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Research article

$\nu-$Conull FDK-spaces

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Abstract: In this paper, we introduced the concepts of (strongly) ν-conull FDK-spaces, which can be regarded as double-indexed versions of FK-spaces (sequence space with coordinate functionals), by utilizing the notion of ν-convergence for double sequences. We provided fundamental characterizations of these new spaces and established several inclusion relations among them. Furthermore, we investigated the conditions under which the summability domain $E_A^{(\nu)}$ is (strongly) ν-conull, thereby providing new insights into its structural and topological properties.

Keywords: double sequence spaces; FDK-space; distinguished subspace; matrix domain; ν-convergence; separable FDK-space
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1. Introduction

An important class of spaces (E, τ) with interesting applications in Schauder basis theory and summability theory is conull and coregular classifications introduced by Wilansky [1]. The characterizations of these properties without matrices were given by Yurimyae in [2] and by Snyder in [3]. Accordingly, an FK-space (E, τ) is called conull if $e^{(n)}$ is weakly convergent to e in $\sigma(E, E^*)$, where E^* is the topological dual of E and $\sigma(E, E^*)$ is the weak topology on E. In his work, Bennett [4] introduced spaces that exhibit strong connections to conull spaces, thereby furthering the development of summability theory. He examined the relation between wedge and conull FK-spaces and obtained some characterizations of both these classes. Then, some results of Bennett [4] were improved by İnce [5] and Dağadur [6] for all (strongly) conull FK-spaces. These studies motivated us to define the concept of ν-conull FDK-space by using ν-convergence for double sequences, where ν represents one of the notions of Pringsheim, bounded and regular convergence. The motivation for this research lies in extending the well-established theory of FK-spaces to the broader class of FDK-spaces, thereby developing a more comprehensive understanding of convergence, completeness, and transformation behavior in the context of double sequence spaces.

In Section 3, we introduce the key generalizations of conull and wedge spaces to double sequences, define weak and strong ν-convergence, and prove basic equivalences that mirror the classical single-sequence theory. In Section 4, we explore the relationship between ν-conullity and inclusion of bounded-variation spaces, proving criteria for when an FDK-space inherits strong or weak conullity. In Section 5, using gliding-hump methods, we show that certain nontrivial elements exist in ν-wedge FDK-spaces and that some natural subspaces are non-separable. In Section 6, we apply our framework to summability domains $E_A^{(\nu)}$, deriving necessary and sufficient conditions for ν-conullity in terms of matrix rows/columns and compactness properties.

2. Notations and preliminaries

Let Ω denote the set of all double sequences with the vector space operations defined coordinatewise. Any linear subspace $E \subseteq \Omega$ is referred to as a double sequence space.

A subspace E of the vector space Ω is called a DK-space, if all the seminorms $r_{kl} : E \to \mathbb{R}, x \mapsto |x_{kl}|$ $(k, l \in \mathbb{N})$ are continuous. An FDK-space is a DK-space with a complete, metrizable, locally convex topology. A normable FDK-space is called a BDK-space [7].

e denotes the double sequence of 1's; (δ^{ij}), $i, j = 1, 2, \ldots$, with 1 in the (i, j)-th position. Also, $\Phi := span\{e^{kl} \ : \ k, l \in \mathbb{N}\}$ and $\Phi_1 := \Phi \cup \{e\}$.

Traditionally, a bounded double sequence means a uniform bounded double sequence. The space of all uniformly bounded double sequences is defined as

$$\mathcal{M}_u := \left\{ x \in \Omega \ : \quad \|x\|_\infty := \sup_{k,l} |x_{kl}| < \infty \right\},$$

which is a BDK-space with a supremum norm [8]. A double sequence $x = (x_{kl})$ is said to be Pringsheim's sense convergent to a (p-convergent to a) if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ : \ k, l > N \Rightarrow |x_{kl} - a| < \varepsilon$ [9]. Also, if $\sup_{k,l} |x_{kl}| < \infty$, or the limits $\lim_k x_{kl}$ $(l \in \mathbb{N})$ and $\lim_l x_{kl}$ $(k \in \mathbb{N})$ exist, then x is said to be boundedly convergent to a in Pringsheim's sense (bp-convergent) and regularly convergent to a (r-convergent). Throughout the paper, ν represents the symbols p, bp, r and, C_ν denotes the space of all ν-convergent double sequences. The set of all null sequences contained in the space C_ν is denoted by $C_{\nu 0}$. Moreover, we consider the following spaces.

$$\mathcal{CS}_\nu := \left\{ x \in \Omega \ : \ \nu - \sum_{k,l}^{m,n} x_{kl} < \infty \right\},$$

$$\mathcal{L}_u := \left\{ x \in \Omega \ : \ \sum_{k,l} |x_{kl}| < \infty \right\},$$

$$\mathcal{L}_\varphi := \{ x \in \mathcal{L}_u \ : \ (x_{kl})_k \in \varphi, \ \forall l \in \mathbb{N} \text{ and } (x_{kl})_l \in \varphi, \ \forall k \in \mathbb{N} \},$$

$$\mathcal{BV} := \left\{ x \in \Omega \ : \ \|x\|_{\mathcal{BV}} := \sum_{k,l} |x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| < \infty \right\},$$

$$\mathcal{BS} := \left\{ x \in \Omega \ : \ \sup_{m,n} \left| \sum_{k,l}^{m,n} x_{kl} \right| < \infty \right\}.$$

The above double sequence spaces were also studied in [10–16].

The set of all continuous linear functionals on a space E is denoted by E' and called the dual spaces of E. Recall that $\alpha, \beta(\nu)$, γ, and the f-duals of a subset E of Ω are defined as follows:

$$\begin{aligned}
E^{\alpha} &:= \{x = (x_{kl}) \ : \ xy \in \mathcal{L}_u \ , \ \ \forall y = (y_{kl}) \in E\}\,, \\
E^{\beta(\nu)} &:= \{x = (x_{kl}) \ : \ xy \in \mathcal{CS}_\nu \ , \ \ \forall y = (y_{kl}) \in E\}\,, \\
E^{\gamma} &:= \{x = (x_{kl}) \ : \ xy \in \mathcal{BS} \ , \ \ \forall y = (y_{kl}) \in E\}\,, \\
E^{f} &:= \left\{(f(\delta^{kl})) \ : \ \forall f \in E'\right\},
\end{aligned}$$

respectively, where $xy = (x_{kl}y_{kl})$.

Let E be a sequence space. $x \in E$ is said to have AK(ν) if $x = \nu - \sum_{k,l} x_{kl}\delta^{kl}$. E is said to be an AK(ν)-space if each element of E has AK(ν) [17, 18].

Theorem 2.1. *[7] Let ν be a notion of convergence for double sequences such that C_ν is an FDK-space and the limit functional $\nu - \lim$ is continuous on C_ν. If E is an AK(ν)-FDK-space, then for every $f \in E'$, there exists $u \in E^{\beta(\nu)}$ such that*

$$f(x) = \nu - \sum_{k,l} u_{kl}x_{kl} \quad (x \in E).$$

Moreover, every functional f having the representation (2.1) *is in E'.*

Let $A = (a_{mnkl})$ be any four-dimensional matrix. Consider

$$\Omega_A^{(\nu)} := \left\{ x \in \Omega \mid \forall m, n \in \mathbb{N} \ : \ [Ax]_{mn} := \nu - \sum_{k,l} a_{mnkl}x_{kl} \text{ exists} \right\}.$$

The map

$$A : \Omega_A^{(\nu)} \to \Omega, \ \ x \mapsto Ax := ([Ax]_{mn})_{m,n}$$

is called a matrix map of type ν. The summability domain of a matrix $A = (a_{mnkl})$ is defined as

$$E_A^{(\nu)} = \{x \in \Omega : Ax \text{ exists and } Ax \in E\}.$$

Also, $\xi_A^{(mn)} := \{a_{mnkl}\}_{k,l=1}^{\infty,\infty}$ is called the (k, l)-th row of the matrix A, and $\zeta_A^{(kl)} := \{a_{mnkl}\}_{m,n=1}^{\infty,\infty}$ is called the (k, l)-th column of the matrix A.

In the following result, Zeltser [7] describes the topology of the space $E_A^{(\nu)}$ and the spaces C_ν and E are FDK-spaces.

Theorem 2.2. *[7] Let ν be some notion of convergence for double sequences such that C_ν is an FDK-space and let $\{t_k : k \in \mathbb{N}\}$ be a system of seminorms, defining the FDK topology of C_ν. Let $A = (a_{mnkl})$ be a four-dimensional matrix and E be an FDK-space with the FDK topology generated by a system of seminorms $\{\varrho_k : k \in \mathbb{N}\}$.*

i. The space $E_A^{(\nu)}$ is an FDK-space and the FDK topology is generated by the system of seminorms $\{r_{mn} : m, n \in \mathbb{N}\} \cup \{t_r \circ A_{mn} : r, m, n \in \mathbb{N}\} \cup \{\varrho_r \circ A : r \in \mathbb{N}\}$, where

$$A_{mn}(x) := \left(\sum_{k=1}^{s} \sum_{l=1}^{t} a_{mnkl}x_{kl} \right)_{s,t} \quad (x \in E_A^{(\nu)}).$$

ii. The topological dual $(E_A^{(\nu)})'$ *consists of all linear functionals* f *of the form*

$$f(x) = g(x) + h(Ax) \quad (x \in E_A^{(\nu)})$$

with certain $g \in (\Omega_A^{(\nu)})'$ *and* $h \in E'$.

iii. If C_ν *and* E *are separable, then* $E_A^{(\nu)}$ *is separable.*

In [19] the authors defined the ν-wedgeness for any FDK-space as follows.

