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Research article

A fast second order PDE approach for the space-time fractional parabolic problems

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Abstract: We study a fast second-order PDE approach for solving the space-time parabolic equations with fractional diffusion and Caputo fractional time derivative. To localize the space fractional elliptic operator, we map a Dirichlet boundary condition to a Neumann condition via an extension problem on the semi-infinite cylinder. For the equivalent extension problem, we use the fast L2-1_σ method based on the sum-of-exponentials to speed up the evaluation of the time fractional Caputo derivative, and the tensor product finite element method to discretize the spatial direction on the truncated cylinder domain. Then, the stability and α-robust error estimates of the fully discrete scheme are derived. Finally, the numerical experiments are presented to demonstrate the effectiveness of our scheme.

Keywords: fractional derivative; weight Sobolev space; fast L2-1σ scheme; stability; α-robust error analysis
Mathematics Subject Classification: 65M12, 65M22, 65M60

1. Introduction

Let Ω be an open and bounded domain of $\mathbb{R}^d$ ($d \geq 1$), with the boundary $\partial\Omega$, given $s \in (0, 1)$, we shall consider the following space-time fractional parabolic problem

$$\begin{cases} \partial_t^\alpha \mathfrak{u} + (-\Delta)^s \mathfrak{u} = f, & \text{in } \Omega \times (0, T], \\ \mathfrak{u} = 0, & \text{on } \partial\Omega \times (0, T], \\ \mathfrak{u}(0) = \mathfrak{u}_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where ∂_t^α stands for the left-sided Caputo fractional derivative of order α with respect to the time t, which is defined by

$$\partial_t^\alpha \mathfrak{u}(t) = \int_0^t w_{1-\alpha}(t-s)\mathfrak{u}'(s)ds, \quad 0 < \alpha < 1, \tag{1.2}$$

where Γ is the Gamma function and $w_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1-\alpha)$. Here $(-\Delta)^s$ is the fractional power of the second-order elliptic operator. Moreover, the left Riemann–Liouville fractional integral is defined by

$$I^{\alpha}\mathfrak{u}(t) = \int_0^t w_{\alpha}(t-s)\mathfrak{u}(s)ds,$$

Then, we have $\partial_t^{\alpha}\mathfrak{u}(t) = (I^{1-\alpha}\partial_t\mathfrak{u})(t)$.

To address the nonlocality of the fractional Laplace operator $(-\Delta)^s$, our method is to transform the problem (1.1) into an equivalent Caffarelli-Silvestre extension problem as follows based on the following equivalent reformulation of problem (1.1):

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla u) = 0, & \text{in } C \times (0,T), \\ \partial_t^{\alpha} u + \frac{1}{d_s}\partial_{\nu}^{1-2s} u = f, & \text{on } (\Omega \times \{0\}) \times (0,T), \\ u = 0, & \text{on } \partial_L C \times (0,T), \\ u|_{t=0} = \mathfrak{u}_0, & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.3}$$

Here $C = \Omega \times (0,\infty)$ is the semi-infinite cylinder, $\partial_L C = \partial\Omega \times [0,\infty)$ is the boundary of C and $d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$. When y is defined as the extended variable in the extended dimension $\mathbb{R}^{d+1}$ of problem (1.3), we have the external normal derivative of u at $\Omega \times \{0\}$

$$\partial_{\nu}^{1-2s} u = -\lim_{y\to 0^+} y^{1-2s}u_y = d_s(-\Delta)^s\mathfrak{u}. \tag{1.4}$$

For the detailed process of transforming the problem (1.1) into the problem (1.3), please refer to references [1,2]. Then, the trace $\mathfrak{u}(x,t) = u(\cdot,0,t)$ is the solution of the space-time fractional parabolic problem (1.1). The main objective of this work is to find the solution of (1.1) by using the extension problem (1.3).

Additionally, the design of an efficient skill to deal numerically with the Caputo fractional derivative ∂_t^{α} is not an easy task. In the past decade, there have been several well-known schemes that have been developed and analyzed via finite difference methods under the assumption that the solution is sufficiently smooth, such as the L1 and L2-1_σ schemes [3, 4], L2 scheme [5], and the convolution quadrature methods [6]. However, the solutions to the time-fractional problems are weak singularities at $t \to 0$. This inspires researchers to design improved finite difference schemes on the graded meshes to overcome the singularity of time, but this theoretical analysis is very difficult; see [7, 8]. Another important feature is the storage problem due to the nonlocality of the time fractional derivatives. To be precise, all of the aforementioned works require $O(N)$ storage and $O(N^2)$ computational cost when the time step is N, which is too costly. Jiang et al. [9] introduced the sum-of-exponentials (SOE) approximation to accelerate the efficient evaluation of the Caputo derivative. This reduces the computational cost of the L1 format to N and the storage capacity to NT under a consistent grid, while maintaining the same accuracy as the L1 format, which reduces the storage and computational cost of the L1 scheme on the uniform meshes to $O(N_p)$, or $O(N_pN)$, here $(N_p \ll N)$, while maintaining almost the same accuracy as the L1 scheme. Along this way, the fast L2 and L2-1_σ schemes are present and analyzed under the uniform meshes [10, 11]. Furthermore, Liao et al. [12] study the fast L1 scheme on the grade meshes for solving the nonlinear time fractional diffusion equations. Based on the fast L2-1_σ on the nonuniform meshes, Liu et al. [13] proposed a fast scheme for solving the nonlinear time fractional diffusion equations. Wang et al. [14] obtain the unconditional optimal bounds of the fast scheme for the time fractional biharmonic equations. Quan et al. [15] prove that a bilinear form associated with the fast L2-1_σ formula is positive semidefinite for all time, and derive the uniform global-in-time H^1-stability of the fast scheme for the time fractional diffusion equations.

Noting that the expansion direction of problem (1.3) is a semi-infinite domain, a direct application of a numerical approximation to the extended problem is not feasible. As a remedy, the exponential decay of u in direction $y \to +\infty$ can be applied such that a truncation of the semi-infinite cylinder to $\Omega\times(0,Y)$ becomes possible, and the height Y of the truncated cylinder needs to be chosen dependent on the mesh parameter to ensure the convergence of the numerical scheme, see [16, 17]. By the aforementioned results, numerical schemes such as finite difference methods [18], finite element methods [16, 19], and spectral methods [20] have been developed for the Caffarelli-Silvestre extension problem of the fractional diffusion equation. However, only a small amount of work has been done on the problem (1.3). Nochetto et al. [2] propose and analyze an implicit fully-discrete scheme via the tensor product finite elements in space and an implicit finite difference discretization in time, and the stability and error estimates of this scheme are proved. Hu et al. [21, 22] studied the finite difference methods to approximate the temporal and spatial directions discretization for the 1D and 2D problems, and the convergence and error estimate of this scheme are shown. The above work has low accuracy in terms of time and does not consider the computational storage for time discretization. To our knowledge, there is no work on the second-order time numerical format for the problem (1.3). However, the expansion problem is a high-dimensional space problem, which is expensive for computational cost of the numerical scheme. Therefore, designing an efficient numerical scheme is crucial.

In order to improve the computational efficiency of numerical approximation for the extended problem (1.3), we use the sum-of-exponentials (SOE) technique to speed up the evaluation of the nonuniform L2-1_σ scheme in the temporal direction and the tensor product finite element method for the spatial direction. It is worth noting that when we use the nonuniform L2-1_σ scheme to discretize the Caputo fractional derivative, the error results may blow up at $\alpha \to 1^-$ [23], so we adopt an improved fractional Grönwall inequality from [14] to obtain an α-robust error estimate. Due to the space elliptic operator of the expansion problem (1.3) being degenerative, the height Y of the truncated cylinder needs to be chosen depending on the mesh parameter to obtain an optimally convergent error. This technique was already pursued in [17, 19] using a discretization with the tensor product finite elements in the extended direction. Here, we shall adapt this approach to the parabolic case.

The main contributions of our work are as follows:

1. The numerical solution of the space-time fractional parabolic equations is obtained by constructing a fast nonuniform L2-1_σ scheme with the tensor product finite element method for the equivalent extended problem. This numerical scheme has the advantages of high accuracy and low computational storage, and can effectively handle the singularity of the solution at $t \to 0$.
2. We prove the stability of the numerical scheme both in the $\mathring{H}_L^1(y^a, C_Y)$ and $L^2(\Omega)$ norms under some constraints on the time step ratio, and obtain an α-robust error estimate by the fractional Grönwall inequality. We note that this theoretical analysis framework is also applicable to the nonlinear space-time fractional equations [13, 14].

The outline of this paper is as follows. In Section 2, we introduce the fast L2-1_σ scheme for the Caputo derivative and its basic lemmas. In Section 3, we establish a fast L2-1_σ fully discrete scheme for the Caffarelli-Silvestre extension problem. In Section 4, the stability and error estimates of the fully discrete scheme are proved. In Section 5, we present a numerical experiment to support our theoretical results.

