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Research article

Existence of ground state solutions for the general Choquard equation with Riesz fractional derivative operator

Sarah Abdullah Qadha[1,2,*], Muneera Abdullah Qadha[1,2], Haibo Chen[1], Mohamed Abdalla[3] and Mohammed Z. Alqarni[4]

[1] School of Mathematics and Statistics, Central South University, Changsha 410083, China
[2] Department of Mathematics, Faculty of Education at Al-Mahweet, Sana'a University, Al-Mahweet, Yemen
[3] Mathematics Department, Faculty of Science South Valley University, Qena 83523, Egypt
[4] Mathematics Department, Faculty of Science King Khalid University, Abha, Saudi Arabia

* **Correspondence:** Email: sarah22258@yahoo.com, sarah-qadha@csu.edu.cn.

Abstract: This work examines the existence of a ground state solution for the following general nonlinear Choquard equation:

$$-\Delta u + u = \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u)\right) q(u), \ \ in\ \mathbb{R}^N,$$

where $N \geq 3$, $\mathcal{Q}$ is the primitive function of q, $\mathcal{Q} \in C^1(\mathbb{R};\mathbb{R})$ fulfils the general Berestycki–Lions conditions, and $(-\Delta)_E^{\frac{-\alpha}{2}}$ is the equivalent Riesz fractional operator of order $\alpha \in (0,2)$. In this case, the Riesz potential has not previously been investigated. The existence of a solution is established through the application of variational techniques. This modification not only expands the theoretical understanding of such equations but also opens up new avenues for practical applications, particularly in fields such as quantum mechanics and astrophysics.

Keywords: existence; general Choquard equation; variational methods; ground state solution; equivalent Riesz Potential
Mathematics Subject Classification: 35J60, 35J20

1. Introduction and Preliminaries

The Choquard equation (CE), initially presented by Choquard in 1976, originally described the behaviour of a single electron interacting with its electrostatic potential in a neutralising background. CE appears in various fields of physics, such as self-gravitating matter [1], modelling of one-component plasma [2] and quantum mechanics [3]. Recently, CE has garnered significant academic interest owing to its emergence in multiple physical contexts. Now, we consider the equation

$$-\Delta u + u = \big(I_\alpha * \mathcal{Q}(\mathrm{u})\big)q(u), \quad in\ \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3, \alpha \in (0, N)$, $q := \acute{\mathcal{Q}}, \mathcal{Q} \in C^1(\mathbb{R}; \mathbb{R})$, and I_α represents the Riesz potential, defined for all $x \in \mathbb{R}^N \backslash \{0\}$ as

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}|x|^{N-\alpha}}.$$

Equation (1.1) represents a semilinear elliptic problem with a nonlocal nonlinearity. The Choquard–Pekar equation receives considerable attention in specific cases in which $\alpha = 2$, $N = 3$ and $\mathcal{Q}(\zeta) = \frac{\zeta^2}{2}$ [2,3]. It is also referred to as the Newton–Schrödinger equation or the stationary Hartree equation [1]. In this context, Lieb, Lions and Menzala demonstrated the existence of solutions using variational methods [2,4,5], as well as through techniques based on ordinary differential equations [1,6,7]. In a more general case in which $\mathcal{Q}(\zeta) = \frac{\zeta^p}{p}$, Eq (1.1) has a solution if and only if $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ ([8], p. 457; [9], Theorem 1; [10], Lemma 2.7). Until now, existence results have only been available in the following conditions:

- When the Riesz potential

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}|x|^{N-\alpha}},$$

  considerable studies can be referred to [11–14].
- When the Riesz potential $I_\alpha = ln\frac{1}{|x|}$, the authors, via asymptotic approximation, investigated the positive solutions of the planar logarithmic CE [15].
- In the case in which $I_\alpha = \frac{1}{|x|^\zeta}$, with $0 < \zeta < min\{N, 4s\}, 0 < s < 1$, the authors investigated ground state solutions (GSSs) of Pohožaev type for fractional CEs with general nonlinearities by combining the deformation lemma with the constrained variational method [16].
- When $I_\alpha = \frac{1}{|x-y|^\zeta}$, with $0 < \zeta < min\{N, 4\}, N \geq 3$, the author investigated the GSS for a class of CEs involving a general critical growth term whilst applying the Pohožaev constraint [17].

In this work, we study a new case for the Riesz potential to examine the existence of GSS. This modification not only expands the theoretical understanding of such equations but also opens up new avenues for practical applications, particularly in fields such as quantum mechanics and astrophysics.

