. Note that $d_{a\odot}^b$ on A is same as $d_{(a,b)}$ on L_m except for the range of values for a, b which is A and L_m, respectively.

Corollary 4.4. *Let $a, b \in A$ with $b \leq a$. Let $d_{a\odot}$ be a principal $(\odot, \vee)$-derivation on A and $k \in \mathbb{N}_+$. Then $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$, where $d_n = (d_{a\odot}^b)^n$ for any $n \leq k$.*

Proof. The proof is similar to that of Theorem 4.2. □

To prove Proposition 4.6, we need first to show that $d^n \in \mathrm{Der}(L_m)$ for every $d \in \mathrm{Der}(L_m)$.

Lemma 4.5. $(\mathrm{Der}(L_m), \circ)$ *is a semigroup.*

Proof. Let $d, d' \in \mathrm{Der}(L_m)$. Since the associative law is guaranteed by composite operations of mappings, it is sufficient to prove that $d \circ d' \in \mathrm{Der}(L_m)$.

There exist $b \leq a \leq 1, f \leq e \leq 1$ such that $d = d_{(a,b)}, d' = d_{(e,f)}$. There are only two cases:

Case 1. $f < 1$. We have

$$d \circ d'(x) = \begin{cases} a \odot f, & \text{if } x = 1; \\ a \odot e \odot x, & \text{otherwise.} \end{cases}$$

Since $f \leq e$ and $\odot$ is order-preserving (Lemma 2.1 (3)), $a \odot f \leq a \odot e$. Hence, $d \circ d' \in \mathrm{Der}(L_m)$.

Case 2. $f = 1$. In this case, $f \leq e$ implies $e = 1$. Hence,

$$d \circ d'(x) = \begin{cases} b, & \text{if } x = 1; \\ a \odot x, & \text{otherwise.} \end{cases}$$

Then $d \circ d' = d \in \mathrm{Der}(L_m)$.

Therefore, $(\mathrm{Der}(L_m), \circ)$ is a semigroup. □

It is clear that the subsemigroup generated by d of $\mathrm{Der}(L_m)$ is finite since $\mathrm{Der}(L_m)$ is finite. We next introduce the period and index of a generator d of $D_k = (d^n)_{n=0}^k$ on L_m.

As noted in [13], the index and period of d can be defined based on the repetitions that occur among its compositions. We define the index of d, denoted by l, as the least element of non-empty set

$$\{x \in \mathbb{N} \mid (\exists y \in \mathbb{N}) \;\; d^x = d^y, x \neq y\},$$

and the period of d, denoted by r, as the least element of the non-empty set

$$\left\{x \in \mathbb{N} \mid d^{l+x} = d^l\right\}.$$

Thus, there exist index l and period r such that

$$d^l = d^{l+r},$$

for any $d \in \mathrm{Der}(L_m)$.

Now, let us calculate l and r of d.

Proposition 4.6. *Let $d = d_{(a,b)} \in \mathrm{Der}(L_m)$, where $a = \frac{i}{m-1}, b = \frac{j}{m-1}, 0 \leq j \leq i \leq m-1$. Let $k \in \mathbb{N}_+$ and $D_k = (d_n)_{n=0}^k$ by $d_n = d^n$ for any $0 \leq n \leq k$. Then the period r of d is 1 and the index l of d is*

$$l = \begin{cases} 1, & \text{if } \; a = 1; \\ i, & \text{if } \; b < a < 1; \\ i+1, & \text{if } \; b = a < 1. \end{cases}$$

Proof. The cases $d = \mathrm{Id}_{L_m}$ or $d = \mathbf{0}_{L_m}$ are clear. We have known that $b \leq a$, so there are only two cases according to the value of a.

Case 1. If $a = 1$ (i.e., $i = m-1$), then

$$d(x) = \begin{cases} b, & \text{if } \; x = 1; \\ x, & \text{otherwise,} \end{cases} \quad \text{and} \quad d_n(x) = \begin{cases} b, & \text{if } x = 1; \\ x, & \text{otherwise.} \end{cases}$$

This shows that $d^n = d^1$ for all $n \geq 1$. Therefore, the sequence stabilizes immediately after $n = 1$, giving index $l = 1$ and period $r = 1$.

Case 2. If $a < 1$ (i.e., $i < m-1$), then

$$d(x) = \begin{cases} b, & \text{if } x = 1; \\ a \odot x, & \text{otherwise,} \end{cases} \quad \text{and} \quad d_n(x) = \begin{cases} a^{n-1} \odot b, & \text{if } x = 1; \\ a^n \odot x, & \text{otherwise.} \end{cases}$$

In this case, $d(x) < x$, and so d must be nilpotent. Denote $M = \left\{n \in \mathbb{N} \mid a^{n-1} \odot b = 0\right\}$ and $N = \left\{n \in \mathbb{N} \mid a^n \odot \frac{m-2}{m-1} = 0\right\}$. To explore the index, we consider the following two cases:

Subcase 2.1: $b < a$ (i.e., $j < i$).