Definition 2.1. *[19] Let* $(E, \tau) \supset \Phi$ *be a DK-space.* (E, τ) *is called a* ν*-wedge FDK-space, if the sequence* (δ^{ij}) *is* ν*-convergent to* 0 *in* τ.

Definition 2.2. *[19] Let* $(E, \tau) \supset \Phi$ *be a DK-space.* (E, τ) *is called a weak* ν*-wedge FDK-space, if the sequence* (δ^{ij}) *is weak* ν*-convergent to* 0 *in* τ.

With these preliminaries in place, we are now equipped to extend the notions of conull and wedge spaces to double sequences and study their structural properties in detail.

3. Main results

In this section, ν-conullity is defined for an FDK-space including Φ_1. In addition, some important results have been obtained on this subject.

Definition 3.1. *Let* $E \supset \Phi_1$ *be an FDK-space. The space* E *is called a* ν*-conull FDK-space if the sequence* $(e^{(mn)})$ *is weakly* ν*-convergent to* e*; that is, for all* $f \in E'$,

$$f(e) = \nu - \lim_{m,n} \sum_{k,l=1}^{m,n} f(\delta^{kl}).$$

Definition 3.2. *Let* $E \supset \Phi$ *be an FDK-space. The space* E *is called a strongly* ν*-conull FDK-space if the sequence* $(e^{(mn)})$ *is* ν*-convergent to* e*; that is,*

$$e = \nu - \lim_{m,n} \sum_{k,l=1}^{m,n} \delta^{kl}.$$

Clearly, each ν-conull FDK-space is also strongly ν-conull. Additionally, there is a relationship between (weak) ν-wedge and (strongly) ν-conull FDK-spaces as follows.

$$\begin{array}{ccccc} & & \nu\text{-conull FDK} & & \\ & \nearrow & & \searrow & \\ \text{strongly } \nu\text{-conull FDK} & & & & \text{weak } \nu\text{-wedge FDK} \\ & \searrow & & \nearrow & \\ & & \nu\text{-wedge FDK} & & \end{array}$$

In fact, let E be a strong ν-conull FDK-space. Then we have $e^{(kl)} \to e$. Hence $q(e^{(kl)} - e) \to 0$ $(k, l \to \infty)$ for any seminorm q in τ. For $k, l \geq 2$, using the following equation

$$\delta^{kl} \;=\; e^{(k,l)} - e^{(k,l-1)} - e^{(k-1,l)} + e^{(k-1,l-1)},$$

we have

$$\begin{aligned} q\left(\delta^{kl}\right) &= q\left(e^{(k,l)} - e^{(k,l-1)} - e^{(k-1,l)} + e^{(k-1,l-1)} + e - e + e - e\right) \\ &\leq q\left(e^{(k,l)} - e\right) + q\left(e^{(k,l-1)} - e\right) + q\left(e^{(k-1,l)} - e\right) + q\left(e^{(k-1,l-1)} - e\right). \end{aligned}$$

It is clear that since $q\left(\delta^{kl}\right) \to 0$, $k, l \to \infty$, E is a ν-wedge space.

Now, let us consider the surjection mapping $S^{(2)} : \Omega \to \Omega$,

$$(S^{(2)}x)_{mn} = \sum_{k,l=1}^{m,n} x_{kl}.$$

Clearly, $(S^{(2)})^{-1} : \mathcal{M}_u \to \mathcal{BS}$,

$$((S^{(2)})^{-1}x)_{mn} = x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}.$$

Theorem 3.1. *i. (E, τ) is a strongly ν-conull FDK-space iff the space $(S^{(2)})^{-1}(E)$ is a ν-wedge FDK-space.*

ii. (E, τ) is a ν-conull FDK-space iff the space $(S^{(2)})^{-1}(E)$ is a weak ν-wedge FDK-space.

Proof. i) Neccessary. Let the topology τ be generated by the seminorms $\{P_{mn}\}$. Then a topology with the set of seminorms $\{q_{mn}\}$ makes $(S^{(2)})^{-1}(E)$ is an FDK-space such that

$$q_{mn}(x) := P_{mn}(S^{(2)}(x)).$$

By hypothesis, $P_{mn}\left(e - e^{(mn)}\right) \to 0$ $(m, n \to \infty)$. Since $(S^{(2)})^{-1}(e - e^{(mn)}) = \delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}$, we get

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}(e - e^{(mn)}). \tag{3.1}$$

So we can say $q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m, n \to \infty)$. In this case, we have $(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m, n \to \infty)$ according to the topology of the space $(S^{(2)})^{-1}(E)$. As $\delta^{m+1,1} \to 0$ and $\delta^{1,n+1} \to 0$, $\delta^{m+1,n+1} \to 0$ hold, $(S^{(2)})^{-1}(E)$ is a ν-wedge FDK-space.

Sufficient. Assume that $(S^{(2)})^{-1}(E)$ is a ν-wedge FDK-space. Then we have $q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m, n \to \infty)$. By Eq (3.1), we obtain $P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right) \to 0$ $(m, n \to \infty)$. So E is a strongly ν-conull FDK-space.

ii) Let (E, τ) be a ν-conull FDK-space and let us define the topology of $(S^{(2)})^{-1}(E)$ as the proof of (i). Then $q_{mn}(x) := P_{mn}(S^{(2)}(x))$ and we have (3.1). Since E is a ν-conull FDK-space, $P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right) \to 0$ (weak) $(m, n \to \infty)$. Hence $\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1} \to 0$ (weak) $(m, n \to \infty)$. Consequently $\delta^{m+1,n+1} \to 0$ (weak) $(m, n \to \infty)$ is obtained. That is, $(S^{(2)})^{-1}(E)$ is a weak ν-wedge FDK-space.

The other part of the proof follows in the same way as the proof of (i). □

In the following, with the help of the transformation $S^{(2)}$, we define a new double sequence space.

$x \in \mathcal{BV}(\varphi)$ means that

$$x = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1l} & b & b & \dots \\ x_{21} & \ddots & & & g & g & \dots \\ \vdots & & & \vdots & & & \dots \\ x_{k1} & \dots & \dots & x_{kl} & h & h & \dots \\ a & d & \dots & f & c & c & \dots \\ a & d & \dots & f & c & \ddots & \\ \vdots & \vdots & \dots & \vdots & \vdots & & \end{pmatrix}$$

for any $a, b, c, d, f, g, h \in \mathbb{R}$. Now we shall give one of the interesting results.

Proposition 3.2. $S^{(2)}(\mathcal{L}_\varphi) = \mathcal{BV}(\varphi)$.

Proof.

$$\begin{aligned} S^{(2)}(\mathcal{L}_\varphi) &= \{S^{(2)}(x) \ : \ x \in \mathcal{L}_\varphi\} \\ &= \{S^{(2)}(x) \ : \ \sum |x_{kl}| < \infty, \forall l \in \mathbb{N}\ (x_{kl})_k \in \varphi, \forall k \in \mathbb{N}\ (x_{kl})_l \in \varphi\} \\ &= \Big\{x \ : \ \sum |((S^{(2)})^{-1}(x))_{kl}| < \infty, \forall l \in \mathbb{N}\ (((S^{(2)})^{-1}(x))_{kl})_k \in \varphi, \\ &\qquad \forall k \in \mathbb{N}\ (((S^{(2)})^{-1}(x))_{kl})_l \in \varphi\Big\}. \end{aligned}$$

Let us prove the last part of the above equation. Assume that $\forall l \in \mathbb{N}\ (((S^{(2)})^{-1}(x))_{kl})_k \in \varphi$. Then

$$\begin{pmatrix} x_{11} & x_{12} - x_{11} & \dots & \dots & \dots \\ x_{21} - x_{11} & x_{22} - x_{12} - x_{21} + x_{11} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & x_{kl} - x_{k,l-1} - x_{k-1,l} + x_{k-1,l-1} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}_k \in \varphi.$$

This means that every column sequence of the matrix is finite. So we get the following system:

$$\begin{gathered} (x_{11}, x_{21} - x_{11}, \dots, x_{k1} - x_{k-11}, \dots) \in \varphi, \\ (x_{12} - x_{11}, x_{22} - x_{12} - x_{21} + x_{11}, \dots, x_{k2} - x_{k-1,2} - x_{k1} + x_{k-1,1}, \dots) \in \varphi, \\ \vdots \\ (x_{1l} - x_{1,l-1}, x_{2l} - x_{1,l-1} - x_{l1} + x_{1,l-1}, \dots, x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}, \dots) \in \varphi. \end{gathered}$$

Assume that $x_{01} = 0$. Considering the first row, we obtain

$$\exists k_1\ \forall k > k_1 : x_{k1} - x_{k-1,1} = 0 \Leftrightarrow x_{k1} = x_{k-1,1}. \tag{3.2}$$

Then from (3.2), we have

$$\exists k_2\ \forall k > k_2 : x_{k2} - x_{k-1,2} - x_{k1} + x_{k-11} = 0 \Leftrightarrow x_{k2} = x_{k-1,2}.$$

