Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25606–25623.
DOI: 10.3934/math.20251134
Received: 21 August 2025
Revised: 18 October 2025
Accepted: 03 November 2025
Published: 06 November 2025

Research article

T-pedal ruled surface with the Frenet frame of the original curve in E^3

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Abstract: This paper presents a geometric study of three types of ruled surfaces generated from the tangent, normal, and binormal unit vectors of unit speed space curves. Using the T-pedal curve construction as a foundation, we analyze these surfaces through their fundamental geometric forms, including curvature properties, the striction curve geometry, and the distribution parameter. The theoretical framework is used to analyze problems in computational geometry and shape modeling, with results relevant to both mathematical research and engineering applications. The work establishes fundamental geometric insights while providing tools for applied shape modeling and analysis.

Keywords: Frenet frame; space curves; T-pedal curve; ruled surfaces; developable surface; minimal surface
Mathematics Subject Classification: 53A04, 53A55, 53A17

1. Introduction

Pedal curves represent a fundamental object of study in differential geometry, characterized by their distinctive construction as the locus of orthogonal projections from a fixed pedal point onto the tangent lines of a generator curve. These curves exhibit profound geometric duality with their originating curves, sharing conceptual parallels with other dual curve families, including evolutes, involutes, and Bertrand partner curves [10]. Modern geometric analysis has significantly expanded the classical understanding of pedal curves through rigorous investigation of their singularities, curvature properties, and generalizations to alternative geometries. While their mathematical foundations were established in seminal works by Newton and Leibniz, contemporary research continues to reveal their theoretical importance and practical utility in applied mathematics, computer-aided geometric design, and mechanical engineering applications.

The systematic study of pedal curves has evolved significantly through distinct phases of mathematical investigation. Early foundational work by Bukcu and Karakus (2008) established rigorous normal forms for pedal curve singularities in S^n, providing crucial classification tools for their degenerate cases [1]. Subsequent research by Tuncer et al. (2018) expanded the theoretical framework through detailed analysis of pedal and contrapedal curve pairs in Euclidean plane geometry, particularly examining their interaction with curve fronts [15]. The field advanced substantially with Li et al. (2023), who studied studied the notions of pedal curves, contrapedal curves, and B-Gauss maps of non-lightlike regular curves in Minkowski 3-space [9]. Most recently, Kaya (2024) developed comprehensive differential geometric aspects of pedal curves on surfaces [8], while Canli et al. (2024) made dual contributions through their investigations of Frenet-frame derived pedal curves and their Smarandache variants, demonstrating novel connections with alternative moving frames [2, 3]. This evolution highlights how research on pedal curves has advanced from classical studies of singularities to modern applications in diverse geometric settings.

Ruled surfaces represent a fundamental class of surfaces in differential geometry, formed by the continuous motion of a straight line called the generator along a space curve known as the directrix. These surfaces exhibit a unique combination of geometric simplicity and mathematical depth, making them valuable for both theoretical analysis and practical applications. Their linear structure enables efficient parameterization and exact solutions to geometric problems while facilitating construction in engineering and architectural design. Classical studies primarily focused on developable surfaces with zero Gaussian curvature, but modern research has expanded to include non-developable cases with complex curvature properties and singularities [4, 14]. Recent computational advances have enabled innovative applications in robotic path planning, computer-aided design, and structural optimization. This continued investigation demonstrates how ruled surfaces remain essential mathematical objects that bridge abstract theory with engineering practice, offering both analytical tractability and functional versatility across multiple disciplines. The combination of their geometric properties and practical applications continues to advance research in both theoretical and applied mathematics.

While ruled surfaces have been studied since classical differential geometry, contemporary research has produced significant new developments. Ivanov's (2021) analysis of normal ruled surfaces provided important insights into their mechanical properties and structural behavior [7]. The following year, Masal and Azak (2022) developed innovative Bishop frame constructions for ruled surfaces in Euclidean space, creating new analytical frameworks [11]. Subsequent work by Pal and Kumar (2023) expanded the classification of ruled-like surfaces through novel geometric characterizations [12]. Building on established geometric foundations, recent research has significantly advanced both theoretical and computational aspects of ruled surfaces. Pan et al. (2025) developed innovative methods for piecewise ruled approximation of freeform surfaces [13], while Elsharkawy et al. (2025) introduced novel quasi-ruled surfaces [5], and the geometric properties of Smarandache ruled surfaces by integral binormal curves in Euclidean space [6]. These parallel developments demonstrate the field's progression toward increasingly sophisticated mathematical formulations coupled with practical computational implementations.

We introduce a new class of ruled surfaces generated by the T-pedal curve, where the Frenet frame of the original curve is chosen as the ruling direction. The study discusses the geometric properties of the tangent, normal, and binormal ruled surfaces derived from this construction, focusing on their curvature behavior, striction curve geometry, distribution parameters, and the three fundamental forms.

Furthermore, the conditions under which these surfaces become developable or minimal are examined. Illustrative examples are presented to illustrate and verify the obtained results.

This paper is structured as follows:

- **Section 2:** Establishes the geometric foundations, including Frenet-frame theory, T-pedal curves, and ruled surface preliminaries.
- **Section 3:** Presents three fundamental ruled surfaces generated from the Frenet apparatus of a unit-speed curve $\alpha(s)$, where the T-pedal curve $\alpha_T(s)$ serves as base curve. The analysis includes the *tangent ruled surface* determined by $\mathbf{T}(s)$, the *normal ruled surface* constructed via $\mathbf{N}(s)$, and the *binormal ruled surface* generated by $\mathbf{B}(s)$. Each surface undergoes complete geometric characterization through its fundamental forms, curvature properties, striction curve geometry, and distribution parameter.
- **Section 4:** Demonstrates practical applications of these surfaces in geometric modeling and computational design.
- **Section 5:** Summarizes key findings in particle dynamics and differential geometry, proposing relativistic extensions and stochastic modeling as future directions.

2. Preliminary

This section briefly outlines the essential geometric concepts. For comprehensive treatments, we refer to established texts [4, 14].

Let $\alpha(s)$ be a unit-speed curve in $\mathbb{E}^3$ parameterized by arc length s. The set $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the Frenet frame along the curve $\alpha(s)$, where $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$ are the tangential, normal, and binormal unit vector fields, respectively, given by

$$\mathbf{T}(s) = \alpha'(s), \quad \mathbf{N}(s) = \frac{\alpha''(s)}{||\alpha''(s)||}, \quad \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s), \tag{2.1}$$

where differentiation with respect to s is indicated by the superposed dash.

