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Research article

$\mathbb{N}^d$-Indexed persistence modules, higher dimensional partitions and rank invariants

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Abstract: We study decomposable $\mathbb{N}^d$-indexed persistence modules via higher dimensional partitions. Their barcodes are defined in terms of the extended interior of the corresponding Young diagrams. For two decomposable $\mathbb{N}^d$-indexed persistence modules, we present a necessary and sufficient condition, in terms of the partitions, for their rank invariants to be the same. This generalizes the well-known fact that for an $\mathbb{N}$-indexed persistence module, its barcode and its rank invariant determine each other, i.e., the rank invariant is a complete invariant.

Keywords: multiparameter persistence; $\mathbb{N}^d$-indexed persistence modules; higher dimensional partitions; rank invariants; barcodes; Young diagrams
Mathematics Subject Classification: 05A17, 14C05, 55N31

1. Introduction

A fundamental structure theorem asserts that every one-dimensional persistence module admits a unique decomposition up to isomorphisms [5, 7]. This gives rise to the concept of barcodes which plays a pivotal role in topological data analysis. It is known from the pioneering work [6] that such structure theorem is no longer true for higher dimensional persistence modules. Extensive research has been devoted to higher dimensional persistence modules in recent years (see [2,4,9] and the references therein). Higher dimensional persistence modules have found important applications in the study of noisy point cloud data and time-varying data [8, 10, 13].

In this paper, we study $\mathbb{N}^d$-indexed persistence modules over a field k via d-dimensional partitions, where $\mathbb{N}$ denotes the set of nonnegative integers. To motivate our concepts to be introduced below, let us look at the case $d = 1$. It is well-known that the barcode of an $\mathbb{N}$-indexed persistence module over k is a multiset consisting of some intervals of the form $[a, b) = \mathbb{T}_a[0, b - a)$ where $a \in \mathbb{N}$, $b \in \mathbb{N} \sqcup \{+\infty\}$, and $\mathbb{T}_a : \mathbb{N} \to \mathbb{N}$ is the translation by a. The closed interval $[0, b - a]$ is precisely the Young diagram of the 1-dimensional partition $(b - a)_O$ corresponding to $b - a$ where $O \in \mathbb{N}^0 = \{O\}$, while the interval

$[0, b-a)$ may be regarded as the extended interior of the Young diagram $[0, b-a]$.

For a general integer $d \geq 1$, a d-dimensional partition λ is an array

$$\lambda = (\lambda_{i_1,\ldots,i_{d-1}})_{i_1,\ldots,i_{d-1}}$$

of $\lambda_{i_1,\ldots,i_{d-1}} \in \mathbb{N} \sqcup \{+\infty\}$ indexed by $(i_1, \ldots, i_{d-1}) \in \mathbb{N}^{d-1}$ such that

$$\lambda_{i_1,\ldots,i_{d-1}} \geq \lambda_{j_1,\ldots,j_{d-1}},$$

if $i_1 \leq j_1, \ldots, i_{d-1} \leq j_{d-1}$. For a d-dimensional partition λ, the *extended interior* of its Young diagram $D_\lambda \subset (\mathbb{R}^+)^d$ is the region

$$D_\lambda^{\text{ei}} = D_\lambda - \overline{(\partial D_\lambda) \cap (\mathbb{R}^+)^d}.$$

Set $D_\lambda^{\text{int}} = D_\lambda^{\text{ei}} \cap \mathbb{N}^d$, which is the set of integral points in D_λ^{ei}. Geometrically, D_λ^{ei} is obtained from the Young diagram D_λ by removing its boundary in $(\mathbb{R}^+)^d$, and D_λ^{int} consists of all the integral points in D_λ^{ei}.

Example 1.1. For the 2-dimensional partitions $\lambda = (3, 3, 1)$, the extended interior D_λ^{ei} of D_λ is illustrated by Figure 1 below. Note that

$$D_\lambda^{\text{int}} = \{(0,0), (1,0), (2,0), (0,1), (1,1), (0,2), (1,2)\},$$

which consists of all the integral points in D_λ^{ei}.

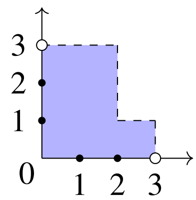

Figure 1. D_λ^{ei} for a 2-dimensional partition of $n = 7$.

Define the $\mathbb{N}^d$-indexed persistence module $\mathbf{k}_\lambda$ by

$$(\mathbf{k}_\lambda)_x = \begin{cases} k, & \text{if } x \in D_\lambda^{\text{int}}; \\ 0, & \text{otherwise.} \end{cases}$$

For $x, y \in \mathbb{N}^d$ with $x \leq y$, the morphism $(\mathbf{k}_\lambda)_{x,y} : (\mathbf{k}_\lambda)_x \to (\mathbf{k}_\lambda)_y$ is the identity map Id_k if $x, y \in D_\lambda^{\text{int}}$, and 0 otherwise.

We define that an $\mathbb{N}^d$-indexed persistence module M admits a *barcode* if

$$M \cong \bigoplus_{i \in \Lambda} \mathbb{T}_{a^{(i)}} \mathbf{k}_{\lambda^{(i)}},$$

where the index set Λ is finite, and for each $i \in \Lambda$, $a^{(i)} \in \mathbb{N}^d$, $\mathbb{T}_{a^{(i)}} : \mathbb{N}^d \to \mathbb{N}^d$ is the translation associated to $a^{(i)}$, and $\lambda^{(i)}$ is a d-dimensional partition with nonzero size $|\lambda^{(i)}| \neq 0$. In this case, the *barcode* $\mathfrak{B}_M$ of M is defined to be the multiset whose elements are

$$\mathbb{T}_{a^{(i)}}(D_{\lambda^{(i)}}^{\text{ei}}), \quad i \in \Lambda.$$

Criterions and algorithms for determining whether an $\mathbb{N}^d$-indexed persistence module admits a barcode were investigated in [11].

Since a general higher dimensional persistence module may not admit a barcode, the rank invariant was introduced in [6] as an alternative discrete invariant. The rank invariant of an $\mathbb{N}^d$-indexed persistence module M is the function $\mathrm{Rank}^M : (\mathbb{N}^d)^{\leq} \to \mathbb{N}$ given by

$$\mathrm{Rank}^M(x,y) = \mathrm{Rank}(M_{x,y}),$$

where $(\mathbb{N}^d)^{\leq} = \{(x,y) \in \mathbb{N}^d \times \mathbb{N}^d | x \leq y\}$. Indeed, Carlsson and Zomorodian [6, Theorem 12] proved that when $d = 1$, the barcode and the rank invariant determine each other, i.e., the rank invariant is a complete invariant. However, when $d > 1$, no prior necessary and sufficient conditions for determining the rank invariant are known. Our main result in this paper provides a necessary and sufficient condition (in terms of the parts in the partition) for determining the rank invariant when the $\mathbb{N}^d$-indexed persistence module admits a barcode. When $d = 1$, our necessary and sufficient condition exactly says that the barcode and the rank invariant determine each other.