2. Fast L2-1_σ scheme for the Caputo derivative

In this section, we shall introduce a fast L2-1_σ scheme to approximate the time Caputo derivative. Firstly, let us review the L2-1_σ scheme. This numerical scheme was first proposed by Alikhanov [4], so it is also known as the Alikhanov scheme. Let N be a positive integer, we consider a nonuniform time mesh

$$t_k = \left(\frac{k}{N}\right)^r T, \quad k = 0, 1, 2 \cdots, N,$$

where the mesh parameter $r \geq 1$ is chosen by the user. For $0 \leq \sigma \leq 1$, we define the off-set time level as $t_{n-\sigma} = (1-\sigma)t_n + \sigma t_{n-1}$. For simplicity, we shall write $u^{n-\sigma} := u(t_{n-\sigma})$ and $u^{n,\sigma} := (1-\sigma)u^n + \sigma u^{n-1}$ for any function u. Giving $\sigma = \alpha/2$ here and after and set the time step $\tau_k = t_k - t_{k-1}$, the time Caputo derivative (1.2) can be approximated by L2-1_σ scheme.

$$\begin{aligned}
\partial_t^\alpha u(t_{n-\sigma}) &= \int_0^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma} - s)\, u'(s) ds \\
&= \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_{n-\sigma} - s)\, u'(s) ds + \int_{t_{n-1}}^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma} - s)\, u'(s) ds \\
&\approx \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_{n-\sigma} - s)\, (\Pi_{2,k} u)'(s) ds \\
&\quad + \int_{t_{n-1}}^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma} - s)\, (\Pi_{1,n} u)'(s) ds =: D_N^\alpha u^{n-\sigma},
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
\Pi_{2,k} u =& \frac{(t-t_k)(t-t_{k+1})}{(t_{k-1}-t_k)(t_{k-1}-t_{k+1})} u^{k-1} + \frac{(t-t_{k-1})(t-t_{k+1})}{(t_k - t_{k-1})(t_k - t_{k+1})} u^k \\
&+ \frac{(t-t_{k-1})(t-t_k)}{(t_{k+1}-t_{k-1})(t_{k+1}-t_k)} u^{k+1}. \\
\Pi_{1,n} u =& \frac{t-t_n}{t_{n-1}-t_n} u^{n-1} + \frac{t-t_{n-1}}{t_n - t_{n-1}} u^n.
\end{aligned}$$

Let the step size ratio be $\rho_k = \tau_k/\tau_{k+1}$ and $\nabla_\tau u^k = u^k - u^{k-1}$ for $k = 0, 1, 2, \cdots, N$, the discrete fractional derivative in (2.1) can be reformulated as

$$\begin{aligned}
D_N^\alpha u^{n-\sigma} &= \sum_{k=1}^{n-1} (a_{n-k}^{(n)} \nabla_\tau u^k + \rho_k b_{n-k}^{(n)} \nabla_\tau u^{k+1} - b_{n-k}^{(n)} \nabla_\tau u^k) + a_0 \nabla_\tau u^n \\
&= \sum_{k=1}^{n-1} A_{n-k}^{(n)} \nabla_\tau u^k + A_0^{(n)} \nabla_\tau u^n
\end{aligned} \tag{2.2}$$

with

$$a_0^{(n)} = \frac{1}{\tau_n}\int_{t_{n-1}}^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma}-s)ds, \quad a_{n-k}^{(n)} = \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_{n-\sigma}-s)ds,$$

$$b_{n-k}^{(n)} = \frac{2}{\tau_k(\tau_k+\tau_{k+1})}\int_{t_{k-1}}^{t_k}(s-t_{k-1/2})w_{1-\alpha}(t_{n-\sigma}-s)ds,$$

and

$$A_{n-k}^{(n)} = \begin{cases} a_0^{(n)} + \rho_{n-1}b_1^{(n)}, & k=n, \\ a_{n-k}^{(n)} + \rho_{k-1}b_{n-k+1}^{(n)} - b_{n-k}^{(n)}, & 2\le k\le n-1, \\ a_{n-1}^{(n)} - b_{n-1}^{(n)}, & k=1. \end{cases}$$

It is known that the computational complexity of the L2-1_σ scheme is huge, so we consider the fast L2-1_σ scheme based on the sum-of-exponentials technique to approximate the kernel $t^{-\alpha}$. Its idea mainly adopts the following lemmas.

Lemma 2.1. *For the given parameters $\alpha, \epsilon, \hat{\tau}$ and T, there exists a family of points s_i and weight ω_i $(i = 1, 2, \cdots, N_p)$ such that*

$$\left| t^{-\alpha} - \sum_{i=1}^{N_p} \omega_i e^{-s_i t} \right| \le \epsilon, \quad \forall\, t \in [\hat{\tau}, T], \tag{2.3}$$

where

$$N_p = O\left(\log\frac{1}{\epsilon}(\log\log\frac{1}{\epsilon} + \log\frac{T}{\hat{\tau}}) + \log\frac{1}{\hat{\tau}}(\log\log\frac{1}{\epsilon} + \log\frac{1}{\hat{\tau}})\right).$$

Therefore, the history part in (2.1) can be written as

$$\begin{aligned} &\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} w_{1-\alpha}\left(t_{n-\sigma}-s\right)\left(\Pi_{2,k}u\right)'(s)ds \\ &\approx \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{N_p}\int_0^{t_{n-1}}\left(\Pi_{2,k}u\right)'(s)\omega_i e^{-s_i(t_{n-\sigma}-s)}ds := \sum_{i=1}^{N_p} H_i(t_{n-1}), \end{aligned} \tag{2.4}$$

where $H_i(t_0) = 0$ and

$$H_i(t_{n-1}) = e^{-s_i\tau_{n-\sigma}}H_i(t_{n-2}) + \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-2}}^{t_{n-1}}\left(\Pi_{2,k}u\right)'(s)\omega_i e^{-s_i(t_{n-\sigma}-s)}ds. \tag{2.5}$$

Combining (2.1) and (2.4), the fast L2-1_σ scheme can be represented as

$$D_F^\alpha u^{n-\sigma} = a_0^{(n)}\nabla_\tau u^n + \sum_{i=1}^{N_p} H_i(t_{n-1}), \tag{2.6}$$

where $H_i(t_{n-1})$ can be calculated by the recurrence formula (2.5). Obviously, the fast L2-1_σ format (2.5) is more computationally efficient than the standard L2-1_σ format (2.2) and has reduced the storage and computational cost from $O(N)$ and $O(N^2)$ to $O(N_p)$ and $O(NN_p)$.

For subsequent theoretical analysis, we equivalently reformulate (2.6) into the following convolution form

$$D_F^{\alpha}u^{n-\sigma} = B_0^{(n)}u^n + \sum_{i=1}^{n-1}(B_{n-i}^{(n)} - B_{n-i-1}^{(n)})u^i - B_{n-1}^{(n)}u^0, \tag{2.7}$$

where

$$B_{n-k}^{(n)} = \begin{cases} a_0^{(n)} + \sum\limits_{i=1}^{N_p} \rho_{n-1}\tilde{b}_1^{(n)}, & k = n, \\ \sum\limits_{i=1}^{N_p} e^{-s_i(t_{n-\sigma}-t_{k+1-\sigma})}(\tilde{a}_{n-k}^{(k+1)} + e^{-s_i\tau_{k+1-\sigma}}\rho_{k-1}\tilde{b}_{n-k+1}^{(k)} - \tilde{b}_{n-k}^{(k+1)}), & 2 \le k \le n-1, \\ \sum\limits_{i=1}^{N_p} e^{-s_i(t_{n-\sigma}-t_{2-\sigma})}(\tilde{a}_{n-1}^{(2)} - \tilde{b}_{n-1}^{(2)}), & k = 1, \end{cases}$$

with

$$\tilde{a}_{n-k}^{(k+1)} = \frac{\omega_i}{\tau_k\Gamma(1-\alpha)}\int_{t_{k-1}}^{t_k} e^{-s_i(t_{k+1-\sigma}-s)}ds,$$

$$\tilde{b}_{n-k}^{(k+1)} = \frac{2\omega_i}{\tau_k(\tau_k+\tau_{k+1})\Gamma(1-\alpha)}\int_{t_{k-1}}^{t^k}(s - t_{k-1/2}e^{-s_i(t_{k+1-\sigma}-s)})ds.$$

Now, we define a sequence of discrete complementary convolution kernels $\{P_j^{(n)}\}_{j=1}^n$ by

$$P_0^{(n)} = \frac{1}{B_0^{(n)}}, \quad P_j^{(n)} = \frac{1}{B_0^{(n-j)}}\sum_{k=0}^{j-1}\left(B_{j-k-1}^{(n-k)} - B_{j-k}^{(n-k)}\right)P_{n-k}^{(n)}, \quad 1 \le j \le n-1.$$

