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Research article

Generalized structural conditions for a Calderón-Zygmund theory on double phase problems

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Abstract: In this paper, we considered a general class of double-phase problems of the form

$$-\mathrm{div}(A(x, Du)) = -\mathrm{div}(B(x, F)).$$

Here, the ellipticity, growth, and continuity assumptions on the main operator A were described using an auxiliary vector field G. The proposed structural condition allowed us to treat, in a single setting, different structural forms that have appeared in earlier studies. Under this generalized structure, we derived Calderón-Zygmund estimates for the gradients of solutions.

Keywords: double phase problems; discontinuous coefficients; Calderón-Zygmund theory; nonlinear elliptic equations; calculus of variations
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1. Introduction

In this paper, we study the structure conditions on the main operator $A(x, z)$ that allow for the Calderón-Zygmund estimates to be established for the elliptic equation

$$\mathrm{div} A(x, Du) = \mathrm{div} B(x, F) \quad \text{in } \Omega \tag{1.1}$$

with a prototype of the form

$$\mathrm{div}(p|Du|^{p-2}Du + qa(x)|Du|^{q-2}Du) = \mathrm{div}(p|F|^{p-2}F + qa(x)|F|^{q-2}F) \quad \text{in } \Omega. \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain and the modulating coefficient $a(x)$ is a nonnegative α-Hölder continuous function on Ω, while p and q are universal constants satisfying

$$1 < p < q \quad \text{and} \quad \frac{q}{p} \le 1 + \frac{\alpha}{n}.$$

The model equation (1.2) is the Euler-Lagrange equation associated with the functional

$$v \mapsto \mathcal{F}(v) - \int_\Omega \langle (|F|^{p-2}F + a(x)|F|^{q-2}F), Dv \rangle \, dx,$$

where

$$\mathcal{F}(v) = \int_\Omega (|Dv|^p + a(x)|Dv|^q) dx. \tag{1.3}$$

The origin of the functional (1.3) can be traced back to Zhikov [30], where they were introduced to model the homogenization of strongly anisotropic medium and to illustrate the Lavrentiev phenomenon. The broader concept of integrals with nonstandard growth was previously explored by Marcellini [22, 23], who introduced the terminology "nonstandard growth" or "(p, q) growth" to describe such phenomena. However, the functional (1.3) in the precise form made its initial appearance in the paper [15] by Esposito, Leonetti, and Mingione, where it was presented as one of several model cases satisfying nonstandard growth conditions. Nonstandard growth problems have found relevance in various physical contexts, including the modeling of electrorheological fluids, image processing, and the description of composite materials with heterogeneous microstructures. For a comprehensive overview and related applications of nonstandard growth problems, we refer to [24] and the references therein.

The pioneering study on the regularity of the minimizers of double phase problems began with Colombo and Mingione [10, 12] and was later extended in [3, 25, 29]. Such results have since catalyzed a vibrant area of research into the regularity of related models, including, for example, viscosity solutions to double phase problems [16, 18], multi-phase problems [20], partial regularity for double phase problems [26], and broader nonuniformly elliptic problems [14, 17]. In particular, the Calderón-Zygmund type estimates for the double phase problems were also established by Colombo and Mingione [11], namely, they proved that

$$H(x, F) \in L^\gamma_{\mathrm{loc}}(\Omega) \quad \Rightarrow \quad H(x, Du) \in L^\gamma_{\mathrm{loc}}(\Omega),$$

holds for all $\gamma \in (1, \infty)$. Following this seminal work, research on Calderón-Zygmund estimates for the double phase problem has been actively pursued, with notable contributions in [2, 8, 13, 28], and other related studies.

While many studies on Calderón-Zygmund estimates have been conducted for double-phase problems, more extensive work exists for p-Laplace type equations (see [1,4,5,9]). A notable feature in this latter field is that such estimates can be established even with discontinuous coefficients, allowing for a certain degree of irregularity with respect to the x-variable. More recently, various studies have attempted to incorporate this feature into the x-variable of the main operator $A(x, z)$ for double-phase problems (see [6, 7, 27]).

The primary challenge in this direction is how to properly introduce a discontinuity with respect to the x-variable. This difficulty stems from the structural difference between the operators.

The p-Laplace operator is autonomous with respect to the x-variable, which makes it relatively straightforward to impose a discontinuity assumption with respect to the x-variable on the main operator. In contrast, the principal part of the operator in a double-phase problem inherently has a dependence on the x-variable due to the presence of the modulating function $a(x)$, making it significantly more difficult to impose an assumption regarding discontinuities with respect to the x-variable.