Theorem 1.2. *Let $d \geq 1$. Let M and N be $\mathbb{N}^d$-indexed persistence modules admitting the barcodes*

$$M = \bigoplus_{i \in \Lambda_1} \mathbb{T}_{a^{(i)}} \mathbf{k}_{\lambda^{(i)}} \qquad \text{and} \qquad N = \bigoplus_{\ell \in \Lambda_2} \mathbb{T}_{b^{(\ell)}} \mathbf{k}_{\mu^{(\ell)}},$$

where $|\lambda^{(i)}| \neq 0$ and $|\mu^{(\ell)}| \neq 0$ for all $i \in \Lambda_1$ and $\ell \in \Lambda_2$. Then, $\mathrm{Rank}^M = \mathrm{Rank}^N$ if and only if for every $(i_1, \ldots, i_{d-1}) \in \mathbb{N}^{d-1}$, the two multisets

$$\left\{(a^{(i)}, (\lambda^{(i)})_{i_1,\ldots,i_{d-1}}) | i \in \Lambda_1 \text{ and } (\lambda^{(i)})_{i_1,\ldots,i_{d-1}} > 0\right\}, \tag{1.1}$$

and

$$\left\{(b^{(\ell)}, (\mu^{(\ell)})_{i_1,\ldots,i_{d-1}}) | \ell \in \Lambda_2 \text{ and } (\mu^{(\ell)})_{i_1,\ldots,i_{d-1}} > 0\right\} \tag{1.2}$$

are equal.

The main idea in the proof of Theorem 1.2 is to use induction on the sizes of M and N. We remark that when $d > 1$, under the conditions of Theorem 1.2, $\mathrm{Rank}^M = \mathrm{Rank}^N$ does not imply that M and N have the same barcode. In other words, when $d > 1$, the rank invariant is not a complete invariant for decomposable $\mathbb{N}^d$-indexed persistence modules. It would be interesting to see how to strengthen the assumption $\mathrm{Rank}^M = \mathrm{Rank}^N$ in Theorem 1.2 so that the decomposable $\mathbb{N}^d$-indexed persistence modules M and N are guaranteed to have the same barcode.

The paper is organized as follows: In Section 2, higher dimensional partitions and Young diagrams are reviewed. We define d-dimensional barcodes via the extended interiors of Young diagrams. Section 3 is devoted to $\mathbb{N}^d$-indexed persistence modules. In Section 4, we prove Theorem 1.2 (= Theorem 4.7).

2. Higher dimensional partitions and barcodes

Definition 2.1. Let $\mathbb{N}$ be the set of nonnegative integers. Let $d \geq 1$ be an integer.

(i) When $d \geq 2$, a d-dimensional partition λ is an array

$$\lambda = (\lambda_{i_1,\dots,i_{d-1}})_{i_1,\dots,i_{d-1}}$$

of $\lambda_{i_1,\dots,i_{d-1}} \in \mathbb{N} \sqcup \{+\infty\}$ indexed by $(i_1, \dots, i_{d-1}) \in \mathbb{N}^{d-1}$ such that

$$\lambda_{i_1,\dots,i_{d-1}} \geq \lambda_{j_1,\dots,j_{d-1}},$$

if $i_1 \leq j_1, \dots, i_{d-1} \leq j_{d-1}$. For $n \in \mathbb{N} \sqcup \{+\infty\}$, define the unique 1-dimensional partition of n to be $\lambda = (n)_O$ indexed by $O \in \mathbb{N}^0 = \mathbb{R}^0 = \{O\}$.

(ii) The *size* $|\lambda|$ of a partition λ is defined to be $|\lambda| = \sum_{i_1,\dots,i_{d-1}} \lambda_{i_1,\dots,i_{d-1}}$. If $|\lambda| = n \in \mathbb{N} \sqcup \{+\infty\}$, then λ is called a partition of n and denoted by $\lambda \vdash n$.

(iii) For $n \in \mathbb{N} \sqcup \{+\infty\}$, the set of d-dimensional partitions of n is denoted by $\mathcal{P}_d(n)$. Define $P_d(n)$ to be the number of d-dimensional partitions of n.

Remark 2.2. The ordinary partitions are 2-dimensional partitions (of nonnegative integers) in our sense.

One immediately sees that the generating function for $P_2(n)$ is given by

$$\sum_{n=0}^{+\infty} P_2(n)q^n = \prod_{n=1}^{+\infty} \frac{1}{1-q^n}, \tag{2.1}$$

where q is a formal variable. A well-known result of McMahon [1] states that

$$\sum_{n=0}^{+\infty} P_3(n)q^n = \prod_{n=1}^{+\infty} \frac{1}{(1-q^n)^n}. \tag{2.2}$$

There is no analogous formula for $P_d(n)$ when $d > 3$.

Fix $n \in \mathbb{N}$ and the field $k = \mathbb{C}$. The group $(k^*)^d$ acts on $k[t_1, \dots, t_d]$ via

$$(k_1, \dots, k_d)(t_1, \dots, t_d) = (k_1t_1, \dots, k_dt_d),$$

where $k^* = k - \{0\}$ and $(k_1, \dots, k_d) \in (k^*)^d$. It induces a $(k^*)^d$-action on the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^d_k)$ parametrizing length-n 0-dimensional closed subschemes of $\mathbb{A}^d_k = \operatorname{Spec} k[t_1, \dots, t_d]$ (see [12]). A d-dimensional partition $\lambda = (\lambda_{i_1,\dots,i_{d-1}})_{i_1,\dots,i_{d-1}}$ of n determines a $(k^*)^d$-invariant ideal

$$I = \langle t_1^{i_1} \cdots t_{d-1}^{i_{d-1}} t_d^{\lambda_{i_1,\dots,i_{d-1}}} | (i_1, \dots, i_{d-1}) \in \mathbb{N}^{d-1} \rangle$$

of $k[t_1, \dots, t_d]$ with co-length n (i.e, $\dim_k k[t_1, \dots, t_d]/I = n$). In this way, the set of d-dimensional partitions of n is in one-to-one correspondence with the set of $(k^*)^d$-invariant ideals of $k[t_1, \dots, t_d]$ with co-length n, which in turn is in one-to-one correspondence with the set of $(k^*)^d$-invariant points in $\mathrm{Hilb}^n(\mathbb{A}^d_k)$.

Definition 2.3. Let λ be a d-dimensional partition, and $a = (a_1, \dots, a_d) \in \mathbb{N}^d$.

(i) The Young diagram D_λ of λ is the subset of $\mathbb{R}^d$ obtained by stacking $\lambda_{i_1,\dots,i_{d-1}}$ d-dimensional unit boxes over the $(d-1)$-dimensional rectangular region in $\mathbb{R}^{d-1} \subset \mathbb{R}^d$ spanned by the vertices

$$(i_1, \dots, i_{d-1}), (i_1, \dots, i_{d-1}) + \mathbf{e}_1, \dots, (i_1, \dots, i_{d-1}) + \mathbf{e}_{d-1},$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_{d-1}\}$ denotes the standard basis of $\mathbb{R}^{d-1}$ and $\mathbb{R}^{d-1}$ is embedded in $\mathbb{R}^d$ by taking the last coordinate to be 0.

(ii) The set of integral points inside D_λ is defined by

$$D_\lambda^{\text{int}} = \{(i_1, \ldots, i_{d-1}, h) \in \mathbb{N}^d | h < \lambda_{i_1,\ldots,i_{d-1}}\} \subset D_\lambda. \tag{2.3}$$

The extended interior of D_λ is defined to be

$$D_\lambda^{\text{ei}} = D_\lambda - \overline{(\partial D_\lambda) \cap (\mathbb{R}^+)^d}. \tag{2.4}$$

(iii) Define $\mathbb{T}_a : \mathbb{R}^d \to \mathbb{R}^d$ to be the translation of $\mathbb{R}^d$ by a.

