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Research article

Third Hankel determinant for a subclass of bi-univalent functions related to balloon-shaped domain

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Abstract: In this paper, we consider a subclass of bi-univalent functions, denoted by $\mathbb{CS}^*_{\beth}(\gamma)$, which is defined via a balloon-shaped domain associated with the function $\frac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}$. We prove that this class is non-empty using illustrative mappings. We investigate upper bounds for the second- and third-order Hankel determinants, focusing on the functional $H_3(1)$. In addition, we estimate the Taylor coefficients up to order $l = 5$, which are essential in obtaining bounds for the determinants. The balloon shape introduces new analytic features that enrich the behavior of the functions in this class. Several examples and special cases are offered to illustrate the flexibility of the outcomes.

Keywords: Hankel determinant; unit disk $\mathcal{U}$; analytic; bi-univalent functions; Fekete-Szegö inequality
Mathematics Subject Classification: 30C45

1. Introduction

Hankel determinants originate from Hankel matrices, where the entries along each anti-diagonal are identical. The q-th Hankel determinant corresponds to a $(q+1)\times(q+1)$ determinant constructed from a sequence, and has applications in both combinatorics and the theory of analytic functions. In the context of geometric function theory, bi-univalent functions that are analytic and injective in the unit disk, with analytic and injective inverses, and are of particular interest. Estimating their coefficients remains a central challenge, with tools such as the FeketeSzegö functional being widely employed.

Especially, the second and third Hankel determinants have received attention in several subclasses to extract geometric insights [1–4].

Let $\mathcal{A}$ denote the class of functions analytic in the open unit disk $\mathcal{U} := \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$, and expressed as

$$f(\varsigma) = \varsigma + \sum_{l=2}^{\infty} a_l \varsigma^l. \tag{1.1}$$

According to the Koebe one-quarter theorem (see [5]), any univalent function $f \in \mathcal{A}$ maps $\mathcal{U}$ onto a region containing a disk of radius $\frac{1}{4}$ centered at the origin. Consequently, such a function possesses an inverse f^{-1}, which is analytic in at least the disk $|w| < r_0(f)$ where $r_0(f) \geq \frac{1}{4}$, and satisfies the identities

$$f^{-1}(f(\varsigma)) = \varsigma \quad \text{for } \varsigma \in \mathcal{U}, \qquad f(f^{-1}(w)) = w \quad \text{for } |w| < r_0(f).$$

The inverse f^{-1} admits the expansion

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is called bi-univalent if both f and f^{-1} are univalent in $\mathcal{U}$. The family of such functions is usually denoted by $\daleth$. The study of class $\daleth$ gained considerable momentum following the foundational work by Srivastava and collaborators [6]. Some classical examples of bi-univalent functions include:

$$\frac{\varsigma}{1-\varsigma}, \quad -\log(1-\varsigma), \quad \frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right).$$

It is worth noting that the class of univalent functions is strictly larger than the class of bi-univalent functions. Indeed, certain classical univalent functions, such as the Koebe function, $\varsigma - \frac{\varsigma^2}{2}$, and $\frac{\varsigma}{1-\varsigma^2}$, are not bi-univalent, since their inverses fail to preserve univalence within the unit disk $\mathcal{U}$ [6,7].

When studying bi-univalent functions, a major issue is the estimation of initial coefficients in their Taylor expansions. An early result is due to Lewin [8], who established the bound $|a_2| \leq 1.51$. This was later improved by Brannan and Clunie [9] to $|a_2| \leq \sqrt{2}$, and subsequently refined by Netanyahu [10] to $|a_2| \leq \frac{4}{3}$. In pursuit of sharper bounds, various subclasses of $\daleth$ have been introduced, particularly those defined via starlikeness and convexity constraints [11,12]. Among the key quantities examined is the Fekete-Szegö functional of the form $|a_3 - \eta a_2^2|$, which has received significant attention across different subclasses [13].

Given two functions $f, g \in \mathcal{A}$, we say that f is subordinate to g, written $f(\varsigma) \prec g(\varsigma)$, if there exists a function $u(\varsigma)$ analytic in $\mathcal{U}$, satisfying $u(0) = 0$ and $|u(\varsigma)| < 1$, such that

$$f(\varsigma) = g(u(\varsigma)).$$

When g is univalent in $\mathcal{U}$, this implies that:

$$f(0) = g(0), \quad \text{and} \quad f(\mathcal{U}) \subseteq g(\mathcal{U}) \quad \text{(see [7,14])}.$$

A function $f \in \mathcal{S}$ is said to be starlike if

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \frac{1+\varsigma}{1-\varsigma}, \quad \varsigma \in \mathcal{U}.$$

This is equivalent to requiring that

$$\Re\left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\right) > 0, \quad \text{for all } \varsigma \in \mathcal{U},$$

which geometrically ensures that $f(\mathcal{U})$ is starlike with respect to the origin. For a detailed discussion, refer to [15, 16].

The q-th Hankel determinant was provided by Noonan and Thomas in 1976 [17]. This determinant is constructed from the Taylor coefficients of an analytic function by forming a $q \times q$ Hankel matrix starting at the coefficient a_l, and then taking its determinant. Formally, it is defined as

$$H_q(l) = \begin{vmatrix} a_l & a_{l+1} & \cdots & a_{l+q-1} \\ a_{l+1} & a_{l+2} & \cdots & a_{l+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l+q-1} & a_{l+q} & \cdots & a_{l+2q-2} \end{vmatrix}, \quad l, q \in \mathbb{N}, \quad a_1 = 1. \tag{1.3}$$

The Hankel determinant encodes relations among the coefficients and has been extensively studied, particularly for large l in connection with boundary behavior of analytic functions [18, 19]. More recently, attention has shifted to smaller indices such as $H_2(2)$, especially within subclasses of bi-univalent functions [1, 20, 21].

For illustrative purposes, when $q = 2$ and $l = 1$, the determinant simplifies to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \tag{1.4}$$

and for $l = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{1.5}$$

Such determinants arise frequently in function theory due to their ability to capture nonlinear dependencies among the Taylor coefficients.

Among these, the functional $H_2(1) = a_3 - a_2^2$, commonly referred to as the Fekete-Szegö functional, plays a prominent role in bounding coefficient estimates. Recent investigations by Shakir et al. [20] have extended the study of bounds for $H_2(2)$ in various subclasses (see also [22, 23]).