This work examines the existence of GSS for the following general nonlinear CE (GNCE):

$$-\Delta u + u = \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(\mathrm{u})\right) q(u), \quad in\ \mathbb{R}^N, \tag{1.2}$$

Where $N \geq 3$, $\mathcal{Q}$ represents the primitive function of q, $\mathcal{Q} \in C^1(\mathbb{R};\mathbb{R})$ satisfies the general Berestycki–Lions conditions [18], and $(-\Delta)_E^{\frac{-\alpha}{2}}$ represents the equivalent Riesz fractional operator of order $\alpha \in (0,2)$ defined by

$$(-\Delta)_E^{\frac{-\alpha}{2}} u = -\frac{1}{\left|\Gamma\left(\frac{\alpha}{2}\right)\right|} \int_0^\infty (e^{t\Delta}u - u)\, t^{-1+\frac{\alpha}{2}} dt.$$

In a more general case in which $\mathcal{Q}(s) = \frac{s^p}{p}$, Eq (1.2) has a solution if and only if $\frac{N+\alpha}{N} < p < \frac{N-\alpha}{N-2}$. In this context, we demonstrate the existence of solutions to Eq (1.2) under the assumption that the nonlinearity $q \in C(\mathbb{R};\mathbb{R})$ satisfies the required growth condition:

(H_1) A constant, $c > 0$, exists. Accordingly, the inequality $|sq(s)| \leq c\left(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}\right)$ holds for all $s \in \mathbb{R}$.

(H_2) The antiderivative that for all $s \in \mathbb{R}$, $\mathcal{Q}: s \to \int_0^s q(\mathfrak{d})\, d\mathfrak{d}$ satisfies subcritical properties, evidenced by

$$\lim_{s\to 0} \frac{\mathcal{Q}(s)}{|s|^{\frac{N+\alpha}{N}}} = 0, \qquad \lim_{|s|\to\infty} \frac{\mathcal{Q}(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0,$$

(H_3) A nontrivial condition is met with the existence of $s_0 \in \mathbb{R}\backslash\{0\}$. Consequently, $\mathcal{Q}(s_0) \neq 0$.

The innovative aspects of this study can be summarised as follows: Previous research defined the Riesz potential

$$I_\alpha(x) = (-\Delta)^{\frac{-\alpha}{2}} = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{N}{2}} |x|^{N-\alpha}}.$$

In our investigation, we replace I_α with $(-\Delta)_E^{\frac{-\alpha}{2}}$, where $(-\Delta)_E^{\frac{-\alpha}{2}}$ represents the equivalent Riesz fractional operator

$$I_\alpha = (-\Delta)_E^{\frac{-\alpha}{2}} u = -\frac{1}{\left|\Gamma\left(\frac{\alpha}{2}\right)\right|} \int_0^\infty (e^{t\Delta}u - u)\, t^{-1+\frac{\alpha}{2}} dt.$$

This substitution leads to challenges in establishing the necessary conditions for the validity of the Pohožaev identity. Our principal contribution does not entail the development of a new method. Instead, our research focuses on demonstrating the existence of a GSS under specific conditions when the Riesz potential is altering. Our work is significant in providing valuable insights into ground state phenomena within this context.

The purpose of this article is to address a gap in the existing literature. Previous studies have not examined this particular class of nonlinearities in the context of the Riesz potential; consequently, the equation introduced here is novel. It has potential applications across a wide range of phenomena involving long-range interactions, including water waves, dislocations in crystals, anomalous diffusion, and non-local quantum theories. Moreover, the framework developed in this work opens new avenues for future research into related properties across diverse branches of mathematics, such as partial differential equations, potential theory, harmonic analysis, semigroup theory, function spaces, and probability theory.

The paper is organised as follows. In the first section, we present essential theorems and definitions. The second section is devoted to demonstrating that the functional $\mathcal{J}(u)$ exhibits a mountain pass geometry, which is then used to derive a corresponding Palais–Smale sequence. In the third section, we establish the necessary conditions for the existence of GSS, followed by the sufficient conditions ensuring its existence.

In this section, we present some fundamental symbols which we will use. $H^1(\mathbb{R}^N)$ is the usual Sobolev space. For all $u, v \in H^1(\mathbb{R}^N)$, the inner product within $H^1(\mathbb{R}^N)$ is defined as

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)dx,$$

whilst the norm within $H^1(\mathbb{R}^N)$ is represented as

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2).$$

Theorem 1.1. (Hardy–Littlewood–Sobolev inequality (HLSI)) Suppose $s \in \left(1, \frac{N}{\alpha}\right)$, then for every $\nu \in L^s(\mathbb{R}^N)$, $\left((-\Delta)_E^{\frac{-\alpha}{2}} * \nu\right) \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} \left|(-\Delta)_E^{\frac{-\alpha}{2}} * \nu\right|^{\frac{Ns}{N-\alpha s}} \leq G\left(\int_{\mathbb{R}^N} |\nu|^s\right)^{\frac{N}{N-\alpha s}}, \tag{1.3}$$

where $G > 0$ is a constant that depends only on α, N and ζ. This result is obtained by combining the HLSI with the weak Young inequality, as stated in [19].