Remark 2.4. For a d-dimensional partition λ, we have $D_\lambda^{\text{int}} = D_\lambda^{\text{ei}} \cap \mathbb{N}^d$.

When $d = 2$ and $\lambda \vdash n \in \mathbb{N}$, D_λ is the usual Young diagram (up to some rotation) of the 2-dimensional (i.e, usual) partition λ. When $\lambda = (+\infty)_{i_1,\ldots,i_{d-1}}$ (i.e., $\lambda_{i_1,\ldots,i_{d-1}} = +\infty$ for every $(i_1, \ldots, i_{d-1}) \in \mathbb{N}^{d-1}$), we have $D_\lambda = \mathbb{R}^d$.

Example 2.5. Recall from Definition 2.1 (i) the notation $(n)_O$ for $n \in \mathbb{N} \sqcup \{+\infty\}$. In Figures 2 and 3 below, we present two sets of examples of $\mathbb{T}_a(D_\lambda)$ for $d = 1$ and $d = 2$, respectively.

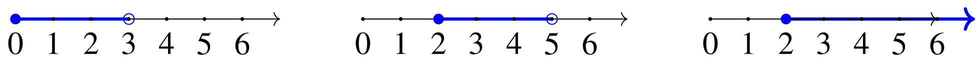

Figure 2. For the 1-dimensional partitions $\lambda = (3)_O$ and $\mu = (+\infty)_O$, the left is D_λ, the middle is $\mathbb{T}_2(D_\lambda)$, and the right is $\mathbb{T}_2(D_\mu)$.

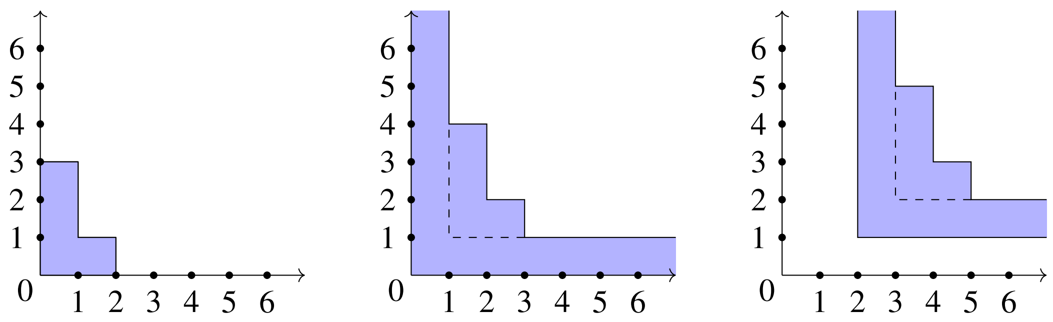

Figure 3. For the 2-dimensional partitions $\lambda = (3, 1)$ and $\mu = (+\infty, 4, 2, 1, 1, \ldots)$, the left is D_λ, the middle is D_μ, and the right is $\mathbb{T}_{(2,1)}(D_\mu)$.

Example 2.6. Figure 4 illustrates the Young diagram of a 3-dimensional partition of $n = 29$ as $\lambda = (\lambda_{(a,b)})$, $(a, b) \in \mathbb{N}^3$, where $\lambda_{(a,b)}$ is defined by

$$\lambda_{(a,b)} = \begin{cases} 4, & (a, b) = (0, 0); \\ 3, & (a, b) \in \{(0, 1), (1, 0), (1, 1), (2, 0)\}; \\ 2, & (a, b) \in \{(0, 2), (1, 2), (2, 1), (3, 0), (3, 1)\}; \\ 1, & (a, b) \in \{(4, 0), (1, 3), (0, 3)\}. \end{cases} \tag{2.5}$$

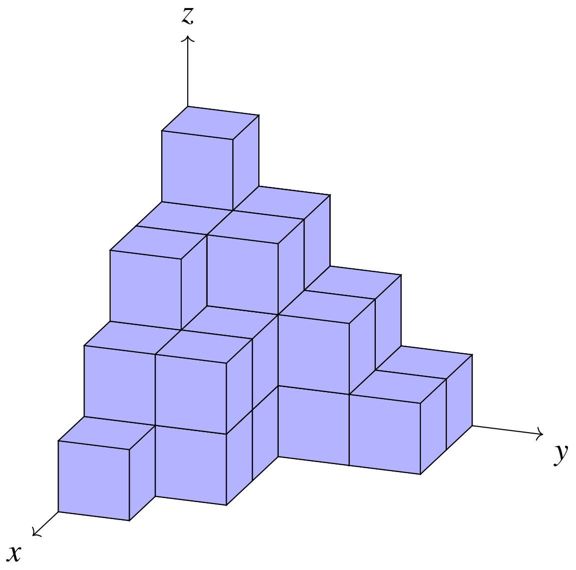

Figure 4. D_λ for a 3-dimensional partition of $n = 29$.

Definition 2.7. (i) A multiset is defined to be an ordered pair (A, m), where A is a set and $m : A \to \mathbb{Z}^+$ is a function from A to the set of positive integers. Informally, a multiset is a set where elements are allowed to have multiple copies.

(ii) For $d \geq 1$, a d-dimensional barcode is a multiset consisting of

$$\mathbb{T}_{a^{(i)}}(D^{\mathrm{ei}}_{\lambda^{(i)}}), \quad i \in \Lambda,$$

where Λ is an index set, and for each $i \in \Lambda$, $a^{(i)} \in \mathbb{N}^d$ and $\lambda^{(i)}$ is a d-dimensional partition.

3. $\mathbb{N}^d$-indexed persistence modules

Definition 3.1. (i) A partially ordered set (or poset for short) is a set P together with a partial ordering $\leq$ satisfying

- $a \leq a$ for all $a \in P$ (reflexivity);
- $a \leq b$ and $b \leq a$ imply $a = b$ (anti-symmetry);
- $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).

(ii) The poset category associated to a poset P is the category whose objects are the elements of P, and for $a, b \in P$, $\mathrm{Hom}(a, b)$ consists of one element if $a \leq b$ and is the empty set if $a \nleq b$. By abusing notations, we also use P to denote the poset category associated to a poset P.

Definition 3.2. (i) Fix a field k and a poset $(P, \leq)$. A P-indexed persistence module over k is a functor

$$M : P \to \mathbf{Vec}_k,$$

where $\mathbf{Vec}_k$ is the category of finite dimensional vector spaces over k (and P denotes the poset category associated to P).

(ii) The image of $p \in P$ is denoted by M_p. For $p, q \in P$ with $p \leq q$, the unique morphism from M_p to M_q corresponding to $p \leq q$ is denoted by $M_{p,q}$.

(iii) Define $\mathrm{gr}(m) = p$ if $m \in M_p$.

In the rest of the paper, we will assume $P = \mathbb{N}^d$ and the field k will be implicit. A persistence module is meant to be a $\mathbb{N}^d$-indexed persistence module over k.