In the present work, we extend the analysis to the third Hankel determinant $H_3(1)$, corresponding to $q = 3$ and $l = 1$, defined by the determinant

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Given the normalization $a_1 = 1$ for functions in $\mathcal{A}$, this determinant can be expanded as

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \tag{1.6}$$

Utilizing the triangle inequality yields the estimate

$$|H_3(1)| \le |a_3| \cdot |a_2 a_4 - a_3^2| + |a_4| \cdot |a_4 - a_2 a_3| + |a_5| \cdot |a_3 - a_2^2|. \tag{1.7}$$

Such an upper bound is particularly useful when examining subclasses of analytic functions where control over coefficient magnitudes is essential.

This study primarily focuses on two objectives: first, to derive upper bounds for the second-order Hankel determinant,

$$H_2(2) = a_2a_4 - a_3^2, \tag{1.8}$$

and second, to establish estimates for the third-order determinant $H_3(1)$, relying on inequality (1.7). Accomplishing these goals requires determining sharp bounds for the quantities

$$|a_3 - a_2^2|, \quad |a_3|, \quad |a_4|, \quad |a_5|.$$

2. Definitions, lemmas, and examples

Recently, several works have focused on analytic functions associated with the balloon-shaped domain. Ahmad et al. [24] introduced and studied such functions, while Khan et al. [25] obtained sharp coefficient bounds and upper estimates for the second Hankel determinant in this setting. In a related direction, Shakir et al. [20] investigated the third Hankel determinant for bi-univalent functions in the crescent-shaped domain. Motivated by these developments, the present paper considers a new subclass of bi-univalent functions related to the balloon-shaped domain and establishes upper bounds for the third Hankel determinant. In this work, we introduce and study a novel subclass of bi-univalent functions, denoted by $\mathbb{CS}^*_{\underline{\circ}}(\gamma)$, associated with the balloon-shaped domain characterized by the analytic function $\frac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}$. This class serves as a unified framework encompassing both starlike and convex bi-univalent functions. Our principal objective is to establish sharp estimates for the initial Taylor coefficients $|a_2|$ and $|a_3|$, analyze the Fekete Szegö functional, and derive bounds for the third Hankel determinant $H_3(1)$, thereby gaining deeper insight into the geometric and analytic properties of functions in this class.

Definition 2.1. Let f be a bi-univalent function defined on $\mathcal{U}$. The function f belongs to the class $\mathbb{CS}^*_{\underline{\circ}}(\gamma)$ if it satisfies the conditions

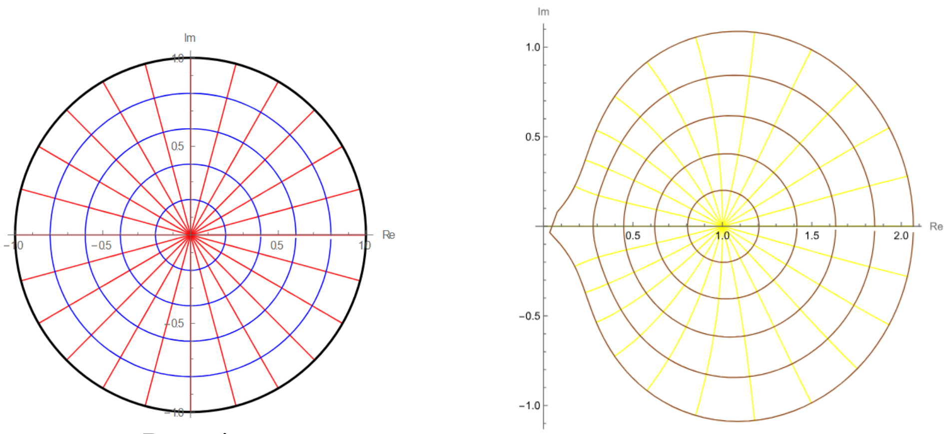
$$\frac{\varsigma f'(\varsigma) + \gamma\varsigma^2 f''(\varsigma)}{\gamma\varsigma f'(\varsigma) + (1-\gamma)f(\varsigma)} \prec \frac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}, \quad \varsigma \in \mathcal{U},$$

and

$$\frac{wg'(w) + \gamma w^2 g''(w)}{\gamma w g'(w) + (1-\gamma)g(w)} \prec \frac{2\sqrt{1+w}}{1+e^{-w}}, \quad w \in \mathcal{U},$$

where $0 \leq \gamma \leq 1$ and $g = f^{-1}$ denotes the inverse of f.

Figure 1 illustrates the transformation of the domain to the codomain under the mapping $f(\varsigma) = \dfrac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}$. The left panel shows the original unit disk, while the right panel depicts the resulting balloon-shaped domain.



Domain

Co-domain

Figure 1. Grid representation of the domain and codomain under the mapping $f(\varsigma) = \dfrac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}$.

Example 2.1. When setting $\gamma = 0$, the class $\mathbb{CS}^*_{\beth}(0)$ reduces to the standard class of starlike bi-univalent functions $\mathbb{S}^*$, characterized by the subordination relations

$$\frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \frac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}, \quad \varsigma \in \mathcal{U},$$

and

$$\frac{wg'(w)}{g(w)} \prec \frac{2\sqrt{1+w}}{1+e^{-w}}, \quad w \in \mathcal{U}.$$

Example 2.2. In the case $\gamma = 1$, the class $\mathbb{CS}^*_{\beth}(1)$ coincides with the convex-type class $\mathbb{C}_{\beth}$, and is defined via the conditions

$$1 + \frac{\varsigma f''(\varsigma)}{f'(\varsigma)} \prec \frac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}, \quad \varsigma \in \mathcal{U},$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \frac{2\sqrt{1+w}}{1+e^{-w}}, \quad w \in \mathcal{U}.$$

For more on the balloon-shaped domain, see [24, 25].

A more general approach to starlike functions was presented by Ma and Minda [26], who introduced the family

$$\mathcal{S}^*(\vartheta) = \left\{ f \in \mathcal{A} : \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \vartheta(\varsigma), \quad \varsigma \in \mathcal{U} \right\},$$

where ϑ is an analytic with a positive real part in $\mathcal{U}$, normalized by $\vartheta(0) = 1$ and $\vartheta'(0) > 0$. Depending on the form of $\vartheta(\varsigma)$, several subclasses can be derived. For instance, taking $\vartheta(\varsigma) = \frac{1+\varsigma}{1-\varsigma}$ retrieves the classical class $\mathcal{S}^*$ of starlike functions.