Lemma 1.1. For every $u \in H^1(\mathbb{R}^N)$, by the upper bound *(H_1)* on $\mathcal{Q}$, the HLSI and Sobolev embedding theorem

$$\int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u)\right)\mathcal{Q}(u) \leq C_1 \left(\int_{\mathbb{R}^N} |\mathcal{Q}(u)|^{\frac{2N}{N+\alpha}}\right)^{1+\frac{\alpha}{N}} \leq C_2 \left(\int_{\mathbb{R}^N} |u|^2 + |u|^{\frac{2N}{N-2}}\right)^{1+\frac{\alpha}{N}}$$
$$\leq C_3 \left(\int_{\mathbb{R}^N} |u|^2\right)^{1+\frac{\alpha}{N}} + \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1+\frac{\alpha+2}{N-2}}.$$

Hence, $\xi > 0$, such that if

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \leq \xi,$$

then

$$\int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u)\right)\mathcal{Q}(u) \leq \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2.$$

Proposition 1.1. ([14], Proposition 3.1). Let $q \in C(\mathbb{R};\mathbb{R})$ fulfil condition *(H_1)*, and suppose $u \in H^1(\mathbb{R}^N)$ represent a solution of Eq (1.2). Then, $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ for all $p \geq 1$.

Lemma 12.2. ([9]**,** Lemma 2.3). Let $q \in [1,\infty)$. If $\frac{1}{2}-\frac{1}{N} \leq \frac{1}{q} \leq \frac{1}{2}$, then for all $u \in W^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |u|^q \leq C\left(\sup_{x\in\mathbb{R}^N}\int_{B_1(x)} |u|^q\right)^{1-\frac{2}{q}}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\right).$$

Definition 1.1. (Equivalent Riesz fractional operator, [20]). Let $\mathfrak{I}$ be any of the spaces L^p, $p \in (0,\infty)$. Let $u \in \mathfrak{I}$ and $\alpha \in (0,2)$, then

$$(-\Delta)_E^{\frac{-\alpha}{2}} u = -\frac{1}{\left|\Gamma\left(\frac{\alpha}{2}\right)\right|}\int_0^\infty (e^{t\Delta}u - u)\, t^{-1+\frac{\alpha}{2}} dt,$$

with Bochner's integral of an $\mathfrak{I}$-valued function.

2. Main results

The energy functional associated with Eq (1.2) is given by

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2) - \frac{1}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(u)\right)Q(u)dx. \tag{2.1}$$

The derivative of the energy functional $\mathcal{J}(u)$ is given by

$$\langle \acute{\mathcal{J}}(u), u\rangle = \int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2) - \int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(u)\right)q(u)u dx. \tag{2.2}$$

We refer to any weak solution u of Eq (1.2), belonging to the Sobolev space $H^1(\mathbb{R}^N)\backslash\{0\}$, as a GSS of Eq (1.2) if it minimises the functional $\mathcal{J}$ amongst all nonzero solutions.

Hence, we can define the mountain pass level of $\mathcal{J}(u)$

$$c = \inf_{\gamma\in\Gamma}\max_{\tau\in[0,1]} \mathcal{J}(\gamma(\tau)) > 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \mathcal{J}(\gamma(1)) < 0\}.$$

Now, we recall the Nehari manifold

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \langle \acute{\mathcal{J}}(u), u\rangle = 0\}.$$

Let $c_0 = \inf_{u\in\mathcal{N}} \mathcal{J}(u)$. Moreover, by adopting an argument similar to that presented in Chapter 4 of [21], we establish the following result:

$$c = \inf_{\gamma\in\Gamma}\max_{\tau\in[0,1]} \mathcal{J}(\gamma(\tau)) = c_0 = \inf_{u\in\mathcal{N}} \mathcal{J}(u) = c^* = \inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}}\max_{\tau\geq 0} \mathcal{J}(u_\tau).$$

In the following step, we assume that the functional $\mathcal{J}(u)$ possesses mountain pass geometry to construct a Palais–Smale sequence for it.

Theorem 2.1. Suppose that the functional $\mathcal{J}(u)$ possesses a mountain pass. Then, $\mathcal{J}(u)$ must fulfil the following requirements:
(1) $\sigma, \mathscr{b} > 0$ exists such that $\mathcal{J}(u)|_{\beta_\sigma} \geq \mathscr{b} > 0,$ for all $u \in \beta_\sigma = \{u \in H^1(\mathbb{R}^N): \|u\| = \sigma\}$;
(2) for any $u \in H^1(\mathbb{R}^N)\backslash\{0\},$ $\tau \in (0, \infty)$ exists such that $\|u_\tau\| > \sigma$ and $\mathcal{J}(u_\tau) < 0,$ where σ is given in (1).

