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Non-uniform dependence for the inviscid Boussinesq equations in Besov spaces

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Abstract: This paper addresses the initial value problem to the inviscid Boussinesq equations in $\mathbb{R}^2$. We rigorously show that the data-to-solution map fails to be uniformly continuous within a broad class of nonhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^2)$ cited in [20] (i.e., $s > 1 + \frac{2}{p}, 1 < p < \infty, 1 \le r \le \infty$ or $s = 1 + \frac{2}{p}, 1 < p < \infty, r = 1$). This result partially extends the nowhere uniform continuity previously demonstrated by Inci [9] in the Sobolev spaces $H^m(\mathbb{R}^2)$ with $m > 2$. Our proof leverages the interaction between terms of low and high frequencies. Besides, the linearized system to the inviscid Boussinesq equations plays a pivotal role in the construction of appropriate approximate solutions.

Keywords: Boussinesq equations; Cauchy problem; non-uniform continuous dependence; Besov spaces; approximate solutions
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1. Introduction and main results

Let us consider the inviscid Boussinesq system in $\mathbb{R}^2$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \theta e_2, & x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^2,\ t > 0, \\ (u, \theta)|_{t=0} = (u_0, \theta_0), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

with $\nabla \cdot u_0 = 0$. Here, $u = (u_1, u_2)$, θ, and P denote the velocity field of the fluid, the temperature and the scalar pressure, respectively, and $e_2 = (0, 1)^\top$ is the unit vector. The system achieved widespread adoption for its ability to model the dynamics of both the atmosphere and the ocean, which are domains characterized by the prominent influence of rotation and stratification [4, 15, 16].

In the Hadamard sense [6], a Cauchy problem exhibits local well-posedness in a Banach space X if, for every initial datum in X, a unique solution is guaranteed to exist on a time interval $[0, T]$, belongs to $C([0, T]; X)$, and continuously depends on the given initial datum. The continuity properties of this solution map for the evolution system are crucial to the theory of well-posedness, as discontinuous dependence may lead to the generation of fallacious or non-physical solutions. One of the pioneering concerns about this local-in-time theory was given by Kato [10], who showed that the solution map failed to be uniformly continuous for the inviscid Burgers equation in $H^m(\mathbb{R})$ with $m > 3/2$.

Concerning the incompressible Euler equations, Himonas and Misiołek [7] first proved the non-uniform continuity in $H^m(\mathbb{T}^d)$ and $H^m(\mathbb{R}^d)$ with $m > 0$. Then Bourgain and Li [3] determined the outcome for the borderline case $m = 0$. As an extension to [7], Tang and Liu [18] constructed a family of periodic solutions and showed the nonuniform continuity of the data-to-solution map in $B^s_{2,r}(\mathbb{T}^d)$ for any $s \in \mathbb{R}$ and $r \in [1, \infty]$. Subsequently, Pastrana [17] broadened the periodic result of [18] to encompass cases where $p \neq 2$ and investigated the non-periodic case by employing the approximate solutions technique as developed in [7]. More recently, Li et al. [12] improved upon prior results in Besov spaces and studied the situation within the whole space.

Regarding to the inviscid Boussinesq equations (1.1), there have been extensive studies (see e.g., [2, 5, 8, 11, 13, 14, 20]). The local-in-time Hadamard well-posedness of (1.1) is well-known in $H^m(\mathbb{R}^2)$ with $m > 2$(see [5]). Using a geometric approach, Inci [9] proved that this corresponding data-to-solution map is nowhere locally uniformly continuous. Subsequently, Yuan [20] and Bie et al. [2] established the local existence and blow-up criteria for (1.1) in the nonhomogeneous Besov spaces, which we recall here.

Theorem 1.1 ([2, 20]). *Assume that (s, p, r) satisfies*

$$s > 1 + \frac{2}{p}, 1 < p < \infty, 1 \leq r \leq \infty \text{ or } s = 1 + \frac{2}{p}, 1 < p < \infty, r = 1 \tag{1.2}$$

and the initial data (u_0, θ_0) belongs to

$$B_R := \{(U, \Theta) \in B^s_{p,r} \times B^s_{p,r} : \|(U, \Theta)\|_{B^s_{p,r}} \leq R, \nabla \cdot U = 0\},$$

for any $R > 0$. Then, there exists some $T = T(R, s, p, r) > 0$ such that the system (1.1) admits a unique solution $(u, \theta) \in C([0, T], B^s_{p,r} \times B^s_{p,r})$ that satisfies the following:

$$\sup_{t\in[0,T]} \|(u, \theta)(t)\|_{B^s_{p,r}} \leq C\|(u_0, \theta_0)\|_{B^s_{p,r}}.$$

Based on this local result, our objective is to establish the non-uniform continuity of the data-to-solution map in Besov spaces. The following describes our main result.

Theorem 1.2. *Assume that (s, p, r) satisfies condition (1.2). Then, the data-to-solution map $(u_0, \theta_0) \mapsto (u, \theta)(t)$ for the system (1.1) is not uniformly continuous from any bounded subset in $B^s_{p,r} \times B^s_{p,r}$ into $C([0, T], B^s_{p,r} \times B^s_{p,r})$. More precisely, there exist two sequences of solutions $(u_{i,n}, \theta_{i,n})(i = 1, 2)$ such that*

- $\|(u_{1,n}, \theta_{1,n})(t)\|_{B^s_{p,r}} + \|(u_{2,n}, \theta_{2,n})(t)\|_{B^s_{p,r}} \lesssim 1,$
- $\lim_{n\to\infty} \|(u_{1,n}, \theta_{1,n})(0) - (u_{2,n}, \theta_{2,n})(0)\|_{B^s_{p,r}} = 0,$

- $\liminf_{n\to\infty} \|(u_{1,n},\theta_{1,n})(t)-(u_{2,n},\theta_{2,n})(t)\|_{B^s_{p,r}} \gtrsim t,$

for any $t\in[0,T^*]$ with $T^*>0$ sufficiently small.