For a unit-speed curve, the Frenet-Serret equations are given by

$$\begin{aligned} \mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s), \\ \mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s), \\ \mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s), \end{aligned} \tag{2.2}$$

where the curvature and torsion of the curve $\alpha(s)$ are given by

$$\kappa(s) = ||T'(s)||, \quad \tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle.$$

The T-pedal curve α_T of a regular unit-speed curve $\alpha(s)$ with respect to a fixed point P in $\mathbb{E}^3$ is given by

$$\alpha_T = \alpha(s) + \langle P - \alpha(s), \mathbf{T}(s) \rangle \mathbf{T}(s),$$

where $\alpha(s)$ is the original curve, P is the pedal point, and $\mathbf{T}(s)$ denotes the unit tangent vector of $\alpha(s)$. The scalar quantity $\langle P - \alpha(s), \mathbf{T}(s) \rangle$ represents the signed projection of the vector from the curve point

$\alpha(s)$ to the pedal point P onto the tangent direction $\mathbf{T}(s)$. Geometrically, this term gives the directed distance from the point $\alpha(s)$ along the tangent line to the foot of the perpendicular dropped from P onto that tangent line.

Thus, the point $\alpha_T(s)$ lies on the tangent line to the original curve at $\alpha(s)$, positioned such that it corresponds to the orthogonal projection of the pedal point P onto this tangent. As the point of contact $\alpha(s)$ moves along the curve, the locus of all such projected points forms the T-pedal curve. The shape and position of the T-pedal curve depend entirely on the location of the pedal point P.

In the special case where the pedal point P coincides with the origin $O(0,0,0)$, the scalar projection simplifies to

$$u(s) = -\langle \alpha(s), \mathbf{T}(s) \rangle,$$

and the expression of the T-pedal curve reduces to

$$\alpha_T = \alpha(s) + u(s)\mathbf{T}(s). \tag{2.3}$$

This formulation provides a simpler representation, where $u(s)$ can be interpreted as the signed distance from the curve point $\alpha(s)$ to the foot of the perpendicular dropped from the origin onto the tangent line at that point, see Figure 1.

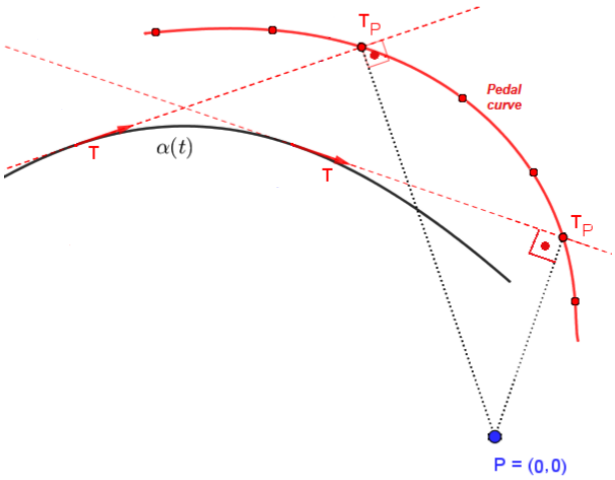

Figure 1. The regular curve (black), T-pedal curve (red).

The unit tangent vector $\mathbf{T_1}$ of the T-pedal curve associated with the original unit-speed curve $\alpha(s)$ can be expressed in terms of the Frenet frame of $\alpha(s)$ as follows [2, 3]:

$$\mathbf{T_1} = \omega_1 (1 + u)\, \mathbf{T} + \omega_1 u\kappa \mathbf{N}, \tag{2.4}$$

where

$$\omega_1 = \frac{1}{\sqrt{(1 + u')^2 + (u\kappa)^2}}.$$

Example 2.1. *[2] The pedal curve of the ellipse $\alpha(t) = (2\cos t, \sin t)$, in $\mathbb{E}^2$, with respect to the origin $O(0,0)$, is given by the following relation, see Figure 2:*

$$\alpha_T(t) = \left(\frac{2\cos t}{1 + 3\sin^2 t}, \frac{4\sin t}{1 + 3\sin^2 t} \right).$$

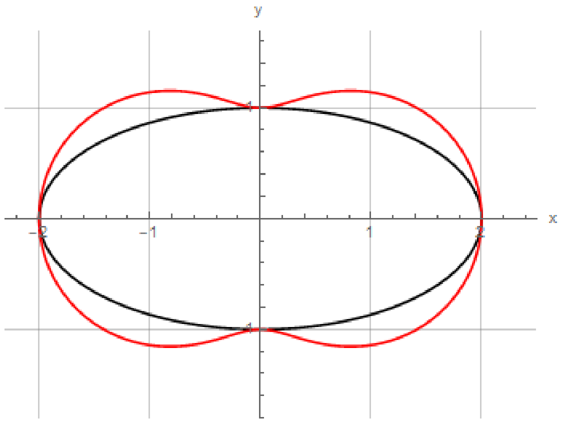

Figure 2. The T-pedal curve (red) of the ellipse (black) according to the origin.

As X moves along $\alpha(s)$, its associated T-pedal curve α_T generates a ruled surface given by the regular parametrization

$$Q(s,v) = \alpha_T + vX(s), \tag{2.5}$$

where the curve α_T is the base curve, and $X(s)$ the ruling of the surface Q [5, 12].
The striction curve and distribution parameter of Q can be written as follows:

$$\beta_X(s) = \alpha_T - \frac{\langle \mathbf{T_1}, X' \rangle}{\|X'\|^2} X(s), \tag{2.6}$$

and

$$\lambda_X(s) = \frac{\det(\mathbf{T_1}, X, X')}{\|X'\|^2}. \tag{2.7}$$

The standard unit normal vector field $\mathbf{n}$ on a surface Q can be defined by

$$\mathbf{n} = \frac{Q_s \times Q_v}{\|Q_s \times Q_v\|}, \tag{2.8}$$

where Q_s and Q_v are the partial derivatives of the surface Q with respect to s and v.

The geometry of the ruled surface $Q(s,v)$ is described by its fundamental forms [11].
The first fundamental form (FFF) is given by

$$I = E\,ds^2 + 2F\,ds\,dv + G\,dv^2, \tag{2.9}$$

where $E = \langle Q_s, Q_s \rangle$, $F = \langle Q_s, Q_v \rangle$, and $G = \langle Q_v, Q_v \rangle$.
The second fundamental form (SFF) is given by

$$II = L\,st^2 + 2M\,ds\,dv + N\,dv^2, \tag{2.10}$$

where $L = \langle Q_{ss}, \mathbf{n} \rangle$, $M = \langle Q_{sv}, \mathbf{n} \rangle$, and $N = \langle Q_{vv}, \mathbf{n} \rangle$.
The third fundamental form (TFF) is given by

$$III = e\,ds^2 + 2f\,ds\,dv + g\,dv^2, \tag{2.11}$$

where $\quad e = \langle \mathbf{n}_s, \mathbf{n}_s \rangle, \quad f = \langle \mathbf{n}_s, \mathbf{n}_v \rangle$, and $\quad g = \langle \mathbf{n}_v, \mathbf{n}_v \rangle$.
Here, $\mathbf{n}_s$ and $\mathbf{n}_v$ are the partial derivatives of the unit normal vector $\mathbf{n}$.