By the theoretical basis provided in [23], we shall obtain that the kernel $P_j^{(n)}$ satisfies the following three properties [14]:

$$\sum_{j=k}^{n} B_{j-k}^{(j)}P_{n-j}^{(n)} = 1 \quad \text{for } 1 \le n \le N. \tag{2.8}$$

$$\sum_{j=1}^{n} P_{n-j}^{(n)}t_{j-\sigma}^{-\alpha} \le \frac{2^{1+r\alpha}T^{\alpha-l_N}e^r t_N^{l_N}\Gamma(1+l_N-\alpha)}{\Gamma(1+l_N)}, \quad l_N = 1/\ln N. \tag{2.9}$$

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \le \frac{2t_n^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.10}$$

An important feature of the solution u to the problem (1.3) is singularity near the initial time $t = 0$, which usually satisfies the following properties in terms of time t:

$$|\partial_t^l u(t)| \le C(1 + t^{\alpha-l}) \quad \text{for } l = 0, 1, 2, 3. \tag{2.11}$$

We point out that the C generally means a constant in this paper. Then, we can get the following error estimates [14, 24]:

$$|\partial_t^{\alpha}u(t^{n-\sigma}) - D_F^{\alpha}u^{n-\sigma}| \le C(t_{n-\sigma}^{-\alpha}N^{-\min\{3-\alpha,r\alpha\}} + \epsilon), \tag{2.12}$$

$$|u(t^{n-\sigma}) - u^{n,\sigma}| \le Ct_{n-\sigma}^{-\alpha}N^{-\min\{2,r\alpha\}} \tag{2.13}$$

for $n = 1, 2, \cdots, N$.

We shall introduce two useful lemmas, which play a significant role in the subsequent theory.

Lemma 2.2. *[14] For any sequence $\{u\}_{n=1}^{N}$, it holds:*

$$(D_F^{\alpha} u^{n-\sigma}, u^{n,\sigma}) \geq \frac{1}{2} D_F^{\alpha} \|u^{n-\sigma}\|^2. \tag{2.14}$$

Here $(\cdot,\cdot)$ represents the inner product, which is defined as $(u,v)_\Omega = \int_\Omega uvdx$; we shall adopt this definition in this paper.

Lemma 2.3. *[14] Let λ_i be the nonnegative constants with $0 \leq \sum_{i=1}^{n} \lambda_i \leq \Lambda$, where Λ is a positive constant. Assume that the nonnegative sequences $\{w^k\}_{k=0}^{N}$, $\{\xi^n\}_{n=1}^{N}$, and $\{\eta^n\}_{n=1}^{N}$ satisfy*

$$D_F^{\alpha}(w^{n-\sigma})^2 \leq \sum_{i=1}^{n} \lambda_i (w^{i,\sigma})^2 + \xi^n w^{n,\sigma} + (\eta^n)^2 \quad \text{for } n \geq 1. \tag{2.15}$$

If the maximum time step satisfies $\tau \leq [2\Gamma(2-\alpha)\Lambda]^{-1/\alpha}$, we can get

$$w^n \leq E_\alpha(2\Lambda t_n^\alpha)\left[w^0 + \max_{1\leq k\leq n} \sum_{j=1}^{k} P_{k-j}^{(k)}(\xi^j + \eta^j) + \max_{1\leq j\leq n}\{\eta^j\}\right] \quad \text{for } 1 \leq n \leq N. \tag{2.16}$$

3. Fully discrete FE scheme for the Caffarelli-Silvestre extension problem

In this section, we shall state the fully discrete scheme for the Caffarelli-Silvestre extension problem. To deal with the nonuniformly elliptic operator, we consider the weighted Sobolev spaces with $|y|^{-a}$, $a \in (-1,1)$. Let $D \subset \mathbb{R}^d \times (0,+\infty)$ be an open region, we then denote $L^2(|y|^a, D)$ to be the space of all measurable functions defined on D such that

$$\|w\|_{L^2(|y|^a,D)}^2 = \int_D |y|^a w^2 < \infty.$$

In a similar way, we also define space

$$H^1(y^a, D) := \{w \in L^2(y^a, D) : \ |\nabla w| \in L^2(y^a, D)\}$$

and its equipped norm is

$$\|w\|_{H^1(y^a,D)} = \left(\|w\|_{L^2(y^a,D)}^2 + \|\nabla w\|_{L^2(y^a,D)}^2 \right)^2. \tag{3.1}$$

Due to $a \in (-1,1)$, we have that $|y|^a$ belong to the so-called Muckenhoupt class [25].

To study the problem (1.3), we introduce the space

$$\mathring{H}_L^1(y^a, C) := \left\{ w \in H^1(y^a, C) : w = 0 \text{ on } \partial_L C \right\}. \tag{3.2}$$

The following weighted Poincaré inequality holds

$$\|w\|_{L^2(y^a,C)} \lesssim \|\nabla w\|_{L^2(y^a,C)}, \quad \forall\, w \in \mathring{H}_L^1(y^a, C). \tag{3.3}$$

Thus, the seminorm on $\mathring{H}_L^1(y^a, C)$ is equivalent to (3.1). For any $w \in \mathring{H}_L^1(y^a, C)$, $\mathrm{tr}_\Omega w$ stands for its trace onto $\Omega \times \{0\}$, i.e., $\mathrm{tr}_\Omega w(t) = w(\cdot, 0, t)$, which holds [16, 26]

$$\mathrm{tr}_\Omega \mathring{H}_L^1(y^a, C) = H^s(\Omega), \quad \|\mathrm{tr}_\Omega w\|_{H^s(\Omega)} \leq C_{\mathrm{tr}_\Omega} \|w\|_{\mathring{H}_L^1(y^a,C)}$$

with $a = 1 - 2s$, and a is uniformly set to $1 - 2s$ in the subsequent theoretical analysis.

Define

$$W := \{w \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\Omega)) : \partial_t^\alpha w \in L^2(0,T;H^s(\Omega))\}.$$

$$V := \{w \in L^2(0,T;\mathring{H}_L^1(y^a,C)) : \partial_t^\alpha \mathrm{tr}_\Omega w \in L^2(0,T;H^{-s}(\Omega))\}.$$

As a result, given $f \in L^2(0,T;H^{-s}(\Omega))$, a function $\mathfrak{u} \in W$ solves problem (1.1) if and only if the function $u \in V$ solves (1.3) [2]. At the same time, there is uniqueness in the problem (1.1) and (1.3) as follows:

Lemma 3.1. *[2] Let $s \in (0,1), \alpha \in (0,1), f \in L^2(0,T;H^{-s}(\Omega))$, and $\mathfrak{u}_0 \in L^2(\Omega)$. Then, problem* (1.1) *and* (1.3) *have a unique solution.*

A weak form of (1.3) reads: If each $t \in (0,T]$, find $u \in V$ such that

$$\begin{cases} (tr_\Omega \partial_t^\alpha u, tr_\Omega \phi) + a(u,\phi) = (f, tr_\Omega \phi), \quad \forall\, \phi \in \mathring{H}_L^1(y^a, C). \\ \mathrm{tr}_\Omega u(0) = \mathfrak{u}_0. \end{cases} \tag{3.4}$$

Here the bilinear form is

$$a(u,\phi) = \frac{1}{d_s}\int_C y^{1-2s}\nabla u \cdot \nabla \phi dxdy. \tag{3.5}$$

Remark 3.1. *[2] The initial datum $\mathfrak{u}_0$ of problem* (1.1)*determines only $u(0)$ on $\Omega \times (0,\infty)$ in a trace sense.*

Since the fact that the solution $u(t)$ in the problem (3.4) is located in the infinite region C, we cannot directly calculate it using the finite element method. However, the solution $u(t)$ of the problem (3.4) decays exponentially in y [2], that is

$$\|\nabla u\|_{L^2(0,T;L^2(y^a,\Omega\times(Y,\infty)))} \le Ce^{-\sqrt{\lambda_1}Y/2},$$

where $Y > 1$, λ_1 is the first eigenvalue of the Dirichlet Laplace operator, and the constant C depends only on the initial value u_0 and the right-hand side f.

As per the above proposition of u, we can truncate C to $C_Y = \Omega\times(0,Y)$ for a suitable Y, and consider the following problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla v) = 0, & \text{in } C_Y \times (0,T), \\ \partial_t^\alpha v + \frac{1}{d_s}\partial_\nu^{1-2s} v = f, & \text{on } (\Omega\times\{0\})\times(0,T), \\ v = 0, & \text{on } (\partial_L C_Y \cup \Omega_Y)\times(0,T), \\ v|_{t=0} = \mathfrak{u}_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{3.6}$$

where $\Omega_Y = \Omega \times Y$ and $Y \ge 1$ is sufficiently large.