Remark 1.3. This theorem partially covers the nowhere uniform continuity given by Inci [9] in the Sobolev spaces $H^m(\mathbb{R}^2)$ with $m>2$. It is worth noting that the nonuniform continuity presented in Theorem 1.2 is confined to a bounded set near the origin. Significantly, if r is finite, then this restriction can be removed. By adjusting the construction of the initial sequences, we can obtain the nowhere uniform continuity in the Besov spaces (i.e., for any initial data $(u_0,\theta_0)\in B^s_{p,r}$, the data-to-solution map, within any neighborhood $U(u_0,\theta_0)\subseteq B^s_{p,r}\times B^s_{p,r}$ of (u_0,θ_0), is not uniformly continuous). However, we shall not continue pursuing this issue here.

Sketch of the proof. Inspired by [12] and [19], the proof of nonuniform continuity depends on the interaction between terms of low and high frequencies.

First, we modify the construction of the initial data in [12] and set two bounded sequences of the initial data $(0,f_n)$ and (g_n,f_n) (see (3.1)–(3.2)), which stay arbitrarily close in $B^s_{p,r}$ and generate solutions $(u_{1,n},\theta_{1,n})$ and $(u_{2,n},\theta_{2,n})$ to (1.1), respectively. Then we claim that $(u_{1,n},\theta_{1,n})$ and $(u_{2,n},\theta_{2,n})$ remain apart for any $t>0$ sufficiently small, i.e.,

$$\liminf_{n\to\infty} \|(u_{1,n},\theta_{1,n})(t)-(u_{2,n},\theta_{2,n})(t)\|_{B^s_{p,r}} \gtrsim t, \tag{1.3}$$

which definitely leads to the desired non-uniform continuity.

To achieve (1.3), we shall approximate solutions to (1.1) using its linearized system (see (3.9)) and apply the fact that the interaction between terms of low and high frequencies $g_n\cdot\nabla f_n$ would not be small as $n\to\infty$ (see Lemma 3.2).

Remark 1.4. Compared with [12], we shall approximate solutions to (1.1) by using solutions to the corresponding linearized system (see (3.9)), which brings an obstacle to proving that the mentioned approximate solutions can approximate solutions to (1.1).

Organization of this paper. Section 2 covers relevant notations and basic results from the Littlewood-Paley theory. Section 3 is devoted to constructing approximate solutions for (1.1) and presenting fundamental lemmas which involve the linearized system and error estimates. Relying on these error estimates, the proof of Theorem 1.2 is subsequently delivered in Section 4.

2. Preliminaries

First, we present some notations which shall be used throughout this paper.

- The symbol $A\lesssim(\gtrsim)B$ represents that there exists a constant $c>0$ independent of A and B such that $A\le(\ge)cB$.
- Let X be a Banach space endowed with the norm $\|\cdot\|_X$ and $I\in\mathbb{R}$. The notation $C(I,X)$ denotes the continuous maps on I with values in X. For $f_1,\cdots,f_N\in X$, we use the following simplified notation:
$$\|(f_1,\cdots,f_N)\|_X := \|f_1\|_X+\cdots+\|f_N\|_X.$$

- For all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^2)$, the Fourier transform of f is defined by the following:

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^2} f(x)e^{-ix\cdot\xi}\,dx, \text{ for any } \xi \in \mathbb{R}^2.$$

The inverse Fourier transform recovers f from $\hat{f}$, which is defined by the following:

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) := (2\pi)^{-2}\int_{\mathbb{R}^2} \hat{f}(\xi)e^{ix\cdot\xi}\,d\xi, \text{ for any } x \in \mathbb{R}^2.$$

- For $m \in \mathbb{R}$, the standard Sobolev space $H^m(\mathbb{R}^2)$ consists of all tempered distributions f such that

$$\|f\|_{H^m} := (\int_{\mathbb{R}^2}(1+|\xi|^2)^m|\hat{f}(\xi)|^2\,d\xi)^{1/2} < \infty.$$

We proceed by introducing the theory of Littlewood-Paley and the nonhomogeneous Besov spaces, along with some relevant properties. For a detailed exposition, one can be directed to [1].

Proposition 2.1 (Littlewood-Paley decomposition). *Let $\mathcal{B} := \{\xi \in \mathbb{R}^2 : |\xi| \leq \frac{4}{3}\}$ be a ball in $\mathbb{R}^2$ and $C := \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ be an annulus in $\mathbb{R}^2$. There exist two smooth radial functions, χ and φ, valued in the interval* $[0,1]$ *such that*

- *χ is supported in $\mathcal{B}$,*
- *φ is supported in C,*
- $\chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1$ *for any $\xi \in \mathbb{R}^2$, and*
- $\frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0}\varphi^2(2^{-j}\xi) \leq 1$ *for any $\xi \in \mathbb{R}^2$.*

In particular, the ensuing Littlewood-Paley decomposition is well-defined as follows:

$$f = \sum_{j\geq -1}\Delta_j f,$$

for any tempered distributions f, where the nonhomogeneous dyadic blocks Δ_j are defined by the following:

$$\Delta_j f := \begin{cases} 0, & \text{if } j \leq -2,\\ \chi(D)f = \mathcal{F}^{-1}(\chi\mathcal{F}f), & \text{if } j = -1,\\ \varphi(2^{-j}D)f = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), & \text{if } j \geq 0. \end{cases}$$

Definition 2.2 (Besov spaces). *Let $\sigma \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^\sigma_{p,r}(\mathbb{R}^2)$ consists of all tempered distributions f such that*

$$\|f\|_{B^\sigma_{p,r}} < \infty,$$

where

$$\|f\|_{B^\sigma_{p,r}} := \begin{cases} (\sum_{j\geq -1} 2^{j\sigma r}\|\Delta_j f\|^r_{L^p})^{1/r}, & \text{if } 1 \leq r < \infty,\\ \sup_{j\geq -1} 2^{j\sigma}\|\Delta_j f\|_{L^p}, & \text{if } r = \infty. \end{cases}$$

Lemma 2.3 (Properties of Besov spaces)**.**

(1) *Let s_1 and s_2 be real numbers such that $s_1 < s_2$, $\theta \in (0,1)$ and $1 \le p, r \le \infty$. Then the following holds:*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|f\|^{\theta}_{B^{s_1}_{p,r}} \|f\|^{1-\theta}_{B^{s_2}_{p,r}}. \tag{2.1}$$