Theorem 3.3. *([6, Theorem 1]) (Correspondence) The category of $\mathbb{N}^d$-indexed persistence modules over k is equivalent to the category of $\mathbb{N}^d$-graded $k[t_1, \ldots, t_d]$-modules.*

Theorem 3.4. *([6, Theorem 2]) (Realization) Let $k = \mathbb{F}_p$ for some prime p, let q be a positive integer, and let M be an $\mathbb{N}^d$-graded $k[x_1, \ldots, x_d]$-module. Then there is an $\mathbb{N}^d$-filtered finite simplicial complex X so that $H_q(X, k) \cong M$ as $\mathbb{N}^d$-persistence modules.*

Denote the standard basis of $\mathbb{R}^d$ by

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}. \tag{3.1}$$

Given an $\mathbb{N}^d$-indexed persistence module $M : \mathbb{N}^d \to \mathbf{Vec}_k$, the associated $\mathbb{N}^d$-graded $k[t_1, \ldots, t_d]$-module is

$$\bigoplus_{z \in \mathbb{N}^d} M_z, \tag{3.2}$$

on which $k[t_1, \ldots, t_d]$ acts via $t_i \cdot m = M_{z, z+\mathbf{e}_i}(m)$ for every $m \in M_z$ and $1 \leq i \leq d$.

Morphisms between persistence modules, their kernels and images, submodules, and quotient modules are defined component-wise in the usual way.

Definition 3.5. Let M be an $\mathbb{N}^d$-indexed persistence module.

(i) Let $S \subset \cup_{z \in \mathbb{N}^d} M_z$ be a subset. The submodule $\langle S \rangle$ of M generated by S is the submodule such that for each $z \in \mathbb{N}^d$, $\langle S \rangle_z$ consists of all the elements

$$\sum_{i=1}^{n} c_i \cdot M_{\mathrm{gr}(s_i), z}(s_i),$$

where $c_1, \ldots, c_n \in k$ and $s_1, \ldots, s_n \in S$ with $\mathrm{gr}(s_i) \leq z$ for each i. If $\mathrm{gr}(s) \nleq z$ for every $s \in S$, then we put $\langle S \rangle_z = 0$.

(ii) A subset $S \subset \cup_{z \in \mathbb{N}^d} M_z$ is a set of generators for M if $M = \langle S \rangle$.

(iii) The persistence module M is finitely generated if there exists a finite set of generators for M.

(iv) Fix $a \in \mathbb{N}^d$. The translation $\mathbb{T}_a M$ of M by a is the persistence module given by

$$(\mathbb{T}_a M)_x = \begin{cases} M_{x-a}, & \text{if } x \geq a, \\ 0, & \text{otherwise,} \end{cases} \qquad (\mathbb{T}_a M)_{x,y} = \begin{cases} M_{x-a, y-a}, & \text{if } a \leq x \leq y, \\ 0, & \text{otherwise,} \end{cases}$$

for $x, y \in \mathbb{N}^d$ with $x \leq y$.

For a d-dimensional partition λ, recall from Definition 2.3 (ii) the set D_λ^{int} of integral points inside the Young diagram D_λ.

Definition 3.6. (i) For a d-dimensional partition λ, define the $\mathbb{N}^d$-indexed persistence module $\mathbf{k}_\lambda$ by

$$(\mathbf{k}_\lambda)_x = \begin{cases} k, & \text{if } x \in D_\lambda^{\text{int}}, \\ 0, & \text{otherwise}, \end{cases} \qquad (\mathbf{k}_\lambda)_{x,y} = \begin{cases} \mathrm{Id}_k, & \text{if } x, y \in D_\lambda^{\text{int}} \text{ and } x \le y, \\ 0, & \text{otherwise}, \end{cases}$$

where $x, y \in \mathbb{N}^d$ with $x \le y$. When $\lambda = (+\infty)_{i_1,\ldots,i_{d-1}}$, we set $\mathbf{k}_\lambda = \mathbf{k}$. When $|\lambda| \neq 0$, define $\mathbf{1}_\lambda \in (\mathbf{k}_\lambda)_O = k$ to be the multiplicative identity.

(ii) An $\mathbb{N}^d$-indexed persistence module F is free if there exists a multiset A of elements in $\mathbb{N}^d$ such that

$$F \cong \bigoplus_{a \in A} \mathbb{T}_a \mathbf{k}.$$

(iii) A *presentation* of a persistence module M is a morphism $f : F \to F'$ of free modules such that $M \cong F'/\mathrm{im}(f)$.

The following combines [9, Proposition 6.40] and [9, Proposition 6.43].

Proposition 3.7. *Every persistence module M has a presentation. Moreover, if M is finitely generated, then there exists a presentation $f : F \to F'$ of M such that both the free modules F and F' are finitely generated.*

Next, we study the case $F' = \mathbb{T}_a \mathbf{k}$ where $a \in \mathbb{N}^d$.

Lemma 3.8. *Let $a \in \mathbb{N}^d$. Then, every quotient of the persistence module $\mathbb{T}_a \mathbf{k}$ is equal to $\mathbb{T}_a \mathbf{k}_\lambda$ for some d-dimensional partition λ.*

Proof. It suffices to prove that every quotient of the persistence module $\mathbf{k}$ is equal to $\mathbf{k}_\lambda$ for some d-dimensional partition λ. Let Q be a quotient of $\mathbf{k}$. Let $\mathcal{I}$ be the submodule of $\mathbf{k}$ such that $Q = \mathbf{k}/\mathcal{I}$. By (3.2), the persistence module $\mathbf{k}$ corresponds to the $\mathbb{N}^d$-graded $k[t_1, \ldots, t_d]$-module $k[t_1, \ldots, t_d]$ itself. Thus, the submodule $\mathcal{I}$ of $\mathbf{k}$ corresponds to an $\mathbb{N}^d$-graded ideal I of $k[t_1, \ldots, t_d]$. Being $\mathbb{N}^d$-graded, the ideal I of $k[t_1, \ldots, t_d]$ is generated by monomials. Define a d-dimensional partition $\lambda = (\lambda_{i_1,\ldots,i_{d-1}})_{i_1,\ldots,i_{d-1}}$ as follows:

$$\lambda_{i_1,\ldots,i_{d-1}} = \begin{cases} \min\{b | t_1^{i_1} \cdots t_{d-1}^{i_{d-1}} t_d^b \in I\}, & \text{if } t_1^{i_1} \cdots t_{d-1}^{i_{d-1}} t_d^b \in I \text{ for some } b \in \mathbb{N}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Indeed, note that if $t_1^{i_1} \cdots t_d^{i_d} \in I$, then $t_1^{j_1} \cdots t_d^{j_d} \in I$ whenever $i_1 \le j_1, \ldots, i_d \le j_d$. It follows that $\lambda_{i_1,\ldots,i_{d-1}} \ge \lambda_{j_1,\ldots,j_{d-1}}$ whenever $i_1 \le j_1, \ldots, i_{d-1} \le j_{d-1}$.