Our core idea is to model the discontinuity in the x-variable by assuming that the main operator $A(x,z)$ can be decomposed into two variables: One that modulates the growth and another that allows the discontinuity. To this end, we assume that there exists a function $G : \Omega \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$G(x,x,z) = A(x,z), \quad \forall (x,z) \in \Omega \times \mathbb{R}^n \tag{1.4}$$

with the following growth, ellipticity, and continuity assumptions:

$$\begin{cases} |G(x,y,z)||z| + |\partial_z G(x,y,z)||z|^2 \le LH(x,z), \\ \nu \dfrac{H(x,z)}{|z|^2}|\xi|^2 \le \langle \partial_z G(x,y,z)\xi, \xi \rangle, \\ |G(x_1,y,z) - G(x_2,y,z)| \le L|a(x_1) - a(x_2)||z|^{q-1}, \end{cases} \tag{1.5}$$

for any $x, y, x_1, x_2 \in \Omega$, $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$. Here, the constants ν and L satisfy $0 < \nu \le L$, and the function

$$H(x,t) = t^p + a(x)t^q \quad {}^\forall (x,t) \in \Omega \times \mathbb{R}^+. \tag{1.6}$$

To simplify notation, we use $H(x,z)$ to mean $H(x,|z|)$ for any $(x,z) \in \Omega \times \mathbb{R}^n$.

While the existence of such a function G may appear to be an overly strong or unnatural assumption, we demonstrate in Section 2 that G can be obtained from the main terms of models considered in earlier studies on double phase problems with discontinuities, which justifies our assumption.

In what follows, for any function f depending on two spatial variables $x, y \in \Omega$ and the gradient variable $z \in \mathbb{R}^n$, we use the notation for the integral average of the function f on any set E with positive measure

$$(f)_{E,y}(x,z) := \fint_E f(x,y,z)\,dy = \frac{1}{|E|}\int_E f(x,y,z)\,dy.$$

On the other hand, if f depends on one spatial variable $x \in \Omega$ and the gradient variable $z \in \mathbb{R}^n$

$$(f)_E(z) := \fint_E f(x,z)\,dx = \frac{1}{|E|}\int_E f(x,z)\,dy.$$

Definition 1.1. *Let $R > 0$ and $\delta \in (0, \frac{1}{8})$. We say G is (δ, R)-vanishing with respect to y-variable if for any $0 < r \le R$ and $B_r(x_0) \subset \Omega$,*

$$\fint_{B_r(x_0)} \Theta(G; B_r(x_0))(x)\,dx \le \delta, \tag{1.7}$$

where

$$\Theta(G; B_r(x_0))(x) := \sup_{z \in \mathbb{R}^n \setminus \{0\}} \left[\frac{|G(x,x,z) - (G)_{B_r(x_0),y}(x,z)|}{|z|^{p-1} + a(x)|z|^{q-1}} \right].$$

The (δ, R)-vanishing condition stems from the notion of a small BMO coefficient, which essentially asserts that the oscillation of the coefficient is suitably small in the integral average sense. See [9] for the definition of BMO space. Thus, Definition 1.1 effectively imposes a small BMO-type condition on the auxiliary vector field G with respect to the y-variable.

It is worth mentioning that

$$0 \le \Theta(G; B_r(x_0)) \le 2L \tag{1.8}$$

holds whenever $B_r(x_0) \subset \Omega$ from $(1.5)_1$.

We finally assume that for all $x \in \Omega$ and $z \in \mathbb{R}^n$,

$$|B(x,z)| \le L\big(|z|^{p-1} + a(x)|z|^{q-1}\big). \tag{1.9}$$

In what follows, for simplicity of notation, we use the abbreviation

$$\texttt{data} = n, p, \nu, L, \Omega, \|\alpha\|_{C^\alpha}.$$

Let us now introduce our main theorem.

