AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25469–25488.
DOI: 10.3934/math.20251128
Received: 02 August 2025
Revised: 10 October 2025
Accepted: 29 October 2025
Published: 05 November 2025

Research article

On explicit periodic solutions in three-dimensional difference systems

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Abstract: This paper focuses on the existence and analytical formulation of closed-form solutions for a three-dimensional system of nonlinear difference equations. The proposed system possesses a mathematical architecture that encapsulates complex nonlinear interactions among three mutually dependent variables. Through the application of systematic analytical transformations, the original system was reduced to a set of solvable recurrence relations, thereby allowing the derivation of explicit closed-form expressions with a high degree of analytical precision. Furthermore, numerical examples revealed that even minute perturbations in the system parameters or initial conditions can induce significant variations in oscillatory patterns, highlighting the system's structural sensitivity and rich dynamical diversity. This paper, therefore, constitutes a natural and essential extension of previously studied two-dimensional frameworks toward more sophisticated three-dimensional models, which provide a more faithful representation of interdependent relationships within discrete nonlinear systems.

Keywords: system of difference equations; nonlinear difference equations; closed form; periodicity
Mathematics Subject Classification: Primary 39A10; Secondary 40A05

1. Introduction

Difference equations, as advanced mathematical tools, provide a robust framework for modeling and analyzing dynamical systems that evolve over discrete time intervals. Their significance extends beyond theoretical exploration to practical applications across various scientific domains, including biology, economics, and engineering. The study of difference equations has gained considerable attention due to their capacity to simplify and transform complex systems into solvable forms, facilitating a deeper understanding of underlying dynamics. Building upon these foundational ideas, several recent studies have examined discrete nonlinear systems from diverse perspectives, aiming to analyze their periodic, steady-state, and alternating behaviors. For instance, Abo-Zeid (2019, [1]) and Abo- Zeid & Cinar (2013, [2]) investigated the general dynamics of third-order rational difference equations, emphasizing the significant influence of quadratic terms on the emergence of complex and unpredictable dynamical patterns. In a similar vein, Elsayed (2023, [3, 4]) explored the periodic characteristics and explicit solutions of nonlinear rational systems, demonstrating that nonlinear interdependencies among terms can generate long-term periodicities or parameter-conditioned stabilities. Meanwhile, the works of Gümüş and collaborators ([5–9]) focused on the stability and boundary behaviors of discrete systems with time delays, revealing that the inclusion of delay effects or nonlinear structures produces a wide spectrum of dynamical behaviors, ranging from stable recurrences to quasi-periodic oscillations. Furthermore, Gümüş and Abo-Zeid ([5–7]) identified the existence of forbidden sets—regions of unsolvability that underscore the importance of defining the conditions required to ensure theoretical consistency and mathematical integrity in discrete models. Complementary studies by Kara (2023, [10]) and Oğul & Simşek (2023, [11]) investigated exponential and rational forms of higher-order difference equations, showing that increasing the order of recurrence or the degree of interaction introduces additional analytical complexity, including intricate rotation patterns and modified stability characteristics. Likewise, the seminal contributions of Zhang et al. (2006, [12]) and Zhang et al. (2012, [13]) provided closed-form solutions and established stability and boundary conditions for third-order rational discrete systems with rational structures, which remain a cornerstone in the field. Collectively, these works reveal that the prevailing body of research predominantly centers on two-dimensional systems or those examined solely through numerical simulations, whereas three-dimensional nonlinear discrete systems that are treated analytically and yield closed-form solutions remain relatively underexplored and continue to represent a promising direction for future mathematical investigation.

Applications of difference equations span from fundamental linear cases to intricate nonlinear systems that exhibit rich dynamical behaviors. Many such systems can be described using well-known numerical sequences, including Fibonacci numbers, (co)balancing numbers, Jacobsthal numbers, and Mersenne numbers (see, e.g., [14–16]). These sequences often emerge in recursive models that capture population dynamics, financial systems, and combinatorial structures, highlighting the versatility of discrete-time models.

Previous studies, such as those conducted by Wisnowski and Schumacher [17], have demonstrated the impact of competition within seed banks on biodiversity, showcasing how difference equations serve as powerful tools for modeling ecological interactions. Their mathematical models illustrate complex behaviors that arise in competitive environments, underscoring the growing scientific interest in rational difference equations, which often exhibit richer and more intricate dynamics than their linear counterparts. The widespread presence of difference equations in biological models further reinforces their fundamental role in advancing research across disciplines.

A particularly intriguing example is the Riccati difference equation, which has been extensively investigated due to its broad applications in control theory, mathematical finance, and population dynamics. Similarly, Sharawi and Roma [18] explored the impact of various harvesting strategies on a discrete Beverton-Holt model, demonstrating how deterministic environments influence ecological sustainability. Their findings highlight the necessity of transitioning toward more complex and generalized models to capture real-world phenomena with greater accuracy. It is noteworthy that a number of recent investigations have explored the multiple factors influencing the dynamical behavior of discrete-time nonlinear systems, with particular emphasis on stability, constraints, and periodicity. For instance, Almatrafi and Alzubaidi (2022, [19]) examined periodic solutions, equilibrium points, and stability conditions of eighth-order rational difference equations using rigorous iterative numerical schemes, thereby illustrating how such models can effectively capture complex real-world behaviors. Similarly, Aljoufi et al. (2023, [20]) conducted a comprehensive dynamical analysis of generalized-order discrete-time equations, focusing on periodicity, stability, and oscillatory phenomena, and derived explicit solutions employing absolute value operators and mathematical iteration. Collectively, these studies underscore that discrete nonlinear models constitute a powerful and versatile framework for the analysis of dynamical systems across natural sciences and engineering applications. These scholarly efforts align closely with the objective of the present study, which is to analyze a three-dimensional nonlinear discrete system exhibiting interactive dependencies among its components. By emphasizing the reciprocal relationships between variables, this research contributes to a deeper mathematical understanding of discrete systems with complex coupling structures.

This paper is motivated by the intention to broaden the mathematical framework established for two-dimensional difference systems, particularly the model proposed by Hassani et al. (2024, [21]), which examined a two-dimensional nonlinear second-order system. In the present study, we introduce a three-dimensional extension of that model by incorporating an additional variable, thereby enabling the representation of interactive dynamics among three interdependent components rather than two. This extension substantially enriches the system's dynamical complexity and allows for a more comprehensive exploration of multi-component interactions in discrete-time settings. Such a generalization constitutes a natural and necessary step toward a deeper understanding of high-dimensional discrete systems, which are known for their pronounced sensitivity to initial conditions and parameter variations. These characteristics make them particularly suitable for modeling a wide range of nonlinear and time-discrete phenomena observed in physical, biological, and economic contexts. Moreover, by employing rational difference equations within this extended framework, the study effectively captures inverse and conditional relationships among variables while preserving the system's nonlinear structure and maintaining its analytical tractability.

Motivated by these considerations, this research examines a novel system governed by the following set of recursive equations:

$$\begin{aligned} r_{m+1} &= \rho\frac{r_{m-1}}{t_m-\delta}s_m+\varepsilon, \forall m \geq 0,\\ s_{m+1} &= \sigma\frac{s_{m-1}}{r_m-\varepsilon}t_m+\lambda,\\ t_{m+1} &= \tau\frac{t_{m-1}}{s_m-\lambda}r_m+\delta, \end{aligned} \tag{1.a}$$

where $\rho, \varepsilon, \sigma, \lambda, \tau, \delta$, and the initial values r_{-1}, r_0, s_{-1}, s_0, t_{-1}, t_0 are nonzero real numbers. It should be noted that the system (1.a) is well-defined only under specific non-degeneracy conditions ensuring that all denominators remain nonzero throughout the iteration process. In particular, the system is defined by the following conditions:

$$t_m \neq \delta\ ,\ r_m \neq \varepsilon\ ,\ s_m \neq \lambda, \forall m \geq 0. \tag{1.b}$$

This constraint guarantees that each recursive relation in (1.a) is meaningful and that the division operations are well-defined at every iteration step. This system represents a significant generalization of previous work, particularly extending the two-dimensional framework discussed in [21] while maintaining analytical tractability through its recursive structure. It is worth emphasizing that the study of nonlinear systems in discrete spaces has attracted growing attention in recent years, owing to their wide-ranging applications across the natural sciences and economics. Recent contributions, such as the work by Althagafi et al. (2025, [22]) on the stability analysis of biological rhythms using three-dimensional difference systems, have revealed that quadratic systems possess strong potential to model biological rhythms and to analyze the boundedness and persistence properties of biological models. In addition, Althagafi, H. (2025, [23]) provided further insight into the role of discrete nonlinear dynamics in modeling neural activity and synchronization phenomena. Similarly, Althagafi et al. (2024, [24]) investigated three-dimensional nonlinear systems reducible to two-dimensional linear forms, demonstrating the existence of closed-form solutions and periodic behaviors associated with animal and social dynamics. Moreover, related approaches have been employed in financial time-series analysis, particularly in stochastic models with periodic thresholds and Markov-switching ([25–27]), to describe volatility dynamics and structural transitions in financial markets.

The novelty of this study resides in the formulation and analytical investigation of a three-dimensional nonlinear discrete system that serves as a generalization of a previously examined two-dimensional model. In contrast to earlier approaches, the proposed framework effectively captures the multivariate coupling effects inherent in complex discrete dynamics, while simultaneously allowing the derivation of explicit closed-form solutions through rigorous analytical procedures. The principal contributions of this work can be summarized as follows: (i) Extending previously established solvable two-dimensional models to a higher-dimensional setting, thereby enriching the analytical landscape of nonlinear discrete systems; (ii) Deriving precise analytical conditions governing the solvability and periodicity of the system, ensuring a deeper understanding of its structural behavior.

The remainder of this paper is structured as follows: Section 2 establishes the mathematical framework and presents the analytical methodology adopted for the investigation of the proposed system of recursive equations (1.a). Section 3 provides the detailed derivation of the closed-form solutions and discusses the analytical conditions for stability and periodicity. Section 4 presents a set of counterexamples and numerical illustrations designed to validate the theoretical results and to highlight the impact of parameter variations on system dynamics. Section 5 introduces the Results and Discussion, where the key analytical and numerical findings are interpreted and compared. Finally, Section 6 concludes the paper by summarizing the main contributions and outlining potential directions for future research.