Table 1 outlines a collection of notable subclasses derived by selecting specific functions for $\vartheta(\varsigma)$, along with their associated references.

Table 1. Representative subclasses of $\mathcal{S}^*(\vartheta)$ with defining analytic functions $\vartheta(\varsigma)$.

Class	Reference	$\vartheta(\varsigma)$
γ_L^*	Sokól et al. [15]	$\sqrt{1+\varsigma}$
γ_{cre}^*	Raina et al. [27]	$\varsigma + \sqrt{1+\varsigma^2}$
γ_{car}^*	Sharma et al. [28]	$1+\frac{4}{3}\varsigma+\frac{2}{3}\varsigma^2$
$\mathcal{S}_{\text{sin}}^*$	Cho et al. [29]	$1+\sin\varsigma$
$\mathcal{S}_{\vartheta}^*$	Kumar et al. [30]	$1+\sin^{-1}\varsigma$
$\mathcal{S}_{\text{nep}}^*$	Wani et al. [16]	$1+\varsigma-\frac{1}{3}\varsigma^3$

Let $\mathbb{P}$ be the class of analytic functions in $\mathcal{U}$ such that $p(0)=1$ and have a strictly positive real part throughout $\mathcal{U}$. In other words, a function $p(\varsigma)\in\mathbb{P}$ satisfies:

$$p(0)=1 \quad \text{and} \quad \Re\left(p(\varsigma)\right)>0 \quad \text{for all } \varsigma\in\mathcal{U}.$$

Lemma 2.1. [31] Suppose that $p(\varsigma)\in\mathbb{P}$ and has the power series representation

$$p(\varsigma)=1+p_1\varsigma+p_2\varsigma^2+p_3\varsigma^3+\cdots.$$

Then, for every $\ell\in\mathbb{N}$, the following coefficient bound holds:

$$|p_\ell|\le 2. \tag{2.1}$$

Lemma 2.2. [32] Let $p(\varsigma)\in\mathbb{P}$ be given by the expansion

$$p(\varsigma)=1+p_1\varsigma+p_2\varsigma^2+p_3\varsigma^3+\cdots.$$

Then, there exist complex numbers τ and ς, with $|\tau|\le 1$ and $|\varsigma|\le 1$, such that the coefficients satisfy the relations:

$$2p_2=p_1^2+\tau(4-p_1^2), \tag{2.2}$$

$$4p_3=p_1^3+2(4-p_1^2)p_1\tau-p_1(4-p_1^2)\tau^2+2(4-p_1^2)(1-|\tau|^2)\varsigma. \tag{2.3}$$

Remark 2.1. The function class $\mathbb{CS}_{\underline{\gimel}}^*(\gamma)$ is nonempty. Notably, the following analytic functions are members of this class:

$$f_1(\varsigma)=\sqrt{1+\varsigma},\quad f_2(\varsigma)=\frac{2}{1+e^{-\varsigma}},\quad f_3(\varsigma)=\sqrt{1+\tanh\varsigma}.$$

Proof. To confirm the inclusion of the functions f_1, f_2, and f_3 in the class $\mathbb{CS}_{\underline{\gimel}}^*(\gamma)$, it suffices to show that the images of $\mathcal{U}$ under these functions are entirely contained within the balloon-shaped domain defined by the subordination condition associated with the class.

As can be observed from Figures 2–4, each of the given functions maps the unit disk $\mathcal{U}$ into a subset of the balloon-shaped domain. This geometric containment implies that the associated subordination condition

$$\frac{\varsigma f'(\varsigma)+\gamma\varsigma^2 f''(\varsigma)}{\gamma\varsigma f'(\varsigma)+(1-\gamma)f(\varsigma)}\prec\frac{2\sqrt{1+\varsigma}}{1+e^{-\varsigma}}$$

is satisfied. Therefore, each of the functions f_1, f_2, and f_3 belongs to the class $\mathbb{CS}_{\underline{\gimel}}^*(\gamma)$, establishing that the class is indeed nonempty. □

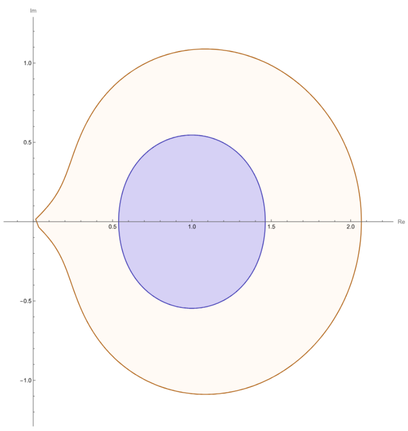


Figure 2. Image of $f_1(\varsigma) = \frac{2}{1+e^{-\varsigma}}$.

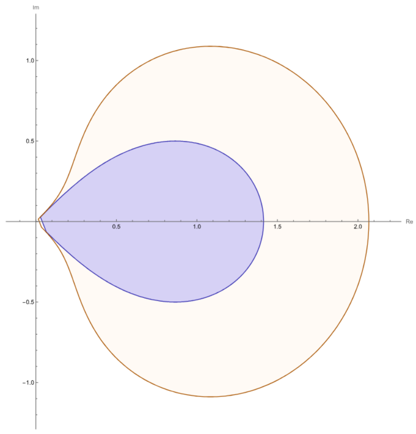


Figure 3. Image of $f_2(\varsigma) = \sqrt{1+\varsigma}$.

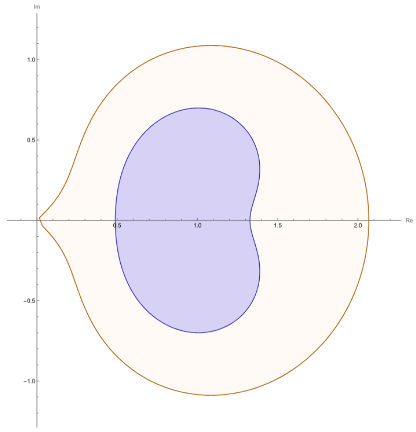


Figure 4. Image of $f_3(\varsigma) = \sqrt{1+\tanh\varsigma}$.

As shown in Figures 2–4, each function maps the unit disk $\mathcal{U}$ entirely into the balloon-shaped domain. Hence, the images confirm that the class $\mathbb{CS}^*_{\beth}(\gamma)$ is nonempty.