Proof. (1) By the definition of norm, the HLSI in Theorem 1.1, the upper bound (H_1) on Q and Lemma 1.1, we obtain

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2) - \frac{1}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(u)\right)Q(u)dx \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2}C_1\left(\int_{\mathbb{R}^N}|Q(u)|^{\frac{2N}{N+\alpha}}\right)^{1+\frac{\alpha}{N}} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\frac{1}{4}\int_{\mathbb{R}^N}|\nabla u|^2 + |u|^2\right) \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{8}\|u\|^2 \\ &\geq \frac{3}{8}\sigma,\end{aligned}$$

then $\mathcal{J}(u) \geq \mathscr{b} > 0$ for all $\|u\| = \sigma$ small enough.

(2) For any nonzero $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and any positive value of $\tau,$ we consider a function u within a family of functions $u_\tau \in H^1(\mathbb{R}^N),$ defined by

$$u_\tau = v(\tfrac{x}{\tau}), \text{ for all } x \in \mathbb{R}^N.$$

For this family, the following scaling identities hold for each $\tau > 0$:

$$\int_{\mathbb{R}^N}|\nabla u_\tau|^2 = \tau^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2.$$

$$\int_{\mathbb{R}^N}|u_\tau|^2 = \tau^N\int_{\mathbb{R}^N}|v|^2.$$

A comprehensive derivation of these results is provided in [14], specifically in Proposition 2.1. The last term can be reformulated as

$$\begin{aligned}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(u_\tau)\right)Q(u_\tau) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}Q(u_\tau(x))\,Q(u_\tau(y))(-\Delta)_E^{\frac{-\alpha}{2}}u_\tau(x-y)dxdy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}Q(v(\tfrac{x}{\tau}))\,Q(v(\tfrac{y}{\tau}))\left(e^{t\Delta}v\left(\frac{x}{\tau}-\frac{y}{\tau}\right) - v\left(\frac{x}{\tau}-\frac{y}{\tau}\right)\right)(t)^{-1+\frac{\alpha}{2}}dxdy.\end{aligned}$$

Let $z = \frac{x}{\tau}$ and $w = \frac{y}{\tau}$. Then

$$\begin{aligned}&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}Q(v(z))\,Q(v(w))(e^{t\Delta}v(z-w) - v(z-w))(t)^{-1+\frac{\alpha}{2}}\tau^{2N}dzdw \\ &= \tau^{2N}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(v)\right)Q(v).\end{aligned}$$

Thus,

$$\mathcal{J}(u_\tau) = \frac{\tau^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{\tau^N}{2}\int_{\mathbb{R}^N}|v|^2 - \frac{\tau^{2N}}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(v)\right)\mathcal{Q}(v).$$

For $\tau > 0$ large enough, $\lim_{\tau\to+\infty} \mathcal{J}(u_\tau) < 0$.

Theorem 2.2. Given $q \in C(\mathbb{R};\mathbb{R})$ satisfying condition (H_1) and $u \in H^1(\mathbb{R}^N)$ that solves Eq (1.2), the following Pohožaev identities $\mathcal{P}$ hold:

$$\mathcal{P}(u) = \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N}|u|^2 - \frac{3N}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u)\right)\mathcal{Q}(u). \tag{2.3}$$

Proof. According to Proposition 1.1, $u \in W_{loc}^{2,2}(\mathbb{R}^N)$. Select $\mathfrak{D} \in C_c^1(\mathbb{R}^N)$ such that $\mathfrak{D} = 1$ in the neighbourhood around 0. For $\eta \in (0,\infty)$ and $x \in \mathbb{R}^N$, define the function $\delta_\eta \in W^{1,2}(\mathbb{R}^N)$ as

$$\delta_\eta = \mathfrak{D}(\eta x)x \cdot \nabla u(x).$$

Utilising δ_η as the test function in the equation yields

$$\int_{\mathbb{R}^N}\nabla u.\nabla\delta_\eta + \int_{\mathbb{R}^N}u\delta_\eta = \int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u)\right)\left(q(u)\delta_\eta\right).$$

The evaluation of the left-hand side can be realised through integration by parts for each $\eta > 0$ and by invoking Lebesgue's dominated convergence. Given that $u \in W^{1,2}(\mathbb{R}^N)$, the following limits are established:

$$\lim_{\eta\to 0}\int_{\mathbb{R}^N}\nabla u.\nabla\,\delta_\eta = -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2.$$

$$\lim_{\eta\to 0}\int_{\mathbb{R}^N}u\delta_\eta = -\frac{N}{2}\int_{\mathbb{R}^N}|u|^2.$$