(2) *For $\sigma > \frac{2}{p}$ with $1 \le p, r \le \infty$, or $\sigma = \frac{2}{p}$ with $r = 1, 1 \le p \le \infty$, there holds the following:*

$$B^{\sigma}_{p,r} \hookrightarrow L^{\infty}.$$

(3) *For $\sigma > 0$ and $1 \le p, r \le \infty$, the space $L^\infty \cap B^\sigma_{p,r}$ is an algebra, and a constant $C = C(\sigma, p, r) > 0$ exists such that*

$$\|fg\|_{B^\sigma_{p,r}} \le C(\|f\|_{L^\infty}\|g\|_{B^\sigma_{p,r}} + \|g\|_{L^\infty}\|f\|_{B^\sigma_{p,r}}).$$

The following lemma is devoted to the 2-D transport equation:

$$\begin{cases} \partial_t f + v \cdot \nabla f = g, \\ f|_{t=0} = f_0, \end{cases} \tag{2.2}$$

where $v : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ and $f_0 : \mathbb{R}^2 \to \mathbb{R}$ are given.

Lemma 2.4. *Let $1 \le p, r \le \infty$, either*

$$\sigma > -2\min\{\frac{1}{p}, 1 - \frac{1}{p}\}$$

or

$$\sigma > -1 - 2\min\{\frac{1}{p}, 1 - \frac{1}{p}\}, \text{ if } \nabla \cdot v = 0.$$

For any smooth solution to (2.2), there exists a constant $C = C(\sigma, p, r) > 0$ such that

$$\sup_{s\in[0,t]} \|f(s)\|_{B^\sigma_{p,r}} \le C\exp(CV_p(v,t))(\|f_0\|_{B^\sigma_{p,r}} + \int_0^t \|g(\tau)\|_{B^\sigma_{p,r}}\, d\tau),$$

where

$$V_p(v,t) = \begin{cases} \int_0^t \|\nabla v(s)\|_{B^{\frac{2}{p}}_{p,\infty}\cap L^\infty}\, d\tau, & \text{if } \sigma < 1 + \frac{2}{p}, \\ \int_0^t \|\nabla v(s)\|_{B^\sigma_{p,r}}\, d\tau, & \text{if } \sigma = 1 + \frac{2}{p} \text{ and } r > 1, \\ \int_0^t \|\nabla v(s)\|_{B^{\sigma-1}_{p,r}}\, d\tau, & \text{if } \sigma > 1 + \frac{2}{p} \text{ or } \{\sigma = 1 + \frac{2}{p} \text{ and } r = 1\}. \end{cases}$$

3. Approximate solutions

3.1. Construction of approximate solutions

Let $\hat{\phi} \in C_0^\infty(\mathbb{R})$ be a fixed, even, and non-negative real-valued function which satisfies the following:

$$\hat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{16}, \\ 0, & |\xi| \ge \frac{1}{8}. \end{cases}$$

Define

$$f_n(x_1, x_2) = 2^{-ns}\phi(x_1)\sin(2^n x_1)\phi(x_2) \tag{3.1}$$

and

$$g_n(x_1, x_2) = 2^{-n}\nabla^{\perp}(\phi(x_1)\phi(x_2)). \tag{3.2}$$

Then we collect some useful lemmas on the properties of f_n and g_n which will be used later.

Lemma 3.1. *Assume that (s, p, r) satisfies condition (1.2) and f_n is defined in (3.1). Then, for any $\sigma \in \mathbb{R}$ and $k \in \{0, 1\}$, the following holds:*

$$\|\nabla^k f_n\|_{L^\infty} \lesssim 2^{n(k-s)} \tag{3.3}$$

and

$$\|f_n\|_{B^{\sigma}_{p,r}} \lesssim 2^{n(\sigma-s)}. \tag{3.4}$$

Proof. It is obvious that (3.3) can be directly obtained from the definition of f_n. It suffices to prove (3.4). Note that

$$\widehat{f_n}(\xi_1, \xi_2) = 2^{-ns-1}i[\hat{\phi}(\xi_1 + 2^n) - \hat{\phi}(\xi_1 - 2^n)]\hat{\phi}(\xi_2),$$

which implies that

$$\operatorname{supp}\widehat{f_n} \subseteq \{\xi \in \mathbb{R}^2 : 2^n - \frac{1}{4} \leq |\xi| \leq 2^n + \frac{1}{4}\},$$

and

$$\Delta_j f_n = \begin{cases} f_n, & \text{if } j = n, \\ 0, & \text{if } j \neq n. \end{cases}$$

Thus, we deduce that

$$\begin{aligned} \|f_n\|_{B^{\sigma}_{p,r}} &= 2^{\sigma n}\|f_n\|_{L^p} \\ &= 2^{n(\sigma-s)}\|\phi(x_1)\sin(2^n x_1)\phi(x_2)\|_{L^p} \\ &\lesssim 2^{n(\sigma-s)}, \end{aligned}$$

which is the desired formula (3.4). □

Lemma 3.2. *Assume that (s, p, r) satisfies condition (1.2). Let f_n and g_n be defined in (3.1) and (3.2), respectively. Then, for any $\sigma \in \mathbb{R}$, the following holds:*

$$\|g_n\|_{B^{\sigma}_{p,r}} \lesssim 2^{-n}. \tag{3.5}$$

Furthermore, there exists a constant $M > 0$ such that

$$\liminf_{n\to\infty} \|g_n \cdot \nabla f_n\|_{B^{s}_{p,r}} \geq M. \tag{3.6}$$

Proof. Since (3.5) can be directly obtained from the definition of g_n, we shall only focus on proving (3.6). Note that

$$\operatorname{supp}\widehat{g_n} \subseteq \{\xi \in \mathbb{R}^2 : 0 \leq |\xi| \leq \frac{1}{4}\}$$

and

$$\operatorname{supp}\widehat{f_n} \subseteq \{\xi \in \mathbb{R}^2 : 2^n - \frac{1}{4} \leq |\xi| \leq 2^n + \frac{1}{4}\},$$

which obviously imply that

$$\text{supp}\, \widehat{g_n \cdot \nabla f_n} \subseteq \{\xi \in \mathbb{R}^2 : 2^n - \frac{1}{2} \leq |\xi| \leq 2^n + \frac{1}{2}\}$$

and

$$\Delta_j(g_n \cdot \nabla f_n) = \begin{cases} g_n \cdot \nabla f_n, & \text{if } j = n, \\ 0, & \text{if } j \neq n. \end{cases}$$