We have

$$I = \bigoplus_{\substack{(i_1,\ldots,i_{d-1}) \in \mathbb{N}^{d-1} \\ \lambda_{i_1,\ldots,i_{d-1}} \neq +\infty}} \bigoplus_{b \ge \lambda_{i_1,\ldots,i_{d-1}}} k \cdot t_1^{i_1} \cdots t_{d-1}^{i_{d-1}} t_d^b. \tag{3.3}$$

As a vector space, the quotient $k[t_1, \ldots, t_d]/I$ is equal to

$$\bigoplus_{(i_1,\ldots,i_{d-1}) \in \mathbb{N}^{d-1}} \bigoplus_{b < \lambda_{i_1,\ldots,i_{d-1}}} k \cdot t_1^{i_1} \cdots t_{d-1}^{i_{d-1}} t_d^b. \tag{3.4}$$

Via the correspondence (3.2), the quotient Q of $\mathbf{k}$, which corresponds to the quotient $k[t_1, \ldots, t_d]/I$ of $k[t_1, \ldots, t_d]$, is equal to $\mathbf{k}_\lambda$. □

Definition 3.9. An $\mathbb{N}^d$-indexed persistence module M admits a *barcode* if

$$M \cong \bigoplus_{i\in\Lambda} \mathbb{T}_{a^{(i)}}\mathbf{k}_{\lambda^{(i)}}, \tag{3.5}$$

where for each $i \in \Lambda$, $a^{(i)} \in \mathbb{N}^d$ and $\lambda^{(i)}$ is a d-dimensional partition with $|\lambda^{(i)}| \neq 0$. In this case, the barcode $\mathfrak{B}_M$ of M is defined to be the multiset whose elements are

$$\mathbb{T}_{a^{(i)}}(D^{\mathrm{ei}}_{\lambda^{(i)}}), \quad i \in \Lambda.$$

Lemma 3.10. *Assume that an $\mathbb{N}^d$-indexed persistence module M admits a barcode $\mathfrak{B}_M$. Then,*

$$\mathrm{Rank}(M_{x,y}) = |\{S \in \mathfrak{B}_M | x, y \in S\}| \tag{3.6}$$

for all $x, y \in \mathbb{N}^d$ satisfying $x \leq y$.
Proof. We may assume $M = \bigoplus_{i\in\Lambda} \mathbb{T}_{a^{(i)}}\mathbf{k}_{\lambda^{(i)}}$ so that

$$\mathfrak{B}_M = \{\mathbb{T}_{a^{(i)}}(D^{\mathrm{ei}}_{\lambda^{(i)}})\}_{i\in\Lambda}$$

as multisets. Let $x, y \in \mathbb{N}^d$ satisfying $x \leq y$. Then,

$$\mathrm{Rank}(M_{x,y}) = \sum_{i\in\Lambda} \mathrm{Rank}((\mathbb{T}_{a^{(i)}}\mathbf{k}_{\lambda^{(i)}})_{x,y}). \tag{3.7}$$

Set

$$\Lambda_1 = \{i \in \Lambda | x, y \in \mathbb{T}_{a^{(i)}}(D^{\mathrm{int}}_{\lambda^{(i)}})\}.$$

By Definition 3.6 (i) and Definition 3.5 (iv),

$$\mathrm{Rank}((\mathbb{T}_{a^{(i)}}\mathbf{k}_{\lambda^{(i)}})_{x,y}) = \begin{cases} 1, & \text{if } x, y \in \mathbb{T}_{a^{(i)}}(D^{\mathrm{int}}_{\lambda^{(i)}}); \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\mathrm{Rank}(M_{x,y}) = |\Lambda_1| = |\{\mathbb{T}_{a^{(i)}}(D^{\mathrm{int}}_{\lambda^{(i)}}) | x, y \in \mathbb{T}_{a^{(i)}}(D^{\mathrm{int}}_{\lambda^{(i)}})\}|$. By Remark 2.4,

$$D^{\mathrm{int}}_{\lambda} = D^{\mathrm{ei}}_{\lambda} \cap \mathbb{N}^d$$

for every d-dimensional partition λ. Hence,

$$\mathrm{Rank}(M_{x,y}) = |\{\mathbb{T}_{a^{(i)}}(D^{\mathrm{ei}}_{\lambda^{(i)}}) \in \mathfrak{B}_M | x, y \in \mathbb{T}_{a^{(i)}}(D^{\mathrm{ei}}_{\lambda^{(i)}})\}|. \qquad \square$$

The following theorem has been proved in [11].

Theorem 3.11. *([11]) Let M be an $\mathbb{N}^d$-indexed persistence module. Assume that there are two isomorphisms*

$$M \cong \bigoplus_{i\in\Lambda_1} \mathbb{T}_{a^{(i)}}\mathbf{k}_{\lambda^{(i)}} \qquad \text{and} \qquad M \cong \bigoplus_{\ell\in\Lambda_2} \mathbb{T}_{b^{(\ell)}}\mathbf{k}_{\mu^{(\ell)}}$$

with $|\lambda^{(i)}| \neq 0$ and $|\mu^{(\ell)}| \neq 0$ for all $i \in \Lambda_1$ and $\ell \in \Lambda_2$. Then, as multisets,

$$\{(\lambda^{(i)}, a^{(i)})\}_{i\in\Lambda_1} = \{(\mu^{(\ell)}, b^{(\ell)})\}_{\ell\in\Lambda_2}. \tag{3.8}$$

In particular, the barcode $\mathfrak{B}_M$ of M in Definition 3.9 is well-defined.

Lemma 3.10 indicates that if an $\mathbb{N}^d$-indexed persistence module M admits a barcode $\mathfrak{B}_M$, then $\mathfrak{B}_M$ is a *good* barcode in the sense of [9, Definition 10.11]; moreover, by Theorem 3.11, the barcode $\mathfrak{B}_M$ of M is well-defined.

Example 3.12. Consider the $\mathbb{N}^2$-indexed persistence module M defined by

$$M = \mathbf{k}_{\lambda^{(1)}} \oplus \mathbf{k}_{\lambda^{(2)}} \oplus \mathbf{k}_{\lambda^{(3)}},$$

where $\lambda^{(1)} = (3, 2, 2)$, $\lambda^{(2)} = (2, 1)$, and $\lambda^{(3)} = (3, 3)$ are 2-dimensional partitions. By Definition 3.9, the barcode $\mathfrak{B}_M$ of M is the multiset consisting of $D^{\mathrm{ei}}_{\lambda^{(1)}}$, $D^{\mathrm{ei}}_{\lambda^{(2)}}$, and $D^{\mathrm{ei}}_{\lambda^{(3)}}$, which are illustrated in Figure 5.

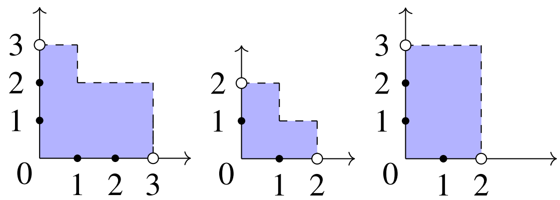

Figure 5. From left to right, $D^{\mathrm{ei}}_{\lambda}$ for $\lambda^{(1)} = (3, 2, 2)$, $\lambda^{(2)} = (2, 1)$, and $\lambda^{(3)} = (3, 3)$.

Note that the three Young diagrams $D_{\lambda^{(1)}}, D_{\lambda^{(2)}}$, and $D_{\lambda^{(3)}}$ are the three shaded regions in Figure 5 together with their respective dashed boundaries.

4. The rank invariant

In this section, we study the relation between the rank invariant which is defined in [6] and the $\mathbb{N}^d$-indexed persistence modules which admit barcodes. We will prove a necessary and sufficient condition for two $\mathbb{N}^d$-indexed persistence modules admitting barcodes to have the same rank invariant.

The following definitions are adopted from [6] (see also [9]).

Definition 4.1. (i) Given a poset P, define

$$P^{\leq} = \{(x, y) \in P \times P | x \leq y\}.$$

(ii) The rank invariant of an $\mathbb{N}^d$-indexed persistence module M is the function $\mathrm{Rank}^M : (\mathbb{N}^d)^{\leq} \to \mathbb{N}$ given by

$$\mathrm{Rank}^M(x, y) = \mathrm{Rank}(M_{x,y}).$$

Lemma 4.2. *Assume that an $\mathbb{N}^d$-indexed persistence module M admits a barcode $\mathfrak{B}_M$. Then,*

$$\mathrm{Rank}^M(x, y) = |\{S \in \mathfrak{B}_M | x, y \in S\}| \tag{4.1}$$

for every $(x, y) \in (\mathbb{N}^d)^{\leq}$.