The Gaussian curvature K and mean curvature H of the ruled surface are given by

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{2.12}$$

3. Ruled surfaces using T-pedal curves

This section has three subsections that introduce three different types of ruled surfaces generated by the T-pedal curve, where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the Frenet frame for the unit speed curve $\alpha(s)$. We also discuss their fundamental properties.

Definition 1. *Let α_T be the T-pedal curve of unit speed curve $\alpha(s)$ with Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Then, the parametric representations of the ruled surfaces Q^T, Q^N, and Q^B using Eq (2.5) are*

$$Q^T(s, v) = \alpha_T + v\mathbf{T}(s), \tag{3.1}$$

$$Q^N(s, v) = \alpha_T + v\mathbf{N}(s), \tag{3.2}$$

$$Q^B(s, v) = \alpha_T + v\mathbf{B}(s). \tag{3.3}$$

These ruled surfaces, known as the tangent ruled surface, normal ruled surface, and binormal ruled surface are constructed using the tangent, normal, and binormal vectors $\mathbf{T}(s), \mathbf{N}(s)$, and $\mathbf{B}(s)$ of a unit-speed space curve $\alpha(s)$, respectively. Each of these surfaces takes the T-pedal curve $\alpha_T(s)$ of $\alpha(s)$ as its base curve, with the corresponding Frenet vector field serving as the ruling direction.

3.1. Q^T tangent ruled surface

Definition 2. *The tangent ruled surface Q^T is represented parametrically by*

$$Q^T(s, v) = \alpha(s) + (u(s) + v)\,\mathbf{T}(s), \tag{3.4}$$

where the expression is derived using Eqs (3.1) and (2.3).

Theorem 3.1. *The FFF of the surface Q^T is given by*

$$I = [(u')^2 + (u + v)^2\kappa^2]\,ds^2 + 2u'\,ds\,dv + \,dv^2. \tag{3.5}$$

Proof. From Eq (3.4), the first partial derivatives of $Q^T(s, v)$ using Eq (2.2), are given by

$$Q^T_s = u'\mathbf{T} + (u + v)\kappa\mathbf{N}, \qquad Q^T_v = \mathbf{T}(s). \tag{3.6}$$

From Eq (2.9), the coefficients of the FFF are given by

$$E = \langle Q^T_s, Q^T_s \rangle = (u')^2 + (u + v)^2\kappa^2, \qquad F = \langle Q^T_s, Q^T_v \rangle = u', \qquad G = \langle Q^T_v, Q^T_v \rangle = 1. \tag{3.7}$$

□

Theorem 3.2. *The SFF of the surface Q^T is given by*

$$II = -(u+v)\kappa\tau\, ds^2. \tag{3.8}$$

Proof. From Eq (3.6), the cross product is given by

$$Q_s^T \times Q_v^T = -(u+v)\kappa\mathbf{B}.$$

By taking the norm, we get

$$\|Q_s^T \times Q_v^T\| = (u+v)\kappa.$$

Thus, the unit normal vector can be defined by

$$\mathbf{n}^T = \frac{Q_s^T \times Q_v^T}{\|Q_s^T \times Q_v^T\|} = \frac{-(u+v)\kappa\mathbf{B}}{(u+v)\kappa} = -\mathbf{B}. \tag{3.9}$$

From Eq (3.6), the second partial derivatives are given by

$$Q_{ss}^T = \left(u'' - (u+v)\kappa^2\right)\mathbf{T} + \left(\kappa + 2u'\kappa + (u+v)\kappa'\right)\mathbf{N} + (u+v)\kappa\tau\mathbf{B},$$
$$Q_{vv}^T = 0,$$
$$Q_{vs}^T = \kappa\mathbf{N}.$$

From Eq (2.10), the coefficients of the SFF are given by

$$L = \langle Q_{ss}^T, \mathbf{n}^T\rangle = -(u+v)\kappa\tau, \qquad M = \langle Q_{sv}^T, \mathbf{n}^T\rangle = 0, \qquad N = \langle Q_{vv}^T, \mathbf{n}^T\rangle = 0. \tag{3.10}$$

□

Theorem 3.3. *The Gaussian curvature K and the mean curvature H for the surface Q^T are respectively given by:*

$$K = 0, \qquad H = \frac{-\tau}{2(u+v)\kappa}. \tag{3.11}$$

Proof. By using Eqs (3.7) and (3.10), then

$$LN - M^2 = 0, \qquad EG - F^2 = (u+v)^2\kappa^2, \qquad LG + EN - 2MF = -(u+v)\kappa\tau,$$

and by substitution into Eq (2.12), we deduce the result. □

Corollary 3.1. *(a) The surface Q^T is always developable, as the Gaussian curvature vanishes* ($K = 0$). *(b) The surface Q^T is both developable and minimal if and only if the base curve $\alpha(s)$ is planar* ($\tau = 0$).

Theorem 3.4. *The TFF of the surface Q^T is given by*

$$III = \tau^2\, ds^2. \tag{3.12}$$

Proof. From Eq (3.9), the first partial derivatives of $\mathbf{n}$ are given by

$$\mathbf{n}_s^T = \tau \mathbf{N}, \qquad \mathbf{n}_v^T = 0.$$

From Eq (2.11), the coefficients of the TFF are given by

$$e = \langle \mathbf{n}_s^T, \mathbf{n}_s^T \rangle = \tau^2, \qquad f = \langle \mathbf{n}_s^T, \mathbf{n}_v^T \rangle = 0, \qquad g = \langle \mathbf{n}_v^T, \mathbf{n}_v^T \rangle = 0.$$

□

Theorem 3.5. *The striction curve $\beta_T(s)$ for the surface Q^T is given by*

$$\beta_T(s) = \alpha(s) + (1 - \omega_1) u \mathbf{T}.$$

Proof. From Eq (2.6), the striction curve is defined by

$$\beta_T(s) = \alpha_T - \frac{\langle \mathbf{T_1}, \mathbf{T}' \rangle}{\|\mathbf{T}'\|^2} \mathbf{T}.$$

By using Eqs (2.2) and (2.4), we have

$$\frac{\langle \mathbf{T_1}, \mathbf{T}' \rangle}{\|\mathbf{T}'\|^2} = \frac{\omega_1 u \kappa^2}{\kappa^2} = \omega_1 u.$$

Thus, by using Eq (2.3), we can deduce the result. □

Theorem 3.6. *For the tangent ruled surface Q^T, the distribution parameter λ_T vanishes.*

Proof. From Eq (2.7), the distribution parameter defined by

$$\lambda_T = \frac{\det(\mathbf{T_1}, \mathbf{T}, \mathbf{T}')}{\|\mathbf{T}'\|^2}.$$

By using Eqs (2.2) and (2.4), we have

$$\det(\mathbf{T_1}, \mathbf{T}, \mathbf{T}') = 0.$$

Thus, we can deduce the result. □

3.2. Q^N normal ruled surface

Definition 3. *The normal ruled surface Q^N is represented parametrically by*

$$Q^N(s, v) = \alpha(s) + u(s)\mathbf{T}(s) + v\mathbf{N}(s), \tag{3.13}$$

where the expression is derived using Eqs (3.2) and (2.3).