2. Discussion of the methodology adopted

The methodology employed in this study is specifically formulated to capture the intrinsic nonlinear interactions governing the proposed three-dimensional discrete system. Unlike conventional linearization or reductionist techniques that often oversimplify the underlying dynamics by neglecting higher-order dependencies, the present approach preserves the complete nonlinear structure of the recursive relations. This methodological choice ensures that critical dynamical features, such as sensitivity to initial conditions, nonlinear coupling effects, and parameter-induced transitions, are faithfully represented throughout the analytical process.

The investigation commences with a direct examination of the system's recursive formulation, where algebraic techniques are utilized to derive necessary and sufficient conditions guaranteeing the well-defined nature of the sequences (r_m), (s_m), and (t_m). Particular attention is given to avoiding singular configurations in which denominators vanish, since such singularities correspond to degenerate states that disrupt recurrence and may lead to indeterminate or divergent trajectories. Consequently, their exclusion is a fundamental requirement for maintaining the theoretical consistency and stability of the model.

Moreover, the adopted framework integrates analytical reasoning with numerical experimentation, allowing a comprehensive understanding of how parameter variations influence system evolution. The analytical component elucidates the internal structure of the system, whereas the numerical analysis validates these findings, and illustrates the corresponding dynamical behaviors, and reveals its sensitivity to initial perturbations. This combined analytical–numerical strategy has demonstrated remarkable effectiveness in recent studies of nonlinear discrete systems (see, for instance, Althagafi et al. [22–24]), making it particularly well-suited for multidimensional discrete models arising in biological, physical, and economic contexts, where nonlinearity and parameter sensitivity play decisive roles.

The insights derived from this methodological discussion naturally lead to the analysis presented in the next section, where theoretical predictions are validated through numerical simulations.

3. Solution of system (1.a)

This section delves into the process of solving system (1.a) under specific initial conditions. Consider a sequence (r_m, s_m, t_m) for $m + 1 \geq 0$ that represents a solution to system (1.a). It is crucial to note that the system becomes undefined if any of the initial conditions, particularly when $k \in \{0, 1\}$, is set to zero. Similarly, the system fails to be defined if any one of the conditions $r_0 = \lambda$, or $s_0 = \delta$, or $t_0 = \varepsilon$ holds. To ensure the existence of a well-defined solution (r_m, s_m, t_m) for $m + 1 \geq 0$ within system (1.a), the following conditions must be met:

$$r_m s_m t_m \neq 0,\, r_0 \neq \lambda,\ s_0 \neq \delta,\ t_0 \neq \varepsilon \text{ for } m + 1 \geq 0. \tag{2.a}$$

From this point forward, we will assume that the condition $r_{-k}s_{-k}t_{-k} \neq 0$ for $k \in \{0, 1\}$ is satisfied, ensuring that the solution meets the requirements outlined in (2.a). This assumption is critical for maintaining the integrity of the solution, allowing us to explore its properties under well-defined circumstances.

To facilitate the analysis of system (1.a), we introduce a change of variables. Define

$$\widehat{r_m} = \frac{r_m - \varepsilon}{s_{m-1}},\ \widehat{s_m} = \frac{s_m - \lambda}{t_{m-1}},\ \widehat{t_m} = \frac{t_m - \delta}{r_{m-1}} \text{ for } m \geq 0. \tag{2.b}$$

Under this transformation, system (1.a) is reformulated as follows:

$$\widehat{r_{m+1}} = \rho\frac{1}{\widehat{t_m}},\quad \widehat{s_{m+1}} = \sigma\frac{1}{\widehat{r_m}},\quad \widehat{t_{m+1}} = \tau\frac{1}{\widehat{s_m}},\ \forall m \geq 0 \tag{2.c}$$

enabling a more structured approach to finding solutions. Consequently, we define a new variable $\widetilde{r_m}$ as follows: $\widetilde{r_m} := \widehat{r_m}\widehat{s_m}\widehat{t_m}$ $\forall m \geq 0$. With this substitution, system (2.c) evolves into: $\widetilde{r_{m+1}} = \rho\sigma\tau\widetilde{r_m}^{-1}$

$\forall m \geq 0$. The solution to this recurrence relation can be expressed as:

$$\widetilde{r}_m \quad = \frac{\rho\sigma\tau}{\widetilde{r}_0}\delta_m(2p-1) + \widetilde{r}_0\delta_m(2p), \forall m \geq 0 \tag{2.d}$$

where the symbol $\delta_m(n)$ represents the Kronecker delta, which is defined as follows:

$$\delta_m(n) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

From (2.d), we find:

$$\widetilde{r}_{2m} = \widetilde{r}_0 = \widehat{r}_{2m}\widehat{s}_{2m}\widehat{t}_{2m}, \ \widetilde{r}_{2m+1} = \frac{\rho\sigma\tau}{\widetilde{r}_0} = \widehat{r}_{2m+1}\widehat{s}_{2m+1}\widehat{t}_{2m+1}, \forall m \geq 0. \tag{2.e}$$

Specifically, in system (2.c), we arrive at

$$\widehat{r}_{m+1} = \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r}_{m-2}}, \quad \widehat{s}_{m+1} = \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s}_{m-2}}, \ \widehat{t}_{m+1} = \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t}_{m-2}}, \forall m \geq 2. \tag{2.f}$$

Therefore, we can define the new sequence as follows:

$$\widehat{r}_m = \begin{cases} \widehat{r}_0 & \text{if } m = 6k, k \geq 0, \\ \widehat{r}_1 & \text{if } m = 6k+1, \\ \widehat{r}_2 & \text{if } m = 6k+2, \\ \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r}_0} & \text{if } m = 6k+3, \\ \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r}_1} & \text{if } m = 6k+4, \\ \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r}_2} & \text{if } m = 6k+5, \end{cases} \qquad \widehat{s}_m = \begin{cases} \widehat{s}_0 & \text{if } m = 6k, k \geq 0, \\ \widehat{s}_1 & \text{if } m = 6k+1, \\ \widehat{s}_2 & \text{if } m = 6k+2, \\ \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s}_0} & \text{if } m = 6k+3, \\ \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s}_1} & \text{if } m = 6k+4, \\ \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s}_2} & \text{if } m = 6k+5. \end{cases}$$

Further, from (2.e), we derive

$$\widehat{t}_{2m} = \frac{\widehat{r}_0\widehat{s}_0\widehat{t}_0}{\widehat{r}_{2m}\widehat{s}_{2m}}, \ \widehat{t}_{2m+1} = \frac{\rho\sigma\tau}{\widehat{r}_0\widehat{s}_0\widehat{t}_0}\frac{1}{\widehat{r}_{2m+1}\widehat{s}_{2m+1}}, \forall m \geq 0.$$

From this, the sequence $\left(\widehat{t}_m\right)$ can be described as follows:

$$\widehat{t}_m = \begin{cases} \widehat{t}_0 & \text{if } m = 6k, k \geq 0, \\ \widehat{t}_1 & \text{if } m = 6k+1, \\ \widehat{t}_2 & \text{if } m = 6k+2, \\ \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t}_0} & \text{if } m = 6k+3, \\ \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t}_1} & \text{if } m = 6k+4, \\ \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t}_2} & \text{if } m = 6k+5. \end{cases}$$

Additionally, from (2.b), we establish

$$r_m = \widehat{r}_m s_{m-1} + \varepsilon,\ s_m = \widehat{s}_m t_{m-1} + \lambda,\ t_m = \widehat{t}_m r_{m-1} + \delta \text{ for } m \geq 0. \tag{2.g}$$

By substituting m with $6m+k$, $0 \leq k \leq 5$, into (2.g), we derive

$$\begin{cases} r_{6m} = \widehat{r_0} s_{6m-1} + \varepsilon, & s_{6m} = \widehat{s_0} t_{6m-1} + \lambda, & t_{6m} = \widehat{t_0} r_{6m-1} + \delta, \\ r_{6m+1} = \widehat{r_1} s_{6m} + \varepsilon, & s_{6m+1} = \widehat{s_1} t_{6m} + \lambda, & t_{6m+1} = \widehat{t_1} r_{6m} + \delta, \\ r_{6m+2} = \widehat{r_2} s_{6m+1} + \varepsilon, & s_{6m+2} = \widehat{s_2} t_{6m+1} + \lambda, & t_{6m+2} = \widehat{t_2} r_{6m+1} + \delta, \\ r_{6m+3} = \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r_0}} s_{6m+2} + \varepsilon, & s_{6m+3} = \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s_0}} t_{6m+2} + \lambda, & t_{6m+3} = \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t_0}} r_{6m+2} + \delta, \\ r_{6m+4} = \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r_1}} s_{6m+3} + \varepsilon, & s_{6m+4} = \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s_1}} t_{6m+3} + \lambda, & t_{6m+4} = \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t_1}} r_{6m+3} + \delta, \\ r_{6m+5} = \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r_2}} s_{6m+4} + \varepsilon, & s_{6m+5} = \frac{\sigma\tau}{\rho}\frac{1}{\widehat{s_2}} t_{6m+4} + \lambda, & t_{6m+5} = \frac{\rho\tau}{\sigma}\frac{1}{\widehat{t_2}} r_{6m+4} + \delta. \end{cases}$$