Proof. The proof follows directly from Definition 4.1 (ii) and Lemma 3.10. □

Lemma 4.3. *Let $d \geq 1$. Let M and N be $\mathbb{N}^d$-indexed persistence modules admitting the barcodes*

$$M = \bigoplus_{i\in\Lambda_1} \mathbf{k}_{\lambda^{(i)}} \qquad \text{and} \qquad N = \bigoplus_{\ell\in\Lambda_2} \mathbf{k}_{\mu^{(\ell)}}$$

with $|\lambda^{(i)}|, |\mu^{(\ell)}| \neq 0$. If $\mathrm{Rank}^M = \mathrm{Rank}^N$, then for every $(i_1, \ldots, i_{d-1}) \in \mathbb{N}^{d-1}$, the two multisets

$$\{\lambda^{(i)}_{i_1,\ldots,i_{d-1}} | i \in \Lambda_1 \text{ and } \lambda^{(i)}_{i_1,\ldots,i_{d-1}} > 0\}, \tag{4.2}$$

and

$$\{\mu^{(\ell)}_{i_1,\ldots,i_{d-1}} | \ell \in \Lambda_2 \text{ and } \mu^{(\ell)}_{i_1,\ldots,i_{d-1}} > 0\} \tag{4.3}$$

are equal.

Proof. Fix $(i_1, \ldots, i_{d-1}) \in \mathbb{N}^{d-1}$. Let $a_1, \ldots, a_s$ be the distinct values in the multiset (4.2) with $a_1 > \ldots > a_s$, and let $m_1, \ldots, m_s$ be the multiplicities of $a_1, \ldots, a_s$, respectively, in the multiset (4.2). Similarly, let $b_1, \ldots, b_t$ be the distinct values in the multiset (4.3) with $b_1 > \ldots > b_t$, and let $n_1, \ldots, n_t$ be the multiplicities of $b_1, \ldots, b_t$, respectively, in the multiset (4.3).

Without loss of generality, assume $a_1 \geq b_1$. By (4.1),

$$\mathrm{Rank}^M((i_1, \ldots, i_{d-1}, 0), (i_1, \ldots, i_{d-1}, h)) = \begin{cases} m_1, & \text{if } a_2 \leq h < a_1; \\ 0, & \text{if } h \geq a_1. \end{cases}$$

Since $\mathrm{Rank}^M = \mathrm{Rank}^N$, we conclude that

$$\mathrm{Rank}^N((i_1, \ldots, i_{d-1}, 0), (i_1, \ldots, i_{d-1}, h)) = \begin{cases} m_1, & \text{if } a_2 \leq h < a_1; \\ 0, & \text{if } h \geq a_1. \end{cases}$$

It follows that $b_1 = a_1$ and $n_1 = m_1$.

Next, without loss of generality, assume $a_2 \geq b_2$. By (4.1) again,

$$\mathrm{Rank}^M((i_1, \ldots, i_{d-1}, 0), (i_1, \ldots, i_{d-1}, h)) = \begin{cases} m_1 + m_2, & \text{if } a_3 \leq h < a_2; \\ m_1, & \text{if } a_2 \leq h < a_1; \\ 0, & \text{if } h \geq a_1. \end{cases}$$

Since $\mathrm{Rank}^M = \mathrm{Rank}^N$, we conclude that

$$\mathrm{Rank}^N((i_1, \ldots, i_{d-1}, 0), (i_1, \ldots, i_{d-1}, h)) = \begin{cases} m_1 + m_2, & \text{if } a_3 \leq h < a_2; \\ m_1, & \text{if } a_2 \leq h < a_1; \\ 0, & \text{if } h \geq a_1. \end{cases}$$

It follows that $b_2 = a_2$ and $n_2 = m_2$.

Repeating the above arguments, we see that $s = t$, $a_i = b_i$, and $m_i = n_i$ for all $1 \leq i \leq s = t$. Therefore, the two multisets (4.2) and (4.3) are equal. □

Unfortunately, under the conditions in Lemma 4.3, the two multisets $\{\lambda^{(i)} | i \in \Lambda_1\}$ and $\{\mu^{(\ell)} | \ell \in \Lambda_2\}$ may not be the same. An example is given below.

Example 4.4. Let $M, N : \mathbb{N}^2 \to \mathbf{Vec}_k$ be persistence modules given by

$$M = \mathbf{k}_{\lambda^{(1)}} \oplus \mathbf{k}_{\lambda^{(2)}} \qquad \text{and} \qquad N = \mathbf{k}_{\mu^{(1)}} \oplus \mathbf{k}_{\mu^{(2)}}$$

where $\lambda^{(1)} = (2)$, $\lambda^{(2)} = (1^2)$, $\mu^{(1)} = (1)$, and $\mu^{(2)} = (2, 1)$ are 2-dimensional partitions. Intuitively, M and N can be illustrated by

$$M = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ k & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ k & \longrightarrow & 0 \end{array} \bigoplus \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

and

$$N = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ k & \longrightarrow & 0 \end{array} \bigoplus \begin{array}{ccccc} 0 & \longrightarrow & 0 & & \\ \uparrow & & \uparrow & & \\ k & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

where all the nontrivial maps are the identity maps. Then,

$$\mathrm{Rank}^M(x, y) = \mathrm{Rank}^N(x, y) = \begin{cases} 1, & \text{if } x = (0, 0) \text{ and } y = (1, 0); \\ 1, & \text{if } x = (0, 0) \text{ and } y = (0, 1); \\ 0, & \text{otherwise .} \end{cases}$$

So, $\mathrm{Rank}^M = \mathrm{Rank}^N$ as asserted by Lemma 4.3. However, the two multisets $\{\lambda^{(1)}, \lambda^{(2)}\}$ and $\{\mu^{(1)}, \mu^{(2)}\}$ are not equal. By Theorem 3.11, the two $\mathbb{N}^2$-indexed persistence modules M and N are not isomorphic.

Our next goal is to present a necessary and sufficient condition for two $\mathbb{N}^d$-indexed persistence modules admitting barcodes to have the same rank invariants. We start with the projections $\mathfrak{p}$ and $\mathfrak{q}$.

Definition 4.5. For $d \geq 2$, we define $\mathfrak{p} : \mathbb{N}^d \to \mathbb{N}^{d-1}$ by

$$\mathfrak{p}(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1}),$$

and define $\mathfrak{q} : \mathbb{N}^d \to \mathbb{N}$ by

$$\mathfrak{q}(x_1, \ldots, x_d) = x_d.$$

For $d = 1$, we define $\mathfrak{p} : \mathbb{N} \to \mathbb{N}^0 = \{O\}$ by $\mathfrak{p}(x) = O$.