So by continuing, we get

$$\exists k_t\ \forall k > k_t : x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1} = 0 \Leftrightarrow x_{kl} = x_{k-1,l},$$

which means that the terms of the sequence $((x_{kl})_k)$ $\forall l \in \mathbb{N}$ are constant and equal to each other for $k > k_t$. If the same steps are applied for $(((S^{(2)})^{-1}(x))_{kl})_l \in \varphi$, we get

$$\exists l_t \ \forall l > l_t : x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1} = 0 \Leftrightarrow x_{kl} = x_{k,l-1},$$

which means that the terms of the sequence $((x_{kl})_l)$ $\forall k \in \mathbb{N}$ are constant and equal to each other for $l > l_t$. By (3.3) and (3.3), we get $x \in \mathcal{BV}(\varphi)$. □

Before giving our main theorem, let us consider the following space. Let $s = (s_m)$, $t = (t_n)$ be two strictly increasing sequences of nonnegative integers with $s_1 = 0$, $t_1 = 0$.

$$m|(s,t)| = \left\{ x \in \Omega \ : \ \sup_{m,n} \sum_{k=s_m+1}^{s_{m+1}} \sum_{l=t_n+1}^{t_{n+1}} |x_{kl}| < \infty \right\},$$

which is a BDK-space with the following norm:

$$||x||_{m|(s,t)|} = \sup_{m,n} \sum_{k=s_m+1}^{s_{m+1}} \sum_{l=t_n+1}^{t_{n+1}} |x_{kl}|.$$

Theorem 3.3. *For any E FDK-space, the following conditions are equivalent.*

i. E is strongly ν-conull,

ii. for $z \in C_{\nu 0}$,

$$S^{(2)}(z^\alpha) = \left\{ x \in \Omega \ : \ \sum_{k,l=1}^{\infty,\infty} |x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}||z_{kl}| < \infty \right\} \subseteq E,$$

iii. E contains the space $M|(s,t)|$ *for some s, t and the inclusion map* $I : M|(s,t)| \to E$ *is compact,*

iv. $E \supseteq \mathcal{BV}(\varphi)$ *and the inclusion map* $I : \mathcal{BV}(\varphi) \to E$ *is compact,*

where

$$z^\alpha := \left\{ y \in \Omega \ : \ \sum_{k,l} y_{kl} z_{kl} \right\},$$

$$M|(s,t)| := \left\{ x \in \Omega \ : \ \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}| < \infty \right\}.$$

Proof. In the proof of this theorem, we apply the technique introduced in [20].

($i \Rightarrow ii$) If the space E is a strongly ν-conull space, by Theorem 3.1, $(S^{(2)})^{-1}(E)$ is a ν-wedge space. Hence, for $z_0 \in C_{\nu 0}$, $z^\alpha \subseteq (S^{(2)})^{-1}(E)$ and

$$S^{(2)}(z^\alpha) \subseteq S^{(2)}(S^{(2)})^{-1}(E) = E.$$

On the other hand,

$$S^{(2)}(z^\alpha) \ = \ \left\{ S^{(2)}(y) \ : \ y \in z^\alpha \right\}$$

$$
\begin{aligned}
&= \left\{ S^{(2)}(y) \ : \ \sum_{k,l=1}^{\infty,\infty} |y_{kl} z_{kl}| < \infty, \ \ y \in \Omega \right\} \\
&= \left\{ x \ : \ \sum_{k,l=1}^{\infty,\infty} \left| ((S^{(2)})^{-1}(x))_{kl} \right| |z_{kl}| < \infty, \ \ x \in \Omega \right\} \\
&= \left\{ x \ : \ \sum_{k,l=1}^{\infty,\infty} |x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}||z_{kl}| < \infty, \ \ x \in \Omega \right\}.
\end{aligned}
$$

With the above equation, the proof is complete.

($ii \Rightarrow iii$) Suppose that $S^{(2)}(z^{\alpha}) \subset E$ for $z \in C_{\nu 0}$. Then $z^{\alpha} \subset (S^{(2)})^{-1}(E)$ and

$$
m|(s,t)| \subseteq z^{\alpha} \subseteq (S^{(2)})^{-1}(E)
$$

hold [19]. Thus, the inclusion map $I : m|(s,t)| \to (S^{(2)})^{-1}(E)$ is compact. Since $M|(s,t)| = S^{(2)}(m|(s,t)|) \subseteq E$, the inclusion map $S^{(2)} \circ I \circ (S^{(2)})^{-1} : M|(s,t)| \to E$ is compact. Let us show that $M|(s,t)| = S^{(2)}(m|(s,t)|)$.

$$
\begin{aligned}
S^{(2)}(m|(s,t)|) &= \left\{ S^{(2)}(x) \ : \ x \in m|(s,t)| \right\} \\
&= \left\{ S^{(2)}(x) \ : \ \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}| < \infty \right\} \\
&= \left\{ y \ : \ \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |((S^{(2)})^{-1}(y))_{k,l}| < \infty, \ y \in \Omega, \ y_{00} = y_{01} = y_{10} = 0 \right\} \\
&= \left\{ y \ : \ \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |y_{kl} - y_{k-1,l} - y_{k,l-1} + y_{k-1,l-1}| < \infty, \ y \in \Omega, \ y_{00} = y_{01} = y_{10} = 0 \right\} \\
&= M|(s,t)|.
\end{aligned}
$$

($iii \Rightarrow iv$) Since $\mathcal{BV}(\varphi) \subset S^{(2)}(m|(s,t)|)$ and from the hypothesis we have $\mathcal{BV}(\varphi) \subset E$. Thus, the inclusion map $I : \mathcal{BV}(\varphi) \to S^{(2)}(m|(s,t)|)$ is continuous, and the inclusion map $I : S^{(2)}(m|(s,t)|) \to E$ is compact.

($iv \Rightarrow i$) Let $E \supseteq \mathcal{BV}(\varphi)$ and the inclusion map $I : \mathcal{BV}(\varphi) \to E$ be compact. Then the set $A = \{e - e^{(mn)} : m, n, \ldots\}$ is a bounded subset of $\mathcal{BV}(\varphi)$. Thus, the set $I(A) = A$ is relatively compact on E. Hence, the topology of coordinat-wise convergence on A and the topology τ are coincident. According to the topology generated by the seminorms of $r_{mn}(x) = |x_{mn}|$ $(m, n = 1, 2, \ldots)$,

$$
r_{kl}(e - e^{(mn)}) = \begin{cases} 0, & (k,l) < (m,n) \\ 1, & (k,l) \geq (m,n) \end{cases}
$$

and $r_{kl}(e - e^{(mn)}) \to 0$ $(m, n = 1, 2, \ldots)$, so $e - e^{(mn)} \to 0$ $(m, n = 1, 2, \ldots)$ on (E, τ). This completes the proof. □

The following result demonstrates that the space obtained in the intersection can vary depending on the chosen notion of convergence.

Theorem 3.4. *i. Let E_n be strongly $p-$conull FDK-spaces. Then $\bigcap E_n = \mathcal{BV}(\varphi)$.*
ii. Let E_n be strongly ν-conull FDK-spaces for $\nu \in \{bp, r\}$. Then $\bigcap E_n = \mathcal{BV}$.

Proof. For $z \in C_{\nu 0}$, $S^{(2)}(z^\alpha)$ is strongly ν-conull and $\mathcal{BV}(\varphi) \subset E$. Then, we obtain

$$\begin{aligned} \bigcap\{S^{(2)}(z^\alpha) \ : \ z \in C_{\nu 0}\} &= S^{(2)}\left(\bigcap\{z^\alpha \ : \ z \in C_{\nu 0}\}\right) \\ &= S^{(2)}(C^\alpha_{\nu 0}) = \begin{cases} S^{(2)}(\mathcal{L}_u) = \mathcal{BV}, & \nu \in \{bp, r\} \\ S^{(2)}(\mathcal{L}_\varphi) = \mathcal{BV}(\varphi), & \nu = p. \end{cases} \end{aligned}$$

It is clear that the equality $S^{(2)}(\mathcal{L}_\varphi) = \mathcal{BV}(\varphi)$ holds from Proposition 3.2. The first equality is obtained as follows.

$$\begin{aligned} S^{(2)}(\mathcal{L}_u) &= \{S^{(2)}(x) \ : \ x \in \mathcal{L}_u\} \\ &= \{S^{(2)}(x) \ : \ \sum |x_{kl}| < \infty\} \\ &= \{x \ : \ \sum |((S^{(2)})^{-1}(x))_{kl}| < \infty\} \\ &= \{x \ : \ \sum |x_{kl} - x_{k-1,l} - x_{k.l-1} + x_{k-1,l-1}| < \infty\} \\ &= \mathcal{BV}. \end{aligned}$$

□

Theorem 3.5. *An FDK-space E is ν-conull iff $\mathcal{BV}(\varphi) \subset E$, and moreover, the inclusion map $I : \mathcal{BV}(\varphi) \to E$ is weakly compact.*

Proof. Let E be a ν-conull FDK-space. By Theorem 3.1 the space $(S^{(2)})^{-1}(E)$ is a weak ν-wedge space. Using the fact that $S^{(2)}$ is a bijection and a topological isomorphism, we identify $(S^{(2)})^{-1}(E)$ with E. Hence E is a weak ν-wedge space. So $\mathcal{L}_\varphi \subset E$, and $J : \mathcal{L}_\varphi \to E$ is compact. Moreover, since $S^{(2)}(\mathcal{L}_\varphi) \subset S^{(2)}(E) = E$ and $S^{(2)}(\mathcal{L}_\varphi) = \mathcal{BV}(\varphi)$, we have $\mathcal{BV}(\varphi) \subset E$ and the inclusion map $I : \mathcal{BV}(\varphi) \to E$ is weakly compact because it is obtained from the compact map $J : \mathcal{L}_\varphi \to E$ conjugation with the topological isomorphism $S^{(2)}$.