3. Upper bounds for coefficients in the function class $\mathbb{CS}^*_{\beth}(\gamma)$

Theorem 3.1. Let $f(\varsigma)$, given by the expansion (1.1), belong to the class $\mathbb{CS}^*_{\beth}(\gamma)$ for $0 \le \gamma \le 1$. Then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{751 + \gamma\,(2253 + 16(69 - 25\gamma)\gamma)}{(1+\gamma)^4(1+3\gamma)}. \tag{3.1}$$

Proof. Suppose $f \in \mathbb{CS}^*_{\beth}(\gamma)$. Then there exist Schwarz functions $u(\varsigma)$ and $v(w)$, analytic in $\mathcal{U}$, with $u(0) = v(0) = 0$, such that

$$|u(\varsigma)| < 1, \quad |v(w)| < 1,$$

and the subordination conditions

$$\frac{\varsigma f'(\varsigma) + \gamma\varsigma^2 f''(\varsigma)}{\gamma\varsigma f'(\varsigma) + (1-\gamma)f(\varsigma)} = \frac{2\sqrt{1+u(\varsigma)}}{1+e^{-u(\varsigma)}}, \tag{3.2}$$

$$\frac{wg'(w) + \gamma w^2 g''(w)}{\gamma w g'(w) + (1-\gamma)g(w)} = \frac{2\sqrt{1+v(w)}}{1+e^{-v(w)}}, \tag{3.3}$$

hold, where $g = f^{-1}$.

Now, define

$$p(\varsigma) := \frac{1+u(\varsigma)}{1-u(\varsigma)} = 1 + p_1\varsigma + p_2\varsigma^2 + p_3\varsigma^3 + \cdots,$$

and

$$q(w) := \frac{1+v(w)}{1-v(w)} = 1 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots,$$

which are both analytic in $\mathcal{U}$ with positive real parts, i.e., $p, q \in \mathbb{P}$. Then,

$$u(\varsigma) = \frac{p(\varsigma)-1}{p(\varsigma)+1} = \frac{1}{2}\left[p_1\varsigma + \left(p_2 - \frac{1}{2}p_1^2\right)\varsigma^2 + \frac{1}{4}\left(p_1^3 - 4p_1p_2 + 4p_3\right)\varsigma^3 + \cdots\right],$$

and similarly,

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[c_1 w + \left(c_2 - \frac{1}{2}c_1^2\right)w^2 + \frac{1}{4}\left(c_1^3 - 4c_1c_2 + 4c_3\right)w^3 + \cdots\right].$$

Substituting the above into (3.2) and (3.3), and expanding, yields:

$$\begin{aligned}\frac{2\sqrt{1+u(\varsigma)}}{1+e^{-u(\varsigma)}} = 1 &+ \frac{1}{2}p_1\varsigma + \left(\frac{1}{2}p_2 - \frac{7}{32}p_1^2\right)\varsigma^2 \\ &+ \left(\frac{17}{192}p_1^3 - \frac{7}{16}p_1p_2 + \frac{1}{2}p_3\right)\varsigma^3 \\ &+ \left(-\frac{203}{6144}p_1^4 + \frac{17}{64}p_1^2p_2 - \frac{7}{32}p_2^2 - \frac{7}{16}p_1p_3 + \frac{1}{2}p_4\right)\varsigma^4 + \cdots,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\frac{2\sqrt{1+v(w)}}{1+e^{-v(w)}} = 1 &+ \frac{1}{2}c_1 w + \left(\frac{1}{2}c_2 - \frac{7}{32}c_1^2\right)w^2 \\ &+ \left(\frac{17}{192}c_1^3 - \frac{7}{16}c_1c_2 + \frac{1}{2}c_3\right)w^3 \\ &+ \left(-\frac{203}{6144}c_1^4 + \frac{17}{64}c_1^2c_2 - \frac{7}{32}c_2^2 - \frac{7}{16}c_1c_3 + \frac{1}{2}c_4\right)w^4 + \cdots.\end{aligned} \tag{3.5}$$

On the other hand, using the series expansion for f and its inverse g, we compute:

$$\begin{aligned}\frac{\varsigma f'(\varsigma) + \gamma z^2 f''(\varsigma)}{\gamma\varsigma f'(\varsigma) + (1-\gamma)f(\varsigma)} =& 1 + (1+\gamma)a_2\varsigma + \left[2a_3(1+2\gamma) - a_2^2(1+\gamma)^2\right]\varsigma^2 \\ &+ \left[3a_4(1+3\gamma) + a_2^3(1+\gamma)^3 - 3a_2a_3(1+\gamma)(1+2\gamma)\right]\varsigma^3 \\ &+ \Big[4a_2^2a_3(1+\gamma)^2(1+2\gamma) - a_2^4(1+\gamma)^4 - 4a_2a_4(1+\gamma)(1+3\gamma) \\ &- 2a_3(1+2\gamma)^2 + 4a_5(1+4\gamma)\Big]\varsigma^4 + \cdots\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{wg'(w) + \gamma w^2 g''(w)}{\gamma w g'(w) + (1-\gamma)g(w)} =& 1 - (1+\gamma)a_2 w + \left[-2a_3(1+2\gamma) + a_2^2(3+6\gamma-\gamma^2)\right]w^2 \\ &+ \left[-3a_4(1+3\gamma) + 6a_2a_3(2+6\gamma-\gamma^2) - a_2^3(10+30\gamma-9\gamma^2+\gamma^3)\right]w^3 \\ &+ \Big[-4a_5(1+4\gamma) + 4a_2a_4(5+20\gamma-3\gamma^2) + 2a_3^2(5+20\gamma-4\gamma^2) \\ &- 4a_2^2a_3(15+60\gamma-18\gamma^2+2\gamma^3) \\ &+ a_2^4(35+140\gamma-58\gamma^2+12\gamma^3-\gamma^4)\Big]w^4 + \cdots\end{aligned} \tag{3.7}$$