M and N represent the nilpotency indices for the sequences at $x = 1$ and at the largest non-unit element $x = \frac{m-2}{m-1}$, respectively. Since $b < a$, we have $N \subseteq M$. The minimal element of N is i, because $a^i = (\frac{i}{m-1})^i = 0$ when multiplied by any element less than 1. For $n \geq i$, we have $d^n(x) = 0$ for all $x < 1$, and $d^n(1) = a^{n-1} \odot b = 0$ (since $n - 1 \geq i - 1 \geq j$ and $b = \frac{j}{m-1}$). Thus, $d^n = \mathbf{0}_{L_m}$ for all $n \geq i$, and d^i is the first zero map in the sequence. Hence, $l = \min N = i$ and $r = 1$.

Subcase 2.2: $b = a$ (i.e., $j = i$).

In this subcase, we have $d^n(1) = a^{n-1} \odot a = a^n$. Hence, $M = \{n \in \mathbb{N} \mid a^n = 0\}$. Since $a = b \leq \frac{m-2}{m-1}$, we have $M \subseteq N$. The minimal element of M is $i + 1$, because $a^i = (\frac{i}{m-1})^i > 0$ but $a^{i+1} = 0$. For $n \geq i + 1$, $d^n(1) = 0$ and $d^n(x) = 0$ for all $x < 1$. Thus, $d^n = \mathbf{0}_{L_m}$ for all $n \geq i + 1$, and d^{i+1} is the first zero map. Hence, $l = \min M = i + 1$ and $r = 1$.

In all cases, period $r = 1$ because once the sequence reaches the zero map, it remains constant.

□

5. Fixed point sets of power $(\odot, \vee)$-higher derivations

In this section, we discuss the set of fixed points of power $(\odot, \vee)$-higher derivations and show that it is a lattice ideal in the case of principal $(\odot, \vee)$-higher derivations. Note the condition is different from [3, Theorem 3.12], where d_n is an increasing function for all $n \in I$ of a higher derivation $D = (d_n)_{n \in I}$ of length n on a lattice.

Definition 5.1. Let A be an MV-algebra. If $D_k = (d_n)_{n=0}^k \in \mathrm{HD}_k(A)$ is power, define the set of all fixed points of D_k by Fix_{D_k}:

$$\mathrm{Fix}_{D_k}(A) = \{x \in A \mid d_n(x) = x \ \text{ for all } n \leq k\}.$$

Then, $0 \in \mathrm{Fix}_{D_k}(A)$, and $\mathrm{Fix}_{D_k}(A)$ is a downset of A by Proposition 3.3 (4).

Proposition 5.2. *Let A be an MV-algebra and $a \in A$. If $D_{\{k,a\}} = (d_{a^n \odot})_{n=0}^k$ is a principal $(\odot, \vee)$-higher derivation on A, then $\mathrm{Fix}_{D_{\{k,a\}}}(A)$ is a lattice ideal of A.*

Proof. It can be verified that $0 \in \mathrm{Fix}_{D_{\{k,a\}}}(A)$ by Proposition 2.8 (1). By Definition 5.1, $\mathrm{Fix}_{D_{\{k,a\}}}(A)$ is a downset of A.

Furthermore, for any $x, y \in \mathrm{Fix}_{D_{\{k,a\}}}(A)$, we have $d_{a^n \odot}(x) = x$, $d_{a^n \odot}(y) = y$; it follows that

$$d_{a^n \odot}(x \vee y) = a^n \odot (x \vee y) = (a^n \odot x) \vee (a^n \odot y) = d_{a^n \odot}(x) \vee d_{a^n \odot}(y) = x \vee y$$

by Lemma 2.1 (5). Hence, we get $x \vee y \in \mathrm{Fix}_{D_{\{k,a\}}}(A)$.

Therefore, $\mathrm{Fix}_{D_{\{k,a\}}}(A)$ is a lattice ideal of A. □

6. Conclusions

In this paper, the power $(\odot, \vee)$-higher derivations, principal $(\odot, \vee)$-higher derivations, and isotone $(\odot, \vee)$-higher derivations on an MV-algebra A have been studied. We have obtained all relationships among them as concluded in Figure 1. Moreover, we have obtained 5 equivalent conditions for a power $(\odot, \vee)$-higher derivation $D_k = (d_n)_{n=0}^k$ to be isotone. For typical MV-chains C and L_m, we have constructed explicitly some power $(\odot, \vee)$-higher derivations. Furthermore, we have discussed the sets of fixed points of power $(\odot, \vee)$-higher derivations. This work opens several avenues for future research:

1) Characterize when an isotone $(\odot, \vee)$-higher derivation is power. The challenge lies in bridging the inherent gap between isotonicity and the power condition, which requires profound insight.
2) Determine whether non-principal power higher derivations exist whose fixed point set is a lattice ideal. The difficulty stems from the more intricate interaction between the derivation's algebraic structure and the order structure of its fixed points in the non-principal case.
3) Connect MV-algebraic derivations with recent advances in fractional calculus and impulsive systems [17, 18, 22], where the iterative nature of higher derivations might provide algebraic tools for studying controllability and stability in logical networks.

Author contributions

Xueting Zhao: Conceptualization, methodology, validation, writing-original draft preparation, writing-review and editing, funding acquisition; Yichuan Yang: Conceptualization, methodology, validation, writing-review and editing, supervision, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work is supported by CNNSF (Grants: 12571487, 12171022), the Youth Research Special Project of NCUT (No. 2025NCUTYRSP059), and the North China University of Technology Research Fund Program for Young Scholars (No. 11005136025XN076-091).

Conflict of interest

The authors declare that they have no conflicts of interest.

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