Theorem 3.7. *The FFF of the surface Q^N is given by*

$$I = [(1 + u' - v\kappa)^2 + (u\kappa)^2 + (v\tau)^2]\, ds^2 + 2u\kappa\, ds\, dv + \, dv^2. \tag{3.14}$$

Proof. From Eq (3.13), the first partial derivatives of $Q^N(s,v)$ using Eq (2.2), are given by

$$Q^N_s = (1+u'-v\kappa)\mathbf{T} + u\kappa\mathbf{N} + v\tau\mathbf{B}, \qquad Q^N_v = \mathbf{N}. \tag{3.15}$$

From Eq (2.9), the coefficients of the FFF are given by

$$E = \langle Q^N_s, Q^N_s\rangle = (1+u'-v\kappa)^2 + (u\kappa)^2 + (v\tau)^2, \quad F = \langle Q^N_s, Q^N_v\rangle = u\kappa, \quad G = \langle Q^N_v, Q^N_v\rangle = 1. \tag{3.16}$$

□

Theorem 3.8. *The SFF of the surface Q^N is given by*

$$II = L\,ds^2 + 2M\,ds\,dv + N\,dv^2, \tag{3.17}$$

where

$$\begin{aligned} L &= \langle Q^N_{ss}, \mathbf{n}\rangle = \frac{-v\tau\left(u'' - v\kappa' - u\kappa^2\right) + (u\kappa\tau + v\tau')(1+u'-v\kappa)}{\sqrt{(v\tau)^2 + (1+u'-v\kappa)^2}}, \\ M &= \langle Q^N_{sv}, \mathbf{n}\rangle = \frac{\tau(1+u')}{\sqrt{(v\tau)^2 + (1+u'-v\kappa)^2}}, \qquad N = \langle Q^N_{vv}, \mathbf{n}\rangle = 0. \end{aligned} \tag{3.18}$$

Proof. From Eq (3.15), the cross product is given by

$$Q^N_s \times Q^N_v = -v\tau\mathbf{T} + (1+u'-v\kappa)\mathbf{B}.$$

By taking the norm, we get

$$\|Q^N_s \times Q^N_v\| = \sqrt{(v\tau)^2 + (1+u'-v\kappa)^2}.$$

Thus, the unit normal vector can be defined by

$$\mathbf{n}^N = \frac{Q^N_s \times Q^N_v}{\|Q^N_s \times Q^N_v\|} = \frac{-v\tau\mathbf{T} + (1+u'-v\kappa)\mathbf{B}}{\sqrt{(v\tau)^2 + (1+u'-v\kappa)^2}}. \tag{3.19}$$

From Eq (3.15), the second partial derivatives are given by

$$Q^N_{ss} = \left(u'' - v\kappa' - u\kappa^2\right)\mathbf{T} + \left(\kappa + 2u'\kappa + u\kappa' - v(\kappa^2+\tau^2)\right)\mathbf{N} + (u\kappa\tau + v\tau')\,\mathbf{B},$$

$$Q^N_{vv} = 0,$$

$$Q^N_{vs} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

From Eq (2.10), we can deduce the coefficients of the SFF.

□

Theorem 3.9. *The Gaussian curvature K and the mean curvature H for the normal ruled surface Q^N are given respectively by:*

$$\begin{aligned} K &= -\frac{\tau^2(1+u')^2}{\left((1+u'-v\kappa)^2 + (v\tau)^2\right)^2}, \\ H &= \frac{-v\tau\left(u'' - v\kappa' - u\kappa^2\right) + (u\kappa\tau + v\tau')(1+u'-v\kappa) - 2u\kappa\tau(1+u')}{2\left((1+u'-v\kappa)^2 + (v\tau)^2\right)^{3/2}}. \end{aligned}$$

Proof. By using Eqs (3.16) and (3.18), then

$$LN - M^2 = -\frac{\tau^2(1+u')^2}{(1+u'-v\kappa)^2+(v\tau)^2},$$

$$EG - F^2 = (1+u'-v\kappa)^2+(v\tau)^2,$$

$$LG + EN - 2MF = \frac{-v\tau\left(u''-v\kappa'-u\kappa^2\right)+(u\kappa\tau+v\tau')(1+u'-v\kappa)-2u\kappa\tau(1+u')}{\sqrt{(1+u'-v\kappa)^2+(v\tau)^2}},$$

and by substitution into Eq (2.12), we deduce the result. □

Corollary 3.2. *(a) The surface Q^N is both developable and a miniamal surface if and only if the base curve $\alpha(s)$ is planar ($\tau = 0$).*

Theorem 3.10. *The TFF of the surface Q^N is given by*

$$III = e\, ds^2 + 2f\, dsdv + g\, dv^2, \tag{3.20}$$

with coefficients

$$e = \left(\frac{-v\tau'}{D_1} - \frac{A_1 D_{1s}}{D^2}\right)^2 + \left(\frac{-\tau(1+u')}{D_1}\right)^2 + \left(\frac{u''-v\kappa'}{D_1} - \frac{C_1 D_{1s}}{D^2}\right)^2,$$

$$f = \left(\frac{-v\tau'}{D_1} - \frac{A_1 D_{1s}}{D_1^2}\right)\left(\frac{-\tau}{D_1} - \frac{A_1(v\tau^2-\kappa C_1)}{D_1^3}\right) + \left(\frac{u''-v\kappa'}{D_1} - \frac{C D_{1s}}{D_1^2}\right)\left(\frac{-\kappa}{D_1} - \frac{C_1(v\tau^2-\kappa C_1)}{D_1^3}\right),$$

$$g = \left(\frac{-\tau}{D_1} - \frac{A_1(v\tau^2-\kappa C_1)}{D_1^3}\right)^2 + \left(\frac{-\kappa}{D_1} - \frac{C_1(v\tau^2-\kappa C_1)}{D_1^3}\right)^2,$$

where

$$A_1 = -v\tau, \quad C_1 = 1+u'-v\kappa, \quad D_1 = \sqrt{A^2+C^2},$$

and D_{1s} is the partial derivatives of D_1 with respect to s.