This implies

$$\begin{aligned} r_{6m} &= \tau^2 \frac{\widehat{r_0}}{\widehat{t_1}\widehat{s_2}} r_{6m-3} + \frac{\sigma\tau}{\rho}\frac{\widehat{r_0}}{\widehat{s_2}}\delta + \widehat{r_0}\lambda + \varepsilon \\ &= \rho\sigma\tau r_{6m-6} + \left(\rho\sigma\tau\frac{1}{\widehat{t_1}} + \frac{\sigma\tau}{\rho}\frac{\widehat{r_0}}{\widehat{s_2}}\right)\delta + \left(\rho\sigma\tau\frac{1}{\widehat{t_1}\widehat{s_2}} + \widehat{r_0}\right)\lambda + \left(\tau^2\frac{\widehat{r_0}}{\widehat{t_1}\widehat{s_2}} + 1\right)\varepsilon, \end{aligned}$$

$$\begin{aligned} r_{6m+1} &= \frac{\rho\tau}{\sigma}\frac{\widehat{r_1}\widehat{s_0}}{\widehat{t_2}} r_{6m-2} + \widehat{r_1}\widehat{s_0}\delta + \widehat{r_1}\lambda + \varepsilon \\ &= \rho\sigma\tau r_{6m-5} + \left(\rho\sigma\tau\frac{1}{\widehat{t_2}} + \widehat{r_1}\widehat{s_0}\right)\delta + \left(\rho^2\frac{\widehat{s_0}}{\widehat{t_2}} + \widehat{r_1}\right)\lambda + \left(\frac{\rho\tau}{\sigma}\frac{\widehat{r_1}\widehat{s_0}}{\widehat{t_2}} + 1\right)\varepsilon, \end{aligned}$$

$$\begin{aligned} r_{6m+2} &= \widehat{r_2}\widehat{s_1}\widehat{t_0} r_{6m-1} + \widehat{r_2}\widehat{s_1}\delta + \widehat{r_2}\lambda + \varepsilon \\ &= \rho\sigma\tau r_{6m-4} + \left(\sigma^2\widehat{t_0} + \widehat{r_2}\widehat{s_1}\right)\delta + \left(\frac{\rho\sigma}{\tau}\widehat{s_1}\widehat{t_0} + \widehat{r_2}\right)\lambda + \left(\widehat{r_2}\widehat{s_1}\widehat{t_0} + 1\right)\varepsilon, \end{aligned}$$

$$\begin{aligned} r_{6m+3} &= \frac{\rho\sigma}{\tau}\frac{\widehat{s_2}\widehat{t_1}}{\widehat{r_0}} r_{6m} + \frac{\rho\sigma}{\tau}\frac{\widehat{s_2}}{\widehat{r_0}}\delta + \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r_0}}\lambda + \varepsilon \\ &= \rho\sigma\tau r_{6m-3} + \left(\sigma^2\widehat{t_1} + \frac{\rho\sigma}{\tau}\frac{\widehat{s_2}}{\widehat{r_0}}\right)\delta + \frac{\rho\sigma}{\tau}\left(\widehat{s_2}\widehat{t_1} + \frac{1}{\widehat{r_0}}\right)\lambda + \left(\frac{\rho\sigma}{\tau}\frac{\widehat{s_2}\widehat{t_1}}{\widehat{r_0}} + 1\right)\varepsilon, \end{aligned}$$

$$\begin{aligned} r_{6m+4} &= \sigma^2\frac{\widehat{t_2}}{\widehat{r_1}\widehat{s_0}} r_{6m+1} + \sigma^2\frac{1}{\widehat{r_1}\widehat{s_0}}\delta + \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r_1}}\lambda + \varepsilon \\ &= \rho\sigma\tau r_{6m-2} + \sigma^2\frac{1}{\widehat{r_1}\widehat{s_0}}\left(\widehat{t_2}\widehat{r_1}\widehat{s_0} + 1\right)\delta + \frac{1}{\widehat{r_1}}\left(\sigma^2\frac{\widehat{r_1}\widehat{t_2}}{\widehat{s_0}} + \frac{\rho\sigma}{\tau}\right)\lambda + \left(\sigma^2\frac{\widehat{t_2}}{\widehat{r_1}\widehat{s_0}} + 1\right)\varepsilon, \end{aligned}$$

$$\begin{aligned} r_{6m+5} &= \rho\sigma\tau\frac{1}{\widehat{r_2}\widehat{s_1}\widehat{t_0}} r_{6m+2} + \sigma^2\frac{1}{\widehat{r_2}\widehat{s_1}}\delta + \frac{\rho\sigma}{\tau}\frac{1}{\widehat{r_2}}\lambda + \varepsilon \\ &= \rho\sigma\tau r_{6m-1} + \sigma^2\frac{1}{\widehat{r_2}\widehat{s_1}}\left(\frac{\rho\tau}{\sigma}\frac{\widehat{r_2}\widehat{s_1}}{\widehat{t_0}} + 1\right)\delta + \frac{1}{\widehat{r_2}}\left(\rho\sigma\tau\frac{\widehat{r_2}}{\widehat{s_1}\widehat{t_0}} + \frac{\rho\sigma}{\tau}\right)\lambda + \left(\rho\sigma\tau\frac{1}{\widehat{r_2}\widehat{s_1}\widehat{t_0}} + 1\right)\varepsilon, \end{aligned}$$

and

$$\begin{aligned} s_{6m} &= \rho^2\frac{\widehat{s_0}}{\widehat{r_1}\widehat{t_2}} s_{6m-3} + \frac{\rho\tau}{\sigma}\frac{\widehat{s_0}}{\widehat{t_2}}\varepsilon + \widehat{s_0}\delta + \lambda \\ &= \rho\sigma\tau s_{6m-6} + \left(\rho\sigma\tau\frac{1}{\widehat{r_1}} + \frac{\sigma\tau}{\sigma}\frac{\widehat{s_0}}{\widehat{t_2}}\right)\varepsilon + \left(\rho\sigma\tau\frac{1}{\widehat{r_1}\widehat{t_2}} + \widehat{s_0}\right)\delta + \left(\rho^2\frac{\widehat{s_0}}{\widehat{r_1}\widehat{t_2}} + 1\right)\lambda, \end{aligned}$$

$$
\begin{aligned}
s_{6m+1} &= \tfrac{\rho\sigma}{\tau}\frac{\widehat{s_1}\widehat{t_0}}{\widehat{r_2}}s_{6m-2} + \widehat{s_1}\widehat{t_0}\varepsilon + \widehat{s_1}\delta + \lambda \\
&= \rho\sigma\tau s_{6m-5} + \left(\rho\sigma\tau\frac{1}{\widehat{r_2}} + \widehat{s_1}\widehat{t_0}\right)\varepsilon + \left(\sigma^2\frac{\widehat{t_0}}{\widehat{r_2}} + \widehat{s_1}\right)\delta + \left(\tfrac{\rho\sigma}{\tau}\frac{\widehat{s_1}\widehat{t_0}}{\widehat{r_2}} + 1\right)\lambda, \\
s_{6m+2} &= \widehat{s_2}\widehat{t_1}\widehat{r_0}s_{6m-1} + \widehat{s_2}\widehat{t_1}\varepsilon + \widehat{s_2}\delta + \lambda \\
&= \rho\sigma\tau s_{6m-4} + \left(\tau^2\widehat{r_0} + \widehat{s_2}\widehat{t_1}\right)\varepsilon + \left(\tfrac{\sigma\tau}{\rho}\widehat{t_1}\widehat{r_0} + \widehat{s_2}\right)\delta + \left(\widehat{s_2}\widehat{t_1}\widehat{r_0} + 1\right)\lambda, \\
s_{6m+3} &= \tfrac{\sigma\tau}{\rho}\frac{\widehat{t_2}\widehat{r_1}}{\widehat{s_0}}s_{6m} + \tfrac{\sigma\tau}{\rho}\frac{\widehat{t_2}}{\widehat{s_0}}\varepsilon + \tfrac{\sigma\tau}{\rho}\frac{1}{\widehat{s_0}}\delta + \lambda \\
&= \rho\sigma\tau s_{6m-3} + \left(\tau^2\widehat{r_1} + \tfrac{\sigma\tau}{\rho}\frac{\widehat{t_2}}{\widehat{s_0}}\right)\varepsilon + \tfrac{\sigma\tau}{\rho}\left(\widehat{t_2}\widehat{r_1} + \frac{1}{\widehat{s_0}}\right)\delta + \left(\tfrac{\sigma\tau}{\rho}\frac{\widehat{t_2}\widehat{r_1}}{\widehat{s_0}} + 1\right)\lambda, \\
s_{6m+4} &= \tau^2\frac{\widehat{r_2}}{\widehat{s_1}\widehat{t_0}}s_{6m+1} + \tau^2\frac{1}{\widehat{s_1}\widehat{t_0}}\varepsilon + \tfrac{\sigma\tau}{\rho}\frac{1}{\widehat{s_1}}\delta + \lambda \\
&= \rho\sigma\tau s_{6m-2} + \tau^2\frac{1}{\widehat{s_1}\widehat{t_0}}\left(\widehat{r_2}\widehat{s_1}\widehat{t_0} + 1\right)\varepsilon + \frac{1}{\widehat{s_1}}\left(\tau^2\frac{\widehat{s_1}\widehat{r_2}}{\widehat{t_0}} + \tfrac{\sigma\tau}{\rho}\right)\delta + \left(\tau^2\frac{\widehat{r_2}}{\widehat{s_1}\widehat{t_0}} + 1\right)\lambda, \\
s_{6m+5} &= \rho\sigma\tau\frac{1}{\widehat{s_2}\widehat{t_1}\widehat{r_0}}s_{6m+2} + \tau^2\frac{1}{\widehat{s_2}\widehat{t_1}}\varepsilon + \tfrac{\sigma\tau}{\rho}\frac{1}{\widehat{s_2}}\delta + \lambda \\
&= \rho\sigma\tau s_{6m-1} + \tau^2\frac{1}{\widehat{s_2}\widehat{t_1}}\left(\tfrac{\rho\sigma}{\tau}\frac{\widehat{s_2}\widehat{t_1}}{\widehat{r_0}} + 1\right)\varepsilon + \frac{1}{\widehat{s_2}}\left(\rho\sigma\tau\frac{\widehat{s_2}}{\widehat{t_1}\widehat{r_0}} + \tfrac{\sigma\tau}{\rho}\right)\delta + \left(\rho\sigma\tau\frac{1}{\widehat{s_2}\widehat{t_1}\widehat{r_0}} + 1\right)\lambda,
\end{aligned}
$$