Thus, there holds

$$\begin{aligned} \|g_n \cdot \nabla f_n\|_{B^s_{p,r}} &= 2^{ns}\|g_n \cdot \nabla f_n\|_{L^p} \\ &\geq 2^{ns}\|2^{-ns}\phi^2(x_1)\cos(2^n x_1)\phi(x_2)\phi'(x_2)\|_{L^p} \\ &\quad - 2 \cdot 2^{ns}\|2^{-n(s+1)}\phi(x_1)\phi'(x_1)\sin(2^n x_1)\phi(x_2)\phi'(x_2)\|_{L^p} \\ &\geq C - C2^{-n}, \end{aligned}$$

from which follows the desired formula (3.6). □

Subsequently, we set

$$U_{1,n}(t,x) := t[f_n e_2 + \nabla(-\Delta)^{-1}\partial_2 f_n] \tag{3.7}$$

and

$$U_{2,n}(t,x) := g_n + t[f_n e_2 + \nabla(-\Delta)^{-1}\partial_2 f_n]. \tag{3.8}$$

It is observed that $U_{i,n}(i = 1, 2)$ solves the following linearized system which corresponds to (1.1):

$$\begin{cases} \partial_t U_{i,n} + \nabla \Pi_{i,n} = f_n e_2, \\ \nabla \cdot U_{i,n} = 0, \end{cases} \tag{3.9}$$

with initial data

$$U_{1,n}|_{t=0} = 0$$

and

$$U_{2,n}|_{t=0} = g_n,$$

respectively.

Applying the standard energy method and using Lemmas 3.1-3.2, we can directly deduce the following estimates for system (3.9).

Lemma 3.3. *Assume that* (s, p, r) *satisfies condition (1.2) and let* $U_{i,n}(i = 1, 2)$ *be defined in (3.7) and (3.8). Then, for any* $\sigma \in \mathbb{R}$ *and* $t \in [0, T]$*, the following holds:*

- $\|U_{1,n}(t)\|_{B^\sigma_{p,r}} \lesssim 2^{n(\sigma-s)}$,
- $\|U_{2,n}(t)\|_{B^\sigma_{p,r}} \lesssim 2^{n\max\{\sigma-s,-1\}}$,
- $\|\nabla^k U_{1,n}\|_{L^\infty} \lesssim 2^{n(k-s)}$, *for* $k = 0, 1$,
- $\|U_{2,n} - g_n\|_{B^\sigma_{p,r}} \lesssim t2^{n(\sigma-s)}$, *and*
- $\|U_{2,n} - U_{1,n}\|_{B^\sigma_{p,r}} \lesssim 2^{-n}$.

3.2. *Estimates of the errors*

Let us denote solutions to the Cauchy problem (1.1) by $(u_{1,n}, \theta_{1,n})$ and $(u_{2,n}, \theta_{2,n})$ with initial data

$$(u_{1,n}, \theta_{1,n})|_{t=0} = (0, f_n)$$

and

$$(u_{2,n}, \theta_{2,n})|_{t=0} = (g_n, f_n),$$

respectively.

Subsequently, we may use $(U_{1,n}, f_n)$ to approximate $(u_{1,n}, \theta_{1,n})$ in $B^s_{p,r}$ (see Lemma 3.4). However, for $t > 0$ sufficiently small, the difference between $(u_{2,n}, \theta_{2,n})$ and $(U_{2,n}, f_n)$ will not tend to zero in $B^s_{p,r}$ as $n \to \infty$ (see Lemma 3.5), which help us to achieve the main nonuniform result.

First, we set the error by the following:

$$\begin{cases} u^{\text{er}}_{1,n} := u_{1,n} - U_{1,n}, \\ \theta^{\text{er}}_{1,n} := \theta_{1,n} - f_n. \end{cases}$$

Then, $(u^{\text{er}}_{1,n}, \theta^{\text{er}}_{1,n})$ satisfies the following:

$$\begin{cases} \partial_t \theta^{\text{er}}_{1,n} + u_{1,n} \cdot \nabla \theta^{\text{er}}_{1,n} = -u^{\text{er}}_{1,n} \cdot \nabla f_n - U_{1,n} \cdot \nabla f_n, \\ \partial_t u^{\text{er}}_{1,n} + u_{1,n} \cdot \nabla u^{\text{er}}_{1,n} + \nabla P' = \theta^{\text{er}}_{1,n} e_2 - u^{\text{er}}_{1,n} \cdot \nabla U_{1,n} - U_{1,n} \cdot \nabla U_{1,n}, \\ \nabla \cdot u^{\text{er}}_{1,n} = 0, \\ (u^{\text{er}}_{1,n}, \theta^{\text{er}}_{1,n})|_{t=0} = (0, 0). \end{cases} \tag{3.10}$$

We assert that the difference between $(U_{1,n}, f_n)$ and $(u_{1,n}, \theta_{1,n})$ converges to zero in $B^s_{p,r}$ as $n \to \infty$ for any $t \in [0, T]$. This is supported by the ensuing lemma.

Lemma 3.4. *Assume that* (s, p, r) *satisfies condition (1.2). For any* $t \in [0, T]$*, the following holds:*

$$\|(u^{er}_{1,n}, \theta^{er}_{1,n})\|_{B^s_{p,r}} \lesssim 2^{\frac{n}{2}(1-s)}.$$

Proof. Recall that $(u_{1,n}, \theta_{1,n})$ denotes the solution to the Cauchy problem (1.1) with the following initial data:

$$(u_{1,n}, \theta_{1,n})|_{t=0} = (0, f_n).$$

It follows from Theorem 1.1 and Lemma 3.1 that $(u_{1,n}, \theta_{1,n}) \in C([0, T], B^s_{p,r} \times B^s_{p,r})$ satisfies

$$\|(u_{1,n}, \theta_{1,n})\|_{B^s_{p,r}} \lesssim \|f_n\|_{B^s_{p,r}} \lesssim 1 \tag{3.11}$$

and

$$\|(u_{1,n}, \theta_{1,n})\|_{B^{s+1}_{p,r}} \lesssim \|f_n\|_{B^{s+1}_{p,r}} \lesssim 2^n. \tag{3.12}$$