Proof. From Eq (3.19), the unit normal vector can be written as

$$\mathbf{n}^N = \frac{A_1\mathbf{T} + C_1\mathbf{B}}{D_1}, \tag{3.21}$$

where

$$A_1 = -v\tau, \quad C_1 = 1+u'-v\kappa, \quad D_1 = \sqrt{A_1^2+C_1^2}.$$

The partial derivatives of $\mathbf{n}$ with respect to s and v are respectively

$$\mathbf{n}_s^N = \left(\frac{-v\tau'}{D_1} - \frac{A_1 D_{1s}}{D_1^2}\right)\mathbf{T} + \left(\frac{-\tau(1+u')}{D_1}\right)\mathbf{N} + \left(\frac{u''-v\kappa'}{D_1} - \frac{C_1 D_{1s}}{D_1^2}\right)\mathbf{B},$$

$$\mathbf{n}_v^N = \left(\frac{-\tau}{D_1} - \frac{A_1(v\tau^2-\kappa C_1)}{D_1^3}\right)\mathbf{T} + \left(\frac{-\kappa}{D_1} - \frac{C_1(v\tau^2-\kappa C_1)}{D_1^3}\right)\mathbf{B}.$$

The coefficients e, f, and g of the third fundamental form are obtained by computing the inner products

$$e = \langle \mathbf{n}_s^N, \mathbf{n}_s^N\rangle, \qquad f = \langle \mathbf{n}_s^N, \mathbf{n}_v^N\rangle, \qquad g = \langle \mathbf{n}_v^N, \mathbf{n}_v^N\rangle,$$

which yield the explicit expressions given in the theorem. □

Theorem 3.11. *The striction curve $\beta_N(s)$ for the surface Q^N is given by*

$$\beta_N(s) = \alpha(s) + u\mathbf{T} + \frac{\omega_1(1+u)\kappa}{\kappa^2+\tau^2}\mathbf{N}.$$

Proof. From Eq (2.6), the striction curve defined by

$$\beta_N(s) = \alpha_T - \frac{\langle \mathbf{T_1}, \mathbf{N}' \rangle}{\|\mathbf{N}'\|^2}\mathbf{N}.$$

By using Eqs (2.2) and (2.4), we have

$$\frac{\langle \mathbf{T_1}, \mathbf{N}' \rangle}{\|\mathbf{N}'\|^2} = -\frac{\omega_1(1+u)\kappa}{\kappa^2+\tau^2}.$$

Thus, by using Eq (2.3), we can deduce the result. □

Theorem 3.12. *The distribution parameter λ_N for the surface Q^N is given by*

$$\lambda_N = \frac{\omega_1(1+u)\tau}{\kappa^2+\tau^2}.$$

Proof. From Eq (2.7), the distribution parameter is defined by

$$\lambda_N = \frac{\det(\mathbf{T_1}, \mathbf{N}, \mathbf{N}')}{\|\mathbf{N}'\|^2}.$$

By using Eqs (2.2) and (2.4), we have

$$\det(\mathbf{T_1}, \mathbf{N}, \mathbf{N}') = \omega_1(1+u)\tau.$$

Thus, we can deduce the result. □

3.3. Q^B binormal ruled surface

Definition 4. *The binormal ruled surface Q^B is represented parametrically by*

$$Q^B(s,v) = \alpha(s) + u(s)\mathbf{T}(s) + v\mathbf{B}(s), \tag{3.22}$$

where the expression is derived using Eqs (3.3) and (2.3).

Theorem 3.13. *The FFF of the surface Q^B is given by*

$$I = [(1+u')^2 + (u\kappa - v\tau)^2]\, ds^2 + \, dv^2. \tag{3.23}$$

Proof. From Eq (3.22), the first partial derivatives of $Q^B(s,v)$ using Eq (2.2), are given by

$$Q^B_s = (1+u')\mathbf{T} + (u\kappa - v\tau)\mathbf{N}, \qquad Q^B_v = \mathbf{B}. \tag{3.24}$$

From Eq (2.9), the coefficients of the FFF are given by

$$E = \langle Q^B_s, Q^B_s \rangle = (1+u')^2 + (u\kappa - v\tau)^2, \quad F = \langle Q^B_s, Q^B_v \rangle = 0, \quad G = \langle Q^B_v, Q^B_v \rangle = 1. \tag{3.25}$$

□

Theorem 3.14. *The SFF of the surface Q^B is given by*

$$II = L\,ds^2 + 2M\,ds\,dv + N\,dv^2, \tag{3.26}$$

where

$$\begin{aligned} L &= \frac{\left(u'' - u\kappa^2 + v\kappa\tau\right)(u\kappa - v\tau) - (\kappa + 2u'\kappa + u\kappa' - v\tau')(1 + u')}{\sqrt{(1 + u')^2 + (u\kappa - v\tau)^2}}, \\ M &= \frac{\tau(1 + u')}{\sqrt{(1 + u')^2 + (u\kappa - v\tau)^2}}, \qquad N = 0. \end{aligned} \tag{3.27}$$

Proof. From Eq (3.24), the cross product is given by

$$Q^B_s \times Q^B_v = (u\kappa - v\tau)\mathbf{T} - (1 + u')\mathbf{N}.$$

By taking the norm, we get

$$\|Q^B_s \times Q^B_v\| = \sqrt{(u\kappa - v\tau)^2 + (1 + u')^2}.$$

Thus, the unit normal vector can be defined by

$$\mathbf{n}^B = \frac{Q^B_s \times Q^B_v}{\|Q^B_s \times Q^B_v\|} = \frac{(u\kappa - v\tau)\mathbf{T} - (1 + u')\mathbf{N}}{\sqrt{(u\kappa - v\tau)^2 + (1 + u')^2}}. \tag{3.28}$$

From Eq (3.24), the second partial derivatives are given by

$$Q^B_{ss} = \left(u'' - u\kappa^2 + v\kappa\tau\right)\mathbf{T} + (\kappa + 2u'\kappa + u\kappa' - v\tau')\,\mathbf{N} + \left(u\kappa\tau - v\tau^2\right)\mathbf{B},$$

$$Q^B_{vv} = 0, \qquad Q^B_{vs} = -\tau\mathbf{N}.$$

From Eq (2.10), we can deduce the coefficients of the SFF. □

Theorem 3.15. *The Gaussian curvature K and the mean curvature H for the binormal ruled surface Q^B are given respectively by:*

$$\begin{aligned} K &= -\frac{\tau^2(1 + u')^2}{\left((1 + u')^2 + (u\kappa - v\tau)^2\right)^2}, \\ H &= \frac{\left(u'' - u\kappa^2 + v\kappa\tau\right)(u\kappa - v\tau) - (\kappa + 2u'\kappa + u\kappa' - v\tau')(1 + u')}{2\left((1 + u')^2 + (u\kappa - v\tau)^2\right)^{3/2}}. \end{aligned}$$