Conversely, if $\mathcal{BV}(\varphi) \supset \mathcal{L}_\varphi$, we obtain $\mathcal{L}_\varphi = (S^{(2)})^{-1}(\mathcal{BV}(\varphi)) \subset (S^{(2)})^{-1}(E) = E$ and the inclusion mapping $I : \mathcal{L}_\varphi \to \mathcal{BV}(\varphi)$ is continuous. Hence $J : \mathcal{L}_\varphi \to E$ is weakly compact. Consequently, $(S^{(2)})^{-1}(E)$ is a weak ν-wedge space, that is, E is a ν-conull FDK-space. □

Corollary 3.6. *i. Let E_n be $p-$conull FDK-spaces. Then $\bigcap E_n = \mathcal{BV}(\varphi)$.*
ii. Let E_n be ν-conull FDK-spaces for $\nu \in \{bp, r\}$. Then $\bigcap E_n = \mathcal{BV}$.

Proof. Let E be a ν-conull FDK-space. By Theorem 3.5, we have the inclusion $\mathcal{BV}(\varphi) \subset E$. If $z \in C_{\nu 0}$, then $S^{(2)}(z^\alpha)$ is strongly ν-conull and so $S^{(2)}(z^\alpha)$ is ν-conull. Let us denote all ν-conull spaces by Y. Hence we obtain that

$$\begin{aligned} Y \subset \bigcup\left\{S^{(2)}(z^\alpha) \ : \ z \in C_{\nu 0}\right\} &= S^{(2)}(C^\alpha_{\nu 0}) \\ &= \begin{cases} S^{(2)}(\mathcal{L}_u) = \mathcal{BV}, & \nu \in \{bp, r\} \\ S^{(2)}(\mathcal{L}_\varphi) = \mathcal{BV}(\varphi), & \nu = p. \end{cases} \end{aligned}$$

□

We used the gliding hump method applied by Bennett [21] to prove the following results.

Theorem 3.7. *Let E be a ν-wedge FDK-space. Then* $E \cap (C_{p0}/\mathcal{BV}) \neq \emptyset$.

Proof. Let us assume that the topology of E is generated by the seminorms $\{p_{mn}\}$ such that

$$\begin{aligned} &|x_{mn}| \leq p_{mn}(x) \leq p_{m+1,n}(x) \leq p_{m+1,n+1}(x) \\ &|x_{mn}| \leq p_{mn}(x) \leq p_{m,n+1}(x) \leq p_{m+1,n+1}(x) \qquad (x \in E, \ \ m, n = 1, 2, \ldots). \end{aligned} \tag{3.3}$$

Since C_p is not a ν-wedge FDK-space, then the subspace $C_p \cap E$ is not a ν-wedge FDK-space by Theorem 2.14 in [19]. So the space $C_p \cap E$ is not closed in E. We know that C_{p0} and C_p are equidimensional so it follows from Theorem 2.14 in [19] that the space $C_{p0} \cap E$ is not closed in E. Hence there exists $x \in C_{p0} \cap E$ such that $p_{mn}(x) < \varepsilon$ and $\|x\|_\infty$ for $\varepsilon > 0$, $\eta > 0$ and positive integers m, n.

To proof this argue, let us suppose that the contrary is true. That is, there exist $\varepsilon > 0$, $\eta > 0$, and $m, n \in \mathbb{Z}^+$ such that if $\|x\|_\infty = \eta$, then $p_{mn}(x) \geq \varepsilon$ for $x \in C_{p0} \cap E$. Then for $0 \neq x \in C_{p0} \cap E$, we have $p_{mn}\left(\frac{\eta x}{\|x\|_\infty}\right) \geq \varepsilon$ so that $\|x\|_\infty \leq (\eta/\varepsilon) p_{mn}(x)$ for all $x \in C_{p0} \cap E$. It follows that $C_{p0} \cap E$ is closed in E, which is a contradiction.

Taking $\varepsilon = \varepsilon_{11} = \frac{1}{2^4}$, $\eta = \eta_{11} = 1$, $m = m_1 = 1$, and $n = n_1 = 1$, we have $x^{(11)}$. Let $k > 1$, $l > 1$ and suppose that $m_1, m_2, \ldots, m_{k-1}, n_1, n_2, \ldots, n_{l-1}$ and $x^{(11)}, \ldots, x^{(k-1,l-1)}$ have been chosen. With $\varepsilon_{kl} = \frac{1}{2^{(k+1)(l+1)}}$ and $\eta_{kl} = \frac{1}{kl}$, choose $m_k > m_{k-1}$ and $n_l > n_{l-1}$ so that

$$|x_{ij}^{(st)}| \leq \frac{1}{2^{(k+1)(l+1)}} \qquad (i \geq m_k, j \geq n_l, 1 \leq s < k, 1 \leq t < l)$$

and choose $x^{(kl)} \in C_{p0} \cap E$ so that

$$p_{m_k, n_l}(x^{(kl)}) < \varepsilon_{kl} \text{ and } \|x^{(kl)}\|_\infty = \frac{1}{kl}\,. \tag{3.4}$$

We obtain by (3.3) the double sequence (x_{kl}) of elements of $C_{p0} \cap E$ so that

$$\begin{aligned} p_{11}(x^{(kl)}) \leq p_{21}(x^{(kl)}) \leq \ldots \leq p_{kl}(x^{(kl)}) < \frac{1}{2^{(k+1)(l+1)}}, \\ p_{11}(x^{(kl)}) \leq p_{12}(x^{(kl)}) \leq \ldots \leq p_{kl}(x^{(kl)}) < \frac{1}{2^{(k+1)(l+1)}}\,. \end{aligned}$$

Let $x = \sum_{k,l=1}^{\infty,\infty} x^{(kl)}$, and the series is clearly convergent in E. We need to show that $x \in C_{p0} \backslash \mathcal{BV}$. For $m_s \leq i \leq m_{s+1}$ and $n_t \leq j \leq n_{t+1}$, we have

$$|x_{ij}^{(st)}| \leq \begin{cases} \frac{1}{2^{(u+1)(v+1)}} & , \quad s < u, t < v \\ \frac{1}{uv} & , \quad s = u, t = v \\ \frac{1}{2^{(s+1)(t+1)}} & , \quad s > u, t > v \end{cases} \qquad (u, v = 1, 2, \ldots), \tag{3.5}$$

such that

$$\|x\| = \left\| \sum_{k,l=1}^{\infty,\infty} x^{(kl)} \right\| \leq \sum_{s,t=1}^{\infty,\infty} |x_{ij}^{(st)}| \leq \sum_{s,t=1}^{u-1,v-1} \frac{1}{2^{(u+1)(v+1)}} + \frac{1}{uv} + \sum_{\substack{s=u+1 \\ t=v+1}}^{\infty,\infty} \frac{1}{2^{(s+1)(t+1)}} \to 0 \quad (u, v \to \infty),$$

so $x \in C_{p0}$.

Now we show that $x \notin \mathcal{BV}$. For every positive integers u, v, there exist $k_u \in (m_u, m_{u+1})$ and $l_v \in (n_v, n_{v+1})$ so that $|x^{(uv)}_{k_u l_v}| = \frac{1}{uv}$ and so, by (3.5),

$$|x_{k_u l_v}| \geq \frac{1}{uv} - \sum_{s,t=1}^{u-1,v-1} \frac{1}{2^{(u+1)(v+1)}} - \sum_{\substack{s=u+1\\ t=v+1}}^{\infty,\infty} \frac{1}{2^{(s+1)(t+1)}} = \frac{1}{uv} - \frac{(u-1)(v-1)+1}{2^{(u+1)(v+1)}} .$$

From (3.4) and (3.5),

$$|x_{m_u n_v}| \geq \sum_{s,t=1}^{u-1,v-1} \frac{1}{2^{(u+1)(v+1)}} + \frac{1}{2^{(u+1)(v+1)}} + \sum_{\substack{s=u+1\\ t=v+1}}^{\infty,\infty} \frac{1}{2^{(s+1)(t+1)}} = \frac{(u-1)(v-1)+2}{2^{(u+1)(v+1)}} .$$

Let us define that

$$y_{uv} = \begin{cases} x_{m_u n_v} & , \quad u \text{ and } v \text{ odd}, \\ x_{k_u l_v} & , \quad u \text{ or } v \text{ even}. \end{cases}$$

Then y is a subsequence of x and $y \notin \mathcal{BV}$. So $x \notin \mathcal{BV}$. □

Theorem 3.8. *Let E be an FDK-space such that $C_p \cap E$ is not closed in E. Then $\mathcal{M}_u \cap E$ is a non-separable subspace of $\mathcal{M}_u$.*

Proof. The result follows as a consequence of Theorem 3.7. From the construction given there, we can choose a family of elements $\{x^{(mn)}\} \subset C_{p0} \cap E$ satisfying $\|x^{(mn)}\|_\infty = 1/(mn)$ and $p_{m,n}(x^{(mn)}) < 2^{-(m+1)(n+1)}$. By slightly modifying the construction, we may normalize each element so that $\|x^{(mn)}\|_\infty = 1$, while keeping their supports pairwise disjoint. That is, for distinct pairs $(m, n) \neq (k, l)$, the supports of $x^{(mn)}$ and $x^{(kl)}$ do not intersect.