Now, we define

$$\mathring{H}_L^1(y^a, C_Y) := \left\{w \in H^1(y^a, C_Y) : w = 0 \text{ on } \partial_L C_Y \cup \Omega_Y\right\}.$$

$$V_Y := \{w \in L^2(0,T;\mathring{H}_L^1(y^a,C_Y)) : \partial_t^\alpha \mathrm{tr}_\Omega w \in L^2(0,T;H^{-s}(\Omega))\}.$$

Then the weak form of problem (3.6) can read: If each $t \in (0,T]$, seeking $v \in V_Y$ such that

$$(\partial_t^\alpha \mathrm{tr}_\Omega v, \mathrm{tr}_\Omega \phi) + a_Y(v,\phi) = (f, \mathrm{tr}_\Omega \phi), \tag{3.7}$$

for all $\phi \in \mathring{H}_L^1(y^a, C_Y)$ and $\mathrm{tr}_\Omega v(0) = \mathrm{u}_0$. Here

$$a_Y(v, \phi) = \frac{1}{d_s}\int_{C_Y} y^{1-2s}\nabla v \cdot \nabla\phi dxdy.$$

The following exponential decay result is given in [2, Lemma 4.3]; it provides us with a basis for numerical discretization.

Lemma 3.2. *(Truncation error) For every $\alpha \in (0,1)$, $Y \geq 1$ and λ_1 are the first eigenvalue of the Dirichlet Laplace operator. Then we have*

$$I^{1-\alpha}\|tr_\Omega(u-v)\|^2_{L^2(\Omega)}(T) + \|\nabla(u-v)\|^2_{L^2(0,T;L^2(y^a,C_Y))} \leq Ce^{-\sqrt{\lambda_1}Y},$$

where the constant C depends only on the initial value u_0 *and the right-hand side* f.

Let $\mathcal{T}_\Omega = \{K\}$ be a uniform mesh of Ω into cell K, and we consider a graded partition $I_Y = \{I\}$ of the interval $[0, Y]$ with node

$$y_k = \left(\frac{k}{M}\right)^\mu Y, \quad k = 0, 1, \cdots, M,$$

where $\mu = \mu(\alpha) > 3/(2s)$. Then, we define a partition $\mathcal{T}_Y$ of C_Y into cells of the form $T = K \times I$. The set of all triangulations $\mathcal{T}_Y$ is denoted by $\mathbb{T}$. Assume that $\#\mathcal{T}_\Omega \approx M^d$; thus, we have $\#\mathcal{T}_Y = M\#\mathcal{T}_\Omega \approx M^{d+1}$. If $\mathcal{T}_\Omega$ is shape regular and quasi-uniform, the element size h_Ω satisfies $h_\Omega \approx (\#\mathcal{T}_\Omega)^{-1/d}$.

For $\mathcal{T}_Y \in \mathbb{T}$, we define the finite element space as

$$\mathbb{V}(\mathcal{T}_Y) = \{W \in C^0(\overline{C}_Y) : W|_T \in P_1(K) \otimes \mathbb{P}_1(I)\ \forall\ T \in \mathcal{T}_Y,\ W|_{\Gamma_D} = 0\}, \tag{3.8}$$

where $\Gamma_D = \partial_L C_Y \cup \Omega \times Y$ is the Dirichlet boundary.

The projection operator plays a crucial role in error analysis. Without a doubt, we also a weight elliptic projector: $G_{\mathcal{T}_Y} : \mathring{H}_L^1(y^a, C_Y) \to \mathbb{V}(\mathcal{T}_Y)$ such that, for $w \in \mathring{H}_L^1(y^a, C_Y)$, is given by

$$a_Y(G_{\mathcal{T}_Y}w, W) = a_Y(w, W), \quad \forall\ W \in \mathbb{V}(\mathcal{T}_Y), \tag{3.9}$$

and the following error estimates hold [2]

$$\|w - G_{\mathcal{T}_Y})\|_{\mathring{H}_L^1(y^a,C_Y)} \leq C|\log \#\mathcal{T}_Y|^s(\#\mathcal{T}_Y)^{-1/(n+1)}, \tag{3.10}$$

$$\|\mathrm{tr}_\Omega(w - G_{\mathcal{T}_Y}w)\|_{L^2(\Omega)} \leq C|\log \#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{-(1+s)/(n+1)}, \tag{3.11}$$

where the constant C depends only on u_0 and f.

Suppose that

$$v_h^0 = \mathcal{I}_{\mathcal{T}_Y}\mathrm{u}_0,$$

where $\mathcal{I}_{\mathcal{T}_Y} = G_{\mathcal{T}_Y} \circ \mathcal{H}_a$ and $\mathcal{H}_a$ is the a-harmonic extension onto C_Y. Thus, the fast L2-1_σ fully discrete scheme of problem (3.7) reads: If $n = 1, 2, \cdots, N$, find $v_h^n \in \mathbb{V}(\mathcal{T}_Y)$ such that

$$(\mathrm{tr}_\Omega D_F^\alpha v_h^{n-\sigma}, \mathrm{tr}_\Omega\phi_h) + a_Y(v_h^{n;\sigma}, \phi_h) = (f^{n-\sigma}, tr_\Omega\phi_h), \tag{3.12}$$

for all $\phi_h \in \mathbb{V}(\mathcal{T}_Y)$ and $\mathrm{tr}_\Omega v_h^0 = \mathrm{tr}_\Omega G_{\mathcal{T}_Y} v(0)$.

4. Stability and error estimates of fully discrete scheme

In this section, let us begin the discussion on the stability and error estimates of the full discrete scheme (3.12). Firstly, we present the stability of the numerical scheme (3.12) in the $L^2(\Omega)$ norm, as detailed in the following theorem.

Theorem 4.1. *Assume that the v_h^n is the fully-discrete solution, if the maximum time step $\tau \leq [2\Gamma(2-\alpha)]^{-1/\alpha}$, then we have*

$$\|tr_\Omega v_h^n\|_{L^2(\Omega)} \leq E_\alpha(2t_n^\alpha)\left[\mathrm{u}_0 + \left(1 + \frac{2t_n^\alpha}{\Gamma(1+\alpha)}\right)\max_{1\leq j\leq n} f^{j-\sigma}\right], \tag{4.1}$$

for each $n = 1, 2, \cdots, N$.

Proof. By taking $\phi_h = v_h^{n,\sigma}$, we have

$$(\mathrm{tr}_\Omega D_F^\alpha v_h^{n-\sigma}, \mathrm{tr}_\Omega v_h^{n,\sigma}) + a_Y(v_h^{n,\sigma}, v_h^{n,\sigma}) = (f^{n-\sigma}, tr_\Omega v_h^{n,\sigma}). \tag{4.2}$$

Applying the Cauchy-Schwartz inequality, we can see that

$$(\mathrm{tr}_\Omega D_F^\alpha v_h^{n-\sigma}, \mathrm{tr}_\Omega v_h^{n,\sigma}) \leq \|f^{n-\sigma}\|\|v_h^{n,\sigma}\|. \tag{4.3}$$

By (2.14) and Young's inequality, we obtain

$$\frac{1}{2}D_F^\alpha\|v_h^{n-\sigma}\|^2 \leq \frac{1}{2}(\|f^{n-\sigma}\|^2 + \|v_h^{n,\sigma}\|^2). \tag{4.4}$$

Based on the fractional Grönwall inequality (2.10), (2.15), and (2.16), we have

$$\begin{aligned}
\|v_h^n\| &\leq E_\alpha(2t_n^\alpha)\left(v_h^0 + \max_{1\leq k\leq n}\sum_{j=1}^{k} P_{k-j}^{(k)} f^{j-\sigma} + \max_{1\leq j\leq n} f^{j-\sigma}\right)\\
&\leq E_\alpha(2t_n^\alpha)\left(v_h^0 + \max_{1\leq k\leq n}(\max_{1\leq j\leq k} f^{j-\sigma})\sum_{j=1}^{k} P_{k-j}^{(k)} + \max_{1\leq j\leq n} f^{j-\sigma}\right)\\
&\leq E_\alpha(2t_n^\alpha)\left(v_h^0 + (\max_{1\leq k\leq n} f^{j-\sigma})\frac{2t_n^\alpha}{\Gamma(1+\alpha)} + \max_{1\leq j\leq n} f^{j-\sigma}\right)\\
&= E_\alpha(2t_n^\alpha)\left[\mathrm{u}_0 + \left(1 + \frac{2t_n^\alpha}{\Gamma(1+\alpha)}\right)\max_{1\leq j\leq n} f^{j-\sigma}\right].
\end{aligned} \tag{4.5}$$

This concludes the proof. □

Next, we shall introduce the following lemma to derive the stability of the fully discrete scheme in the $\mathring{H}_L^1(y^a, C_Y)$ norm.