By matching the coefficients of ς in (3.4) and (3.6), we obtain the system:

$$(1+\gamma)a_2 = \frac{1}{2}p_1, \tag{3.8}$$

$$2a_3(1+2\gamma) - a_2^2(1+\gamma)^2 = \frac{1}{2}p_2 - \frac{7}{32}p_1^2, \tag{3.9}$$

$$3a_4(1+3\gamma) + a_2^3(1+\gamma)^3 - 3a_2a_3(1+\gamma)(1+2\gamma) = \frac{17}{192}p_1^3 - \frac{7}{16}p_2p_1 + \frac{1}{2}p_3, \tag{3.10}$$

and

$$\begin{aligned} &4a_2^2a_3(1+\gamma)^2(1+2\gamma) - a_2^4(1+\gamma)^4 - 4a_2a_4(1+\gamma)(1+3\gamma) - 2a_3(1+2\gamma)^2 + 4a_5(1+4\gamma) \\ &= -\frac{203}{6144}p_1^4 + \frac{17}{64}p_1^2p_2 - \frac{7}{16}p_3p_1 - \frac{7}{32}p_2^2 + \frac{1}{2}p_4. \end{aligned} \tag{3.11}$$

Similarly, equating the coefficients of (3.5) and (3.7) leads to:

$$-(1+\gamma)a_2 = \frac{1}{2}c_1, \tag{3.12}$$

$$-2a_3(1+2\gamma) - a_2^2(-3-6\gamma+\gamma^2) = \frac{1}{2}c_2 - \frac{7}{32}c_1^2, \tag{3.13}$$

$$\begin{aligned} &-3a_4(1+3\gamma) - 6a_2a_3(-2-6\gamma+\gamma^2) - a_2^3(10+30\gamma-9\gamma^2+\gamma^3) \\ &= \frac{17}{192}c_1^3 - \frac{7}{16}c_2c_1 + \frac{1}{2}c_3, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &-4a_5(1+4\gamma) - 4a_2a_4(-5-20\gamma+3\gamma^2) - 2a_3^2(-5-20\gamma+4\gamma^2) \\ &-4a_2^2a_3(15+60\gamma-18\gamma^2+2\gamma^3) - a_2^4(-35-140\gamma+58\gamma^2-12\gamma^3+\gamma^4) \\ &= -\frac{203}{6144}c_1^4 + \frac{17}{64}c_1^2c_2 - \frac{7}{16}c_3c_1 - \frac{7}{32}c_2^2 + \frac{1}{2}c_4. \end{aligned} \tag{3.15}$$

Comparing Eqs (3.8) and (3.12) yields

$$a_2 = \frac{p_1}{2(1+\gamma)} = \frac{-c_1}{2(1+\gamma)}, \tag{3.16}$$

which implies

$$p_1 = -c_1. \tag{3.17}$$

Subtracting (3.13) from (3.9) and applying (3.16), we find

$$a_3 = \frac{p_1^2}{4(1+\gamma)^2} + \frac{p_2 - c_2}{8(1+2\gamma)}. \tag{3.18}$$

Similarly, subtracting (3.14) from (3.10) and using (3.16) and (3.18), we obtain

$$\begin{aligned} a_4 =& \frac{1}{1152}\Bigg(\frac{72p_1^3}{(1+\gamma)^2} - \frac{36(p_1^3 + 5p_1(p_2-c_2))}{(1+\gamma)} + \frac{360p_1(p_2-c_2)}{(1+2\gamma)} \\ &+ \frac{94p_1^3 - 84p_1(c_2+p_2) + 96(p_3-c_3)}{(1+3\gamma)}\Bigg). \end{aligned} \tag{3.19}$$

By substituting (3.16), (3.18), and (3.19), the expression for $a_2a_4 - a_3^2$ takes the form:

$$\begin{aligned} a_2a_4 - a_3^2 = &- \frac{(p_2-c_2)^2(1+\gamma)^4 + 4p_1^2(1+2\gamma)(p_2-c_2)(1+\gamma)^2 + 4p_1^4(1+2\gamma)^2}{64(1+\gamma)^4(1+2\gamma)^2} \\ &+ \frac{p_1}{2(1+\gamma)}\Bigg(\frac{72p_1^3}{(1+\gamma)^2} - \frac{36p_1(p_1^2+5(p_2-c_2))}{(1+\gamma)} \\ &+ \frac{360p_1(p_2-c_2)}{(1+2\gamma)} + \frac{94p_1^3 - 84p_1(p_2+c_2) + 96(p_3-c_3)}{(1+3\gamma)}\Bigg). \end{aligned} \tag{3.20}$$

According to Lemma 2.2 and the relation $p_1 = -c_1$, we have

$$p_2 - c_2 = \frac{4-p_1^2}{2}(x-y), \quad p_2 + c_2 = p_1^2 + \frac{4-p_1^2}{2}(x+y), \tag{3.21}$$

and

$$\begin{aligned} p_3 - c_3 = &\frac{p_1^3}{2} + \frac{(4-p_1^2)p_1}{2}(x+y) - \frac{(4-p_1^2)p_1}{4}(x^2+y^2) \\ &+ \frac{4-p_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w], \end{aligned} \tag{3.22}$$

where $|x|, |y|, |\varsigma|, |w| \le 1$.

Since $p \in \mathbb{P}$, it follows that $|p_1| \le 2$. Denoting $p_1 = p$, we may restrict p to the interval $[0,2]$. Substituting relations (3.21) and (3.22) into the expression for $a_2a_4 - a_3^2$, and setting $\delta = |x| \le 1$ and $\varsigma = |y| \le 1$, we deduce the inequality

$$|a_2a_4 - a_3^2| \le \mathbb{T}_1 + \mathbb{T}_2(\delta+\varsigma) + \mathbb{T}_3(\delta^2+\varsigma^2) + \mathbb{T}_4(\delta+\varsigma)^2 = \mathbb{T}(\delta,\varsigma),$$

where

$$\mathbb{T}_1 = \mathbb{T}_1(p) = \frac{(751+\gamma(2253+16(69-25\gamma)\gamma))\,p^4}{16(1+\gamma)^4(1+3\gamma)} \ge 0, \tag{3.23}$$

$$\mathbb{T}_2 = \mathbb{T}_2(\delta,\varsigma) = \frac{p^2\,(1535+3\gamma(1535+64\gamma))\,(4-p^2)}{32(1+\gamma)^2\,(1+\gamma(5+6\gamma))} \ge 0, \tag{3.24}$$

$$\mathbb{T}_3 = \mathbb{T}_3(\delta^2,\varsigma^2) = \frac{p}{1+\gamma}\left(\frac{96-48p-24p^2+12p^3}{1+3\gamma}\right) \le 0, \tag{3.25}$$

and

$$\mathbb{T}_4 = \mathbb{T}_4(\delta,\varsigma)^2 = \frac{8p^2-p^4-16}{256(1+2\gamma)^2} \ge 0. \tag{3.26}$$

We aim to maximize $\mathbb{T}(\delta,\varsigma)$ over $[0,1]\times[0,1]$ for $p \in [0,2]$. Given the conditions $\mathbb{T}_3 \le 0$ and $\mathbb{T}_3 + 2\mathbb{T}_2 \ge 0$, it follows that $p \in (0,2)$ and $\mathbb{T}(\delta,\delta)\mathbb{T}(\varsigma,\varsigma) - (\mathbb{T}(\delta,\varsigma))^2 < 0$, which implies that $\mathbb{T}$ does not admit a local maximum in the interior of the domain.