A detailed exposition is shown in [22], proof of Proposition 11; [9], proof of Proposition 3.1; and [14], Proposition 3.5. In every $\eta > 0$, the last term can be reformulated via integration by parts as follows:

$$\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u)\right)\left(q(u)\delta_\eta\right) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathcal{Q}ou)(y)(-\Delta)_E^{\frac{-\alpha}{2}}(x-y)\,\mathfrak{D}(\eta x)x.\nabla(\mathcal{Q}ou)(x)dxdy$$

$$= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(-\Delta)_E^{\frac{-\alpha}{2}}(x-y)\left((\mathcal{Q}ou)(y)\mathfrak{D}(\eta x)x.\nabla(\mathcal{Q}ou)(x)\right)dxdy$$

$$+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(-\Delta)_E^{\frac{-\alpha}{2}}(x-y)\left((\mathcal{Q}ou)(x)\mathfrak{D}(\eta y)y.\nabla(\mathcal{Q}ou)(y)\right)dxdy$$

$$= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathcal{Q}ou)(y)(-\Delta)_E^{\frac{-\alpha}{2}}(x-y)\left(N\mathfrak{D}(\eta x) + \eta x\nabla\mathfrak{D}(\eta x)\right)(\mathcal{Q}ou)(x)dxdy$$

$$-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathcal{Q}ou)(y)(-\Delta)_E^{\frac{-\alpha}{2}}(x-y)\left(x\mathfrak{D}(\eta x) - y\mathfrak{D}(\eta y)\right)(\mathcal{Q}ou)(x)dxdy$$

We can therefore utilise Lebesgue's dominated convergence to infer that

$$\lim_{\eta\to 0}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(u)\right)\left(q(u)\delta_\eta\right) = -\frac{3N}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(u)\right)Q(u).$$

Theorem 2.3. Provided that $q \in C(\mathbb{R};\mathbb{R})$ fulfils conditions (H_1) and (H_3), there exists a sequence u_n in $H^1(\mathbb{R}^N)$ such that as $n\to\infty$,

$$\mathcal{J}(u_n)\to c>0, \acute{\mathcal{J}}(u_n)\to 0, \mathcal{P}(u_n)\to 0.$$

Proof. To establish a Pohožaev–Palais–Smale sequence, with reference to Jeanjean ([23], Section 2), we define the mapping

$$\psi\colon \mathbb{R}\times H^1(\mathbb{R}^N)\to H^1(\mathbb{R}^N),$$

for $\lambda\in\mathbb{R}$, $\kappa\in H^1(\mathbb{R}^N)$ and $x\in\mathbb{R}^N$ by

$$\psi(\lambda,\kappa)(x)=\kappa\left(e^{-\lambda}x\right).$$

Then, the functional $\mathcal{J} o\, \psi$ is computed as follows:

$$\mathcal{J}\left(\psi(\lambda,\kappa)\right)=\frac{e^{(N-2)\lambda}}{2}\int_{\mathbb{R}^N}|\nabla\kappa|^2+\frac{e^{N\lambda}}{2}\int_{\mathbb{R}^N}|\kappa|^2-\frac{e^{2N\lambda}}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * Q(\kappa)\right)Q(\kappa).$$

Given hypothesis (H_1), the composition $\mathcal{J} o\, \psi$ is continuously Fréchet differentiable on $\mathbb{R}\times H^1(\mathbb{R}^N)$. We denote the associated family of paths as

$$\tilde{\Gamma}=\left\{\tilde{\gamma}\in C\left([0,1];\mathbb{R}\times H^1(\mathbb{R}^N)\right)\colon \tilde{\gamma}(0)=(0,0)\ and\ \mathcal{J} o\, \psi(\tilde{\gamma}(1)<0\right\}.$$

The mountain pass levels of $\mathcal{J}$ and $\mathcal{J} o\, \psi$ coincide as $\Gamma=\left\{\psi o\tilde{\gamma}\colon \tilde{\gamma}\in\tilde{\Gamma}\right\}$:

$$c=\inf_{\tilde{\gamma}\in\tilde{\Gamma}}\ \sup_{t\in[0,1]}\ (\mathcal{J} o\, \psi)(\tilde{\gamma}(t)).$$

According to a minimax principle ([21], Theorem 2.9), one can identify a sequence $(\lambda_n,\kappa_n)_{n\in N}$ within $\mathbb{R}\times H^1(\mathbb{R}^N)$, such that as $n\to\infty$, the following holds:

$$(\mathcal{J} o\, \psi)(\lambda_n,\kappa_n)\to c,$$

$$(\mathcal{J} o\, \psi)'(\lambda_n,\kappa_n)\to 0\ \text{ in }\ (\mathbb{R}\times H^1(\mathbb{R}^N)^*.$$

Given $u_n=\psi(\lambda_n,\kappa_n)$, then as $n\to\infty$,

$$\mathcal{J}(u_n)\to c>0,\ \acute{\mathcal{J}}(u_n)\to 0, \mathcal{P}(u_n)\to 0. \tag{2.4}$$

Theorem 2.4. Assume that u_n is a sequence in $H^1(\mathbb{R}^N)$, $q\in C(\mathbb{R};\mathbb{R})$ fulfils (H_1) and (H_2), and $\mathcal{J}(u_n)$ is bounded. As $n\to\infty$,

$$\acute{\mathcal{J}}(u_n)\xrightarrow{strongly}0,\ \text{ in }\ \left(H^1(\mathbb{R}^N)\right)',\ \mathcal{P}(u_n)\to 0.$$

Then, either

(1) $u_n\xrightarrow{strongly}0$ in $H^1(\mathbb{R}^N)$, or

(2) there exists a function $u\in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $\acute{\mathcal{J}}(u_n)=0$ and the sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$ of points within $\mathbb{R}^N$ such that, up to the subsequence, $u_n(\cdot-\mathcal{O}_n)\xrightarrow{weakly}u$ within $H^1(\mathbb{R}^N)$ as $n\to\infty$.