Lemma 4.1. *[15, Theorem 3.2] If the nonuniform mesh $\{\tau_k\}$ satisfies that*

$$\begin{aligned}
&\rho_k \geq 0.475329, \quad \epsilon \leq \min_{k\geq 1}\frac{1}{5(1-\alpha)(\sigma\tau_k)^\alpha},\\
&\Delta t \leq \min_{k\geq 2}\sigma\tau_k, \quad T \geq \max_{k\geq 2}(\sigma\tau_{k+1} + \tau_k),
\end{aligned} \tag{4.6}$$

for $k \geq 2$, then for any function w on $[0, \infty) \times C_{\mathcal{T}_Y}$ such that

$$\sum_{k=1}^{n} (D_F^{\alpha} w^{k-\sigma}, \nabla_\tau w^k) \geq \sum_{k=1}^{n} \frac{[\mathbf{B}]_{kk}}{\Gamma(1-\alpha)} \|\nabla_\tau w^k\|^2 > 0, \tag{4.7}$$

where $n \geq 1$ and $\mathbf{B}$ is defined in [15].

By making above assumption about the ratio of the time step, we can obtain the following results.

Theorem 4.2. *Assume that the v_h^n is the fully discrete solution, if $\{\tau_k\}$ satisfies the condition in Lemma 4.1, then we have*

$$\|v_h^n\|_{\mathring{H}_L^1(y^a, C_Y)} \leq \|v_h^0\|_{\mathring{H}_L^1(y^a, C_Y)} + C_f C_{tr_\Omega}, \tag{4.8}$$

for each $n = 1, 2, \cdots, N.$, where C_f is a constant depending on the right-hand f, C_{tr_Ω} is the Sobolev embedding constant depending on Ω and the spatial dimension d.

Proof. For $n \geq 1$, taking $\phi_h = \nabla_\tau v_h^k$ in (3.12) and summing up the derived equations over k, we have

$$\sum_{k=1}^{n} (\mathrm{tr}_\Omega D_F^{\alpha} v_h^{k-\sigma}, \mathrm{tr}_\Omega \nabla_\tau v_h^k) + \sum_{k=1}^{n} a_Y(v_h^{k,\sigma}, \nabla_\tau v_h^k) = \sum_{k=1}^{n} (f^{k-\sigma}, \nabla_\tau v_h^k). \tag{4.9}$$

Duo to

$$\sum_{k=1}^{n} a_Y(v_h^{k,\sigma}, \nabla_\tau v_h^k) = \frac{1}{2} \|v_h^n\|^2_{\mathring{H}_L^1(y^a, C_Y)} - \frac{1}{2} \|v_h^0\|^2_{\mathring{H}_L^1(y^a, C_Y)} + \frac{1-\gamma}{2} \sum_{k=1}^{n} \|\nabla_\tau v_h^k\|^2_{\mathring{H}_L^1(y^a, C_Y)}, \tag{4.10}$$

and

$$\begin{aligned} \sum_{k=1}^{n} (f^{k-\sigma}, \nabla_\tau v_h^k) &= (f^{n-\sigma}, v_h^n) - (f^1, v_h^0) - \sum_{k=2}^{n} (f^{k-\sigma} - f^{k-1-\sigma}, v_h^{k-1}) \\ &\leq C_f \max_{0 \leq k \leq n} \|\mathrm{tr}_\Omega v_h^k\|_{H^s(\Omega)} \\ &\leq C_f C_{\mathrm{tr}_\Omega} \max_{0 \leq k \leq n} \|v_h^k\|_{\mathring{H}_L^1(y^a, C_Y)}. \end{aligned} \tag{4.11}$$

Then, we have

$$\max_{0 \leq n \leq N} \|v_h^n\|^2_{\mathring{H}_L^1(y^a, C_Y)} \leq \|v_h^0\|_{\mathring{H}_L^1(y^a, C_Y)} + 2 C_f C_{\mathrm{tr}_\Omega} \max_{0 \leq n \leq N} \|v_h^n\|_{\mathring{H}_L^1(y^a, C_Y)}, \tag{4.12}$$

which indicates

$$\begin{aligned} \max_{0 \leq n \leq N} \|v_h^n\|_{\mathring{H}_L^1(y^a, C_Y)} &\leq C_f C_{\mathrm{tr}_\Omega} + \sqrt{(C_f C_{\mathrm{tr}_\Omega})^2 + \|v_h^0\|^2_{\mathring{H}_L^1(y^a, C_Y)}} \\ &\leq \|v_h^0\|_{\mathring{H}_L^1(y^a, C_Y)} + 2 C_f C_{\mathrm{tr}_\Omega}. \end{aligned} \tag{4.13}$$

The proof is completed. □

Analogously, we assume that solution v of the extended problem (3.6) satisfies the following condition

$$\|\partial_t^l v\| \leq C(1 + t^{\alpha - l}), \quad l = 0, 1, 2, 3. \tag{4.14}$$

$$\mathrm{tr}_\Omega v(t) \in L^\infty(0, T; H^s(\Omega)), \quad \partial_t^\alpha v \in L^\infty(0, T; \mathring{H}_L^1(y^a, C_Y)). \tag{4.15}$$

Then, we have the following error results:

Theorem 4.3. *Let v be the solution of problem* (3.7) *and v_h^n be the finite element solution of* (3.12)*, Then, there exists constant $C > 0$ such that*

$$\|tr_\Omega(v - v_h^n)\|_{L^2(\Omega)} \leq C\Big(|\log \#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{-(1+s)/(n+1)} + N^{-\min\{2,r\alpha\}} + \epsilon\Big), \tag{4.16}$$

$$\|v - v_h^n\|_{\mathring{H}_L^1(y^a, C_Y)} \leq C\Big(|\log \#\mathcal{T}_Y|^{s}(\#\mathcal{T}_Y)^{-1/(n+1)} + N^{-\min\{2,r\alpha\}} + \epsilon\Big), \tag{4.17}$$

where C is a constant depends only on the initial value u_0 *and the right-hand side f.*

Proof. Firstly, we split the error into the interpolation and approximation error

$$v_n - v_h^n = v^n - G_{\mathcal{T}_Y}v^n + G_{\mathcal{T}_Y}v^n - v_h^n = \rho^n + \theta^n, \tag{4.18}$$

where $\rho^n = v^n - G_{\mathcal{T}_Y}v^n,\ \theta^n = G_{\mathcal{T}_Y}v^n - v_h^n$.

From (3.7) and (2.14) can lead to

$$\begin{aligned}
&(\mathrm{tr}_\Omega D_F^\alpha \theta^{n-\sigma}, \mathrm{tr}_\Omega \phi_h) + a_Y(\theta^{n,\sigma}, \phi_h) \\
&= (\mathrm{tr}_\Omega(D_F^\alpha G_{\mathcal{T}_Y} v^{n-\sigma} - \partial_t^\alpha v^{n-\sigma}), \mathrm{tr}_\Omega\phi_h) - a_Y(v^{n-\sigma} - v^{n,\sigma}, \phi_h) \\
&= a_Y(v^{n,\sigma} - v^{n-\sigma}, \phi_h) + (\mathrm{tr}_\Omega(D_F^\alpha G_{\mathcal{T}_Y} v^{n-\sigma} - D_F^\alpha v^{n-\sigma}), \mathrm{tr}_\Omega\phi_h) \\
&\quad + (\mathrm{tr}_\Omega(D_F^\alpha v^{n-\sigma} - \partial_t^\alpha v^{n-\sigma}), \mathrm{tr}_\Omega\phi_h) \\
&= a_Y(v^{n,\sigma} - v^{n-\sigma}, \phi_h) + (\mathrm{tr}_\Omega D_F^\alpha \rho^{n-\sigma}, \mathrm{tr}_\Omega\phi_h) + (\mathrm{tr}_\Omega(D_F^\alpha v^{n-\sigma} - \partial_t^\alpha v^{n-\sigma}), \mathrm{tr}_\Omega\phi_h).
\end{aligned} \tag{4.19}$$

Also, we have

$$a_Y(v^{n,\sigma} - v^{n-\sigma}, \phi_h) = d_s(\mathrm{tr}_\Omega(\partial_t^\alpha(v^{n-\sigma} - v^{n,\sigma})), \mathrm{tr}_\Omega\phi_h) + d_s(f^{n,\sigma} - f^{n-\sigma}, \mathrm{tr}_\Omega\phi_h). \tag{4.20}$$