Thus, the maximum must be on the boundary of the square. Considering the boundary segment $\delta = 0$ with $\varsigma \in [0,1]$, we define

$$\mathbb{T}(0,\varsigma) = \Psi(\varsigma) = \mathbb{T}_1 + \mathbb{T}_2\varsigma + (\mathbb{T}_3+\mathbb{T}_4)\varsigma^2.$$

We now distinguish two cases:

Case 1: If $\mathbb{T}_3 + \mathbb{T}_4 \geq 0$, then for all $\varsigma \in [0,1]$ and fixed $p \in [0,2)$, then $\Psi'(\varsigma) = \mathbb{T}_2 + 2(\mathbb{T}_3 + \mathbb{T}_4)\varsigma > 0$, showing that Ψ is increasing on the interval. Consequently, the maximum of Ψ over $[0,1]$ is attained at $\varsigma = 1$, yielding

$$\max_{0\leq\varsigma\leq1} \Psi(\varsigma) = \Psi(1) = \mathbb{T}_1 + \mathbb{T}_2 + \mathbb{T}_3 + \mathbb{T}_4.$$

Case 2: Suppose $\mathbb{T}_3 + \mathbb{T}_4 < 0$. Since $2(\mathbb{T}_3 + \mathbb{T}_4) + \mathbb{T}_2 \geq 0$ for $0 < p < 2$ and $0 < \varsigma < 1$, it follows that

$$2(\mathbb{T}_3 + \mathbb{T}_4) + \mathbb{T}_2 < 2(\mathbb{T}_3 + \mathbb{T}_4)\varsigma + \mathbb{T}_2 < \mathbb{T}_2.$$

Hence, $\Psi(\varsigma) > 0$, implying the maximum of Ψ on $[0,1]$ is attained at $\varsigma = 1$, yielding

$$\mathbb{T}(1,\varsigma) = \mathbb{G}(\varsigma) = (\mathbb{T}_3 + \mathbb{T}_4)\varsigma^2 + (\mathbb{T}_2 + 2\mathbb{T}_4)\varsigma + \mathbb{T}_1 + \mathbb{T}_2 + \mathbb{T}_3 + \mathbb{T}_4.$$

From the two cases regarding $\mathbb{T}_3 + \mathbb{T}_4$, it follows that

$$\max \mathbb{G}(\varsigma) = \mathbb{G}(1) = \mathbb{T}_1 + 2\mathbb{T}_2 + 2\mathbb{T}_3 + 4\mathbb{T}_4.$$

Since $\Psi(1) \leq \mathbb{G}(1)$, we conclude

$$\max_{(\delta,\varsigma)\in[0,1]^2} \mathbb{T}(\delta,\varsigma) = \mathbb{T}(1,1).$$

Define the real-valued function H on $(0,2)$ by

$$H(p) = \max \mathbb{T}(\delta,\varsigma) = \mathbb{T}(1,1) = \mathbb{T}_1 + 2\mathbb{T}_2 + 2\mathbb{T}_3 + 4\mathbb{T}_4.$$

Substituting expressions for $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4$ into H, we have

$$\begin{aligned}\mathbb{T}(1,1) = {} & \frac{-16 + 8p^2 - p^4}{64(1+2\gamma)^2} + \frac{2p(96 - 48p - 24p^2 + 12p^3)}{(1+\gamma)(1+3\gamma)} \\ & + \frac{p^2(4-p^2)(1535 + 3\gamma(1535 + 64\gamma))}{16(1+\gamma)^2(1+\gamma(5+6\gamma))} \\ & + \frac{p^4(751 + \gamma(2253 + 16(69 - 25\gamma)\gamma))}{16(1+\gamma)^4(1+3\gamma)}.\end{aligned} \tag{3.27}$$

Elementary analysis shows $H(p)$ is monotonically increasing on $[0,2]$. Hence, the maximum is attained at $p = 2$, yielding

$$\max H(p) = H(2) = \frac{751 + \gamma\,(2253 + 16(69 - 25\gamma)\gamma)}{(1+\gamma)^4(1+3\gamma)}. \tag{3.28}$$

□

Theorem 3.2. Let $f(\varsigma) \in \mathbb{CS}^*_{\beth}(\gamma)$, with $0 \leq \gamma \leq 1$. Then, the following sharp inequality holds:

$$|a_2a_3 - a_4| \leq \begin{cases} \dfrac{48(1+\gamma)^2(19 + 93\gamma + 92\gamma^2) - 8(169 + 1061\gamma + 2133\gamma^2 + 1759\gamma^3 + 662\gamma^4)}{576(1+\gamma)^3(1+5\gamma+6\gamma^2)}, & \text{if } m \leq p \leq 2, \\ \dfrac{1+2\gamma}{3\,(1+5\gamma+6\gamma^2)}, & \text{if } 0 \leq p \leq m, \end{cases} \tag{3.29}$$

where

$$m = \frac{-d_3 \pm \sqrt{d_3^2 - 12k_2(d_1 - d_2)}}{3(d_1 - d_2)}, \tag{3.30}$$

and the constants d_1, d_2, d_3 are given explicitly by:

$$d_1 = -\frac{55 + 165\gamma + 21\gamma^2 + 55\gamma^3}{576(1+\gamma)^3(1+3\gamma)}, \qquad d_2 = \frac{19 + 93\gamma + 92\gamma^2}{96(1 + 6\gamma + 11\gamma^2 + 6\gamma^3)}, \qquad d_3 = \frac{-1}{12 + 36\gamma}.$$