and

$$
\begin{aligned}
t_{6m} &= \sigma^2\frac{\widehat{t_0}}{\widehat{r_2}\widehat{s_1}}t_{6m-3} + \tfrac{\rho\sigma}{\tau}\frac{\widehat{t_0}}{\widehat{r_2}}\lambda + \widehat{t_0}\varepsilon + \delta \\
&= \rho\sigma\tau t_{6m-6} + \left(\rho\sigma\tau\frac{1}{\widehat{s_1}} + \tfrac{\rho\sigma}{\tau}\frac{\widehat{t_0}}{\widehat{r_2}}\right)\lambda + \left(\rho\sigma\tau\frac{1}{\widehat{r_2}\widehat{s_1}} + \widehat{t_0}\right)\varepsilon + \left(\sigma^2\frac{\widehat{t_0}}{\widehat{r_2}\widehat{s_1}} + 1\right)\delta, \\
t_{6m+1} &= \tfrac{\sigma\tau}{\rho}\frac{\widehat{t_1}\widehat{r_0}}{\widehat{s_2}}t_{6m-2} + \widehat{t_1}\widehat{r_0}\lambda + \widehat{t_1}\varepsilon + \delta \\
&= \rho\sigma\tau t_{6m-5} + \left(\rho\sigma\tau\frac{1}{\widehat{s_2}} + \widehat{t_1}\widehat{r_0}\right)\lambda + \left(\tau^2\frac{\widehat{r_0}}{\widehat{s_2}} + \widehat{t_1}\right)\varepsilon + \left(\tfrac{\sigma\tau}{\rho}\frac{\widehat{t_1}\widehat{r_0}}{\widehat{s_2}} + 1\right)\delta, \\
t_{6m+2} &= \widehat{t_2}\widehat{r_1}\widehat{s_0}t_{6m-1} + \widehat{t_2}\widehat{r_1}\lambda + \widehat{t_2}\varepsilon + \delta \\
&= \rho\sigma\tau t_{6m-4} + \left(\rho^2\widehat{s_0} + \widehat{t_2}\widehat{r_1}\right)\lambda + \left(\tfrac{\rho\tau}{\sigma}\widehat{r_1}\widehat{s_0} + \widehat{t_2}\right)\varepsilon + \left(\widehat{t_2}\widehat{r_1}\widehat{s_0} + 1\right)\delta, \\
t_{6m+3} &= \tfrac{\rho\tau}{\sigma}\frac{\widehat{r_2}\widehat{s_1}}{\widehat{t_0}}t_{6m} + \tfrac{\rho\tau}{\sigma}\frac{\widehat{r_2}}{\widehat{t_0}}\lambda + \tfrac{\rho\tau}{\sigma}\frac{1}{\widehat{t_0}}\varepsilon + \delta \\
&= \rho\sigma\tau t_{6m-3} + \left(\rho^2\widehat{s_1} + \tfrac{\rho\tau}{\sigma}\frac{\widehat{r_2}}{\widehat{t_0}}\right)\lambda + \tfrac{\rho\tau}{\sigma}\left(\widehat{r_2}\widehat{s_1} + \frac{1}{\widehat{t_0}}\right)\varepsilon + \left(\tfrac{\rho\tau}{\sigma}\frac{\widehat{r_2}\widehat{s_1}}{\widehat{t_0}} + 1\right)\delta, \\
t_{6m+4} &= \rho^2\frac{\widehat{s_2}}{\widehat{t_1}\widehat{r_0}}t_{6m+1} + \rho^2\frac{1}{\widehat{t_1}\widehat{r_0}}\lambda + \tfrac{\rho\tau}{\sigma}\frac{1}{\widehat{t_1}}\varepsilon + \delta \\
&= \rho\sigma\tau t_{6m-2} + \rho^2\frac{1}{\widehat{t_1}\widehat{r_0}}\left(\widehat{s_2}\widehat{t_1}\widehat{r_0}\lambda + 1\right)\lambda + \frac{1}{\widehat{t_1}}\left(\rho^2\frac{\widehat{s_2}\widehat{t_1}}{\widehat{r_0}} + \tfrac{\rho\tau}{\sigma}\right)\varepsilon + \left(\rho^2\frac{\widehat{s_2}}{\widehat{t_1}\widehat{r_0}} + 1\right)\delta, \\
t_{6m+5} &= \rho\sigma\tau\frac{1}{\widehat{t_2}\widehat{r_1}\widehat{s_0}}t_{6m+2} + \rho^2\frac{1}{\widehat{t_2}\widehat{r_1}}\lambda + \tfrac{\rho\tau}{\sigma}\frac{1}{\widehat{t_2}}\varepsilon + \delta \\
&= \rho\sigma\tau t_{6m-1} + \rho^2\frac{1}{\widehat{t_2}\widehat{r_1}}\left(\tfrac{\sigma\tau}{\rho}\frac{\widehat{t_2}\widehat{r_1}}{\widehat{s_0}} + 1\right)\lambda + \frac{1}{\widehat{t_2}}\left(\rho\sigma\tau\frac{\widehat{t_2}}{\widehat{r_1}\widehat{s_0}} + \tfrac{\rho\tau}{\sigma}\right)\varepsilon + \left(\rho\sigma\tau\frac{1}{\widehat{t_2}\widehat{r_1}\widehat{s_0}} + 1\right)\delta.
\end{aligned}
$$

Based on the comprehensive analysis and discussions provided above, we can deduce the following significant result:

Theorem 3.1. *Assume that the triplet (r_m, s_m, t_m) represents a well-defined solution to system (1.a). Then, we can derive the following explicit solution:*

$$\begin{aligned}
r_{6m} &= (\rho\sigma\tau)^m r_0 + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \left(\rho\sigma \frac{s_0 - \lambda}{t_{-1}} + \tau \frac{r_{-1}(r_0 - \varepsilon)}{(t_0 - \delta) s_{-1}} \right) \delta \right. \\
&\quad \left. + \left(\rho^2 \frac{(s_0 - \lambda) r_{-1}}{(t_0 - \delta) t_{-1}} + \frac{r_0 - \varepsilon}{s_{-1}} \right) \lambda + \left(\tfrac{\tau\rho}{\sigma} \frac{(s_0 - \lambda)(r_0 - \varepsilon) r_{-1}}{(t_0 - \delta) s_{-1} t_{-1}} + 1 \right) \varepsilon \right\},
\end{aligned}$$

$$\begin{aligned}
r_{6m+1} &= (\rho\sigma\tau)^m \left(\rho \frac{r_{-1} s_{-1}}{t_0 - \delta} + \varepsilon \right) + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \rho \left(\sigma^2 \frac{s_{-1}}{r_0 - \varepsilon} + \frac{(s_0 - \lambda) r_{-1}}{(t_0 - \delta) t_{-1}} \right) \delta \right. \\
&\quad \left. + \rho \left(\tfrac{\sigma\rho}{\tau} \frac{(s_0 - \lambda) s_{-1}}{(r_0 - \varepsilon) t_{-1}} + \frac{r_{-1}}{t_0 - \delta} \right) \lambda + \left(\rho^2 \frac{(s_0 - \lambda) r_{-1} s_{-1}}{(r_0 - \varepsilon)(t_0 - \delta) t_{-1}} + 1 \right) \varepsilon \right\},
\end{aligned}$$

$$\begin{aligned}
r_{6m+2} &= (\rho\sigma\tau)^m \left(\tfrac{\rho}{\tau} \frac{s_0 - \lambda}{t_{-1}} \left(\sigma \frac{s_{-1} t_{-1}}{r_0 - \varepsilon} + \lambda \right) + \varepsilon \right) + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \left(\sigma \frac{t_0 - \delta}{r_{-1}} + \tfrac{\rho}{\tau} \frac{(s_0 - \lambda) s_{-1}}{(r_0 - \varepsilon) t_{-1}} \right) \right. \\
&\quad \left. \times \sigma\delta + \tfrac{\rho}{\tau} \left(\sigma^2 \frac{(t_0 - \delta) s_{-1}}{(r_0 - \varepsilon) r_{-1}} + \frac{s_0 - \lambda}{t_{-1}} \right) \lambda + \left(\tfrac{\rho\sigma}{\tau} \frac{(s_0 - \lambda)(t_0 - \delta) s_{-1}}{(r_0 - \varepsilon) r_{-1} t_{-1}} + 1 \right) \varepsilon \right\},
\end{aligned}$$

$$\begin{aligned}
r_{6m+3} &= (\rho\sigma\tau)^m \left(\tfrac{\rho\sigma}{\tau} \frac{s_{-1}}{r_0 - \varepsilon} \left(\tfrac{\sigma}{\rho} \frac{t_0 - \delta}{r_{-1}} \left(\tau \frac{r_{-1} t_{-1}}{s_0 - \lambda} + \delta \right) + \lambda \right) + \varepsilon \right) + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \\
&\quad \times \left\{ \sigma^2 \left(\tau \frac{t_{-1}}{s_0 - \lambda} + \tfrac{1}{\tau} \frac{(t_0 - \delta) s_{-1}}{(r_0 - \varepsilon) r_{-1}} \right) \delta + \left(\sigma^2 \frac{(t_0 - \delta) t_{-1}}{(s_0 - \lambda) r_{-1}} + \tfrac{\rho\sigma}{\tau} \frac{s_{-1}}{r_0 - \varepsilon} \right) \lambda \right. \\
&\quad \left. + \left(\sigma^2 \frac{(t_0 - \delta) s_{-1} t_{-1}}{(r_0 - \varepsilon)(s_0 - \lambda) r_{-1}} + 1 \right) \varepsilon \right\},
\end{aligned}$$

$$\begin{aligned}
r_{6m+4} &= (\rho\sigma\tau)^m \left(\tfrac{\sigma}{\tau} \frac{t_0 - \delta}{r_{-1}} \left(\tfrac{\sigma\tau}{\rho} \frac{t_{-1}}{s_0 - \lambda} \left(\tfrac{\tau}{\sigma} \frac{r_0 - \varepsilon}{s_{-1}} \left(\rho \frac{r_{-1} s_{-1}}{t_0 - \delta} + \varepsilon \right) + \delta \right) + \lambda \right) + \varepsilon \right) \\
&\quad + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \sigma^2 \left(\tfrac{\tau}{\sigma} \frac{r_0 - \varepsilon}{s_{-1}} + \tfrac{1}{\rho} \frac{(t_0 - \delta) t_{-1}}{(s_0 - \lambda) r_{-1}} \right) \delta \right. \\
&\quad \left. + \sigma \left(\tau \frac{(r_0 - \varepsilon) t_{-1}}{(s_0 - \lambda) s_{-1}} + \tfrac{1}{\tau} \frac{t_0 - \delta}{r_{-1}} \right) \lambda + \left(\tfrac{\sigma\tau}{\rho} \frac{(r_0 - \varepsilon)(t_0 - \delta) t_{-1}}{(s_0 - \lambda) r_{-1} s_{-1}} + 1 \right) \varepsilon \right\},
\end{aligned}$$