Taking $\phi_h = \theta^{n,\sigma}$ and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&(\mathrm{tr}_\Omega D_F^\alpha \theta^{n-\sigma}, \mathrm{tr}_\Omega\theta^{n,\sigma}) + a_Y(\theta^{n,\sigma}, \theta^{n,\sigma}) \\
&\leq d_s\|\mathrm{tr}_\Omega(\partial_t^\alpha(v^{n-\sigma} - v^{n,\sigma})\|\|\mathrm{tr}_\Omega\theta^{n,\sigma}\| + d_s\|f^{n,\sigma} - f^{n-\sigma}\|\|\mathrm{tr}_\Omega\theta^{n,\sigma}\| \\
&\quad + \mathrm{tr}_\Omega\|D_F^\alpha\rho^{n-\sigma}\|\|\mathrm{tr}_\Omega\theta^{n,\sigma}\| + \|\mathrm{tr}_\Omega(D_F^\alpha v^{n-\sigma} - \partial_t^\alpha v^{n-\sigma})\|\|\mathrm{tr}_\Omega\theta^{n,\sigma}\|.
\end{aligned} \tag{4.21}$$

From (3.11) and (2.7), we can derive

$$\begin{aligned}
\mathrm{tr}_\Omega\|D_F^\alpha\rho^{n-\sigma}\| &= \mathrm{tr}_\Omega\left\|B_0^{(n)}\rho^n + \sum_{k=1}^{n-1}(B_{n-k}^{(n)} - B_{n-k-1}^{(n)})\rho^k - B_{n-1}^{(n)}\rho^0\right\| \\
&= B_0^{(n)}\|\mathrm{tr}_\Omega\rho^n\| + \sum_{k=1}^{n-1}(B_{n-k-1}^{(n)} - B_{n-k}^{(n)})\|\mathrm{tr}_\Omega\rho^k\| + B_{n-1}^{(n)}\|\mathrm{tr}_\Omega\rho^0\| \\
&\leq \left(B_0^{(n)} + \sum_{k=1}^{n-1}\left(B_{n-k-1}^{(n)} - B_{n-k}^{(n)}\right) + B_{n-1}^{(n)}\right)\left(C_1|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{\frac{-(1+s)}{n+1}}\right) \\
&= 2B_0^{(n)}\left(C_1|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{\frac{-(1+s)}{n+1}}\right) \\
&\leq C_1\frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)}\left(|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{\frac{-(1+s)}{n+1}}\right),
\end{aligned} \tag{4.22}$$

where the fact that $B_0^{(n)} \leq \frac{2\tau_n^{-\alpha}}{\Gamma(2-\alpha)}$ is used. Together the condition (4.15), (2.12), and (2.13), for $j = 1, 2, \cdots, n$ such that

$$\begin{aligned} & d_s\|\mathrm{tr}_\Omega(\partial_t^\alpha(v^{j-\sigma} - v^{j,\sigma})\| + d_s\|f^{j,\sigma} - f^{j-\sigma}\| + \|\mathrm{tr}_\Omega(D_F^\alpha v^{j-\sigma} - \partial_t^\alpha v^{j-\sigma})\| \\ & \leq t_{j-\sigma}^{-\alpha}\left(N^{-\min\{2,r\alpha\}} + N^{-\min\{3-\alpha,r\alpha\}} + \epsilon\right) \\ & \leq C_2 t_{j-\sigma}^{-\alpha}\left(N^{-\min\{2,r\alpha\}} + \epsilon\right). \end{aligned} \tag{4.23}$$

Therefore, from (2.14), (4.22), and (4.23), we get

$$\begin{aligned} \mathrm{tr}_\Omega D_F^\alpha\|\theta^{n-\sigma}\|^2 \leq & \Big[C_1\frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)}\left(|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{\frac{-(1+s)}{n+1}}\right) \\ & + C_2 t_{j-\sigma}^{-\alpha}\left(N^{-\min\{2,r\alpha\}} + \epsilon\right)\Big]\|\theta^{n,\sigma}\|. \end{aligned} \tag{4.24}$$

Using the Grönwall inequality can yield

$$\begin{aligned} \|\mathrm{tr}_\Omega\theta_h^n\| \leq & E_\alpha(2t_n^\alpha)\Big[\|\mathrm{tr}_\Omega\theta^0\| + C_1\frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)}|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{-(1+s)/(n+1)}\max_{1\leq k\leq n}\sum_{j=1}^{k}P_{k-j}^{(k)} \\ & + C_2\left(N^{-\min\{2,r\alpha\}} + \epsilon\right)\max_{1\leq k\leq n}\sum_{j=1}^{k}P_{k-j}^{(k)}t_{j-\sigma}^{-\alpha}\Big]. \end{aligned} \tag{4.25}$$

It from (2.9) and (2.10) that

$$\begin{aligned} \|\mathrm{tr}_\Omega\theta_h^n\| \leq & E_\alpha(2t_n^\alpha)\Big(\|\mathrm{tr}_\Omega\theta^0\| + C_1\frac{8t_n^\alpha\tau_n^{-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{-(1+s)/(n+1)} \\ & + C_2\frac{2^{1+r\alpha}T^{\alpha-l_N}e^r t_N^{l_N}\Gamma(1+l_N-\alpha)}{\Gamma(1+l_N)}N^{-\min\{2,r\alpha\}} + \epsilon\Big). \end{aligned} \tag{4.26}$$

Choosing $\mathrm{tr}_\Omega v_h^0 = \mathrm{u}_0$, we derive $\|\mathrm{tr}_\Omega\theta^0\| = 0$. Thus, we have

$$\|\mathrm{tr}_\Omega\theta_h^n\| \leq C\Big(|\log\#\mathcal{T}_Y|^{2s}(\#\mathcal{T}_Y)^{-(1+s)/(n+1)} + N^{-\min\{2,r\alpha\}} + \epsilon\Big), \tag{4.27}$$

where

$$C = E_\alpha(2t_n^\alpha)\max\left\{C_1\frac{8t_n^\alpha\tau_n^{-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}, C_2\frac{2^{1+r\alpha}T^{\alpha-l_N}e^r t_N^{l_N}\Gamma(1+l_N-\alpha)}{\Gamma(1+l_N)}\right\}. \tag{4.28}$$

Finally, by (3.11) and the triangle inequality, we can derive (4.34).

According to the definition (2.7), we rewrite the $D_F^\alpha\|\theta^{n-\sigma}\|^2$ as

$$\mathrm{tr}_\Omega D_F^\alpha\|\theta^{n-\sigma}\|^2 = B_0^{(n)}\|\mathrm{tr}_\Omega\theta^n\|^2 + \sum_{i=1}^{n-1}(B_{n-i}^{(n)} - B_{n-i-1}^{(n)})\|\mathrm{tr}_\Omega\theta^i\|^2 - B_{n-1}^{(n)}\|\mathrm{tr}_\Omega\theta^0\|^2. \tag{4.29}$$

Due to $B_{n-i}^{(0)} - B_{n-i-1}^{(n)} < 0$ and $\theta_0 = 0$, we have

$$\begin{aligned} & B_0^{(n)}\|\mathrm{tr}_\Omega\theta^n\|^2 + \sum_{i=1}^{n-1}(B_{n-i}^{(n)} - B_{n-i-1}^{(n)})\|\mathrm{tr}_\Omega\theta^i\|^2 - B_{n-1}^{(n)}\|\mathrm{tr}_\Omega\theta^0\|^2 \\ & \geq B_0^{(n)}\|\mathrm{tr}_\Omega\theta^n\|^2 + \sum_{i=1}^{n-1}(B_{n-i}^{(n)} - B_{n-i-1}^{(n)})\|\theta^i\|^2_{\mathring{H}_L^1(y^a,C_Y)} \\ & = D_F^\alpha\|\theta^{n-\sigma}\|^2_{\mathring{H}_L^1(y^a,C_Y)} - B_0^{(n)}\|\theta^n\|^2_{\mathring{H}_L^1(y^a,C_Y)} + B_0^{(n)}\|\mathrm{tr}_\Omega\theta^n\|^2. \end{aligned} \tag{4.30}$$