Proof. Suppose that the initial component of the alternative condition is not satisfied, namely,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 > 0. \tag{2.5}$$

Proof of claim (1). Now, we prove the boundedness of the sequence u_n in $H^1(\mathbb{R}^N)$. For all $n \in \mathbb{N}$,

$$\frac{2N+2}{6N}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{3}\int_{\mathbb{R}^N} |u|^2 = \mathcal{J}(u_n) - \frac{1}{3N}\mathcal{P}(u_n).$$

Given that our assumptions bound the right-hand side, the sequence u_n is bounded within $H^1(\mathbb{R}^N)$.

Proof of claim (2). Now, we prove the nonvanishing of the sequence. For $2 < p < \frac{2N}{N-2}$,

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^p > 0.$$

Through Eqs (2.3) and (2.5), we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n) \right) \mathcal{Q}(u_n) = \liminf_{n\to\infty} \frac{N-2}{3N} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{3}\int_{\mathbb{R}^N} |u|^2 - \frac{2}{3N}\mathcal{P}(u_n) > 0. \tag{2.6}$$

For any $n \in \mathbb{N}$, u_n fulfils the following inequality ([4], Lemma 1.1; [9], Lemma 2.3; [21], Lemma 1.21)

$$\int_{\mathbb{R}^N} |u_n|^2 \le C\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 \right) \left(\sup_{\mathcal{O}\in\mathbb{R}^N} \int_{B_1(\mathcal{O})} |u_n|^p \right)^{1-\frac{2}{p}}.$$

Because $\mathcal{Q}$ is continuous and fulfils (H_1), for each $\varepsilon > 0$, there exists a constant C_ε such that for each $s \in \mathbb{R}$,

$$|\mathcal{Q}(s)|^{\frac{2N}{N+\alpha}} \le \varepsilon\left(|s|^2 + |s|^{\frac{2N}{N-2}} \right) + C_\varepsilon |s|^p.$$

Given that u_n is bounded within $H^1(\mathbb{R}^N)$ and, via the Sobolev embedding, within $L^{\frac{2N}{N-2}}$, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\mathcal{Q}(u_n)|^{\frac{2N}{N+\alpha}} \le C_1\varepsilon + C_\varepsilon \left(\liminf_{n\to\infty} \sup_{\mathcal{O}\in\mathbb{R}^N} \int_{B_1(\mathcal{O})} |u_n|^p \right)^{1-\frac{2}{p}}.$$

Because $\varepsilon > 0$ is arbitrary, if $\liminf_{n\to\infty} \sup_{\mathcal{O}\in\mathbb{R}^N} \int_{B_1(\mathcal{O})} |u_n|^p = 0$, then

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\mathcal{Q}(u_n)|^{\frac{2N}{N+\alpha}} = 0.$$

Furthermore, the HLSI indicates that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n) \right) \mathcal{Q}(u_n) = 0,$$

which is inconsistent with Eq (2.6).

Theorem 2.5. If $(H_1)-(H_3)$ hold, then there is at least one nontrivial solution to Eq (1.2).

Proof. Suppose that for some $2 < p < \frac{2N}{N-2}$,

$$\lim_{n\to\infty} inf \int_{B_1} |u_n|^p > 0.$$

According to Rellich's theorem, up to a subsequence,

$$u_n \xrightarrow{weakly} u, \text{ in } H^1(\mathbb{R}^N).$$

As the sequence u_n is bounded within $H^1(\mathbb{R}^N)$, it is also bounded within $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ by a Sobolev embedding. Therefore, by (H_1), the sequence $(Q \circ u_n)$ is bounded within $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Because

$$u_n \xrightarrow{weakly} u, \text{ in } H^1(\mathbb{R}^N),$$

$$u_n \xrightarrow{weakly} u, \text{ in a.e. } \mathbb{R}^N.$$

According to Q's continuity,

$$(Q \circ u_n) \to (Q \circ u), \text{in a.e.}\, \mathbb{R}^N.$$

Hence,

$$(Q \circ u_n) \xrightarrow{weakly} (Q \circ u), \text{ within } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