Now, applying Lemma 2.4 to system (1.1) and by using (3.11)–(3.12) and Lemma 3.1, we deduce that for any $t \in [0, T]$,

$$\begin{aligned} \|\theta_{1,n}\|_{B^{s-1}_{p,r}} &\lesssim \exp(CV_p(u_{1,n}, t))\|f_n\|_{B^{s-1}_{p,r}} \\ &\lesssim 2^{-n} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\|u_{1,n}\|_{B^{s-1}_{p,r}} &\lesssim \exp(CV_p(u_{1,n},t))(\|f_n\|_{B^{s-1}_{p,r}} + \int_0^t \|(\nabla P, \theta_{1,n}e_2)\|_{B^{s-1}_{p,r}}\, d\tau) \\ &\lesssim 2^{-n} + \int_0^t \|(\nabla P, \theta_{1,n}e_2)\|_{B^{s-1}_{p,r}}\, d\tau.\end{aligned} \tag{3.14}$$

Notice that

$$\begin{aligned}\|\nabla P\|_{B^{s-1}_{p,r}} &= \|\nabla\Delta^{-1}\nabla\cdot(\theta_{1,n}e_2 - u_{1,n}\cdot\nabla u_{1,n})\|_{B^{s-1}_{p,r}} \\ &\lesssim \|\theta_{1,n}\|_{B^{s-1}_{p,r}} + \|u_{1,n}\|_{B^{s-1}_{p,r}}\|u_{1,n}\|_{B^{s}_{p,r}} \\ &\lesssim \|(\theta_{1,n}, u_{1,n})\|_{B^{s-1}_{p,r}},\end{aligned} \tag{3.15}$$

by Lemma 2.3 and (3.11). Combining (3.13)–(3.15), we obtain that for any $t \in [0, T]$,

$$\|(u_{1,n}, \theta_{1,n})\|_{B^{s-1}_{p,r}} \lesssim 2^{-n} + \int_0^t \|(u_{1,n}, \theta_{1,n})\|_{B^{s-1}_{p,r}}\, d\tau,$$

which, along with the Gronwall inequality, leads to the following:

$$\|(u_{1,n}, \theta_{1,n})\|_{B^{s-1}_{p,r}} \lesssim 2^{-n}. \tag{3.16}$$

With the assistance of (3.11)–(3.12) and (3.16), we are now ready to estimate the error term $(\theta^{\mathrm{er}}_{1,n}, u^{\mathrm{er}}_{1,n})$ in $B^s_{p,r}$. Applying Lemma 2.4 again to system (3.10), we obtain the following:

$$\begin{aligned}\|\theta^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}} &\lesssim \exp(CV_p(u_{1,n},t))\int_0^t \|(u^{\mathrm{er}}_{1,n}\cdot\nabla f_n, U_{1,n}\cdot\nabla f_n)\|_{B^{s-1}_{p,r}}\, d\tau \\ &\lesssim \int_0^t \|(u^{\mathrm{er}}_{1,n}\cdot\nabla f_n, U_{1,n}\cdot\nabla f_n)\|_{B^{s-1}_{p,r}}\, d\tau\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}} &\lesssim \exp(CV_p(u_{1,n},t))\int_0^t \|(\nabla P', \theta^{\mathrm{er}}_{1,n}e_2, u^{\mathrm{er}}_{1,n}\cdot\nabla U_{1,n}, U_{1,n}\cdot\nabla U_{1,n})\|_{B^{s-1}_{p,r}}\, d\tau \\ &\lesssim \int_0^t \|(\nabla P', \theta^{\mathrm{er}}_{1,n}e_2, u^{\mathrm{er}}_{1,n}\cdot\nabla U_{1,n}, U_{1,n}\cdot\nabla U_{1,n})\|_{B^{s-1}_{p,r}}\, d\tau.\end{aligned} \tag{3.18}$$

It follows from Lemma 2.3, Lemma 3.1, and Lemma 3.3 that

$$\begin{aligned}\|u^{\mathrm{er}}_{1,n}\cdot\nabla f_n\|_{B^{s-1}_{p,r}} &\lesssim \|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}}\|\nabla f_n\|_{B^{s-1}_{p,r}} \\ &\lesssim \|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}}\|f_n\|_{B^{s}_{p,r}} \\ &\lesssim \|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\|U_{1,n}\cdot\nabla f_n\|_{B^{s-1}_{p,r}} &\lesssim \|U_{1,n}\|_{L^\infty}\|\nabla f_n\|_{B^{s-1}_{p,r}} + \|\nabla f_n\|_{L^\infty}\|U_{1,n}\|_{B^{s-1}_{p,r}} \\ &\lesssim 2^{-ns},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\|u^{\mathrm{er}}_{1,n}\cdot\nabla U_{1,n}\|_{B^{s-1}_{p,r}} &\lesssim \|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}}\|\nabla U_{1,n}\|_{B^{s-1}_{p,r}} \\ &\lesssim \|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}}\|U_{1,n}\|_{B^{s}_{p,r}} \\ &\lesssim \|u^{\mathrm{er}}_{1,n}\|_{B^{s-1}_{p,r}},\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\|U_{1,n}\cdot\nabla U_{1,n}\|_{B^{s-1}_{p,r}} &\lesssim \|U_{1,n}\|_{L^\infty}\|\nabla U_{1,n}\|_{B^{s-1}_{p,r}}+\|\nabla U_{1,n}\|_{L^\infty}\|U_{1,n}\|_{B^{s-1}_{p,r}}\\ &\lesssim 2^{-ns}.\end{aligned}\tag{3.22}$$