This equation combined with (4.21)–(4.23) yields

$$\begin{aligned} D_F^\alpha \|\theta^{n-\sigma}\|^2_{\mathring{H}_L^1(y^a,C_Y)} \leq & \Big(C_1 \frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)} |\log \#\mathcal{T}_Y|^{2s} (\#\mathcal{T}_Y)^{-(1+s)/(n+1)} \\ & + C_2 t_{j-\sigma}^{-\alpha} (N^{-\min\{2,r\alpha\}} + \epsilon)\Big) \|\theta^{n,\sigma}\| + B_0^{(n)} \|\theta^n\|^2_{\mathring{H}_L^1(y^a,C_Y)} \\ \leq & \Big(C_1 \frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)} |\log \#\mathcal{T}_Y|^{2s} (\#\mathcal{T}_Y)^{-(1+s)/(n+1)} \\ & + C_2 t_{j-\sigma}^{-\alpha} (N^{-\min\{2,r\alpha\}} + \epsilon)\Big) \|\theta^{n,\sigma}\|_{\mathring{H}_L^1(y^a,C_Y)} \\ & + C_\theta B_0^{(n)} \|\theta^{n,\sigma}\|^2_{\mathring{H}_L^1(y^a,C_Y)}, \end{aligned} \tag{4.31}$$

where the imposed constant $C_\theta > 0$ is obtained through Taylor expansion in t_n. Also $B_0^{(n)} \leq \frac{\tau_n^{-\alpha}}{\Gamma(2-\alpha)}$. Thus, we can get the following result by the Grönwall inequality

$$\begin{aligned} \|\theta^n\|_{\mathring{H}_L^1(y^a,C_Y)} \leq & E_\alpha(2C_\theta B_0^{(n)} t_n^\alpha) \Big[\|\theta^0\|_{\mathring{H}_L^1(y^a,C_Y)} \\ & + C_1 \frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)} |\log \#\mathcal{T}_Y|^{2s} (\#\mathcal{T}_Y)^{-(1+s)/(n+1)} \max_{1\leq k\leq n} \sum_{j=1}^k P_{k-j}^{(k)} \\ & + C_2 \left(N^{-\min\{2,r\alpha\}} + \epsilon\right) \max_{1\leq k\leq n} \sum_{j=1}^k P_{k-j}^{(k)} t_{j-\sigma}^{-\alpha} \Big]. \end{aligned} \tag{4.32}$$

Similar to (4.26), there is

$$\|\theta^n\|_{\mathring{H}_L^1(y^a,C_Y)} \leq C\Big(|\log \#\mathcal{T}_Y|^{2s} (\#\mathcal{T}_Y)^{-(1+s)/(n+1)} + N^{-\min\{2,r\alpha\}} + \epsilon\Big). \tag{4.33}$$

Here the constant C is consistent with the constant defined in (4.28). Thus, conclusion (4.35) can be obtained through the estimation in (3.10). □

Remark 4.1. *In the framework of Theorem 4.3 and in view of the Lemma 3.2, we have the following error estimates for* $\mathfrak{u}^n, n = 0, 1, 2, \cdots, N$*:*

$$\|\mathfrak{u}^n - tr_\Omega v_h^n\|_{L^2(\Omega)} \leq C\Big(|\log \#\mathcal{T}_Y|^{2s} (\#\mathcal{T}_Y)^{-(1+s)/(n+1)} + N^{-\min\{2,r\alpha\}} + e^{-\sqrt{\lambda_1}Y} + \epsilon\Big), \tag{4.34}$$

$$\|\mathfrak{u}^n - tr_\Omega v_h^n\|_{H^s(\Omega)} \leq C\Big(|\log \#\mathcal{T}_Y|^{s} (\#\mathcal{T}_Y)^{-1/(n+1)} + N^{-\min\{2,r\alpha\}} + e^{-\sqrt{\lambda_1}Y} + \epsilon\Big), \tag{4.35}$$

where C is a constant that depends only on the initial value $\mathfrak{u}_0$ *and the right-hand side* f*.*

5. Numerical experiments

To demonstrate the effectiveness of our proposed numerical scheme, here we present the following numerical example, and the implementation was carried out with the help of the MATLAB software library iFEM. In our computation, we adopt the tensor product element defined in (3.8) for the spatial direction. In addition, we choose $\epsilon = 10^{-12}$ and $r = 3/\alpha$ in the fast L2-1_σ scheme for the time direction.

Example 5.1. *Assume that the domain* $\Omega = [0,1]^2$*, and the initial time* $T = 1$*; we consider the following space-time fractional parabolic problem*

$$\partial_t^\alpha \mathfrak{u} + (-\Delta)^s \mathfrak{u} = f(x_1, x_2, t), \quad x \in \Omega, \quad t \in (0,1]. \tag{5.1}$$

Let $\lambda_{m,n}$ *and* $\varphi_{m,n}$ *be the eigenvalues and eigenfunctions of the 2 D Laplacian with homogeneous Dirichlet boundary conditions. If* $f \in L^2(0,T;H^{-s}(\Omega))$ *and* $\mathfrak{u}_0 \in L^2(\Omega)$*, we have*

$$\mathfrak{u}(x_1, x_2, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathfrak{u}_{m,n}(t)\varphi_{m,n}.$$

Since $\mathfrak{u}(x,0) = \mathfrak{u}_0$*, yields the fractional initial value problem for* $\mathfrak{u}_{m,n}$

$$\partial_t^\alpha \mathfrak{u}_{m,n}(t) + \lambda_{m,n}^s \mathfrak{u}_{m,n}(t) = f_{m,n}(t), \quad \mathfrak{u}_{m,n}(0) = \mathfrak{u}_{0,m,n},$$

with $\mathfrak{u}_{0,m,n} = (\mathfrak{u}_0, \varphi_{m,n})$ *and* $f_{m,n} = (f, \varphi_{m,n})$*. For the solution of the problem* (1.3)*, we can be written as*

$$u(x_1, x_2, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathfrak{u}_{m,n}(t)\varphi_{m,n}(x)\psi_{m,n}(y),$$

where

$$\psi_{m,n}(y) = \frac{2^{1-s}}{\Gamma(s)}(\sqrt{\lambda_{m,n}}y)^s K_s(\sqrt{\lambda_{m,n}}y).$$

Here $K_s(\cdot)$ *denotes the modified Bessel function of the second kind [16]. Once u in the problem* (1.3) *is solved, we can obtain the solution to problem* (1.1) *as*

$$\mathfrak{u}(x_1, x_2, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathfrak{u}_{m,n}(t)\varphi_{m,n}(x)\psi_{m,n}(0) = u(x_1, x_2, 0, t).$$

In our example, we take the following configuration from [16, Section 6.1], let the eigenvalues and eigenfunctions of Laplace operator $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ *be*

$$\lambda_{m,n} = \pi^2(m^2 + n^2), \quad \varphi_{m,n}(x_1, x_2) = \sin(m\pi x_1)\sin(n\pi x_2), \quad m, n \in \mathbb{N}.$$

If we take $\mathfrak{u}(x_1, x_2, t) = t^\alpha \lambda_{2,2}^{-s}\sin(2\pi x_1)\sin(2\pi x_2)$ *in problem* (1.1)*, we have*

$$u(x_1, x_2, y, t) = t^\alpha \frac{2^{1-s}}{\Gamma(s)}\lambda_{2,2}^{-s/2}\sin(2\pi x_1)\sin(2\pi x_2)y^s K_s(\sqrt{\lambda_{2,2}}y)$$

in extend problem (1.3)*. The right-hand sides* f *in* (1.1) *is determined from the choice for* $\mathfrak{u}$*. That is*

$$f = \Gamma(1+\alpha)\lambda_{2,2}^{-s}\sin(2\pi x_1)\sin(2\pi x_2) + t^\alpha \sin(2\pi x_1)\sin(2\pi x_2).$$

To balance the approximation and truncation errors, we choose the truncation parameter as

$$Y = 1 + \frac{1}{3}log(\#\mathcal{T}_\Omega), \quad \text{and} \quad \mu = \begin{cases} 1, & s = 1/2, \\ 3/(2 * s) + 0.01, & \text{otherwise.} \end{cases}$$

This detail can be found in [17]. By the above techniques and taking $s = \alpha = 0.5$, the time step $N = 100$, we show in Tables 1 and 2 that the attenuation value and the truncation error of the extended direction are very small, which indicates that truncating the expansion direction is reasonable.

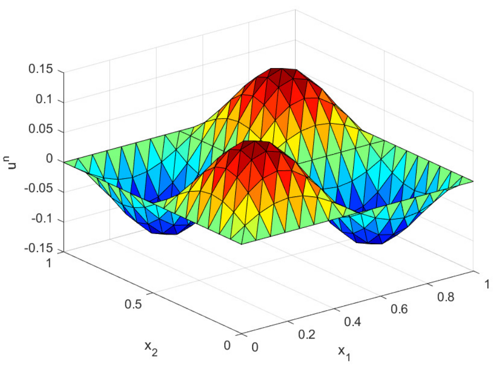
Table 1. The different attenuation values of the extend direction when $s = \alpha = 0.5$.

h	$\#\mathcal{T}_\Omega$	Y	$\max\|u(x_1, x_2, Y, t)\|$
1/4	25	2.7295	1.1264e-09
1/8	81	2.4648	3.4632e-11
1/16	289	2.8889	8.0035e-13
1/32	1089	3.3310	1.5734e-14

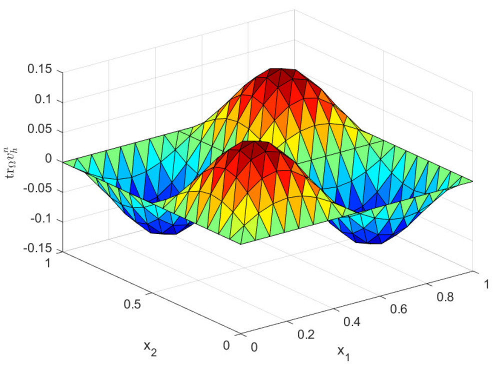
Table 2. The truncation error of the extended direction when $s = \alpha = 0.5$.