$$\begin{aligned}
r_{6m+5} &= (\rho\sigma\tau)^m \left(\sigma \frac{t_{-1}}{s_0 - \lambda} \left(\tfrac{\tau}{\rho} \frac{r_0 - \varepsilon}{s_{-1}} \left(\tfrac{\tau\rho}{\sigma} \frac{r_{-1}}{t_0 - \delta} \left(\tfrac{\rho}{\tau} \widehat{s_0} \left(\sigma \frac{s_{-1} t_{-1}}{r_0 - \varepsilon} + \lambda \right) + \varepsilon \right) + \delta \right) + \lambda \right) + \varepsilon \right) \\
&\quad + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \sigma\tau \left(\rho \frac{r_{-1}}{t_0 - \delta} + \frac{1}{\rho} \frac{(r_0 - \varepsilon) t_{-1}}{(s_0 - \lambda) s_{-1}} \right) \delta \right. \\
&\quad \left. + \left(\rho\tau \frac{(r_0 - \varepsilon) r_{-1}}{(t_0 - \delta) s_{-1}} + \sigma \frac{t_{-1}}{s_0 - \lambda} \right) \lambda + \left(\tau^2 \frac{(r_0 - \varepsilon) r_{-1} t_{-1}}{(s_0 - \lambda)(t_0 - \delta) s_{-1}} + 1 \right) \varepsilon \right\},
\end{aligned}$$

and

$$\begin{aligned}
s_{6m} &= (\rho\sigma\tau)^m s_0 + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \left(\sigma\tau \frac{t_0 - \delta}{r_{-1}} + \rho \frac{s_{-1}(s_0 - \lambda)}{(r_0 - \varepsilon) t_{-1}} \right) \varepsilon \right. \\
&\quad \left. + \left(\sigma^2 \frac{(t_0 - \delta) s_{-1}}{(r_0 - \varepsilon) r_{-1}} + \frac{s_0 - \lambda}{t_{-1}} \right) \delta + \left(\tfrac{\sigma\rho}{\tau} \frac{(t_0 - \delta)(s_0 - \lambda) s_{-1}}{(r_0 - \varepsilon) t_{-1} r_{-1}} + 1 \right) \lambda \right\},
\end{aligned}$$

$$s_{6m+1} = (\rho\sigma\tau)^m \left(\sigma\frac{s_{-1}t_{-1}}{r_0-\varepsilon} + \lambda\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\sigma\left(\tau^2\frac{t_{-1}}{s_0-\lambda} + \frac{(t_0-\delta)\,s_{-1}}{(r_0-\varepsilon)\,r_{-1}}\right)\varepsilon + \sigma\left(\frac{\sigma\tau}{\rho}\frac{(t_0-\delta)\,t_{-1}}{(s_0-\lambda)\,r_{-1}} + \frac{s_{-1}}{r_0-\varepsilon}\right)\delta + \left(\sigma^2\frac{(t_0-\delta)\,s_{-1}t_{-1}}{(s_0-\lambda)\,(r_0-\varepsilon)\,r_{-1}} + 1\right)\lambda\right\},$$

$$s_{6m+2} = (\rho\sigma\tau)^m \left(\frac{\sigma}{\rho}\frac{t_0-\delta}{r_{-1}}\left(\tau\frac{t_{-1}r_{-1}}{s_0-\lambda} + \delta\right) + \lambda\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\left(\tau\frac{r_0-\varepsilon}{s_{-1}} + \frac{\sigma}{\rho}\frac{(t_0-\delta)\,t_{-1}}{(s_0-\lambda)\,r_{-1}}\right) \times\tau\,\varepsilon + \frac{\sigma}{\rho}\left(\tau^2\frac{(r_0-\varepsilon)\,t_{-1}}{(s_0-\lambda)\,s_{-1}} + \frac{t_0-\delta}{r_{-1}}\right)\delta + \left(\frac{\sigma\tau}{\rho}\frac{(t_0-\delta)\,(r_0-\varepsilon)\,t_{-1}}{(s_0-\lambda)\,s_{-1}r_{-1}} + 1\right)\lambda\right\},$$

$$s_{6m+3} = (\rho\sigma\tau)^m \left(\frac{\sigma\tau}{\rho}\frac{t_{-1}}{s_0-\lambda}\left(\frac{\tau}{\sigma}\frac{r_0-\varepsilon}{s_{-1}}\left(\rho\frac{s_{-1}r_{-1}}{t_0-\delta} + \varepsilon\right) + \delta\right) + \lambda\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \times\left\{\tau^2\left(\rho\frac{r_{-1}}{t_0-\delta} + \frac{1}{\rho}\frac{(r_0-\varepsilon)\,t_{-1}}{(s_0-\lambda)\,s_{-1}}\right)\varepsilon + \left(\tau^2\frac{(r_0-\varepsilon)\,r_{-1}}{(t_0-\delta)\,s_{-1}} + \frac{\sigma\tau}{\rho}\frac{t_{-1}}{s_0-\lambda}\right)\delta + \left(\tau^2\frac{(r_0-\varepsilon)\,t_{-1}r_{-1}}{(s_0-\lambda)\,(t_0-\delta)\,s_{-1}} + 1\right)\lambda\right\},$$

$$s_{6m+4} = (\rho\sigma\tau)^m \left(\frac{\tau}{\rho}\frac{r_0-\varepsilon}{s_{-1}}\left(\frac{\rho\tau}{\sigma}\frac{r_{-1}}{t_0-\delta}\left(\frac{\rho}{\tau}\frac{s_0-\lambda}{t_{-1}}\left(\sigma\frac{s_{-1}t_{-1}}{r_0-\varepsilon} + \lambda\right) + \varepsilon\right) + \delta\right) + \lambda\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\tau^2\left(\frac{\rho}{\tau}\frac{s_0-\lambda}{t_{-1}} + \frac{1}{\sigma}\frac{(r_0-\varepsilon)\,r_{-1}}{(t_0-\delta)\,s_{-1}}\right)\varepsilon + \tau\left(\rho\frac{(s_0-\lambda)\,r_{-1}}{(t_0-\delta)\,t_{-1}} + \frac{1}{\rho}\frac{r_0-\varepsilon}{s_{-1}}\right)\delta + \left(\frac{\rho\tau}{\sigma}\frac{(s_0-\lambda)\,(r_0-\varepsilon)\,r_{-1}}{(t_0-\delta)\,s_{-1}t_{-1}} + 1\right)\lambda\right\},$$

$$s_{6m+5} = (\rho\sigma\tau)^m \left(\tau\frac{r_{-1}}{t_0-\delta}\left(\frac{\rho}{\sigma}\frac{s_0-\lambda}{t_{-1}}\left(\frac{\rho\sigma}{\tau}\frac{s_{-1}}{r_0-\varepsilon}\left(\frac{\sigma}{\rho}\frac{t_0-\delta}{r_{-1}}\left(\tau\frac{t_{-1}r_{-1}}{s_0-\lambda} + \delta\right) + \lambda\right) + \varepsilon\right) + \delta\right) + \lambda\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\rho\tau\left(\sigma\frac{s_{-1}}{r_0-\varepsilon} + \frac{1}{\sigma}\frac{(s_0-\lambda)\,r_{-1}}{(t_0-\delta)\,t_{-1}}\right)\varepsilon + \left(\rho\sigma\frac{(s_0-\lambda)\,s_{-1}}{(r_0-\varepsilon)\,t_{-1}} + \tau\frac{r_{-1}}{t_0-\delta}\right)\delta + \left(\rho^2\frac{(s_0-\lambda)\,s_{-1}r_{-1}}{(t_0-\delta)\,(r_0-\varepsilon)\,t_{-1}} + 1\right)\lambda\right\},$$

and

$$t_{6m} = (\rho\sigma\tau)^m\, t_0 + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\left(\tau\rho\frac{r_0-\varepsilon}{s_{-1}} + \sigma\frac{t_{-1}\,(t_0-\delta)}{(s_0-\lambda)\,r_{-1}}\right)\lambda + \left(\tau^2\frac{(r_0-\varepsilon)\,t_{-1}}{(s_0-\lambda)\,s_{-1}} + \frac{t_0-\delta}{r_{-1}}\right)\varepsilon + \left(\frac{\tau\sigma}{\rho}\frac{(r_0-\varepsilon)\,(t_0-\delta)\,t_{-1}}{(s_0-\lambda)\,r_{-1}s_{-1}} + 1\right)\delta\right\},$$

$$t_{6m+1} = (\rho\sigma\tau)^m \left(\tau\frac{t_{-1}r_{-1}}{s_0-\lambda} + \delta\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\tau\left(\rho^2\frac{r_{-1}}{t_0-\delta} + \frac{(r_0-\varepsilon)\,t_{-1}}{(s_0-\lambda)\,s_{-1}}\right)\lambda + \tau\left(\frac{\rho\tau}{\sigma}\frac{(r_0-\varepsilon)\,r_{-1}}{(t_0-\delta)\,s_{-1}} + \frac{t_{-1}}{s_0-\lambda}\right)\varepsilon + \left(\tau^2\frac{(r_0-\varepsilon)\,t_{-1}r_{-1}}{(t_0-\delta)\,(s_0-\lambda)\,s_{-1}} + 1\right)\delta\right\},$$

$$t_{6m+2} = (\rho\sigma\tau)^m \left(\frac{\tau}{\sigma}\frac{r_0-\varepsilon}{s_{-1}}\left(\rho\frac{r_{-1}s_{-1}}{t_0-\delta} + \varepsilon\right) + \delta\right) + \sum_{k=0}^{m-1}(\rho\sigma\tau)^k \left\{\left(\rho\frac{s_0-\lambda}{t_{-1}} + \frac{\tau}{\sigma}\frac{(r_0-\varepsilon)\,r_{-1}}{(t_0-\delta)\,s_{-1}}\right) \times\rho\,\lambda + \frac{\tau}{\sigma}\left(\rho^2\frac{(s_0-\lambda)\,r_{-1}}{(t_0-\delta)\,t_{-1}} + \frac{r_0-\varepsilon}{s_{-1}}\right)\varepsilon + \left(\frac{\rho\tau}{\sigma}\frac{(r_0-\varepsilon)\,(s_0-\lambda)\,r_{-1}}{(t_0-\delta)\,t_{-1}s_{-1}} + 1\right)\delta\right\},$$