Furthermore,

$$\begin{aligned}\nabla P' &= \nabla\Delta^{-1}\nabla\cdot(-u_{1,n}\cdot\nabla u^{\rm er}_{1,n}-\theta^{\rm er}_{1,n}e_2+u^{\rm er}_{1,n}\cdot\nabla U_{1,n}+U_{1,n}\cdot\nabla U_{1,n})\\ &= \nabla\Delta^{-1}\nabla\cdot(-u^{\rm er}_{1,n}\cdot\nabla u_{1,n}-\theta^{\rm er}_{1,n}e_2+u^{\rm er}_{1,n}\cdot\nabla U_{1,n}+U_{1,n}\cdot\nabla U_{1,n}),\end{aligned}\tag{3.23}$$

for $\nabla\cdot u^{\rm er}_{1,n}=\nabla\cdot u_{1,n}=0$. Imitating the proof of (3.21) and using (3.22), we can deduce the following:

$$\begin{aligned}\|\nabla P'\|_{B^{s-1}_{p,r}} &\lesssim \|(u^{\rm er}_{1,n}\cdot\nabla u_{1,n},u^{\rm er}_{1,n}\cdot\nabla U_{1,n},\theta^{\rm er}_{1,n}e_2,U_{1,n}\cdot\nabla U_{1,n})\|_{B^{s-1}_{p,r}}\\ &\lesssim \|(\theta^{\rm er}_{1,n},u^{\rm er}_{1,n})\|_{B^{s-1}_{p,r}}+2^{-ns}.\end{aligned}\tag{3.24}$$

Combining (3.17)–(3.24) yields the following:

$$\|(u^{\rm er}_{1,n},\theta^{\rm er}_{1,n})\|_{B^{s-1}_{p,r}} \lesssim \int_0^t \|(u^{\rm er}_{1,n},\theta^{\rm er}_{1,n})\|_{B^{s-1}_{p,r}}\,d\tau+2^{-ns};$$

this implies the following:

$$\|(u^{\rm er}_{1,n},\theta^{\rm er}_{1,n})\|_{B^{s-1}_{p,r}} \lesssim 2^{-ns}.\tag{3.25}$$

Hence, it follows from the interpolation inequalities (2.1) and (3.25) that

$$\begin{aligned}\|(u^{\rm er}_{1,n},\theta^{\rm er}_{1,n})\|_{B^{s}_{p,r}} &\lesssim \|(u^{\rm er}_{1,n},\theta^{\rm er}_{1,n})\|^{\frac12}_{B^{s-1}_{p,r}}\|(u^{\rm er}_{1,n},\theta^{\rm er}_{1,n})\|^{\frac12}_{B^{s+1}_{p,r}}\\ &\lesssim 2^{\frac{n}{2}(1-s)},\end{aligned}$$

which concludes the proof of this lemma. □

Subsequently, let us introduce another error term as follows:

$$\begin{cases}u^{\rm er}_{2,n}:=u_{2,n}-U_{2,n},\\ \theta^{\rm er}_{2,n}:=\theta_{2,n}-f_n+tV_n,\end{cases}$$

where $V_n:=U_{2,n}\cdot\nabla f_n$. Then, $(u^{\rm er}_{2,n},\theta^{\rm er}_{2,n})$ satisfies the following:

$$\begin{cases}\partial_t\theta^{\rm er}_{2,n}+u_{2,n}\cdot\nabla\theta^{\rm er}_{2,n}=-u^{\rm er}_{2,n}\cdot\nabla f_n+tu_{2,n}\cdot\nabla V_n+t\partial_tV_n,\\ \partial_tu^{\rm er}_{2,n}+u_{2,n}\cdot\nabla u^{\rm er}_{2,n}+\nabla P''=\theta^{\rm er}_{2,n}e_2-tV_ne_2-u^{\rm er}_{2,n}\cdot\nabla U_{2,n}-U_{2,n}\cdot\nabla U_{2,n},\\ \nabla\cdot u^{\rm er}_{2,n}=0,\\ (u^{\rm er}_{2,n},\theta^{\rm er}_{2,n})|_{t=0}=(0,0).\end{cases}\tag{3.26}$$

We claim that $(U_{2,n},f_n)$ cannot approximate $(u_{2,n},\theta_{2,n})$, which is the key to achieving the non-uniform continuity. To be more specific, the following lemma is presented.

Lemma 3.5. *Assume that (s,p,r) satisfies (1.2). For any $t\le 1$, the following holds:*

$$\|(u^{er}_{2,n},\theta^{er}_{2,n})\|_{B^{s}_{p,r}} \lesssim t^2+2^{-n}.$$

Proof. Before proving this lemma, we make some preparations.

Note that

$$\|(V_n, \partial_t V_n)\|_{B^{s-1}_{p,r}} \lesssim \|(U_{2,n}, \partial_t U_{2,n})\|_{B^{s-1}_{p,r}} \|f_n\|_{B^s_{p,r}} \lesssim 2^{-n} \tag{3.27}$$

and

$$\begin{aligned}\|(V_n, \partial_t V_n)\|_{B^{s+k}_{p,r}} &\lesssim \|(U_{2,n}, \partial_t U_{2,n})\|_{L^\infty} \|\nabla f_n\|_{B^{s+k}_{p,r}} + \|(U_{2,n}, \partial_t U_{2,n})\|_{B^{s+k}_{p,r}} \|\nabla f_n\|_{L^\infty} \\ &\lesssim \|(U_{2,n}, \partial_t U_{2,n})\|_{B^{s-1}_{p,r}} \|f_n\|_{B^{s+k+1}_{p,r}} + \|(U_{2,n}, \partial_t U_{2,n})\|_{B^{s+k}_{p,r}} \|\nabla f_n\|_{B^{s-1}_{p,r}} \\ &\lesssim 2^{kn},\end{aligned} \tag{3.28}$$

for $k = 0, 1$, where Lemma 2.3 and Lemmas 3.1-3.3 were used.

Imitating the proof of (3.16), combined with results from Lemmas 3.1-3.2, can lead to the following:

$$\|(u_{2,n}, \theta_{2,n})\|_{B^{s+k}_{p,r}} \lesssim 2^{kn}, \tag{3.29}$$

for $k = -1, 0, 1$.