Since the ∞-norm of each $x^{(mn)}$ equals 1 and their supports are disjoint, we have

$$\|x^{(mn)} - x^{(kl)}\|_\infty = 1 \qquad \text{for all } (m, n) \neq (k, l).$$

Therefore, the set $\{x^{(mn)}\}$ forms a 1-separated family in $\mathcal{M}_u \cap E$. Since this family is uncountable, $\mathcal{M}_u \cap E$ cannot be separable in $\mathcal{M}_u$.

Remark. Condition $\|x^{(mn)} - x^{(kl)}\|_\infty > 1$ does not follow from boundedness assumptions but from the disjointness of supports and normalization $\|x^{(mn)}\|_\infty = 1$. Hence, the argument does not require the sequence family to be bounded in any other sense. □

Corollary 3.9. *Let E be a v-conull FDK-space, and then $\mathcal{M}_u \cap E$ is not separable in $\mathcal{M}_u$.*

Proof. Since $C_p \cap E \subset C_p$, $C_p \cap E$ is not a v-conull FDK-space. So, the space $C_p \cap E$ is not closed in E. Thus, by Theorem 3.8, $\mathcal{M}_u \cap E$ is a not separable space in $\mathcal{M}_u$. □

Theorem 3.10. *Let E be an FDK-space. If $\mathcal{L}_u \cap E$ is not closed in E, then there exists a double sequence summable and not absolutely summable such that E contains it.*

Proof. Consider the mapping $S^{(2)} : E \to F$. By hypothesis, $S^{(2)}(\mathcal{L}_u \cap E) = \mathcal{BV} \cap F$ is not closed in F. If $C_p \cap F = \mathcal{BV} \cap F$, then $C_p \cap F$ is not closed in F and Theorem 3.7 is contradicted. Hence $F \cap \left(C_p \backslash \mathcal{BV}\right)$ is nonempty and then $E \cap (\mathcal{CS}_v \backslash \mathcal{L}_u)$ is nonempty. This proves the theorem. □

Having established the definitions and basic properties of ν-conull and ν-wedge FDK-spaces, we next investigate distinguished subspaces within these spaces, highlighting their structural significance and relation to classical bounded-variation spaces.

4. Distinguished subspaces of FDK-spaces

In this section, we have provided some examples of distinguished subspaces. We examined some of the properties of these spaces and their relationships with each other and with ν-conull FDK-spaces.

We begin with the smallest. The letter S stands for strong (convergence).

Definition 4.1. *Let $E \supset \Phi$ be an FDK-space. Then*

$$S_E^{(\nu)} = S^{(\nu)}(E) = \left\{ x = (x_{kl}) \ : \ x = \nu - \sum_{k,l} x_{kl}\delta^{kl} \right\}.$$

If A is a matrix, $S^{(\nu)}(A) = S^{(\nu)}(C_{\nu A})$ [7].

Thus, E is an AK(ν) space iff $S_E^{(\nu)} = E$. Also, $S_E^{(\nu)} \subset E$ since E is complete.

Definition 4.2. *Let $E \supset \Phi$ be an FDK-space. Then*

$$W_E^{(\nu)} = W^{(\nu)}(E) = \left\{ x = (x_{kl}) \ : \ \forall f \in E', \ f(x) = \nu - \sum_{k,l} x_{kl} f(\delta^{kl}) \right\}.$$

If A is a matrix, $W^{(\nu)}(A) = W^{(\nu)}(C_{\nu A})$ [7, 22].

Theorem 4.1. *If E is an FDK-space that contains Φ, then $\Phi \subset S_E^{(\nu)} \subset W_E^{(\nu)} \subset \overline{\Phi}$.*

Proof. It is sufficient to prove $W_E^{(\nu)} \subset \overline{\Phi}$. Let $f \in E'$ and $f = 0$ on Φ. A glance at the definition of $W_E^{(\nu)}$ just given shows that $f = 0$ on $W_E^{(\nu)}$. Thus, the Hahn-Banach theorem gives the result.

Note that the stronger inclusion $W_E^{(\nu)} \subset \Phi$ holds only when the space E is minimal (that is, $E = \Phi$). In the general case considered here, we have only $W_E^{(\nu)} \subset \overline{\Phi}$, which is consistent with the standard FK-space framework. □

Definition 4.3. *Let E be an FDK-space with $E \supset \Phi$, and then*

$$B_E^+ = B^+(E) = \left\{ x = (x_{kl}) \ : \ \forall f \in E', \ (x_{kl} f(\delta^{kl})) \in \mathcal{BS} \right\}.$$

$B_E = B_E^+ \cap E$. If A is a matrix, $B(A) = B(C_{\nu A})$ [7, 22]. Also, if E is an AB-space, then $B(E) = E$.

Theorem 4.2. *Let $E \supset \Phi$ be an FDK-space. Then for each $z \in B_E^+$ and each continuous seminorm p on E we have $z_{mn} = O(p(\delta^{mn})^{-1})$.*

Proof. For each $z = (z_{mn}) \in B_E^+$

$$|z_{mn}| p(\delta^{mn}) = p(z_{mn}\delta^{mn}) = p\left(z^{(mn)} - z^{(m-1,n)} - z^{(m,n-1)} + z^{(m-1,n-1)}\right) < M,$$

since a continuous seminorm is bounded on bounded sets. □

Theorem 4.3. *Let $E \supset \Phi$ be an FDK space. Then $B_E^+ = E^{f\gamma}$.*

Proof. $z \in B^+$ means $z.u \in \mathcal{BS}$ for each $u \in E^f$. That is exactly the claim. □

This makes it easy to compute B^+ and B. The next result makes it even easier although there is a little less here than meets the eye–namely B will be different for different Y, e.g., Y may be AB and E is not, even if Y is closed in E.

Theorem 4.4. *Let $E \supset \Phi$ be an FDK-space. Then B^+ is equal for all FDK-spaces between $(\overline{\Phi})_E$ and E; i.e., if $(\overline{\Phi})_E \subset Y \subset E$, then $B^+(Y) = B^+(E)$.*

Proof. Since the distinguished subspaces are monotone, we have $B^+(\overline{\Phi}) \subset B^+(Y) \subset B^+(E)$. The first and the last are equal by Theorem 4.3 and 7.2.4 of [23] □

Theorem 4.5. *Let $E \supset \Phi$ be an FDK-space. Then E has AB iff $E^f \subset E^\gamma$ i.e., $E^f = E^\gamma$.*

Proof. Necessity. Using Theorem 4.3, $E \subset B^+(E) = E^{f\gamma}$. Hence $E^\gamma \supset E^{f\gamma\gamma} \supset E^f$. Sufficiency. $B^+(E) = E^{f\gamma} \supset E^{\gamma\gamma} \supset E$. □

Corollary 4.6. *Let $E \supset \Phi$ be an FDK-space. If E has AB, then $E^{\beta(\nu)}$ is closed in E^f.*

Proof. The proof is clear by Theorem 4.5, since $E^{\beta(\nu)}$ is closed in E^f. □

Definition 4.4. *Let E be an FDK-space with $E \supset \Phi$, and then*

$$F_E^{(\nu)+} = F^{(\nu)+}(E) = \left\{x = (x_{kl}) \;\;:\;\; \forall f \in E', \;\; (x_{kl} f(\delta^{kl})) \in \mathcal{CS}_\nu\right\}.$$

$F_E^{(\nu)} = F_E^{(\nu)+} \cap E$. If A is a matrix, $F^{(\nu)}(A) = F^{(\nu)}(C_{\nu A})$ [7, 22].

The letter $F_E^{(\nu)}$ stands for functional (convergence) since $z \in F_E^{(\nu)+}$ if and only if $\{f(z^{(mn)})\}$ is convergent for all $f \in E'$. It is customary to write $z \in F_E^{(\nu)+}$ as z has FAK, i.e., functional AK. If $F^{(\nu)} = E$, then the FDK-space E is called a FAK(ν)-space.

Theorem 4.7. *Let $E \supset \Phi$ be an FDK-space. Then $F_E^{(\nu)+} = E^{f\beta(\nu)}$.*

Proof. The desired result is obtained by replacing $\mathcal{CS}_\nu$ with $\mathcal{BS}$ in Theorem 4.3. □

Corollary 4.8. *Let E be an FDK-space with $E \supset \Phi$. Then $F^{(\nu)+}$ is equal for all FDK-spaces between $(\overline{\Phi})_E$ and E, that is, if $(\overline{\Phi})_E \subset Y \subset E$, then $F^{(\nu)+}(Y) = F^{(\nu)+}(E)$.*

Corollary 4.9. *Let $E \supset \Phi$ be an FDK-space. Then E has FAK iff $E^f \subset E^{\beta(\nu)}$; i.e., $E^f = E^{\beta(\nu)}$.*

Theorem 4.10. *Let E be an FDK-space and $\overline{\Phi}$ has AK(ν). Then $F^{(\nu)+} = (\overline{\Phi})^{\beta(\nu)\beta(\nu)}$.*

Proof. Since $F^{(\nu)+} = E^{f\beta(\nu)} = (\overline{\Phi})^{f\beta(\nu)} = (\overline{\Phi})^{\beta(\nu)\beta(\nu)}$, and the proof is complete. □

Example 4.1. *i. If $E = \mathcal{BV}_0$, then $S_E^{(\nu)} = W_E^{(\nu)} = B_E = F_E^{(\nu)} = F_E^{(\nu)+} = \mathcal{BV}_0$ and $B_E^+ = \mathcal{BV}$.*

ii. Let $E = \mathcal{L}_u \oplus e$. By $\|e^{(mn)}\| = mn$, $B_E^+ = B_E = \mathcal{L}_u$.

iii. If $E = C_{p0}$, then $e \in F_E^{(\nu)+} \backslash F_E^{(\nu)}$. So $F_E^{(\nu)+} = \mathcal{M}_u$ and $F_E^{(\nu)} = C_{bp0}$.