$$t_{6m+3} = (\rho\sigma\tau)^m \left(\tfrac{\rho\tau}{\sigma} \frac{r_{-1}}{t_0 - \delta} \left(\tfrac{\rho}{\tau} \frac{s_0 - \lambda}{t_{-1}} \left(\sigma \frac{t_{-1} s_{-1}}{r_0 - \varepsilon} + \lambda \right) + \varepsilon \right) + \delta \right) + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k$$
$$\times \left\{ \rho^2 \left(\sigma \frac{s_{-1}}{r_0 - \varepsilon} + \tfrac{1}{\sigma} \frac{(s_0 - \lambda) r_{-1}}{(t_0 - \delta) t_{-1}} \right) \lambda + \left(\rho^2 \frac{(s_0 - \lambda) s_{-1}}{(r_0 - \varepsilon) t_{-1}} + \tfrac{\rho\tau}{\sigma} \frac{r_{-1}}{t_0 - \delta} \right) \varepsilon \right.$$
$$\left. + \left(\rho^2 \frac{(s_0 - \lambda) r_{-1} s_{-1}}{(t_0 - \delta)(r_0 - \varepsilon) t_{-1}} + 1 \right) \delta \right\},$$

$$t_{6m+4} = (\rho\sigma\tau)^m \left(\tfrac{\rho}{\sigma} \frac{s_0 - \lambda}{t_{-1}} \left(\tfrac{\rho\sigma}{\tau} \frac{s_{-1}}{r_0 - \varepsilon} \left(\tfrac{\sigma}{\rho} \frac{t_0 - \delta}{r_{-1}} \left(\tau \frac{t_{-1} r_{-1}}{s_0 - \lambda} + \delta \right) + \lambda \right) + \varepsilon \right) + \delta \right)$$
$$+ \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \rho^2 \left(\tfrac{\sigma}{\rho} \frac{t_0 - \delta}{r_{-1}} + \tfrac{1}{\tau} \frac{(s_0 - \lambda) s_{-1}}{(r_0 - \varepsilon) t_{-1}} \right) \lambda \right.$$
$$\left. + \rho \left(\sigma \frac{(t_0 - \delta) s_{-1}}{(r_0 - \varepsilon) r_{-1}} + \tfrac{1}{\sigma} \frac{s_0 - \lambda}{t_{-1}} \right) \varepsilon + \left(\tfrac{\sigma\rho}{\tau} \frac{(t_0 - \delta)(s_0 - \lambda) s_{-1}}{(r_0 - \varepsilon) t_{-1} r_{-1}} + 1 \right) \delta \right\},$$

$$t_{6m+5} = (\rho\sigma\tau)^m \left(\rho \frac{s_{-1}}{r_0 - \varepsilon} \left(\tfrac{\sigma}{\tau} \frac{t_0 - \delta}{r_{-1}} \left(\tfrac{\tau\sigma}{\rho} \frac{t_{-1}}{s_0 - \lambda} \left(\tfrac{\tau}{\sigma} \frac{r_0 - \varepsilon}{s_{-1}} \left(\rho \frac{r_{-1} s_{-1}}{t_0 - \delta} + \varepsilon \right) + \delta \right) + \lambda \right) + \varepsilon \right) \right.$$
$$\left. + \delta \right) + \sum_{k=0}^{m-1} (\rho\sigma\tau)^k \left\{ \rho\sigma \left(\tau \frac{t_{-1}}{s_0 - \lambda} + \frac{1}{\tau} \frac{(t_0 - \delta) s_{-1}}{(r_0 - \varepsilon) r_{-1}} \right) \lambda \right.$$
$$\left. + \left(\sigma\tau \frac{(t_0 - \delta) t_{-1}}{(s_0 - \lambda) r_{-1}} + \rho \frac{s_{-1}}{r_0 - \varepsilon} \right) \varepsilon + \left(\sigma^2 \frac{(t_0 - \delta) t_{-1} s_{-1}}{(r_0 - \varepsilon)(s_0 - \lambda) r_{-1}} + 1 \right) \delta \right\}.$$

Remark 3.1. *The closed-form analytical expressions obtained in this study clearly demonstrate that the stability behavior of the system is governed by the multiplicative factor* $(\rho\sigma\tau)^m$*, which appears in the three interacting sequences* (r_m, s_m, t_m)*. Consequently, the stability characteristics can be directly inferred from the explicit solutions without any need for linearization or numerical approximation. Specifically: The system is asymptotically stable if* $|\rho\sigma\tau| < 1$*; unstable if* $|\rho\sigma\tau| > 1$*; the system exhibits bounded periodic (marginally stable) behavior when* $|\rho\sigma\tau| = 1$*. These analytical criteria provide a precise and transparent characterization of the system's dynamical nature, illustrating how the explicit solution structure itself encapsulates the conditions for stability. This result highlights the mathematical rigor and methodological efficiency of the proposed analytical framework, as it establishes stability properties intrinsically, rather than relying on external numerical or perturbative techniques.*

4. Counterexamples and numerical illustrations

To further clarify and validate the analytical results obtained in the previous section, this section provides a set of counterexamples and numerical illustrations. These examples are designed to demonstrate the accuracy and applicability of the theoretical framework developed for the three-dimensional nonlinear difference system. In particular, they highlight how variations in parameter values and initial conditions can influence the qualitative behavior of the sequences.

Example 4.1. *In this example, we explore a dynamic system characterized by three sequences:* (r_m)*,*

(s_m), *and* (t_m) , *as described in the followinng system:*

$$\begin{aligned} r_{m+1} &= \rho \frac{r_{m-1}}{t_m - 0.45} s_m - 1.35, \; \forall m \geq 0, \\ s_{m+1} &= \sigma \frac{s_{m-1}}{r_m + 1.35} t_m + 1.35, \\ t_{m+1} &= \tau \frac{t_{m-1}}{s_m - 1.35} r_m + 0.45, \end{aligned} \tag{2.h}$$

with $\rho = \sigma = \tau = 1.35$. *The initial values are specified as* $r_0 = -0.7$, $r_{-1} = 1.2$, $s_0 = -0.5$, $s_{-1} = 0.3$, $t_0 = 0.9$, *and* $t_{-1} = -1.5$. *To illustrate the system's behavior over time, we generate plots of* $\{r_m\}$, $\{s_m\}$, *and* $\{t_m\}$ *over multiple iterations. This graphical analysis is presented in Figure* 1.

In Figure 1, the plots reveal extreme instability in the sequences r_m, s_m, *and* t_m, *characterized by rapid and large-magnitude oscillations that worsen over time. The initial values and parameters used in this simulation result in an explosive system where the sequences diverge rapidly. The behavior observed here is indicative of a system that is highly unstable and sensitive to even minor changes in initial conditions.*

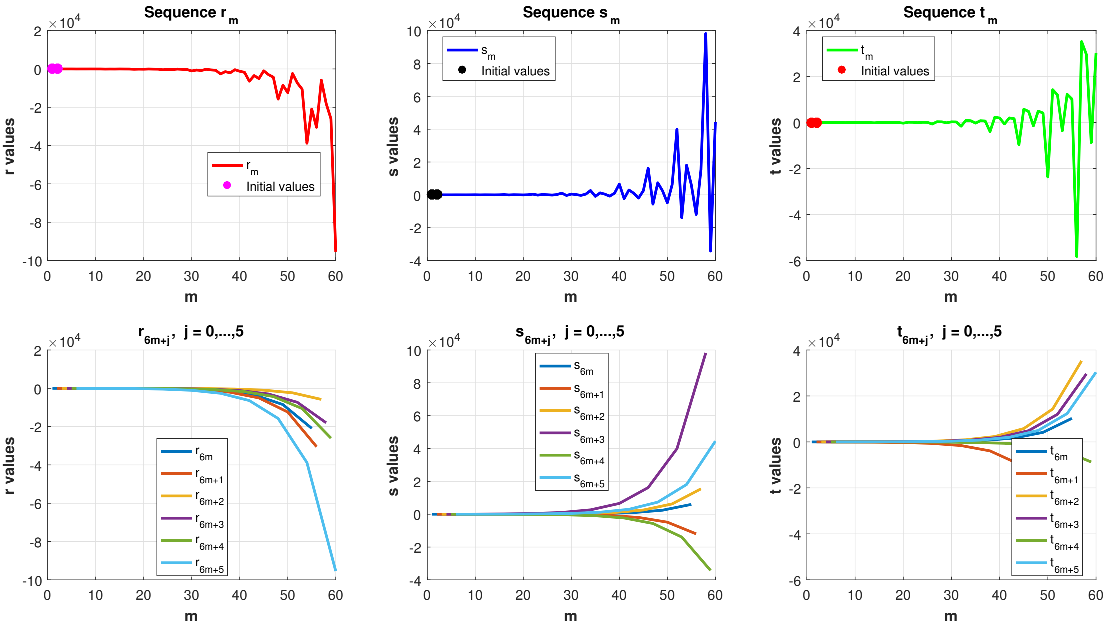

Figure 1. Extreme instability in r_m, s_m, and t_m due to severe parameterization.