Applying $\nabla\cdot$ to $(3.26)_2$, we obtain the following:

$$\begin{aligned}\nabla P'' &= \nabla\Delta^{-1}\nabla \cdot (-u_{2,n} \cdot \nabla u^{\text{er}}_{2,n} + \theta^{\text{er}}_{2,n} e_2 - tV_n e_2 - u^{\text{er}}_{2,n} \cdot \nabla U_{2,n} - U_{2,n} \cdot \nabla U_{2,n}) \\ &= \nabla\Delta^{-1}\nabla \cdot (-u^{\text{er}}_{2,n} \cdot \nabla u_{2,n} - u^{\text{er}}_{2,n} \cdot \nabla U_{2,n} + \theta^{\text{er}}_{2,n} e_2) \\ &\quad + \nabla\Delta^{-1}\nabla \cdot (-tV_n e_2 - g_n \cdot \nabla g_n - (U_{2,n} - g_n) \cdot \nabla U_{2,n} - (U_{2,n} - g_n) \cdot \nabla g_n),\end{aligned}$$

for $\nabla \cdot u_{2,n} = \nabla \cdot u^{\text{er}}_{2,n} = \nabla \cdot U_{2,n} = \nabla \cdot g_n = 0$. It follows from Lemma 2.3, Lemmas 3.1-3.3, and (3.29) that

$$\begin{aligned}\|\nabla P''\|_{B^{s-1}_{p,r}} &\lesssim \|u^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}} \|(u_{2,n}, U_{2,n})\|_{B^s_{p,r}} + \|\theta^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}} + t\|V_n\|_{B^{s-1}_{p,r}} \\ &\quad + \|g_n\|_{B^{s-1}_{p,r}} \|g_n\|_{B^s_{p,r}} + \|U_{2,n} - g_n\|_{B^{s-1}_{p,r}} \|(U_{2,n}, g_n)\|_{B^s_{p,r}} \\ &\lesssim \|(u^{\text{er}}_{2,n}, \theta^{\text{er}}_{2,n})\|_{B^{s-1}_{p,r}} + t2^{-n} + 2^{-2n}\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\|\nabla P''\|_{B^s_{p,r}} &\lesssim \|u^{\text{er}}_{2,n}\|_{B^s_{p,r}} \|(u_{2,n}, U_{2,n})\|_{B^s_{p,r}} + \|\theta^{\text{er}}_{2,n}\|_{B^s_{p,r}} + t\|V_n\|_{B^s_{p,r}} + \|g_n\|^2_{B^s_{p,r}} \\ &\quad + \|U_{2,n} - g_n\|_{B^s_{p,r}} \|(U_{2,n}, g_n)\|_{B^s_{p,r}} + \|U_{2,n} - g_n\|_{B^{s-1}_{p,r}} \|(U_{2,n}, g_n)\|_{B^{s+1}_{p,r}} \\ &\lesssim \|(u^{\text{er}}_{2,n}, \theta^{\text{er}}_{2,n})\|_{B^s_{p,r}} + 2^n \|u^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}} + t + 2^{-2n}.\end{aligned} \tag{3.31}$$

Now, we apply Lemma 2.4 to (3.26) and use (3.27)–(3.31) to obtain the following:

$$\begin{aligned}\|\theta^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}} &\lesssim \exp(CV_p(u_{2,n}, t))\Big(\int_0^t \|u^{\text{er}}_{2,n} \cdot \nabla f_n\|_{B^{s-1}_{p,r}}\, d\tau + \int_0^t \tau\|u_{2,n} \cdot \nabla V_n + \partial_\tau V_n\|_{B^{s-1}_{p,r}}\, d\tau\Big) \\ &\lesssim \int_0^t \|u^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}} \|f_n\|_{B^s_{p,r}}\, d\tau + \int_0^t \tau\|u_{2,n} \cdot \nabla V_n + \partial_\tau V_n\|_{B^{s-1}_{p,r}}\, d\tau \\ &\lesssim \int_0^t \|u^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}}\, d\tau + t^2 2^{-n}\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\|u^{\text{er}}_{2,n}\|_{B^{s-1}_{p,r}} &\lesssim \exp(CV_p(u_{2,n}, t))\Big(\int_0^t \|(\nabla P'', \theta^{\text{er}}_{2,n} e_2, u^{\text{er}}_{2,n} \cdot \nabla U_{2,n}, U_{2,n} \cdot \nabla U_{2,n})\|_{B^{s-1}_{p,r}}\, d\tau \\ &\quad + \int_0^t \tau\|V_n\|_{B^{s-1}_{p,r}}\, d\tau\Big) \\ &\lesssim \int_0^t \|(u^{\text{er}}_{2,n}, \theta^{\text{er}}_{2,n})\|_{B^{s-1}_{p,r}}\, d\tau + t^2 2^{-n} + 2^{-2n}.\end{aligned} \tag{3.33}$$

Then, combining (3.32) and (3.33) leads to

$$\|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^{s-1}_{p,r}} \lesssim \int_0^t \|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^{s-1}_{p,r}}\,d\tau + t^2 2^{-n} + 2^{-2n},$$

which shows

$$\|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^{s-1}_{p,r}} \lesssim t^2 2^{-n} + 2^{-2n}, \tag{3.34}$$

with the aid of the Gronwall inequality.

Applying Lemma 2.4 to (3.26), we have the following:

$$\begin{aligned}
\|\theta_{2,n}^{\mathrm{er}}\|_{B^s_{p,r}} &\lesssim \exp(CV_p(u_{2,n},t))\Big(\int_0^t \|u_{2,n}^{\mathrm{er}}\cdot\nabla f_n\|_{B^s_{p,r}}\,d\tau + \int_0^t \tau\|u_{2,n}\cdot\nabla V_n + \partial_\tau V_n\|_{B^s_{p,r}}\,d\tau\Big)\\
&\lesssim \int_0^t (\|u_{2,n}^{\mathrm{er}}\|_{B^s_{p,r}}\|\nabla f_n\|_{L^\infty} + \|u_{2,n}^{\mathrm{er}}\|_{L^\infty}\|\nabla f_n\|_{B^s_{p,r}})\,d\tau + t^2\\
&\lesssim \int_0^t (\|u_{2,n}^{\mathrm{er}}\|_{B^s_{p,r}} + 2^n\|u_{2,n}^{\mathrm{er}}\|_{B^{s-1}_{p,r}})\,d\tau + t^2
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
\|u_{2,n}^{\mathrm{er}}\|_{B^s_{p,r}} &\lesssim \exp(CV_p(u_{2,n},t))\Big(\int_0^t \|(\nabla P'', \theta_{2,n}^{\mathrm{er}}e_2, u_{2,n}^{\mathrm{er}}\cdot\nabla U_{2,n}, U_{2,n}\cdot\nabla U_{2,n})\|_{B^s_{p,r}}\,d\tau\\
&\quad + \int_0^t \tau\|V_n\|_{B^s_{p,r}}\,d\tau\Big)\\
&\lesssim \int_0^t (\|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^s_{p,r}} + 2^n\|u_{2,n}^{\mathrm{er}}\|_{B^{s-1}_{p,r}})\,d\tau + t^2 + 2^{-2n}.
\end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36) and applying (3.34), we obtain that for $t \le 1$,