Clearly, we can see that $\Phi \subset S_E^{(\nu)} \subset W_E^{(\nu)} \subset F_E^{(\nu)}$. If $\nu = r$, then $F_E^{(\nu)} \subset B_E$. But if $\nu = c$, then there is no inclusion between $F_E^{(\nu)}$ and B_E because $\mathcal{BS}$ and $\mathcal{CS}_\nu$ do not contain each other.

Definition 4.5. *Let E be an FDK-space that includes Φ. E is called a ν-semiconservative space, if $E^f \subset \mathcal{CS}_\nu$. That is, $E \supset \Phi$ and for all $f \in E'$, $\nu-\sum f(\delta^{kl})$ is convergent. Moreover, if a semiconservative FDK-space includes the space $\mathcal{BV}$, then the space is called a variational ν-semiconservative space.*

Theorem 4.11. *Let E be an FDK-space, $E \supset \Phi$, $z \in \Omega$, and let z be invertible. Then*

i. $z \in B_E^+$ iff $z^{-1}E \supset \mathcal{BV}_0$. In particular $e \in B_E^+$ iff $E \supset \mathcal{BV}_0$.

ii. $z \in B_E$ iff $z^{-1}E \supset \mathcal{BV}$. In particular $e \in B_E$ iff $E \supset \mathcal{BV}$.

iii. $z \in F_E^{(\nu)+}$ iff $z^{-1}E$ is ν-semiconservative. In particular $e \in F_E^{(\nu)+}$ iff E is ν-semiconservative.

iv. $z \in F_E^{(\nu)}$ iff $z^{-1}E$ is variational ν-semiconservative. In particular $e \in F_E^{(\nu)}$ iff E is variational ν-semiconservative.

v. $z \in W_E^{(\nu)}$ iff $z^{-1}E$ is ν-conull. In particular $e \in W_E^{(\nu)}$ iff E is ν-conull.

vi. $z \in S_E^{(\nu)}$ iff $z^{-1}E$ is strong ν-conull. In particular $e \in S_E^{(\nu)}$ iff E is strong ν-conull.

Proof. i) Let $f \in (z^{-1}E)'$. Then $f(x) = \alpha x + g(zx)$, $\alpha \in \Phi$, $g \in E'$. In particular, if we take $x = \delta^{mn}$, then

$$f(\delta^{mn}) = \alpha_{mn} + z_{mn}g(\delta^{mn}).$$

So, we get

$$z^{-1}E \supset \mathcal{BV}_0 \Leftrightarrow f(\delta^{mn}) \in \mathcal{BS} \Leftrightarrow (z_{mn}g(\delta^{mn})) \in \mathcal{BS} \Leftrightarrow z \in B_E^+.$$

ii) Necessity.

$$\begin{aligned} z \in B_E &\Rightarrow z \in E \text{ and } z \in B_E^+ \\ &\Rightarrow e \in z^{-1}E \text{ and } z^{-1}E \supset \mathcal{BV}_0 \\ &\Rightarrow z^{-1}E \supset \mathcal{BV}. \end{aligned}$$

Sufficiency. Let $z^{-1}E \supset \mathcal{BV}$. Since $\mathcal{BV}_0$, $z^{-1}E \supset \mathcal{BV}_0$, by (i), $z \in B_E^+$ is satisfied. Also, $e \in \mathcal{BV} \Rightarrow e \in z^{-1}E \Rightarrow z \in E$, so $z \in B_E^+ \cap E = B_E$ is obtained.

iii) Let $f \in (z^{-1}E)'$. Then $f(x) = \alpha x + g(zx)$, $\alpha \in \Phi$, $g \in E'$. For $x = \delta^{mn}$, $f(\delta^{mn}) = \alpha_{mn} + z_{mn}g(\delta^{mn})$ holds. Thus, if $z^{-1}E$ is ν-semiconservative, then $\left(z^{-1}E\right)^f \subset \mathcal{CS}_\nu \Leftrightarrow f(\delta^{mn}) \in \mathcal{CS}_\nu \Leftrightarrow (z_{mn}g(\delta^{mn})) \in \mathcal{CS}_\nu$, and so $z \in F_E^{(\nu)+}$ is obtained.

iv)

$$\begin{aligned} z \in F_E^{(\nu)} &\Leftrightarrow z \in F_E^{(\nu)+} \text{ and } z \in E \\ &\Leftrightarrow \left(z^{-1}E\right)^f \subset \mathcal{CS}_\nu \text{ and } e \in z^{-1}E \\ &\Leftrightarrow z^{-1}E \text{ is variational } \nu-\text{semiconservative}. \end{aligned}$$

v) Sufficiency is clear.

Necessity. To prove the necessary, we assume that $z \in W_E^{(\nu)}$. Then $\forall g \in (z^{-1}E)'$, $g\left(z - z^{(mn)}\right) \to 0$, $(m, n \to \infty)$. Furthermore, each $f \in (z^{-1}E)'$ has the representation $f(x) = \alpha x + g(zx)$, $\alpha \in \Phi$, $g \in E'$. So we obtain

$$f(e - e^{(mn)}) = \alpha(e - e^{(mn)}) + g(z(e - e^{(mn)})) \tag{4.1}$$

$$= \sum_{\substack{k=1\\l=n+1}}^{m,\infty} \alpha_{kl} + \sum_{\substack{k=m+1\\l=1}}^{\infty,\infty} \alpha_{kl} + g(z - z^{(mn)}).$$

Because of $\alpha \in \Phi$, the sum of the series on the right side of the above equation is 0. Taking the limit for $m, n \to \infty$ on the two sides of the above equation, we get $f(e - e^{(mn)}) \to 0$, $\forall f' \in (z^{-1}E)$. This proves necessary.

vi) Necessary. Let $z \in S_E^{(\nu)}$. There is a seminorm q on E such that $q(z - z^{(mn)}) \to 0$ $(m, n \to \infty)$. Furthermore, for the seminorms r_{kl} and $h(x) = q(zx)$,

$$r_{kl}(e - e^{(mn)}) = 0 \qquad ((k,l) < (m,n))$$

and

$$h(e - e^{(mn)}) = q(z(e - e^{(mn)})) = q(z - z^{(mn)}) \to 0$$

are obtained. So, $z^{-1}E$ is strong ν-conull.

Sufficiency. Let $z^{-1}E$ be strong ν-conull. Then $r_{kl}(e - e^{(mn)}) \to 0$ and $h(e - e^{(mn)}) \to 0$. Since $h(x) = q(zx)$, $h(e - e^{(mn)}) = q(z - z^{(mn)}) \to 0$ is obtained. This means $z \in S_E^{(\nu)}$. □

Theorem 4.12. *The distinguished subspaces are monotone, that is, if $E_1 \subset E_2$, then $\Psi(E_1) \subset \Psi(E_2)$ where $\Psi = S_E^{(\nu)}, W_E^{(\nu)}, F_E^{(\nu)}, F_E^{(\nu)+}, B_E, B_E^{+}$. This also holds for $\Psi = \overline{\Phi}$, i.e., $(\overline{\Phi})_{E_1} \subset (\overline{\Phi})_{E_2}$.*

Proof. Since the map $i : E_1 \to E_2$ is continuous, $x^{(mn)} \to x$ in E_1 implies the same in E_2. This claim is for $S_E^{(\nu)}$. If we consider $W_E^{(\nu)}$, it follows that i is weakly continuous at the same time.

Now $z \in F_E^{(\nu)+}, B_E^{+}$ if and only if $(z_{mn} f(\delta^{mn})) \in \mathcal{CS}_\nu, \mathcal{BS}$, respectively, for all $f \in E_1'$, and hence for all $g \in E_2'$ since $g|_{E_1} \in E_1'$. The result follows for $F_E^{(\nu)+}, B_E^{+}$ and so for $F_E^{(\nu)}, B_E$. □

Theorem 4.13. *Let $\supset \Phi$ be an FDK-space. The following assertions are equivalent:*

i) E has FAK(ν);
ii) $E \subset (S_E^{(\nu)})^{\beta(\nu)\beta(\nu)}$;
iii) $E \subset (W_E^{(\nu)})^{\beta(\nu)\beta(\nu)}$;
iv) $E \subset (F_E^{(\nu)})^{\beta(\nu)\beta(\nu)}$;
v) $E^{\beta(\nu)} = (S_E^{(\nu)})^{\beta(\nu)}$;
vi) $E^{\beta(\nu)} = (F_E^{(\nu)})^{\beta(\nu)}$.

Proof. ($ii \Rightarrow iii$) and ($iii \Rightarrow iv$) are clear since $S_E^{(\nu)} \subset W_E^{(\nu)} \subset F_E^{(\nu)}$. If (iv) is true, then $E^{\beta(\nu)} \supset (F_E^{(\nu)})^{\beta(\nu)} = E^{f\beta(\nu)\beta(\nu)} \supset E^f$ so (i) is true by Corollary 4.9. If (i) is true, Theorem 4.10 implies that $S = \overline{\Phi}$ and that (ii) is true. The equivalence of (v), (vi) with the others is clear. □

5. Matrix domains

The original ground space of summability is $C_{\nu A}^{(\nu)}$. In this section, we discuss $E_A^{(\nu)}$. Its properties depend on the choice of E, ν, and A; our procedure will be to fix E and discuss how the properties of $E_A^{(\nu)}$ depend on those of ν and A. This discussion will depend on which E is chosen.