Example 4.2. *In this illustrative example, we consider the system (2.h) with parameter values* $\rho = 0.5$, $\sigma = 0.9$, *and* $\tau = 3.0$. *To examine the system's temporal evolution, we compute and plot the sequences* $\{r_m\}$, $\{s_m\}$, *and* $\{t_m\}$ *over a series of discrete iterations. The resulting trajectories, depicted in Figure 2, provide a clear visualization of the system's dynamic response and highlight the oscillatory and interactive behavior among the three components under the given parameter configuration.*

Figure 2 depicts the temporal evolution of the three sequences $\{r_m\}$, $\{s_m\}$, *and* $\{t_m\}$ *under relatively high parameter values* ($\rho = 0.5$, $\sigma = 0.9$, *and* $\tau = 3.0$). *The trajectories exhibit a rapid and irregular* *increase in amplitude, leading to a numerical explosion in all three variables. The oscillations alternate sharply between positive and negative values, indicating extreme instability resulting from the unbalanced nonlinear coupling among the variables. This behavior suggests that the chosen parameters amplify rather than dampen the recursive dynamics, producing a chaotic-like regime characterized by high sensitivity to initial conditions and long-term unpredictability.*

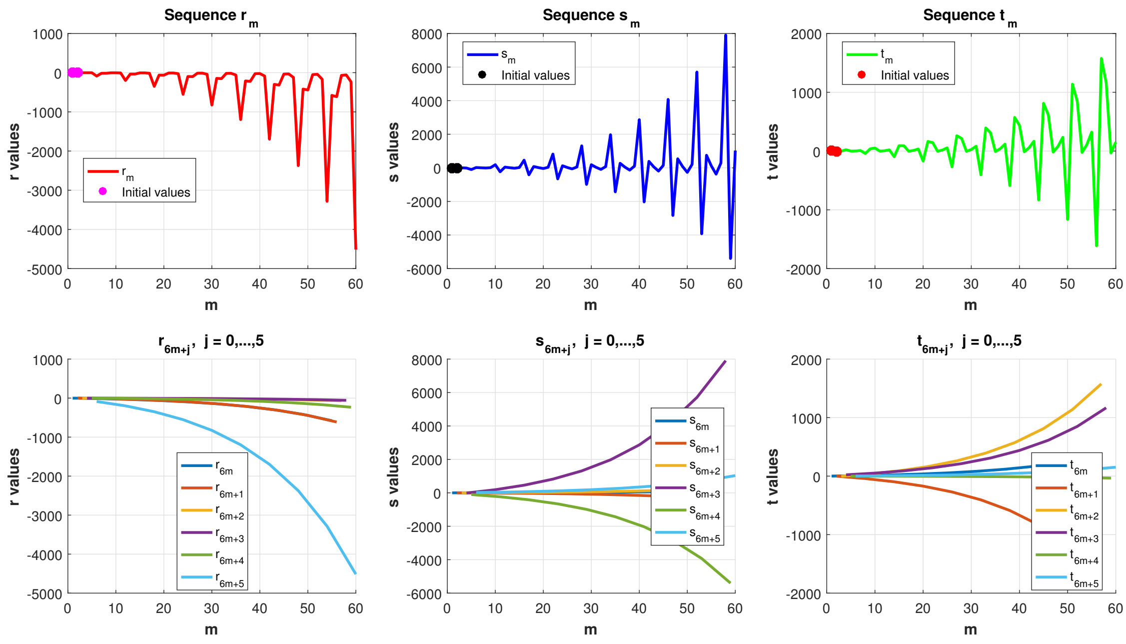

Figure 2. Extreme instability in r_m, s_m, and t_m due to severe parameterization.

Example 4.3. *In this example, we investigate a three-dimensional nonlinear discrete system defined by the sequences* (r_m)*,* (s_m)*, and* (t_m)*, which evolve according to the following recursive relations:*

$$\begin{aligned} r_{m+1} &= \frac{r_{m-1}}{t_m - 0.2} s_m + 0.2, \ \forall m \geq 0, \\ s_{m+1} &= \frac{s_{m-1}}{r_m - 0.2} t_m + 0.2, \\ t_{m+1} &= \frac{t_{m-1}}{s_m - 0.2} r_m + 0.2. \end{aligned} \tag{2.i}$$

The initial values are specified as $r_0 = -0,10$*,* $r_{-1} = 0,10$*,* $s_0 = -0,12$*,* $s_{-1} = 0,08$*,* $t_0 = -0,11$*, and* $t_{-1} = 0,09$*. To illustrate the system's behavior over time, we generate plots of* $\{r_m\}$*,* $\{s_m\}$*, and* $\{t_m\}$ *over multiple iterations. This graphical analysis is presented in Figure* 3*,*

Figure 3 illustrates the bounded and periodic behavior of system (2.i), where the three sequences $\{r_m\}$*,* $\{s_m\}$*, and* $\{t_m\}$ *evolve in a regular and repetitive manner over time. The initial plots show that all values remain confined within a narrow range around zero, indicating bounded stability without divergence or numerical explosion. Moreover, the subsequences* $\left\{r_{6m+j}\right\}$*,* $\left\{s_{6m+j}\right\}$*, and* $\left\{t_{6m+j}\right\}$ *reveal an approximate six-period cycle, in which the oscillation patterns repeat consistently and symmetrically. This behavior signifies that the system attains a state of stable periodic oscillation, reflecting a* *delicate balance between the opposing nonlinear forces within the system. The reciprocal interactions among the variables maintain the trajectories within stable bounds, preventing chaotic divergence and ensuring sustained periodic stability.*

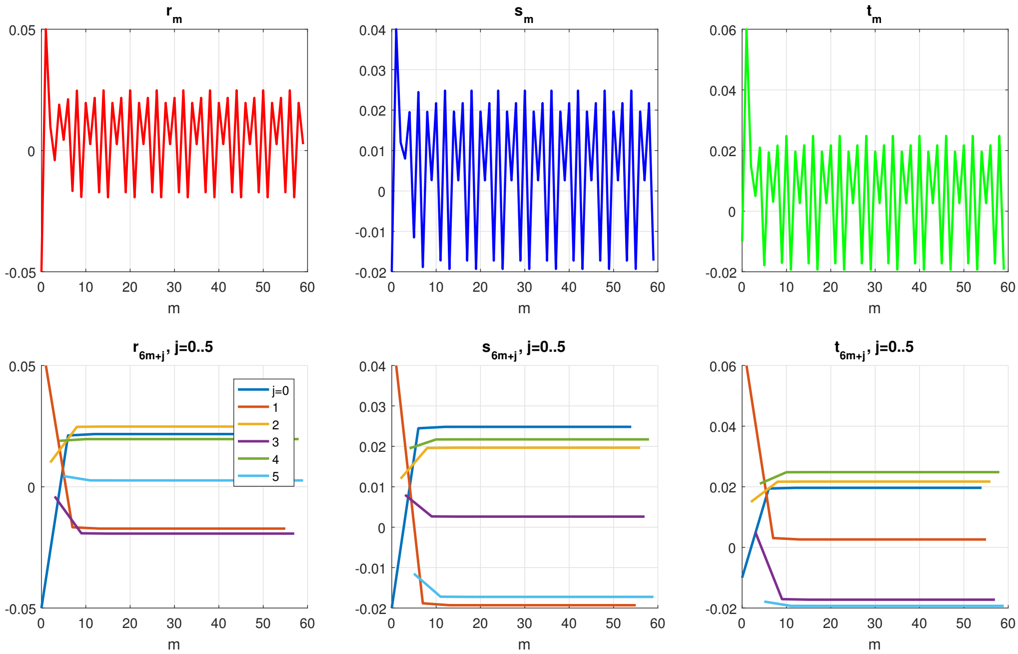

Figure 3. Periodic and bounded dynamics of system (2.i).

Example 4.4. *In this example, we explore a dynamic system characterized by three sequences:* (r_m), (s_m), *and* (t_m) *, as described in the following system:*

$$\begin{aligned} r_{m+1} &= 0.70\frac{r_{m-1}}{t_m - 0.05}s_m + 0.04, \ \forall m \geq 0, \\ s_{m+1} &= 0.75\frac{s_{m-1}}{r_m - 0.04}t_m + 0.04, \\ t_{m+1} &= 0.80\frac{t_{m-1}}{s_m - 0.04}r_m + 0.05. \end{aligned} \tag{2.j}$$

The initial values are specified as $r_0 = -0.10$, $r_{-1} = 0.10$, $s_0 = -0.12$, $s_{-1} = 0.08$, $t_0 = -0.11$, *and* $t_{-1} = 0.09$. *To illustrate the system's behavior over time, we generate plots of* $\{r_m\}$, $\{s_m\}$, *and* $\{t_m\}$ *over multiple iterations. This graphical analysis is presented in Figure 4.*

Figure 4 illustrates the system response of (2.j) under moderately damped parameter values ($\rho = 0.70$, $\sigma = 0.75$, *and* $\tau = 0.80$). *The sequences* $\{r_m\}$, $\{s_m\}$, *and* $\{t_m\}$ *initially exhibit small oscillations around zero, which gradually diminish over time and converge to a constant steady value. This behavior represents asymptotic stability, where oscillations decay progressively with increasing iterations, confirming that the damping effect dominates the system's internal dynamics. Furthermore, the detailed trajectories of the subsequences* $\left\{r_{6m+j}\right\}$, $\left\{s_{6m+j}\right\}$, *and* $\left\{t_{6m+j}\right\}$ *reveal that all paths converge toward the same terminal value, indicating insensitivity to initial conditions and the presence of a fixed* *attractor corresponding to a stable equilibrium point. Such a response highlights the system's inherent capacity to absorb initial perturbations and settle into a dynamically balanced steady state.*

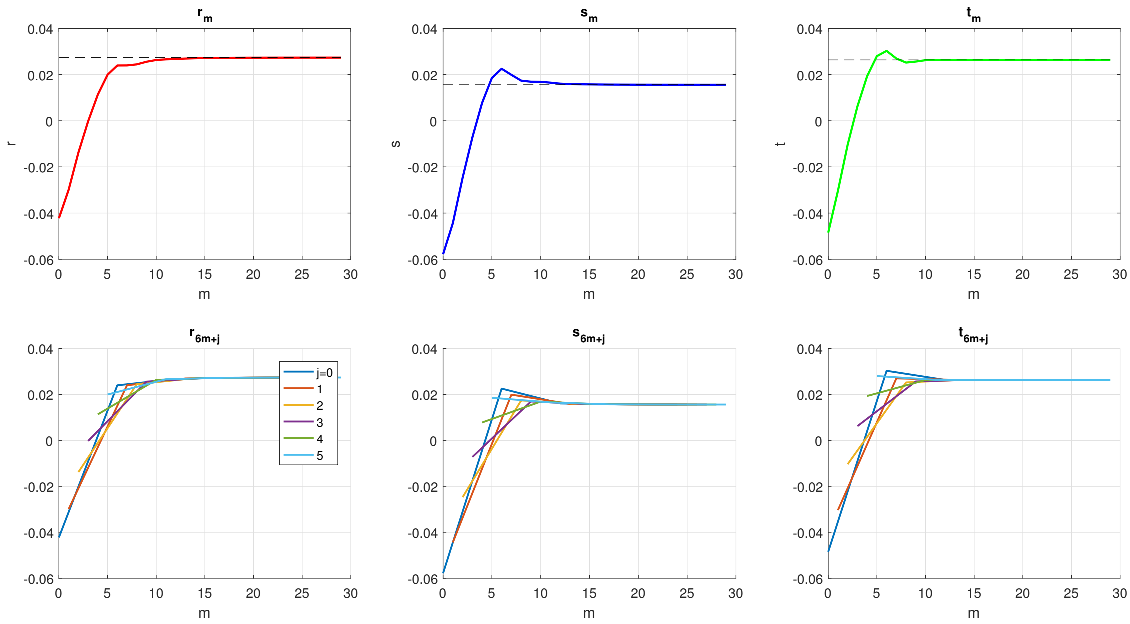

Figure 4. Asymptotically stable behavior of system (2.j).