The Riesz potential is a linear and continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$,

$$\left((-\Delta)_E^{\frac{-\alpha}{2}} * (Q \circ u_n)\right) \xrightarrow{weakly} \left((-\Delta)_E^{\frac{-\alpha}{2}} * (Q \circ u)\right), \text{ in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Conversely, by taking assumption *(*H_1*)* into account and applying Rellich's theorem, we obtain

$$(q \circ u_n) \xrightarrow{strongly} (q \circ u), \text{ in } L^p_{loc}(\mathbb{R}^N),$$

for all $p \in \left[1, \frac{2N}{\alpha+2}\right)$. We conclude that

$$\left((-\Delta)_E^{\frac{-\alpha}{2}} * (Q \circ u_n)\right)(q \circ u_n) \xrightarrow{weakly} \left((-\Delta)_E^{\frac{-\alpha}{2}} * (Q \circ u)\right)(q \circ u) \text{ in } L^p(\mathbb{R}^N),$$

for all $p \in \left[1, \frac{2N}{N+2}\right)$. Specifically, for each $\psi \in C_c^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \psi + \int_{\mathbb{R}^N} u\psi - \int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * (Q \circ u)\right)\big((q \circ u)\psi\big)$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \psi + \int_{\mathbb{R}^N} u_n\psi - \int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * (Q \circ u_n)\right)\big((q \circ u_n)\psi\big) = 0,$$

i.e. u is a weak solution of Eq (1.2).

Theorem 2.6. Consider the assumption that $N \geq 3$ and $\alpha \in (0,2)$. Provided that $q \in C(\mathbb{R}, \mathbb{R})$ fulfils *(*H_1*)–(*H_3*)*, Eq (1.2) possesses a *GSS*.

Proof. Let (u_n) denote a minimising sequence obtained as a consequence of Theorem 2.1, i.e. $(u_n) \subset H^1(\mathbb{R}^N)$ such that

$$\mathcal{J}(u_n) \to c, \quad \acute{\mathcal{J}}(u_n) \to 0,$$

where

$$c = c_o = inf\, \mathcal{J}(u) = c^* = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{\tau \geq 0} \mathcal{J}(u_\tau).$$

Then, we have

$$\begin{aligned} c_o &= \mathcal{J}(u_n) - \frac{1}{4}\langle \acute{\mathcal{J}}(u_n), u_n \rangle = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |u_n|^2) - \frac{1}{2}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n)\right)\mathcal{Q}(u_n) \\ &- \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |u_n|^2) + \frac{1}{4}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n)\right)q(u_n)u_n \\ &= \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |u_n|^2) + \frac{1}{4}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n)\right)[q(u_n)u_n - 2\mathcal{Q}(u_n)] \\ &\geq \frac{1}{4}\|u_n\|^2. \end{aligned}$$

Consequently, (u_n) is bounded. By employing standard methods, we can achieve the convergence of (u_n).

Next, let

$$\delta := \overline{\lim_{n\to\infty}} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2.$$

We claim $\delta > 0$. On the contrary, on the basis of Lions' concentration compactness principle, we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N}{N-2}$. By (H_1), $\mathcal{Q}$ is continuous and satisfies (H_2), for any $\varepsilon > 0$, a constant $C_\varepsilon > 0$ exists such that

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n)\right)q(u_n)(u_n) &\leq C\overline{\lim_{n\to\infty}}\left(\int_{\mathbb{R}^N}|\mathcal{Q}(u_n)|^{\frac{2N}{N+\alpha}}\right)^{1+\frac{\alpha}{N}} \\ &\leq C\overline{\lim_{n\to\infty}}\left(\int_{\mathbb{R}^N}|u_n|^2 + \int_{\mathbb{R}^N}|u_n|^{\frac{2N}{N-2}}\right)^{1+\frac{\alpha}{N}}. \end{aligned}$$

Through Lemma 12.2,

$$\begin{aligned} &\leq C\overline{\lim_{n\to\infty}}\left[\varepsilon\left(\int_{\mathbb{R}^N}|u_n|^2 + \int_{\mathbb{R}^N}|u_n|^{\frac{2N}{N-2}}\right) + C_\varepsilon\int_{\mathbb{R}^N}|u_n|^p\right]^{1+\frac{\alpha}{N}} \\ &\leq C\left[\varepsilon C_1 + C_\varepsilon\overline{\lim_{n\to\infty}}\int_{\mathbb{R}^N}|u_n|^p\right]^{1+\frac{\alpha}{N}} \leq C[\varepsilon C_2]^{1+\frac{\alpha}{N}}. \end{aligned}$$

Considering that ε is arbitrary, we obtain

$$\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n)\right)q(u_n)(u_n) = 0.$$

Combining with $\acute{\mathcal{J}}(u_n) \to 0$, we can generate

$$0 = \langle \acute{\mathcal{J}}(u_n), u_n \rangle = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) - \int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n) \right) q(u_n) u_n dx,$$

which implies that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) = \int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(u_n) \right) q(u_n) u_n dx.$$

Then, we have

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) \to 0,$$

which indicates that $(u_n) \to 0$ in $H^1(\mathbb{R}^N)$. Accordingly, $c_o = 0$, which contradicts the assumption that $c_o > 0$.