Remark 5.1. *In this section, $z \in \Omega$, E is an FDK-spce, and A is a four-dimensional matrix such that $E_A^{(\nu)} \supset \Phi$, i.e., the columns of A belong to E. The subspaces $S_E^{(\nu)}, W_E^{(\nu)}, F_E^{(\nu)}, B_E$ are calculated in the FDK-space $E_A^{(\nu)}$.*

Lemma 5.1. *With the notation of Remark 5.1, $Az^{(ij)} = \sum_{k,l=(1,1)}^{(i,j)} z_{kl}\zeta_A^{(kl)}$,*

$$(Az^{(ij)})_{mn} = \sum_{k,l=(1,1)}^{(i,j)} a_{mnkl}z_{kl} = \left(\sum z_{kl}\zeta_A^{(kl)}\right)_{mn}.$$

Theorem 5.1. *With z, E, A as in Remark 5.1, these are equivalent:*

i) $z \in B^+$,

ii) $\{Az^{(ij)}\}$ is bounded in E,

iii) $E_{A.z}^{(\nu)} \supset \mathcal{BV}_0$,

iv) $\{z_{kl}.g(\zeta_A^{(kl)})\} \in \mathcal{BS}$ for each $g \in E'$.

Also, these are equivalent: $z \in B$ and $E_{A.z}^{(\nu)} \supset \mathcal{BV}$, (ii) and $z \in E_A^{(\nu)}$, and (iv) and $z \in E_A^{(\nu)}$.

Proof. By Theorem 4.11, $z \in B^+ \Leftrightarrow z^{-1}.E_A^{(\nu)} \supset \mathcal{BV}_0 \Leftrightarrow E_{A.z}^{(\nu)} \supset \mathcal{BV}_0$. So we get $i \equiv iii$. Since the (k,l)th column of $A.z$ is $z_{kl}\zeta_A^{(kl)}$, $iii \equiv ii$ is obtained by the last part of 8.6.4 of [23]. Also, (ii) is true iff $g(Az^{(ij)})$ is bounded for each $g \in E'$. This gives $ii \equiv iv$. The second set of equivalences is clear since $z \in E_A^{(\nu)} \Leftrightarrow e \in E_{A.z}^{(\nu)}$. □

Theorem 5.2. *Let E be an FDK-space, A be a four-dimensional matrix, and $\nu \in \{r, c\}$. Then $E_A^{(\nu)}$ is ν-semiconservative iff $\zeta_A^{(kl)} \in E$ and $g(\zeta_A^{(kl)}) \in \mathcal{CS}_\nu$ for each $g \in E'$.*

Proof. Necessity. It is clear that $\zeta_A^{(kl)} \in E$ by being ν-semiconservative. Given g, let $f(x) = g(Ax)$ for $x \in E_A^{(\nu)}$, so $f \in (E_A^{(\nu)})'$ by Theorem 2.2. Then $f(\delta^{kl}) = g(A\delta^{kl}) = g(\zeta_A^{(kl)})$, and the result follows.

Sufficiency. First, $\xi_A^{(mn)} \in \mathcal{CS}_\nu$ by the hypothesis and we can take $g = P_{mn}$ where $P_{mn}(x) = x_{mn}$; this yields $\{g(\zeta_A^{(kl)})\} = \{a_{mnkl}\}$. Hence $\Omega_A^{(\nu)} \supset \mathcal{BV}$.

Now, let $f \in (E_A^{(\nu)})'$. Then by Theorem 2.2, $f(x) = h(x) + g(Ax)$ with $g \in E$, $h \in (\Omega_A^{(\nu)})'$. Also, $h(x) = \nu - \sum_{k,l} u_{kl}x_{kl}$ with $x \in \Omega_A^{(\nu)}$, $u \in (\Omega_A^{(\nu)})^{\beta(\nu)} \subset \mathcal{BV}^{\beta(\nu)} = \mathcal{CS}_\nu$, by Theorem 2.1. Thus

$$\begin{aligned} f(\delta^{kl}) &= h(\delta^{kl}) + g(A\delta^{kl}) \\ &= h(\delta^{kl}) + g(\zeta_A^{(kl)}). \end{aligned}$$

By the hypothesis and the fact that $u \in \mathcal{CS}_\nu$ we have $\{f(\delta^{kl})\} \in \mathcal{CS}_\nu$. Hence $E_A^{(\nu)}$ is ν-semiconservative. □

Theorem 5.3. *With z, E, A as in Remark 5.1, these are equivalent:*

i) $z \in F^{(\nu)+}$,

ii) $\{Az^{(ij)}\}$ is weakly Cauchy in E, i.e., $\{g[Az^{(ij)}]\} \in C_\nu$ for each $g \in E'$,

iii) E_{Az} is ν-semiconservative,

iv) $\{z_{kl}.g(\zeta_A^{(kl)})\} \in \mathcal{CS}_\nu$ for each $g \in E'$.

Proof. $i \equiv iii$ is obtained by Theorem 4.11; $iii \equiv iv$ follows from 9.4.1 of [23]; $ii \equiv iv$ follows from Lemma 5.1. □

Theorem 5.4. *Let E be an FDK-space, $\nu \in \{r, c\}$, and A is a four-dimensional matrix such that $E_A^{(\nu)}$ is variational $\nu-$semiconservative. Then $E_A^{(\nu)}$ is ν-conull iff $\sum g(\zeta_A^{(kl)}) = g(Ae)$ for each $g \in E'$.*

Proof. Necessity. Let $f(x) = g(Ax)$ so $f \in (E_A^{(\nu)})'$ by Theorem 2.2. Then

$$g(Ae) = f(e) = \lim_{m,n} f(e^{(mn)}) = \lim_{m,n} g(Ae^{(mn)}) = \lim_{m,n} g\left(\sum_{k,l=1}^{m,n} a_{mnkl}\right)$$
$$= g\left(\lim_{m,n} \sum_{k,l=1}^{m,n} a_{mnkl}\right) = g\left(\sum_{k,l=1}^{\infty} a_{mnkl}\right) = g(\zeta_A^{(kl)}).$$

Sufficiency. Let $f \in (E_A^{(\nu)})'$. By Theorem 2.2 there are two cases to consider. First, $f(x) = h(x)$, $x \in \Omega_A^{(\nu)}$, $u \in (\Omega_A^{(\nu)})^{\beta(\nu)}$. On the other hand, $E_A^{(\nu)} \supset \mathcal{BV}$ by the definition of a variational ν-semiconservative space. So

$$f(e - e^{(mn)}) = \nu \sum_{\substack{k=m+1\\ l=1}}^{\infty} u_{kl} + \nu - \sum_{\substack{k=1\\ l=n+1}}^{m,\infty} u_{kl} \to 0.$$

Second, $f(x) = g(Ax)$ for which the calculation given in the first part shows $f(e - e^{(mn)}) \to 0$. □

Theorem 5.5. *With z, E, A as in Remark 5.1, these are equivalent:*
i) $z \in W^{(\nu)}$,
ii) $Az^{(ij)} \to Az$ *weakly in* E,
iii) E_{Az} *is* ν*-conull,*
iv) $\sum z_{kl} g(\zeta_A^{(kl)}) = g(Az)$ *for each* $g \in E'$.

Proof. $i \equiv iii$ follows from Theorem 4.11; $ii \equiv iii$ follows from Theorem 9.4.9 of [23]; $ii \equiv iv$ follows from Lemma 5.1. □

Theorem 5.6. *With z, E, A as in Remark 5.1, these are equivalent:*
i) $z \in S^{(\nu)}$,
ii) $Az^{(ij)} \to Az$ *in* E,
iii) E_{Az} *is strongly* ν*-conull,*
iv) $\sum z_{kl}\zeta_A^{(kl)} = Az$ *convergence in* E.

Proof. $i \equiv iii$ is obtained by Theorem 4.11; $ii \equiv iv$ follows from Lemma 5.1.

($i \Rightarrow ii$) $z = \sum z_{kl}\delta^{kl}$ and the map $A : E_A \to E$ is continuous, so $Az = \sum z_{kl} A\delta^{kl} = \sum z_{kl}\zeta_A^{(kl)}$.

($ii \Rightarrow i$) Ω_A has AK(ν) by 4.3.8 of [23], therefore $u(z - z^{(ij)}) \to 0$ for any $z \in \Omega_A$. Thus $z \in S^{(\nu)}$ if $q[A(z - z^{(ij)})] \to 0$ where q is a typical seminorm of Y. But this is simply $Az^{(ij)} \to Az$ in E. □

6. Applications of ν-conull FDK-spaces to summability domains

In the last section, we examined the applications of some of the results we obtained in the main section on summability domains.

Theorem 6.1. *Let C_ν and E be FDK-spaces, $\nu \in \{r, c\}$, and $A = (a_{mnkl})$ is a four-dimensional matrix. Then the following statements are equivalent:*

i) $E_A^{(\nu)}$ *is a strongly* ν*-conull FDK-space;*
ii) $\mathcal{BV}(\varphi) \subset E_A^{(\nu)}$ *and* $A : \mathcal{BV}(\varphi) \to E$ *is compact;*
iii) $\zeta_A^{(kl)} \in E$ $(k, l = 1, 2, \ldots)$ *and* $A(e - \sum \delta^{kl}) \to 0$ *in* E.