Proof. By using Eqs (3.25) and (3.27), then

$$LN - M^2 = -M^2 = -\frac{\tau^2(1 + u')^2}{(1 + u')^2 + (u\kappa - v\tau)^2},$$

$$EG - F^2 = E = (1 + u')^2 + (u\kappa - v\tau)^2,$$

$$\begin{aligned} LG + EN - 2MF &= L \\ &= \frac{\left(u'' - u\kappa^2 + v\kappa\tau\right)(u\kappa - v\tau) - (\kappa + 2u'\kappa + u\kappa' - v\tau')(1 + u')}{\sqrt{(1 + u')^2 + (u\kappa - v\tau)^2}}, \end{aligned}$$

and by substitution into Eq (2.12), we deduce the result. □

Corollary 3.3. *(a) The surface Q^B is developable ($K = 0$) if the base curve $\alpha(t)$ is planar ($\tau = 0$) with mean curvature*

$$H = \frac{(u'' - u\kappa^2)(u\kappa) - (\kappa + 2u'\kappa + u\kappa')(1 + u')}{2\left((1 + u')^2 + (u\kappa)^2\right)^{3/2}}.$$

Theorem 3.16. *The TFF of surface Q^B is given by*

$$III = e\,ds^2, +2f\,dsdv + g\,dv^2, \tag{3.29}$$

with coefficients:

$$\begin{aligned}
e &= \left(\frac{(u'\kappa + u\kappa' - v\tau') - C_2\kappa}{D_2} - \frac{A_2 D_{2s}}{D_2^2}\right)^2 + \left(\frac{A_2\kappa - u''}{D_2} - \frac{C_2 D_{2s}}{D_2^2}\right)^2 + \left(\frac{C_2\tau}{D_2}\right)^2,\\
f &= \left(\frac{(u'\kappa + u\kappa' - v\tau') - C_2\kappa}{D_2} - \frac{A_2 D_{2s}}{D_2^2}\right)\left(\frac{-\tau}{D_2} + \frac{A_2^2\tau}{D_2^3}\right) + \left(\frac{A_2\kappa - u''}{D_2} - \frac{C_2 D_{2s}}{D_2^2}\right)\left(\frac{A_2 C_2\tau}{D_2^3}\right),\\
g &= \left(\frac{-\tau}{D_2} + \frac{A_2^2\tau}{D_2^3}\right)^2 + \left(\frac{A_2 C_2\tau}{D_2^3}\right)^2,
\end{aligned}$$

where

$$A_2 = u\kappa - v\tau, \quad C_2 = -(1 + u'), \quad D_2 = \sqrt{A_2^2 + C_2^2},$$

and D_{2s} is the partial derivative of D_2 with respect to s.

Proof. From Eq (3.28), the unit normal vector can be written as

$$\mathbf{n}^B = \frac{A_2\mathbf{T} + C_2\mathbf{N}}{D_2}, \tag{3.30}$$

where

$$A_2 = u\kappa - v\tau, \quad C_2 = -(1 + u'), \quad D_2 = \sqrt{A_2^2 + C_2^2}.$$

The partial derivatives of $\mathbf{n}$ with respect to t and v are respectively

$$\mathbf{n}_s^B = \left(\frac{(u'\kappa + u\kappa' - v\tau') - C_2\kappa}{D_2} - \frac{A_2 D_{2s}}{D_2^2}\right)\mathbf{T} + \left(\frac{A_2\kappa - u''}{D_2} - \frac{C_2 D_{2s}}{D_2^2}\right)\mathbf{N} + \left(\frac{C_2\tau}{D_2}\right)\mathbf{B},$$

$$\mathbf{n}_v^B = \left(\frac{-\tau}{D_2} + \frac{A_2^2\tau}{D_2^3}\right)\mathbf{T} + \frac{A_2 C_2\tau}{D_2^3}\mathbf{N}.$$

The coefficients e, f, and g of the third fundamental form are obtained by computing the inner products

$$e = \langle \mathbf{n}_s^B, \mathbf{n}_s^B\rangle, \qquad f = \langle \mathbf{n}_s^B, \mathbf{n}_v^B\rangle, \qquad g = \langle \mathbf{n}_v^B, \mathbf{n}_v^B\rangle,$$

which yield the explicit expressions given in the theorem. □

Theorem 3.17. *The striction curve $\beta_B(s)$ for the surface Q^B is given by*

$$\beta_B(s) = \alpha(s) + u\mathbf{T} + \frac{\omega_1 u\kappa}{\tau}\mathbf{B}.$$

Proof. From Eq (2.6), the striction curve is defined by:

$$\beta_B(s) = \alpha_T - \frac{\langle \mathbf{T_1}, \mathbf{B}' \rangle}{||\mathbf{B}'||^2}\mathbf{B}.$$

By using Eqs (2.2) and (2.4), we have

$$\frac{\langle \mathbf{T_1}, \mathbf{B}' \rangle}{||\mathbf{B}'||^2} = -\frac{\omega_1 u\kappa}{\tau}\mathbf{B}.$$

Thus, by using Eq (2.3), we can deduce the result. □

Theorem 3.18. *The distribution parameter λ_T for the binormal ruled surface M_3 is given by*

$$\lambda_B = \frac{\omega_1(1+u)}{\tau}.$$

Proof. From Eq (2.7), the distribution parameter is defined by

$$\lambda_B = \frac{\det(\mathbf{T_1}, \mathbf{B}, \mathbf{B}')}{||\mathbf{B}'||^2}.$$

By using Eqs (2.2) and (2.4), we have

$$\det(\mathbf{T_1}, \mathbf{B}, \mathbf{B}') = \omega_1(1+u)\tau.$$

Thus, we can deduce the result. □

Theorem 3.19. *Among the three surfaces, only Q^T is always developable ($K \equiv 0$). The surfaces Q^N and Q^B are developable if and only if the base curve is planar ($\tau = 0$), creating a hierarchical relationship where tangent-directed rulings naturally produce developable surfaces while normal and binormal directions introduce torsion-dependent curvature.*

4. Computational and illustrative examples

Example 4.1. *Let $\alpha(s)$ be a general helix curve given by the parametrization*

$$\alpha(s) = \left(4\cos\left(\frac{s}{5}\right), 4\sin\left(\frac{s}{5}\right), \frac{3s}{5}\right).$$

By using the equations in (2.1), the Frenet-frame is

$$\mathbf{T}(s) = \left(-\frac{4}{5}\sin\left(\frac{s}{5}\right), \frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right),$$
$$\mathbf{N}(s) = \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right),$$
$$\mathbf{B}(s) = \left(\frac{3}{5}\sin\left(\frac{s}{5}\right), -\frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right).$$

We can deduce the T-pedal curve using Eq (2.3),

$$\alpha_T(s) = \left(4\cos\left(\frac{s}{5}\right) + \frac{36s}{125}\sin\left(\frac{s}{5}\right), 4\sin\left(\frac{s}{5}\right) - \frac{36s}{125}\cos\left(\frac{s}{5}\right), \frac{48s}{125}\right),$$

where $u(s) = -\langle\alpha(s), \mathbf{T}(s)\rangle = -\frac{9s}{25}$.