Proof. From the expressions derived in Eqs (3.16), (3.18), and (3.19), we arrive at the following representation:

$$\begin{aligned} a_2a_3 - a_4 =& \frac{1}{576}\left(\frac{72p_1^3}{(1+\gamma)^3} - \frac{36p_1^3}{(1+\gamma)^2} + \frac{18p_1(-3c_2 + p_1^2 + 3p_2)}{1+\gamma}\right. \\ &\left. + \frac{108p_1(c_2 - p_2)}{1+2\gamma} + \frac{48c_3 + 42c_2p_1 - 47p_1^3 + 42p_1p_2 - 48p_3}{1+3\gamma}\right), \end{aligned} \tag{3.31}$$

where the auxiliary coefficients c_2, c_3, p_2, p_3 satisfy the bounds from Lemma 2.2. Keeping the generality, we set $p_1 = p \in [0, 2]$ and denote $\varrho = |x| \leq 1$, $\xi = |y| \leq 1$. This allows us to estimate:

$$|a_2a_3 - a_4| \leq \mathbb{K}_1 + \mathbb{K}_2(\varrho + \xi) + \mathbb{K}_3(\varrho^2 + \xi^2) =: \mathbb{W}(\varrho, \xi), \tag{3.32}$$

where the coefficients $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$ are defined as follows:

Zeroth-order term:

$$\mathbb{K}_1 = \frac{p^3\left(25 + 75\gamma - 69\gamma^2 + 25\gamma^3\right)}{576(1+\gamma)^3(1+3\gamma)} \geq 0, \tag{3.33}$$

First-order term:

$$\mathbb{K}_2 = \frac{1}{192}\,p\left(\frac{9(-4+p^2)}{1+\gamma} + \frac{18(-4+p^2)}{1+2\gamma} + \frac{16+p^2}{1+3\gamma}\right) \geq 0, \tag{3.34}$$

Second-order term:

$$\mathbb{K}_3 = \frac{-96 + 48p + 24p^2 - 12p^3}{576(1+3\gamma)} \leq 0. \tag{3.35}$$

Proceeding analogously to the approach employed in Theorem 3.2, we observe that the extremal value of the expression $\mathbb{W}(\varrho, \xi)$ is attained at the boundary point $(\varrho, \xi) = (1, 1)$ within the square region $[0, 1] \times [0, 1]$. Hence, we define

$$\psi(p) := \max \mathbb{W}(\varrho, \xi) = \mathbb{K}_1 + 2(\mathbb{K}_2 + \mathbb{K}_3), \tag{3.36}$$

where substituting the expressions for $\mathbb{K}_1, \mathbb{K}_2$, and $\mathbb{K}_3$, we arrive at

$$\psi(p) = d_1p^3 + d_2p(4 - p^2) - d_3(4 - p^2), \tag{3.37}$$

with constants:

$$d_1 = -\frac{55 + 165\gamma + 21\gamma^2 + 55\gamma^3}{576(1+\gamma)^3(1+3\gamma)}, \quad d_2 = \frac{19 + 93\gamma + 92\gamma^2}{96(1 + 6\gamma + 11\gamma^2 + 6\gamma^3)}, \quad d_3 = \frac{-1}{12 + 36\gamma}.$$

To locate the maximum of $\psi(p)$, we compute the first and second derivatives:

$$\psi'(p) = 3(d_1 - d_2)p^2 + 2d_3p + 4d_2, \quad \psi''(p) = 6(d_1 - d_2)p + 2d_3.$$

In the case where $d_1 > d_2$, the derivative $\psi'(p)$ remains positive throughout the interval $[0, 2]$, implying that $\psi(p)$ is increasing. Consequently, the maximal value is done at the right endpoint $p = 2$, yielding

$$|a_2a_3 - a_4| \leq \psi(2) = \frac{48(1+\gamma)^2(19 + 93\gamma + 92\gamma^2) - 8(169 + 1061\gamma + 2133\gamma^2 + 1759\gamma^3 + 662\gamma^4)}{576(1+\gamma)^3(1 + 5\gamma + 6\gamma^2)}.$$

Alternatively, if $d_1 - d_2 < 0$, then we locate the stationary point by solving $\psi'(p) = 0$, giving:

$$p = m = \frac{-d_3 \pm \sqrt{d_3^2 - 12d_2(d_1 - d_2)}}{3(d_1 - d_2)}. \tag{3.38}$$

Depending on the sign and location of m, the monotonicity of $\psi(p)$ is determined. If $m < p \leq 2$, then $\psi'(p) > 0$ and $\psi(p)$ is increasing on $[0, 2]$, again implying a maximum at $p = 2$. Conversely, if $\psi(p)$ is decreasing on the interval, then the extremum occurs at $p = 0$, and we deduce the bound

$$|a_2a_3 - a_4| \leq \psi(0) = \frac{1 + 2\gamma}{3\,(1 + 5\gamma + 6\gamma^2)}.$$

□

4. Estimation of the third Hankel determinant for the class $\mathbb{CS}^*_{\beth}(\gamma)$

In this section, we continue the coefficient estimates for functions belonging to the class $\mathbb{CS}^*_{\beth}(\gamma)$. We first derive upper bounds for the expressions $|a_3 - a_2^2|$ and $|a_3|$, and later extend our analysis to estimate $|a_4|$, $|a_5|$, and compute the third Hankel determinant $H_3(1)$.

Theorem 4.1. Let $f(\varsigma) \in \mathbb{CS}^*_{\beth}(\gamma)$, where $0 \leq \gamma \leq 1$. Then the following bounds hold:

$$|a_3 - a_2^2| \leq \frac{1}{2(1 + 2\gamma)}, \tag{4.1}$$

$$|a_3| \leq \frac{1}{(1+\gamma)^2} + \frac{1}{2(1 + 2\gamma)}. \tag{4.2}$$

Proof. Let $\eta \in \mathbb{C}$, and consider the Fekete-Szegö functional associated with the function $f \in \mathbb{CS}^*_{\beth}(\gamma)$, given by:

$$a_3 - \eta a_2^2 = \frac{(1-\eta)p_1^2}{4(1+\gamma)^2} + \frac{p_2 - c_2}{8(1 + 2\gamma)}.$$

Invoking Lemma 2.1, we estimate the modulus as:

$$|a_3 - \eta a_2^2| \leq \frac{|1-\eta|}{(1+\gamma)^2} + \frac{1}{2(1 + 2\gamma)}.$$

Choosing $\eta = 1$ yields inequality (4.1), thus concluding that $|a_3 - a_2^2| \leq \frac{1}{2(1+2\gamma)}$.