$$\begin{aligned}
\|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^s_{p,r}} &\lesssim \int_0^t (\|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^s_{p,r}} + 2^n\|u_{2,n}^{\mathrm{er}}\|_{B^{s-1}_{p,r}})\,d\tau + t^2 + 2^{-2n}\\
&\lesssim \int_0^t \|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^s_{p,r}}\,d\tau + t^2 + 2^{-n};
\end{aligned}$$

then, using the Gronwall inequality yields the following:

$$\|(u_{2,n}^{\mathrm{er}},\theta_{2,n}^{\mathrm{er}})\|_{B^s_{p,r}} \lesssim t^2 + 2^{-n}.$$

□

4. Proof of Theorem 1.2

Building upon Lemmas 3.4–3.5, the proof of Theorem 1.2 can now commence.

Proof of Theorem 1.2. Define $(u_{1,n},\theta_{1,n})$ and $(u_{2,n},\theta_{2,n})$ as the solutions to the Cauchy problem (1.1), which correspond to the following initial conditions:

$$(u_{1,n},\theta_{1,n})|_{t=0} = (0, f_n)$$

and

$$(u_{2,n}, \theta_{2,n})|_{t=0} = (g_n, f_n),$$

respectively. When $t = 0$, it is obvious that

$$\|u_{2,n}(0) - u_{1,n}(0)\|_{B^s_{p,r}} + \|\theta_{2,n}(0) - \theta_{1,n}(0)\|_{B^s_{p,r}} = \|g_n\|_{B^s_{p,r}} \lesssim 2^{-n} \to 0,$$

as $n \to \infty$.

Notice that

$$\begin{aligned} u_{2,n} - u_{1,n} &= u_{2,n} - U_{2,n} + U_{2,n} - U_{1,n} + U_{1,n} - u_{1,n} \\ &= u^{\mathrm{er}}_{2,n} - u^{\mathrm{er}}_{1,n} + U_{2,n} - U_{1,n}, \end{aligned}$$

$$\begin{aligned} \theta_{2,n} - \theta_{1,n} &= \theta_{2,n} - f_n + tV_n - tV_n + f_n - \theta_{1,n} \\ &= \theta^{\mathrm{er}}_{2,n} - \theta^{\mathrm{er}}_{1,n} - tV_n \end{aligned}$$

and

$$\begin{aligned} \|V_n\|_{B^s_{p,r}} &= \|(U_{2,n} - g_n) \cdot \nabla f_n + g_n \cdot \nabla f_n\|_{B^s_{p,r}} \\ &\geq \|g_n \cdot \nabla f_n\|_{B^s_{p,r}} - \|(U_{2,n} - g_n) \cdot \nabla f_n\|_{B^s_{p,r}} \\ &\geq \|g_n \cdot \nabla f_n\|_{B^s_{p,r}} - \|U_{2,n} - g_n\|_{L^\infty}\|\nabla f_n\|_{B^s_{p,r}} - \|U_{2,n} - g_n\|_{B^s_{p,r}}\|\nabla f_n\|_{L^\infty} \\ &\gtrsim \|g_n \cdot \nabla f_n\|_{B^s_{p,r}} - \|U_{2,n} - g_n\|_{B^{s-1}_{p,r}}\|f_n\|_{B^{s+1}_{p,r}} - \|U_{2,n} - g_n\|_{B^s_{p,r}}\|f_n\|_{B^s_{p,r}} \\ &\gtrsim \|g_n \cdot \nabla f_n\|_{B^s_{p,r}} - t, \end{aligned} \tag{4.1}$$

by Lemma 2.3 and Lemmas 3.1–3.3.

Thus, for $t > 0$, it follows from Lemmas 3.3–3.5 and (4.1) that

$$\begin{aligned} &\|u_{2,n} - u_{1,n}\|_{B^s_{p,r}} + \|\theta_{2,n} - \theta_{1,n}\|_{B^s_{p,r}} \\ &\quad\geq t\|V_n\|_{B^s_{p,r}} - \|U_{2,n} - U_{1,n}\|_{B^s_{p,r}} - \|(u^{\mathrm{er}}_{1,n}, \theta^{\mathrm{er}}_{1,n})\|_{B^s_{p,r}} - \|(u^{\mathrm{er}}_{2,n}, \theta^{\mathrm{er}}_{2,n})\|_{B^s_{p,r}} \\ &\quad\gtrsim t\|g_n \cdot \nabla f_n\|_{B^s_{p,r}} - 2^{-n} - 2^{\frac{n}{2}(1-s)} - t^2, \end{aligned}$$

which, combined with Lemma 3.2, leads to the following:

$$\liminf_{n\to\infty} \|(u_{2,n} - u_{1,n}, \theta_{2,n} - \theta_{1,n})\|_{B^s_{p,r}} \gtrsim t,$$

for $t > 0$ sufficiently small. Thus, Theorem 1.2 was proven. □

5. Conclusions

Initiated by [7], the failure of uniform dependence on the initial data for classical solutions to hyperbolic systems is an interesting area of research. This work creates the non-uniform continuity of the data-to-solution map to the inviscid Boussinesq equations in Besov spaces. Our proof depends on the interaction between terms of low and high frequencies, and the linearized system to the inviscid Boussinesq equations also proves to be a powerful tool in the construction of appropriate approximate solutions.

Looking ahead, the hyperbolic property of the thermal equation creates the possibility of the non-uniform continuity, even when considering the parabolic-hyperbolic coupling induced by adding (1.1) with a dissipated term $-\Delta u$. We are confident that the method developed in this paper can be extended to prove the non-uniform continuity to the dissipated Boussinesq equations in the Sobolev spaces $H^s(\mathbb{R}^2)$ with $s > 1$.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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