Therefore, there exists a positive constant $\delta > 0$ and a sequence $\{y_n\} \in \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_n|^p \geq \frac{\delta}{2} > 0.$$

We set $\mathcal{V}_n(x) = u_n(x + y_n)$,

$$\|\mathcal{V}_n\| = \|u_n\|,$$

$$\int_{B_1(0)} |\mathcal{V}_n|^p dx > \frac{\delta}{2},$$

and

$$\mathcal{J}(\mathcal{V}_n) \to c_o = c, \ \ \acute{\mathcal{J}}(u_n) \to 0.$$

Thus, $\mathcal{V}_0 \neq 0$ exists such that

$$\mathcal{V}_n \to \mathcal{V}_0 \text{ in } H^1(\mathbb{R}^N),$$

$$\mathcal{V}_n \to \mathcal{V}_0 \text{ in } L^p(\mathbb{R}^N), \ \forall p \in \left[2, \tfrac{2N}{N-2}\right),$$

$$\mathcal{V}_n \to \mathcal{V}_0 \text{ a.e. on } \mathbb{R}^N.$$

Then, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$0 = \langle \acute{\mathcal{J}}(\mathcal{V}_n), \psi \rangle = \langle \acute{\mathcal{J}}(\mathcal{V}_0), \psi \rangle,$$

which means that $\mathcal{V}_0$ is a solution of Eq (1.2).

Conversely, the application of Fatou's lemma enables us to obtain

$$c_o = \mathcal{J}(\mathcal{V}_n) - \frac{1}{4}\langle \acute{\mathcal{J}}(\mathcal{V}_n), \mathcal{V}_n \rangle = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \mathcal{V}_n|^2 + |\mathcal{V}_n|^2) - \frac{1}{2}\int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(\mathcal{V}_n) \right) \mathcal{Q}(\mathcal{V}_n)$$
$$- \frac{1}{4}\int_{\mathbb{R}^N} (|\nabla \mathcal{V}_n|^2 + |\mathcal{V}_n|^2) + \frac{1}{4}\int_{\mathbb{R}^N} \left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(\mathcal{V}_n) \right) q(\mathcal{V}_n) \mathcal{V}_n$$

$$= \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla \mathcal{V}_n|^2 + |\mathcal{V}_n|^2) + \frac{1}{4}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(\mathcal{V}_n)\right)[q(\mathcal{V}_n)\mathcal{V}_n - 2\mathcal{Q}(\mathcal{V}_n)]$$
$$\geq \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla \mathcal{V}_0|^2 + |\mathcal{V}_0|^2) + \frac{1}{4}\int_{\mathbb{R}^N}\left((-\Delta)_E^{\frac{-\alpha}{2}} * \mathcal{Q}(\mathcal{V}_0)\right)[q(\mathcal{V}_0)\mathcal{V}_0 - 2\mathcal{Q}(\mathcal{V}_0)]$$
$$= \mathcal{J}(\mathcal{V}_0) - \frac{1}{4}\langle \acute{\mathcal{J}}(\mathcal{V}_0), \mathcal{V}_0\rangle = \mathcal{J}(\mathcal{V}_0).$$

From the definition of c_o, $c_o \leq \mathcal{J}(\mathcal{V}_0)$. Consequently, we can conclude that $\mathcal{V}_0$ is a GSS of Eq (1.2).

Remark 1. This study examines the issue when α is between $(0,2)$, but not when α is greater than 2.

3. Conclusions

In conclusion, our study investigated a GSS for a GNCE. Through rigorous analysis and the application of variational methods, we have successfully established the existence of GSS under certain conditions. Our main contribution consists of a novel alteration of the Riesz potential, an aspect that has not been previously examined in this context. This modification not only expands the theoretical understanding of such equations but also opens new avenues for practical applications, particularly in fields such as quantum mechanics and astrophysics, where nonlocal interactions play a crucial role.

The theoretical implications of our findings extend beyond the realm of mathematical analysis, suggesting broader applications in understanding physical systems with nonlocal interactions. By showcasing the effectiveness of our approach in identifying GSS, we advocate for further exploration of nonlocal effects and their mathematical representations in various scientific domains.

Whilst our study builds upon previous methodologies, our novel approach to modifying the Riesz potential underscores the importance of innovative thinking in advancing mathematical and scientific research. This study serves as a foundation for future research aimed at exploring the complexities of nonlinear Choquard equations and their implications across various disciplines. For instance, subsequent work could investigate the existence of GSSs for magnetic or other variants of the nonlinear Choquard equation when the Riesz potential is equivalent to a Riesz fractional operator.

Author contributions

Sarah Abdullah Qadha: Writing-original draft preparation, Methodology, Review and editing, Investigation; Muneera Abdullah Qadha: Writing-review and editing, Methodology, Investigation; Haibo Chen: Supervision, Writing-review and editing, Investigation; Mohamed Abdalla: Review and editing, Investigation; Mohammed Z. Alqarni: Review and editing, Investigation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest.

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