Consequently, we can deduce the tangent ruled surface Q^T*, normal ruled surface* Q^N *and binormal ruled surface* Q^B*, respectively, given in Figure 3, by*

$$Q^T(s,v) = \left(4\cos\left(\frac{s}{5}\right) + \left(\frac{36s}{125} - \frac{4v}{5}\right)\sin\left(\frac{s}{5}\right), 4\sin\left(\frac{s}{5}\right) + \left(-\frac{36s}{125} + \frac{4v}{5}\right)\cos\left(\frac{s}{5}\right), \frac{48s}{125} + \frac{3v}{5}\right),$$

$$Q^N(s,v) = \left((4-v)\cos(\frac{s}{5}) + \frac{36s}{125}\sin(\frac{s}{5}), (4-v)\sin(\frac{s}{5}) - \frac{36s}{125}\cos(\frac{s}{5}), \frac{48s}{125}\right),$$

$$Q^B(s,v) = \left(4\cos\frac{s}{5} + \left(\frac{36s}{125} + \frac{3v}{5}\right)\sin\frac{s}{5}, 4\sin\frac{s}{5} - \left(\frac{36s}{125} + \frac{3v}{5}\right)\cos\frac{s}{5}, \frac{48s}{125} + \frac{4v}{5}\right).$$

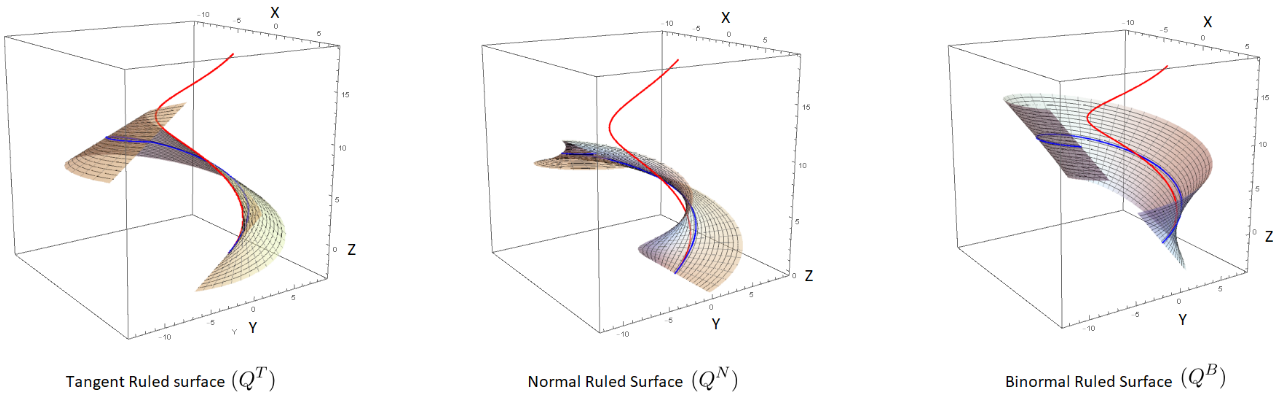

Figure 3. The helix (red), T-pedal curve (blue).

Example 4.2. *Let* $\alpha(s)$ *be a regular curve given by the parametrization*

$$\alpha(s) = \left(\frac{3}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{6}\cos\left(\frac{3s}{2}\right), \frac{3}{2}\sin\left(\frac{s}{2}\right) + \frac{1}{6}\sin\left(\frac{3s}{2}\right), \sqrt{3}\cos\left(\frac{s}{2}\right)\right).$$

By Eq (2.1), the Frenet-frame is

$$\mathbf{T}(s) = \left(-\frac{3}{2}\sin\frac{s}{2} + \sin^3\frac{s}{2}, \cos^3\frac{s}{2}, -\frac{\sqrt{3}}{2}\sin\frac{s}{2}\right),$$

$$\mathbf{N}(s) = \left(-\frac{\sqrt{3}}{2}\cos s, -\frac{\sqrt{3}}{2}\sin s, -\frac{1}{2}\right),$$

$$\mathbf{B}(s) = \left(-\frac{1}{2}\cos\frac{s}{2}\left(1 + 2\sin^2\frac{s}{2}\right), -\sin^3\frac{s}{2}, \frac{\sqrt{3}}{2}\cos\frac{s}{2}\right).$$

We can deduce the T-Pedal curve using Eq (2.3),

$$\alpha_T(s) = \left(\frac{3}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{6}\cos\left(\frac{3s}{2}\right) + \sin s\left(-\frac{3}{2}\sin\left(\frac{s}{2}\right) + \sin^3\left(\frac{s}{2}\right)\right),\ \frac{3}{2}\sin\left(\frac{s}{2}\right) + \frac{1}{6}\sin\left(\frac{3s}{2}\right) + \sin s\cos^3\left(\frac{s}{2}\right),\ \sqrt{3}\cos\left(\frac{s}{2}\right) - \frac{\sqrt{3}}{2}\sin s\sin\left(\frac{s}{2}\right)\right),$$

where $u(s) = -\langle \alpha(s), \mathbf{T}(s)\rangle = \sin s$.

Consequently, we can deduce the tangent ruled surface Q^T*, normal ruled surface* Q^N *and binormal ruled surface* Q^B*, respectively, given in Figure 4, by*

$$Q^T(s,v) = \left(\left(\frac{3}{2}\cos\frac{s}{2} + \frac{1}{6}\cos\frac{3s}{2}\right) + (\sin s + v)\left(-\frac{3}{2}\sin\frac{s}{2} + \sin^3\frac{s}{2}\right), \left(\frac{3}{2}\sin\frac{s}{2} + \frac{1}{6}\sin\frac{3s}{2}\right) + (\sin s + v)\cos^3\frac{s}{2},\ \sqrt{3}\cos\frac{s}{2} + (\sin s + v)\left(-\frac{\sqrt{3}}{2}\sin\frac{s}{2}\right)\right),$$

$$Q^N(s,v) = \left(\left(\frac{3}{2}\cos\frac{s}{2} + \frac{1}{6}\cos\frac{3s}{2}\right) + (\sin s + v)\left(-\frac{\sqrt{3}}{2}\cos s\right), \left(\frac{3}{2}\sin\frac{s}{2} + \frac{1}{6}\sin\frac{3s}{2}\right) + (\sin s + v)\left(-\frac{\sqrt{3}}{2}\sin s\right),\ \sqrt{3}\cos\frac{s}{2} + (\sin s + v)\left(-\frac{1}{2}\right)\right),$$

$$Q^B(s,v) = \left(\left(\frac{3}{2}\cos\frac{s}{2} + \frac{1}{6}\cos\frac{3s}{2}\right) + (\sin s + v)\left(-\frac{1}{2}\cos\frac{s}{2}(1 + 2\sin^2\frac{s}{2})\right), \left(\frac{3}{2}\sin\frac{s}{2} + \frac{1}{6}\sin\frac{3s}{2}\right) + (\sin s + v)\left(-\sin^3\frac{s}{2}\right),\ \sqrt{3}\cos\frac{s}{2} + (\sin s + v)\left(\frac{\sqrt{3}}{2}\cos\frac{s}{2}\right)\right).$$