h	$Y \to 0$	$\max\|\mathfrak{u}(x_1, x_2, t) - u(x_1, x_2, Y, t)\|$
1/16	1e-03	9.9557e-04
1/16	1e-06	9.9999e-07
1/16	1e-09	9.9999e-10
1/16	1e-12	9.9998e-13

It can be seen from Figure 1 that the graphs of the numerical solution $\mathrm{tr}_\Omega v_h^n$ and the exact solution $\mathfrak{u}^n$ are consistent, which indicates that our numerical scheme is feasible. For simplicity, we use "L^2-error" to represent "$\max\limits_{1\le n\le N} \|\mathfrak{u}^n - \mathrm{tr}_\Omega v_h^n\|_{L^2(\Omega)}$" and "$H^s$-error" to represent "$\max\limits_{1\le n\le N} \|\mathfrak{u}^n - \mathrm{tr}_\Omega v_h^n\|_{H^s(\Omega)}$", respectively. We define "Rate 1" and "Rate 2" as the L^2- error convergence rate and H^s-error convergence rate, respectively. Fixed the time step $N = 100$ and $s = 0.5$, taking the mesh size $h = 1/4, 1/8, 1/16, 1/32$, the L^2-error and H^s-error are given in the Table 3, and the spatial convergence orders are shown in Figure 2 and Table 4. It can be seen from them that the L^2-error convergence order is approximately $(\#\mathcal{T}_Y)^{-2/3}$ and the H^s-error convergence order is approximately $(\#\mathcal{T}_Y)^{-1/3}$, which is consistent with the results of theoretical in Remark 4.1.



(a) The exact solution $\mathfrak{u}^n$.

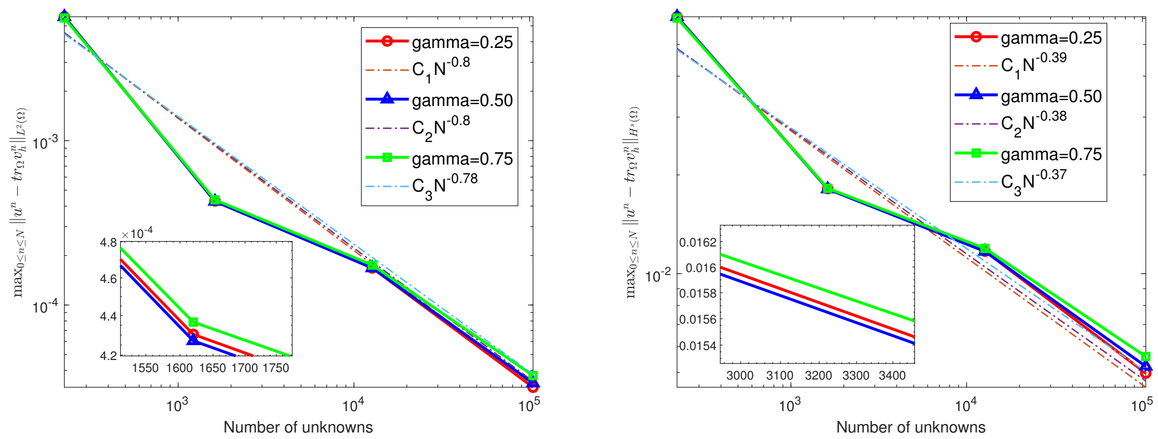


(b) The FEM solution $\mathrm{tr}_\Omega v_h^n$.

Figure 1. The exact solution, FEM solution for $s = \alpha = 0.50$.

Table 3. The error estimates of space directions with $N = 100$ and $s = 0.5$.

$\#\mathcal{T}_Y$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
	L^2-error	H^s-error	L^2-error	H^s-error	L^2-error	H^s-error
225	5.669e-03	6.067e-02	5.677e-03	6.069e-02	5.579e-03	6.014e-02
1620	4.304e-04	1.818e-02	4.271e-04	1.811e-02	4.366e-04	1.820e-02
12716	1.682e-04	1.171e-02	1.682e-04	1.169e-02	1.750e-04	1.194e-02
104544	3.161e-05	4.975e-03	3.351e-05	5.217e-03	3.723e-05	5.590e-03



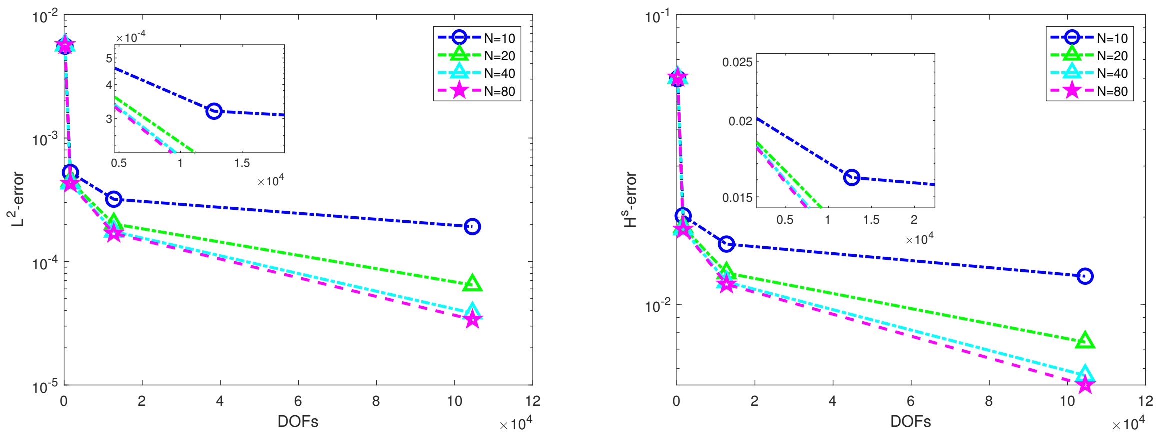
(a) The L^2-error convergence order.
(b) The H^s-error convergence order.

Figure 2. The space convergence order of FEM solution.

Table 4. The convergence rates of space directions with $N = 100$ and $s = 0.5$.

$\#\mathcal{T}_Y$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
	Rate 1	Rate 2	Rate 1	Rate 2	Rate 1	Rate 2
225	–	–	–	–	–	–
1620	1.3060	0.6104	1.3105	0.6126	1.2906	0.6055
12716	0.4561	0.2134	0.4524	0.2123	0.4436	0.2045
104544	0.7934	0.4065	0.7657	0.3831	0.7347	0.3602
Expected result	2/3	1/3	2/3	1/3	2/3	1/3

Meanwhile, the unconditional convergence of the numerical scheme can be confirmed by taking the time step $N = 10, 20, 40, 80$ in Figure 3, we can know that the errors tend to be constant. The numerical results imply that the error estimates hold without certain time-step restrictions dependent on the spatial mesh sizes. Fixed the space mesh size $h = 1/36$, and the corresponding total degrees of freedom are $\#\mathcal{T}_Y = 150590$, Table 5 presents the L^2-errors and error convergence orders under different time step sizes. It can be clearly seen that the time error order is close to 2, which is consistent with our theoretical analysis.



(a) The L^2-error convergence order. (b) The H^s-error convergence order.

Figure 3. The convergence order of FEM solution with $\alpha = s = 0.5$.

Table 5. The error estimates and convergence of time directions with $\#\mathcal{T}_Y = 150590$ and $s = 0.5$.

N	$\alpha = 0.25$ L^2-error	Order	$\alpha = 0.50$ L^2-error	Rate	$\alpha = 0.75$ L^2-error	Order
4	1.997e-03	–	1.900e-03	–	9.715e-04	–
8	7.138e-04	1.484	5.379e-04	1.821	2.379e-04	1.970
16	1.880e-04	1.925	1.260e-04	2.093	5.982e-05	2.051
32	3.068e-05	2.615	2.700e-05	2.223	1.560e-05	1.939
Expected result	–	2	–	2	–	2

6. Conclusions

We develop fast high-order PDE techniques for solving the space-time parabolic problems with the fractional Laplacian. By transforming the original problem into an equivalent extended problem, we then construct a fast L2-1_σ scheme for the time direction and adopt the tensor finite element method for the spatial direction, thereby obtaining a fully discrete numerical scheme. Subsequently, we analyze the stability and error estimates of the numerical scheme. Finally, a numerical example is used to verify the efficiency of the numerical scheme and the correctness of the theoretical analysis. In the future, we will further apply this technique to solve the nonlinear or variable-order fractional PDEs and consider the a posteriori error estimation for the space fractional diffusion equations.

Author contributions

Qingfeng Li: Funding acquisition, conceptualization, writing–original draft, software; Jia Xie: Methodology, supervision, investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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