Remark 4.1. *The figures and numerical results presented in this section embody the physical and dynamical interpretation of the studied system. The three sequences* $\{r_m\}$, $\{s_m\}$, *and* $\{t_m\}$ *represent interdependent variables that capture the evolution of three interacting quantities within a nonlinear discrete framework exhibiting either periodic or steady-state characteristics. In scenarios where unbounded oscillations or numerical explosions occur (as shown in Figures 1 and 2), these phenomena can be interpreted as manifestations of a system driven far from equilibrium or exhibiting strong sensitivity to initial conditions, analogous to physical or biological systems that surpass their stability thresholds due to cumulative nonlinear interactions. Conversely, under stable or quasi-periodic regimes (as illustrated in Figures 3 and 4), the results correspond to the response of a dynamically balanced system, in which the mutual interactions among the variables effectively dissipate initial disturbances, leading to either equilibrium or regular oscillatory motion. Hence, the numerical experiments presented here not only demonstrate abstract mathematical behaviors but also provide a qualitative understanding of the physical essence of discrete nonlinear systems, illustrating how variations in parameters and initial stimuli govern the transition between stability and chaotic dynamics.*

Remark 4.2. *The case where* $t_m = \delta$, $r_m = \varepsilon$, *or* $s_m = \lambda$ *for some m leads to a singularity in system (1.a), since one or more denominators vanish, resulting in undefined or divergent terms. Mathematically, this situation corresponds to the system reaching a degenerate manifold where the recursive relations collapse. From a dynamical perspective, such a state may be interpreted as a critical transition point, indicating the loss of regularity or the onset of instability. In future work, we* *aim to investigate regularization or perturbation-based approaches to handle these singular cases, for instance by introducing small perturbations $\delta \to \delta + \eta_t$, $\varepsilon \to \varepsilon + \eta_r$, and $\lambda \to \lambda + \eta_s$, where η_t, η_r, η_s are infinitesimal parameters. This approach may enable the characterization of limiting behaviors and bifurcation patterns near these singular states.*

Remark 4.3. *In comparison with the studies reviewed in the introduction, the present paper distinguishes itself through both its mathematical formulation and analytical methodology. Unlike most previous research, which primarily focused on one- or two-dimensional models, this study develops a novel three-dimensional nonlinear discrete system that effectively captures the mutual interactions among three interdependent variables. The analytical framework employed here is based on a series of systematic transformations, enabling the derivation of explicit closed-form solutions while preserving the intrinsic nonlinear structure of the system. Moreover, the obtained results reveal that even slight variations in the system's initial conditions or parameter values can induce abrupt transitions in its periodic behavior, emphasizing the model's structural sensitivity and dynamical richness. Consequently, this research represents a natural and rigorous extension of previously investigated two-dimensional frameworks (e.g., [21]) into a three-dimensional setting that is not only more mathematically intricate but also more representative of realistic nonlinear phenomena.*

Remark 4.4. *It is important to note that the proposed analytical framework provides a distinct computational advantage. Once the explicit closed-form expressions for the sequences (r_m), (s_m), and (t_m) are obtained, each subsequent term can be computed directly without the need for iterative or recursive numerical procedures. Consequently, the computation of each time step is achieved in constant time, O(1), in contrast to conventional numerical schemes whose computational cost typically increases linearly or super-linearly with the simulation length. This feature makes the proposed method both mathematically transparent and computationally efficient, thereby enhancing its suitability for long-term dynamical analysis of discrete nonlinear systems and for applications where high precision and low computational complexity are simultaneously required.*

5. Results and discussion

This section highlights the key analytical and numerical findings obtained throughout the study. It aims to demonstrate how the mathematical structure of the proposed system governs and interacts with the dynamic behaviors observed in the numerical simulations. By integrating rigorous analytical derivations with computational experiments, a coherent and comprehensive understanding of the system's evolution was achieved, capturing the transitions between unstable, periodic, and asymptotically stable states within the proposed three-dimensional framework. From an analytical standpoint, the system, defined through the interdependent relationships among the three variables r_m, s_m, and t_m, exhibits a closed sixth-order periodic structure arising from the cyclic coupling of these variables. The obtained closed-form expressions reveal that the system's dynamic behavior is fundamentally governed by the absolute value of the composite product $(\rho\sigma\tau)$, which serves as the principal determinant of the system's stability properties. When the condition $|\rho\sigma\tau| < 1$ is satisfied, the system converges toward an asymptotically stable state, in which oscillations progressively diminish and the sequences r_m, s_m, and t_m approach a fixed equilibrium point. For $|\rho\sigma\tau| = 1$, the system exhibits stable periodic behavior, characterized by a regular recurrence of states over successive iterations, indicating a balance between the internal nonlinear forces and damping effects. Conversely, when

$|\rho\sigma\tau| > 1$, the system becomes highly unstable, with amplitudes that grow rapidly and extreme sensitivity to initial conditions. This regime leads to chaotic-like dynamics, where small perturbations result in significant divergence in the system's trajectories, reflecting a breakdown of equilibrium and loss of predictability.

Numerically, the obtained results were in full agreement with the theoretical predictions. Figures 1 and 2 illustrate that when the system parameters take relatively large values, the sequences experience a clear numerical explosion, reflecting a complete loss of equilibrium and the onset of divergent, unstable dynamics. In contrast, Figure 3 demonstrates a regime of bounded stability accompanied by regular periodic oscillations of approximately sixth order. This pattern confirms the presence of a recurring dynamic balance resulting from the mutual compensations among the interacting variables. Finally, Figure 4 depicts the most evident case of asymptotic stability, where the amplitudes of the three sequences gradually decrease and converge toward a steady numerical equilibrium around zero. This numerical behavior aligns precisely with the analytical stability condition $|\rho\sigma\tau| < 1$, validating the theoretical framework developed in this study.

These results demonstrate that the nonlinear structure of the system does not merely generate random or chaotic-like behavior but is governed by well-defined stability relationships that can be precisely controlled through parameter adjustment. Each of the three variables exerts an indirect influence on the others, producing reciprocal responses that range from amplification to attenuation depending on the chosen parameter values. Consequently, the proposed model exhibits a clear dynamic transition, from numerical explosion, through finite periodicity, to asymptotic stability, reflecting the structural richness of the system and its capacity to represent diverse patterns observed in natural, biological, and socio-economic phenomena. Furthermore, the findings emphasize the computational efficiency and analytical clarity of the proposed framework. The availability of closed-form solutions enables the direct computation of future states without resorting to iterative numerical schemes. This reduces the computational cost to O(1) per iteration, making the approach particularly well-suited for long-term simulations or large-scale discrete system analyses.

Based on the above, the main conclusions can be summarized as follows:

- The compound product ($\rho\sigma\tau$) serves as the principal control parameter governing the system's dynamic behavior and determines the transition between stability, periodicity, and chaos.
- A balanced interaction among the system's variables gives rise to bounded periodicity with an approximate six-period cycle, indicating the presence of structural stability.
- When the parameters are moderately damped, the system exhibits asymptotic stability, converging toward a fixed equilibrium point and demonstrating its ability to absorb initial perturbations.
- The proposed analytical approach not only captures the qualitative dynamics of the system but also facilitates the derivation of accurate numerical solutions with minimal computational effort, thus bridging theoretical rigor and practical applicability.

6. Conclusions

In this paper, we investigated the solvability and periodicity of a three-dimensional system of difference equations, deriving explicit solutions under different parameter conditions. The significance of the results obtained in this paper lies in the elucidation of the precise analytical structure underlying three-dimensional nonlinear discrete systems, representing a natural and rigorous extension of previously investigated two-dimensional models. Through a systematic analytical approach, the original system was transformed into a set of solvable recurrence relations, allowing the derivation of explicit closed-form expressions that clearly demonstrate how mutual nonlinear interactions among variables give rise to periodic and oscillatory dynamics. The numerical experiments further reveal that even slight perturbations in system parameters or initial conditions can induce substantial transitions in oscillatory patterns, thereby confirming the system's structural sensitivity and dynamical richness. Collectively, these findings establish a robust analytical framework that advances the theoretical understanding of discrete nonlinear dynamics and provides a solid foundation for practical applications in biological, physical, and economic modeling, domains in which recursive relationships play a central role in describing complex phenomena.

Future research may focus on a comprehensive investigation of the stability properties and bifurcation structures of the proposed system by introducing small parametric perturbations and examining their influence on the long-term dynamical behavior. Extending the present analytical framework to four- or higher-dimensional nonlinear systems also constitutes a promising direction, as the inclusion of additional coupling mechanisms could give rise to richer and more intricate oscillatory patterns. Moreover, comparative analyses with alternative nonlinear formulations, such as fuzzy difference systems, would provide valuable insights into the robustness and persistence of the obtained periodic solutions. Finally, it is recommended to design and implement specialized numerical algorithms tailored to the proposed system, particularly those employing adaptive time-stepping or invariant-preserving schemes, to further enhance the computational accuracy of the theoretical results and to broaden their applicability to real-world problems in discrete dynamical modeling.

Author contributions

A. Ghezal: Writing-original draft preparation, methodology; N. Attia: Conceptualization; N. Attia and A. Ghezal: Formal analysis, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [KFU253940].

Conflict of interest

The authors declare no conflicts of interest in this paper.

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