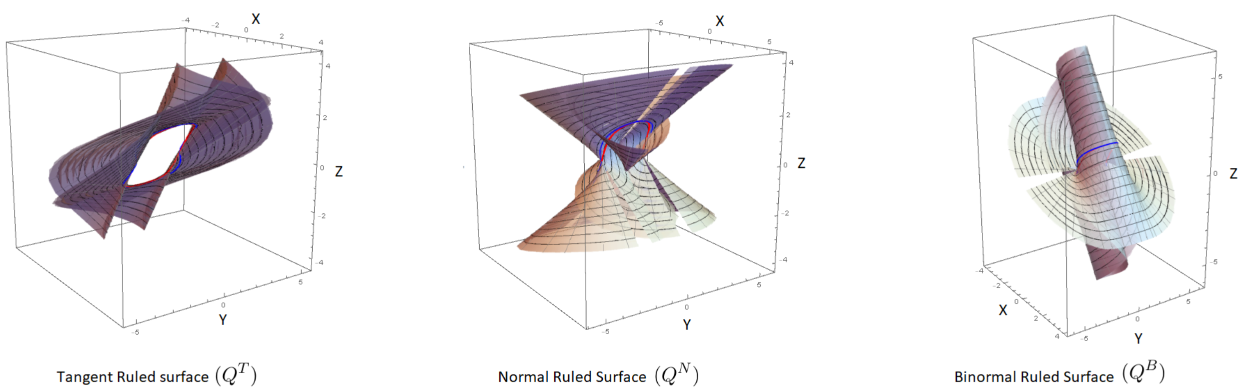

Figure 4. The regular curve (red), T-pedal curve (blue).

5. Conclusions

This work develops a complete differential-geometric framework for three classes of Frenet-frame ruled surfaces generated from unit-speed curves. Through systematic analysis of the T-pedal curve construction, we establish fundamental curvature properties, metric characteristics, and distribution parameters for each surface type. The results yield both theoretical advances in surface geometry and practical computational tools for parametric surface modeling. These contributions not only deepen our knowledge of ruled surface geometry, but also provide a foundation for future investigations in extended mathematical contexts, including relativistic systems and non-Euclidean geometries, demonstrating the continued relevance of classical surface theory in modern interdisciplinary research.

Author contributions

A. Elsharkawy: Conceptualization, methodology, formal analysis, writing – original draft, writing – review & editing, supervision; H. K. Elsayied: Conceptualization, methodology, validation, writing – original draft, writing – review & editing; M. E. Desouky: Formal analysis, investigation, validation, writing – review & editing; C. Cesarano: Conceptualization, resources, writing – review & editing, supervision, project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors extend their appreciation to the anonymous reviewers for their valuable comments and suggestions that helped improve this manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. B. Bukcu, M. K. Karakus, Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n, *Geom. Dedicata,* **133** (2008), 59–66. https://doi.org/10.1007/s10711-008-9233-5
2. D. Canli, S. Şenyurt, F. E. Kaya, Pedal curves obtained from Frenet vector of a space curve and Smarandache curves belonging to these curves, *AIMS Math.,* **9** (2024), 20136–20162. https://doi.org/10.3934/math.2024981
3. D. Canli, S. Şenyurt, F. E. Kaya, L. Grilli, The pedal curves generated by alternative frame vectors and their Smarandache curves, *Symmetry,* **16** (2024), 1012. https://doi.org/10.3390/sym16081012

4. M. P. D. Carmo, *Differential geometry of curves and surfaces*, Prentice Hall, Englewood Cliffs, NJ, 1976.
5. A. Elsharkawy, H. K. Elsayied, A. Refaat, Quasi ruled surfaces in Euclidean 3-space, *Eur. J. Pure Appl. Math.,* **18** (2025), 5710–5710. https://doi.org/10.29020/nybg.ejpam.v18i1.5710
6. A. Elsharkawy, H. Hamdani, C. Cesarano, N. Elsharkawy, Geometric properties of Smarandache ruled surfaces generated by integral binormal curves in Euclidean 3-space, *Part. Differ. Equ. Appl. Math.,* **15** (2025), 101298. https://doi.org/10.1016/j.padiff.2025.101298
7. V. N. Ivanov, Geometry of the normal ruled surfaces, *Struct. Mech. Eng. Constr. Build.,* **17** (2021), 562–575. https://doi.org/10.22363/1815-5235-2021-17-6-562-575
8. F. E. Kaya, Differential geometric aspects of pedal curves on surfaces, *J. Math.,* **2024** (2024), 6237091. https://doi.org/10.1155/2024/6237091
9. M. Li, K. Yao, P. Li, D. Pei, Pedal curves of non-lightlike curves in Minkowski 3-space, *Symmetry,* **14** (2022), 59. https://doi.org/10.3390/sym14010059
10. E. H. Lockwood, *A book of curves*, Cambridge University Press, Cambridge, 1961. https://doi.org/10.1017/CBO9780511569340
11. M. Masal, A. Z. Azak, Ruled surfaces according to Bishop frame in the Euclidean 3-space, *P. Natl. A. Sci. India A,* **89** (2019), 415–424. https://doi.org/10.1007/s40010-018-0546-y
12. B. Pal, S. Kumar, Ruled-like surfaces in three-dimensional Euclidean space, *Ann. Math. Inform.,* **59** (2023), 83–101. https://doi.org/10.33039/ami.2022.12.011
13. Y. Pan, Z. Xu, B. Wang, B. Deng, Piecewise ruled approximation for freeform mesh surfaces, *ACM T. Graphic.,* **44** (2025), 1–18. https://doi.org/10.1145/3730866
14. T. Shifrin, *Differential geometry: A first course in curves and surfaces*, University of Georgia, 2015.
15. O. O. Tuncer, H. Ceyhan, İ. Gök, F. N. Ekmekci, Notes on pedal and contrapedal curves of fronts in the Euclidean plane, *Math. Method. Appl. Sci.,* **41** (2018), 5096–5111. https://doi.org/10.1002/mma.5056