Lemma 4.6. *Let $a \in \mathbb{N}^d$ and λ be a d-dimensional partition with $|\lambda| \neq 0$. Let $(x, y) \in (\mathbb{N}^d)^{\leq}$. Then, $x, y \in \mathbb{T}_a(D^{\mathrm{ei}}_\lambda)$ if and only if $a \leq x$ and $\mathfrak{q}(y - a) < \lambda_{\mathfrak{p}(y-a)}$.*

Proof. Recall the maps $\mathfrak{p}$ and $\mathfrak{q}$ from Definition 4.5. By (2.4), $z \in D_\lambda^{\text{ei}}$ if and only if $O \leq z$ and $\mathfrak{q}(z) < \lambda_{\mathfrak{p}(z)}$. Since $(x, y) \in (\mathbb{N}^d)^{\leq}$, we have $x \leq y$. Thus,

$$\begin{aligned} x, y \in \mathbb{T}_a(D_\lambda^{\text{ei}}) \quad &\text{if and only if} \quad x - a, y - a \in D_\lambda^{\text{ei}}, \\ &\text{if and only if} \quad a \leq x \text{ and } y - a \in D_\lambda^{\text{ei}}, \\ &\text{if and only if} \quad a \leq x \text{ and } \mathfrak{q}(y - a) < \lambda_{\mathfrak{p}(y-a)}. \end{aligned}$$

This completes the proof of the lemma. □

The following is the main result of the paper.

Theorem 4.7. *Let $d \geq 1$. Let M and N be $\mathbb{N}^d$-indexed persistence modules admitting the barcodes*

$$M = \bigoplus_{i \in \Lambda_1} \mathbb{T}_{a^{(i)}} \mathbf{k}_{\lambda^{(i)}} \qquad \text{and} \qquad N = \bigoplus_{\ell \in \Lambda_2} \mathbb{T}_{b^{(\ell)}} \mathbf{k}_{\mu^{(\ell)}}$$

where $|\lambda^{(i)}| \neq 0$ and $|\mu^{(\ell)}| \neq 0$ for all $i \in \Lambda_1$ and $\ell \in \Lambda_2$. Then, $\text{Rank}^M = \text{Rank}^N$ if and only if for every $(i_1, \ldots, i_{d-1}) \in \mathbb{N}^{d-1}$, the two multisets

$$\left\{ (a^{(i)}, (\lambda^{(i)})_{i_1,\ldots,i_{d-1}}) | i \in \Lambda_1 \text{ and } (\lambda^{(i)})_{i_1,\ldots,i_{d-1}} > 0 \right\}, \tag{4.4}$$

and

$$\left\{ (b^{(\ell)}, (\mu^{(\ell)})_{i_1,\ldots,i_{d-1}}) | \ell \in \Lambda_2 \text{ and } (\mu^{(\ell)})_{i_1,\ldots,i_{d-1}} > 0 \right\} \tag{4.5}$$

are equal.

Proof. First of all, assume that the two multisets (4.4) and (4.5) are equal. Fix $(x, y) \in (\mathbb{N}^d)^{\leq}$. By Lemma 4.2,

$$\text{Rank}^M(x, y) = |\{S \in \mathfrak{B}_M | x, y \in S\}|.$$

Since $\mathfrak{B}_M = \{\mathbb{T}_{a^{(i)}}(D_{\lambda^{(i)}}^{\text{ei}})\}_{i \in \Lambda_1}$, we see from Lemma 4.6 that $\text{Rank}^M(x, y)$ is equal to the cardinality of the multiset

$$\left\{ (a^{(i)}, (\lambda^{(i)})_{\mathfrak{p}(y - a^{(i)})}) | i \in \Lambda_1, a^{(i)} \leq x \text{ and } \mathfrak{q}(y - a^{(i)}) < (\lambda^{(i)})_{\mathfrak{p}(y - a^{(i)})} \right\}. \tag{4.6}$$

Similarly, $\text{Rank}^N(x, y)$ is equal to the cardinality of the multiset

$$\left\{ (b^{(\ell)}, (\mu^{(\ell)})_{\mathfrak{p}(y - b^{(\ell)})}) | \ell \in \Lambda_2, b^{(\ell)} \leq x \text{ and } \mathfrak{q}(y - b^{(\ell)}) < (\mu^{(\ell)})_{\mathfrak{p}(y - b^{(\ell)})} \right\}. \tag{4.7}$$

Since the two multisets (4.4) and (4.5) are equal, so are the two multisets (4.6) and (4.7). Hence, $\text{Rank}^M(x, y) = \text{Rank}^N(x, y)$ for every $(x, y) \in (\mathbb{N}^d)^{\leq}$. It follows that $\text{Rank}^M = \text{Rank}^N$.

Conversely, assume that $\text{Rank}^M = \text{Rank}^N$. Without loss of generality, assume that $a^{(1)}$ is a minimal element in the multiset

$$\{a^{(i)} | i \in \Lambda_1\} \cup \{b^{(\ell)} | \ell \in \Lambda_2\}. \tag{4.8}$$

Then, $\text{Rank}^N(a^{(1)}, a^{(1)}) = \text{Rank}^M(a^{(1)}, a^{(1)}) > 0$. By Lemma 4.2, $a^{(1)} \in \mathfrak{B}_N$. Since $\mathfrak{B}_N$ is the multiset consisting of all $\mathbb{T}_{b^{(\ell)}}(D_{\mu^{(\ell)}}^{\text{ei}})$ with $\ell \in \Lambda_2$, we have $a^{(1)} \in \mathbb{T}_{b^{(\ell)}}(D_{\mu^{(\ell)}}^{\text{ei}})$ for some $\ell \in \Lambda_2$. By Lemma 4.6,

$b^{(\ell)} \leq a^{(1)}$ for some $\ell \in \Lambda_2$. Since $a^{(1)}$ is a minimal element in the multiset (4.8), we must have $b^{(\ell)} = a^{(1)}$. Without loss of generality, we may let $\ell = 1$ so that $b^{(1)} = a^{(1)}$. Set

$$\Lambda_1' = \{i \in \Lambda_1 | a^{(i)} = a^{(1)}\}, \quad \widetilde{M} = \bigoplus_{i \in \Lambda_1'} \mathbf{k}_{\lambda^{(i)}},$$

$$M' = \bigoplus_{i \in \Lambda_1'} \mathbb{T}_{a^{(i)}} \mathbf{k}_{\lambda^{(i)}} = \mathbb{T}_{a^{(1)}} \widetilde{M}, \quad M'' = \bigoplus_{i \in \Lambda_1 - \Lambda_1'} \mathbb{T}_{a^{(i)}} \mathbf{k}_{\lambda^{(i)}},$$

and

$$\Lambda_2' = \{\ell \in \Lambda_2 | b^{(\ell)} = b^{(1)} = a^{(1)}\}, \quad \widetilde{N} = \bigoplus_{\ell \in \Lambda_2'} \mathbf{k}_{\mu^{(\ell)}},$$

$$N' = \bigoplus_{\ell \in \Lambda_2'} \mathbb{T}_{b^{(\ell)}} \mathbf{k}_{\mu^{(\ell)}} = \mathbb{T}_{a^{(1)}} \widetilde{N}, \quad N'' = \bigoplus_{\ell \in \Lambda_2 - \Lambda_2'} \mathbb{T}_{b^{(\ell)}} \mathbf{k}_{\mu^{(\ell)}}.$$

We have $M = M' \oplus M''$ and $N = N' \oplus N''$.

Claim. $\mathrm{Rank}^{\widetilde{M}} = \mathrm{Rank}^{\widetilde{N}}$ and $\mathrm{Rank}^{M''} = \mathrm{Rank}^{N''}$.