To obtain (4.2), we combine relation (3.18) with the estimate from Lemma 2.1:

$$|a_3| \leq \left|\frac{p_1^2}{4(1+\gamma)^2}\right| + \left|\frac{p_2 - c_2}{8(1+2\gamma)}\right| \leq \frac{1}{(1+\gamma)^2} + \frac{1}{2(1+2\gamma)}.$$

□

Theorem 4.2. Let $f(\varsigma) \in \mathbb{CS}^*_{\beth}(\gamma)$, where $0 \leq \gamma \leq 1$. Then the following coefficient bounds are satisfied:

$$|a_4| \leq \frac{437 + 2180\gamma + 3229\gamma^2 + 1414\gamma^3}{72(1+5\gamma+6\gamma^2)(1+\gamma)^2}, \tag{4.3}$$

$$|a_5| \leq \frac{5331 + 65185\gamma + 320930\gamma^2 + 827914\gamma^3 + 1218635\gamma^4 + 1041913\gamma^5 + 492344\gamma^6 + 99780\gamma^7 + 864\gamma^8}{288(1+3\gamma)(1+4\gamma)(1+2\gamma)^2(1+\gamma)^4}. \tag{4.4}$$

Proof. To derive the estimate for $|a_5|$, we refer to Eqs (3.11) and (3.15), subtract them, and then incorporate the relations given by (3.16), (3.18), and (3.19). This leads to the identity:

$$\begin{aligned}&128a_3 + (c_2 - p_2)(14c_2 - 17p_1^2 + 14p_2) + 128\Big[-16a_5\gamma + 4a_3\gamma(1+\gamma) + 12a_2(a_4 + 4a_4\gamma)\\&+ a_3^2(5 - 4(-5+\gamma)\gamma) + 2a_2^4(9 + \gamma(36 + \gamma(-13+4\gamma))) - 2a_2^2a_3(16 + \gamma(64 + \gamma(-13+4\gamma)))\Big]\\=&4\Big(128a_5 + 8c_4 + 7p_1(c_3 + p_3) - 8p_4\Big).\end{aligned} \tag{4.5}$$

Solving the equation and simplifying yields an explicit expression for a_5:

$$\begin{aligned}a_5 =&\frac{1}{41472}\Bigg(648(-c_2+p_2) - \frac{4968\,p_1^4}{(1+\gamma)^4} + \frac{3312\,p_1^4}{(1+\gamma)^3} - \frac{24\,p_1^2(36 + 9c_2 + 77p_1^2 - 9p_2)}{(1+\gamma)^2}\\&+ \frac{2\,p_1(1296c_3 - 1205p_1^3 + 18p_1(64 + 127c_2 - p_2) - 1296p_3)}{(1+\gamma)}\\&+ \frac{972(c_2-p_2)^2}{(1+2\gamma)^2} - \frac{3240\,p_1^2(c_2-p_2)}{(1+2\gamma)} + \frac{162\,p_1(-48c_3 - 42c_2p_1 + 47p_1^3 - 42p_1p_2 + 48p_3)}{(1+3\gamma)}\\&- \frac{-972c_2^2 + 512p_1^4 + 9c_2(72 + 313p_1^2 - 36p_2) - 2817p_1^2p_2 + 1296p_2^2}{(1+4\gamma)}\\&- \frac{2268p_1p_3 + 36(72c_4 + (63c_3 - 32p_1)p_1 - 18(p_2 + 4p_4))}{(1+4\gamma)}\Bigg).\end{aligned} \tag{4.6}$$

Applying Lemma 2.1 yields the bound in (4.4). On the other hand, Eq (3.19), when combined with Lemma 2.1, leads directly to the upper bound in (4.3). □

Theorem 4.3. Let $f(\varsigma) \in \mathbb{CS}^*_{\beth}(\gamma)$, with $0 \leq \gamma \leq 1$. Then the third Hankel determinant satisfies the inequality:

$$|H_3(1)| \leq \begin{cases}\mathcal{K}\mathcal{K}_1 + \mathcal{K}_2\left[\frac{48(1+\gamma)^2\left(19+93\gamma+92\gamma^2\right)-8\left(169+1061\gamma+2133\gamma^2+1759\gamma^3+662\gamma^4\right)}{576(1+\gamma)^3(1+5\gamma+6\gamma^2)}\right] + \mathcal{K}_3\mathcal{K}_4, & \text{if } m \leq c \leq 2,\\[2ex] \mathcal{K}\mathcal{K}_1 + \frac{1+2\gamma}{3\left(1+5\gamma+6\gamma^2\right)}\mathcal{K}_2 + \mathcal{K}_3\mathcal{K}_4, & \text{if } 0 \leq c \leq m,\end{cases}$$

where the constants $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$, and m are as defined in (4.2), (3.1), (4.3), (4.4), (4.1), and (3.30), respectively.

Proof. From the general expression for the third Hankel determinant, we write

$$|H_3(1)| = |a_3| \cdot |a_2 a_4 - a_3^2| + |a_4| \cdot |a_4 - a_2 a_3| + |a_5| \cdot |a_3 - a_2^2|.$$

Invoking the triangle inequality together with the previously derived estimates for the involved coefficients yields the desired bound in (1.7). □

5. Conclusions

In this paper, we introduced and studied the subclass $\mathbb{CS}^*_{\underline{3}}(\gamma)$ of bi-univalent functions associated with a balloon-shaped domain. We confirmed that the class is non-empty and examined several key analytic properties. We derived sharp coefficient bounds, including inequalities for $|a_3 - a_2^2|$, $|a_3|$, and extended estimates up to $|a_5|$. Moreover, we established upper bounds for the second and third Hankel determinants, which serve as important measures of the function's complexity. The results reveal how the parameter γ influences the coefficient estimates and determinant bounds, showing that this class bridges well-known starlike and convex subclasses. Future work could explore other subclasses defined by alternative geometric constraints or investigate deeper operator-theoretic aspects to further characterize these bi-univalent functions.

Author contributions

Mohammad El-Ityan and Tariq Al-Hawary contributed to the conceptualization and methodology of the study. Basem Aref Frasin provided supervision and critical revisions. Ibtisam Aldawish coordinated the research, performed formal analysis, and wrote the original draft. All authors reviewed and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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