Proof. ($i \Rightarrow ii$) By Theorem 3.3, $\mathcal{BV}(\varphi) \subset E_A^{(\nu)}$, and the map $I : \mathcal{BV}(\varphi) \to E_A^{(\nu)}$ is compact. So $I \circ A : \mathcal{BV}(\varphi) \to E$ is compact, since the mapping $A : E_A^{(\nu)} \to E$ is continuous.

($ii \Rightarrow iii$) By (ii) and $\delta^{kl} \in \mathcal{BV}(\varphi)$ $(\forall k, l)$, we obtain $\zeta_A^{(kl)} = A(\delta^{kl}) \in E$, $\forall k, l \geq 1$. Furthermore, since $\Omega_A^{(\nu)}$ is an AK(ν) space and $e \in \mathcal{BV}(\varphi) \subset E_A^{(\nu)} \subset \Omega_A^{(\nu)}$, we have $e - \sum_{k,l=1}^{i,j} \delta^{kl} \to 0$ $(i, j \to \infty)$. Also, since the mapping $A : \Omega_A^{(\nu)} \to \Omega$ is continuous, we obtain $A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right) \to 0$ $(i, j \to \infty)$. Then the set $\left\{e - \sum_{k,l=1}^{i,j} \delta^{kl} : \ i, j \geq 1\right\}$ is bounded in $\mathcal{BV}(\varphi)$. Because $A : \mathcal{BV}(\varphi) \to E$ is compact, $\left\{A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right) : \ i, j \geq 1\right\}$ is relatively compact in E. Hence, the coordinat-wise convergence topology and the topology of E are coincident. So, $\left\{A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right)\right\}$ in E also converges to zero.

($iii \Rightarrow i$) If $\zeta_A^{(kl)} \in E$, then $E_A^{(\nu)} \supset \Phi$. Because the sequence $\left\{A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right)\right\}$ converges to zero, $E_A^{(\nu)}$ is a strongly ν-conull FDK-space. □

Theorem 6.2. *Let* C_ν *and* E *be FDK-spaces,* $\nu \in \{r, c\}$*, and* $A = (a_{mnkl})$ *is a four-dimensional matrix. The following statements are equivalent:*
i) $E_A^{(\nu)}$ *is a* ν*-conull FDK-space;*
ii) $\mathcal{BV}(\varphi) \subset E_A^{(\nu)}$ *and* $A : \mathcal{BV}(\varphi) \to E$ *is weakly compact;*
iii) $\zeta_A^{(kl)} \in E$ $(k, l = 1, 2, \ldots)$ *and* $A(e - \sum \delta^{kl}) \to 0$ *(weakly) in* E.

Proof. ($i \Rightarrow ii$) By Theorem 3.5, $\mathcal{BV}(\varphi) \subset E_A^{(\nu)}$ and the map $I : \mathcal{BV}(\varphi) \to E_A^{(\nu)}$ is weakly compact. So $I \circ A : \mathcal{BV}(\varphi) \to E$ is weakly compact, since the mapping $A : E_A^{(\nu)} \to E$ is continuous.

($ii \Rightarrow iii$) By hypothesis, $\delta^{kl} \in \mathcal{BV}(\varphi)$, $\forall k, l \geq 1$, and $\zeta_A^{(kl)} = A(\delta^{kl}) \in E$, $\forall k, l \geq 1$. Furthermore, since $\Omega_A^{(\nu)}$ is an AK(ν) space and $e \in \mathcal{BV}(\varphi) \subset E_A^{(\nu)} \subset \Omega_A^{(\nu)}$, we have $e - \sum_{k,l=1}^{i,j} \delta^{kl} \to 0$ $(i, j \to \infty)$. Also, since the mapping $A : \Omega_A^{(\nu)} \to \Omega$ is continuous, we obtain $A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right) \to 0$ $(i, j \to \infty)$. Then the set $\left\{e - \sum_{k,l=1}^{i,j} \delta^{kl} : \ i, j \geq 1\right\}$ is bounded in $\mathcal{BV}(\varphi)$. Because $A : \mathcal{BV}(\varphi) \to E$ is weakly compact, $\left\{A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right) : \ i, j \geq 1\right\}$ is weakly relatively compact in E. Hence, the coordinat-wise convergence topology and the topology of E are coincident. So $\left\{A\left(e - \sum_{k,l=1}^{i,j} \delta^{kl}\right)\right\}$ in E also converges to zero. □

Theorem 6.3. *Let* A *be a four-dimensional matrix, and* E *be an FDK-space. It is equivalent for the space* $E_A^{(\nu)}$ *to be a* ν*-conull FDK-space and a strongly* ν*-conull FDK-space whenever weak convergence and strong convergence coincide.*

Proof. Let $E_A^{(\nu)}$ be a strongly ν-conull FDK-space. Then the columns of A are in E and $\{A\left(e - \sum \delta^{kl}\right)\} \to 0$. By hypothesis, the columns of A are weakly convergent in E. That is, $E_A^{(\nu)}$ is a ν-conull FDK-space. □

For example, if we choose $E = \mathcal{L}_u$, $\mathcal{BV}$, we obtain the following results by Theorem 6.3, since weak convergence and strong convergence coincide in these spaces.

Theorem 6.4. *Let* A *be a four-dimensional matrix.*
i. $(\mathcal{L}_u)_A$ *is a (strongly)* ν*-conull FDK-space iff*

$$\lim_{k,l} \sum_{m,n} \left| \sum_{i=k, j=1}^{\infty, l} a_{mnij} + \sum_{i=1, j=l}^{\infty, \infty} a_{mnij} \right| = 0.$$

ii. $\mathcal{BV}_A$ is a (strongly) ν-conull FDK-space iff

$$\lim_{k,l}\sum_{m,n}\left|\sum_{i,j=1}^{\infty,\infty}(a_{mnij}-a_{m+1,n,i,j}-a_{m,n+1,i,j}+a_{m+1,n+1,i,j})\right.$$
$$\left.-\sum_{i,j=1}^{k,l}(a_{mnij}-a_{m+1,n,i,j}-a_{m,n+1,i,j}+a_{m+1,n+1,i,j})\right|=0.$$

Proof. (i) Let $E_A^{(\nu)}=(\mathcal{L}_u)_A$ in Theorem 6.1. Then $(\mathcal{L}_u)_A$ is a (strongly) ν-conull FDK-space if and only if $A(e-\sum\delta^{kl})$ converges to zero in $\mathcal{L}_u$, which means

$$\lim_{k,l}\left\|(A(e-\sum\delta^{kl}))_{mn}\right\|_{\mathcal{L}_u}=0\Leftrightarrow\lim_{k,l}\sum_{m,n}\left|\sum_{i=k,j=1}^{\infty,l}a_{mnij}+\sum_{i=1,j=l}^{\infty,\infty}a_{mnij}\right|=0.$$

(ii) Let $E_A^{(\nu)}=\mathcal{BV}_A$ in Theorem 6.1. Then $\mathcal{BV}_A$ is a (strongly) ν-conull FDK-space if and only if $A(e-\sum\delta^{kl})$ converges to zero in $\mathcal{BV}$, which means

$$\lim_{k,l}\left\|(A(e-\sum\delta^{kl}))_{mn}\right\|_{\mathcal{BV}}=0\Leftrightarrow\lim_{k,l}\left\|\sum_{i,j=1}^{\infty}a_{mnij}-\sum_{i,j=1}^{k,l}a_{mnij}\right\|_{\mathcal{BV}}=0$$
$$\Leftrightarrow\lim_{k,l}\sum_{m,n}\left|\sum_{i,j=1}^{\infty,\infty}(a_{mnij}-a_{m+1,n,i,j}-a_{m,n+1,i,j}+a_{m+1,n+1,i,j})\right.$$
$$\left.-\sum_{i,j=1}^{k,l}(a_{mnij}-a_{m+1,n,i,j}-a_{m,n+1,i,j}+a_{m+1,n+1,i,j})\right|=0.$$

□

7. Conclusions

In this paper, we have introduced and studied several new structural properties of FDK-spaces related to ν-conullity and ν-wedge constructions. We established the equivalence between strongly ν-conull FDK-spaces and ν-wedge FDK-spaces, and between ν-conull and weak ν-wedge spaces. These results clarify how the transformation operator $S^{(2)}$ preserves or modifies the topological character of a given space.

In the later sections, we examined the distinguished subspaces associated with an FDK-space E, including $S_E^{(\nu)}$ and $W_E^{(\nu)}$, and discussed their inclusion relations. This analysis highlights that the structure of E, rather than its dual, determines the behavior of these distinguished components. Furthermore, we proved that if E is a ν-wedge FDK-space, then the intersection $E\cap(C_{p0}/\mathcal{BV})$ is nonempty, revealing a nontrivial relation between ν-wedge properties and spaces of bounded variation.

Overall, the results provide a unified framework connecting the notions of ν-conullity, wedge-type constructions, and distinguished subspaces in the setting of double sequence spaces. They also extend classical results from FK-space theory to the broader context of FDK-spaces.

Recent Various kinds of methods have been resolved successfully by building various approaches for convergence in recent times [24]. Future work can consider the preservation of some important physical properties and physical structures with particular conditions and refer to recent work [25].

Author contributions

Şeyda Sezgek: Conceptualization, methodology, validation, formal analysis, writing-original draft, writing-review and editing; İlhan Dağadur: Supervision, methodology, simulation, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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