Proof. Let $(x, y) \in (\mathbb{N}^d)^{\leq}$. Since $\widetilde{M}$ is generated at the origin O,

$$\mathrm{Rank}^{\widetilde{M}}(x, y) = \mathrm{Rank}^{\widetilde{M}}(O, y) = \mathrm{Rank}^{\mathbb{T}_{a^{(1)}}\widetilde{M}}(a^{(1)}, y + a^{(1)}) = \mathrm{Rank}^{M'}(a^{(1)}, y + a^{(1)}).$$

Since $a^{(1)}$ is a minimal element in the multiset $\{a^{(i)} | i \in \Lambda_1\}$, we have $M''_{a^{(1)}, y+a^{(1)}} = 0$ and $\mathrm{Rank}^{M'}(a^{(1)}, y + a^{(1)}) = \mathrm{Rank}^{M}(a^{(1)}, y + a^{(1)})$. Thus,

$$\mathrm{Rank}^{\widetilde{M}}(x, y) = \mathrm{Rank}^{M}(a^{(1)}, y + a^{(1)}). \tag{4.9}$$

Similarly, $\mathrm{Rank}^{\widetilde{N}}(x, y) = \mathrm{Rank}^{N}(a^{(1)}, y + a^{(1)})$. Combining with $\mathrm{Rank}^{M} = \mathrm{Rank}^{N}$ and (4.9), we conclude that $\mathrm{Rank}^{\widetilde{M}}(x, y) = \mathrm{Rank}^{\widetilde{N}}(x, y)$ for every $(x, y) \in (\mathbb{N}^d)^{\leq}$. Therefore, we obtain

$$\mathrm{Rank}^{\widetilde{M}} = \mathrm{Rank}^{\widetilde{N}}. \tag{4.10}$$

Next, we prove that $\mathrm{Rank}^{M''} = \mathrm{Rank}^{N''}$. We have

$$\mathrm{Rank}^{M'}(x, y) = \mathrm{Rank}^{\mathbb{T}_{a^{(1)}}\widetilde{M}}(x, y) = \begin{cases} \mathrm{Rank}^{\widetilde{M}}(O, y - a^{(1)}) & \text{, if } a^{(1)} \leq x; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$\mathrm{Rank}^{N'}(x, y) = \begin{cases} \mathrm{Rank}^{\widetilde{N}}(O, y - a^{(1)},) & \text{if } a^{(1)} \leq x; \\ 0, & \text{otherwise.} \end{cases}$$

By (4.10), $\mathrm{Rank}^{M'}(x, y) = \mathrm{Rank}^{N'}(x, y)$. Since

$$\mathrm{Rank}^{M}(x, y) = \mathrm{Rank}^{M'}(x, y) + \mathrm{Rank}^{M''}(x, y),$$

$$\mathrm{Rank}^{N}(x, y) = \mathrm{Rank}^{N'}(x, y) + \mathrm{Rank}^{N''}(x, y),$$

and $\mathrm{Rank}^{M}(x, y) = \mathrm{Rank}^{N}(x, y)$, we get $\mathrm{Rank}^{M''}(x, y) = \mathrm{Rank}^{N''}(x, y)$ for every $(x, y) \in (\mathbb{N}^d)^{\leq}$. Therefore, $\mathrm{Rank}^{M''} = \mathrm{Rank}^{N''}$. □

We continue the proof of the theorem. Fix $(i_1, \dots, i_{d-1}) \in \mathbb{N}^{d-1}$. Applying Lemma 4.3 to $\widetilde{M}$ and $\widetilde{N}$ with $\mathrm{Rank}^{\widetilde{M}} = \mathrm{Rank}^{\widetilde{N}}$, we see that the two multisets

$$\{(\lambda^{(i)})_{i_1,\dots,i_{d-1}} | i \in \Lambda_1' \text{ and } (\lambda^{(i)})_{i_1,\dots,i_{d-1}} > 0\},$$

and

$$\{(\mu^{(\ell)})_{i_1,\dots,i_{d-1}} | \ell \in \Lambda_2' \text{ and } (\mu^{(\ell)})_{i_1,\dots,i_{d-1}} > 0\}$$

are equal, i.e., the two multisets

$$\left\{(a^{(i)}, (\lambda^{(i)})_{i_1,\dots,i_{d-1}}) | i \in \Lambda_1' \text{ and } (\lambda^{(i)})_{i_1,\dots,i_{d-1}} > 0\right\}, \tag{4.11}$$

and

$$\left\{(b^{(\ell)}, (\mu^{(\ell)})_{i_1,\dots,i_{d-1}}) | \ell \in \Lambda_2' \text{ and } (\mu^{(\ell)})_{i_1,\dots,i_{d-1}} > 0\right\} \tag{4.12}$$

are equal. Applying induction to M'' and N'' with $\mathrm{Rank}^{M''} = \mathrm{Rank}^{N''}$, we conclude that

$$\left\{(a^{(i)}, (\lambda^{(i)})_{i_1,\dots,i_{d-1}}) | i \in \Lambda_1 - \Lambda_1' \text{ and } (\lambda^{(i)})_{i_1,\dots,i_{d-1}} > 0\right\}, \tag{4.13}$$

and

$$\left\{(b^{(\ell)}, (\mu^{(\ell)})_{i_1,\dots,i_{d-1}}) | \ell \in \Lambda_2 - \Lambda_2' \text{ and } (\mu^{(\ell)})_{i_1,\dots,i_{d-1}} > 0\right\} \tag{4.14}$$

are equal. Combining (4.11)–(4.14), we see that the two multisets (4.4) and (4.5) are equal. □

Corollary 4.8. *Let $d \geq 1$. Let M and N be $\mathbb{N}^d$-indexed persistence modules admitting the barcodes*

$$M = \bigoplus_{i \in \Lambda_1} \mathbb{T}_{a^{(i)}} \mathbf{k}_{\lambda^{(i)}} \quad \text{and} \quad N = \bigoplus_{\ell \in \Lambda_2} \mathbb{T}_{b^{(\ell)}} \mathbf{k}_{\mu^{(\ell)}},$$

where $|\lambda^{(i)}| \neq 0$ and $|\mu^{(\ell)}| \neq 0$ for all $i \in \Lambda_1$ and $\ell \in \Lambda_2$. If $\mathrm{Rank}^M = \mathrm{Rank}^N$, then the two multisets

$$\left\{(a^{(i)}, (\lambda^{(i)})_{i_1,\dots,i_{d-1}}) | (i_1, \dots, i_{d-1}) \in \mathbb{N}^{d-1}, i \in \Lambda_1 \text{ and } (\lambda^{(i)})_{i_1,\dots,i_{d-1}} > 0\right\}, \tag{4.15}$$

and

$$\left\{(b^{(\ell)}, (\mu^{(\ell)})_{i_1,\dots,i_{d-1}}) | (i_1, \dots, i_{d-1}) \in \mathbb{N}^{d-1}, \ell \in \Lambda_2 \text{ and } (\mu^{(\ell)})_{i_1,\dots,i_{d-1}} > 0\right\} \tag{4.16}$$

are equal, and $\sum_{i \in \Lambda_1} |\lambda^{(i)}| = \sum_{\ell \in \Lambda_2} |\mu^{(\ell)}|$.

Proof. Follows immediately from Theorem 4.7. □

Remark 4.9. Recall from Definition 2.1 that a 1-dimensional partition is of the form $\lambda = (n)_O$ for some $n \in \mathbb{N} \sqcup \{+\infty\}$. Moreover, every $\mathbb{N}$-indexed persistence module admits a barcode. Therefore, when $d = 1$, Theorem 4.7 recovers the well-known result that the rank invariant and the barcode determine each other uniquely. Unfortunately, when $d > 1$, Example 4.4 shows that the rank invariant of a decomposable $\mathbb{N}^d$-indexed persistence modules does not determine the barcode.

Author contributions

All the authors of this article have contributed equally. All the authors of this article have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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