Theorem 1.2. *Let $u \in W^{1,H(\cdot)}(\Omega)$ be a solution to* (1.1) *with* (1.9). *Assume that there is a function $G : \Omega\times\Omega\times\mathbb{R}^n \to \mathbb{R}^n$ satisfying* (1.4) *and* (1.5). *Suppose that $H(x,F) \in L^\gamma_{\mathrm{loc}}(\Omega)$ for any given $\gamma \in (1,\infty)$. Then, there exist constants $\delta = \delta(\texttt{data}, \|H(x,F)\|_{L^1(\Omega)}, \gamma)$ and $R = R(\texttt{data}, \|H(x,F)\|_{L^1(\Omega)}, \gamma)$, such that if G is (δ, R)-vanishing with respect to y-variable, then $H(x,Du) \in L^\gamma_{\mathrm{loc}}(\Omega)$. Moreover, there exists a constant $c = c(\texttt{data}, \|H(x,F)\|_{L^1(\Omega)}, \gamma)$ such that the estimate*

$$\left(\fint_{B_r} H(x,Du)^\gamma\,dx\right)^{1/\gamma} \le c\fint_{B_r} H(x,Du)\,dx + c\left(\fint_{B_r} H(x,F)^\gamma\,dx\right)^{1/\gamma}$$

holds for all $B_r \subset \Omega$ with $0 < r \le R$.

As confirmed in the next section, Theorem 1.2 shows that specific forms of the structural assumptions previously explored in [6, 7, 27] are covered by our generalized conditions. This not only highlights the broad applicability of our proposed methodology but also confirms its unifying nature. Furthermore, we anticipate that the structural conditions introduced here can be effectively extended to establish the optimal C^1 regularity for solutions to double phase problems involving Dini coefficients, opening a clear path for future investigation.

2. Generality of the structure

This section is dedicated to examining how our generalized model equation extends specific existing formulations. We demonstrate that the assumptions on the equations considered in [7] are captured by our proposed structure. Furthermore, we analyze the extended model in [6]. Specifically, we show that its reduction to double phase problems with constant exponents can be described by the conditions of our generalized equation. This examination helps to understand the significance of our work in relation to model extensions, thereby clarifying how our structure can unify diverse existing formulations.

2.1. The structure conditions in [6] with constant exponents

It is worth mentioning that the original paper [6] considers double phase problems with variable exponents and small BMO-coefficients. Our work introduces a generalized approach whose assumptions unify the diverse forms of double phase problems found in prior research. To illustrate this generality, we specifically examine the constant exponents case of the result in [6].

For the constant exponent case, the study in [6] considers Eq (1.1) with the vector field A defined by

$$A(x,z) = A_1(x,z) + a(x)A_2(x,z). \tag{2.1}$$

Here, each vector field A_i for $i = 1, 2$ is assumed to satisfy

$$\begin{cases} |A_i(x,z)| + |\partial_z A_i(x,z)||z| \le L|z|^{p_i-1}, \\ \nu|z|^{p_i-2}|\xi|^2 \le \langle \partial_z A_i(x,z)\xi, \xi\rangle, \end{cases} \qquad p_i = \begin{cases} p, & i = 1, \\ q, & i = 2, \end{cases} \tag{2.2}$$

for some constants $0 < \nu \le L < \infty$, and every $x \in \Omega, z, \xi \in \mathbb{R}^n$. Moreover, each A_i satisfies the (δ, R)-vanishing condition:

$$\sup_{0<\rho\le R} \sup_{B_r(x_0)\subset\Omega} \fint_{B_r(x_0)} \theta_i(A_i; B_r(x_0))(x)\, dx \le \delta, \tag{2.3}$$

where

$$\theta_i(A_i; B_r(x_0))(x) := \sup_{z\in\mathbb{R}^n\setminus\{0\}} \frac{|A_i(x,z) - (A_i)_{B_r(x_0)}(z)|}{|z|^{p_i-1}}.$$

To discuss how (1.1) with (2.1)–(2.3) follows the conditions (1.4)–(1.7), we define the vector field $G : \Omega\times\Omega\times\mathbb{R}^n \to \mathbb{R}^n$ in (1.5) by

$$G(x,y,z) = A_1(y,z) + a(x)A_2(y,z). \tag{2.4}$$

Under this definition, (1.4) holds, i.e.,

$$G(x,x,z) = A_1(x,z) + a(x)A_2(x,z) = A(x,z).$$

Moreover, (2.2) directly implies (1.5).

We now check that G in (2.4) satisfies (1.7). For any $x \in \Omega$ and $B_r(y_0) \subset \Omega$ with $0 < r \le R$, we have

$$\begin{aligned} \frac{|G(x,x,z) - (G)_{B_r(x_0),y}(x,z)|}{|z|^{p-1} + a(x)|z|^{q-1}} &\le \frac{|A_1(x,z) - (A_1)_{B_r(x_0)}(z)|}{|z|^{p-1} + a(x)|z|^{q-1}} + \frac{a(x)|A_2(x,z) - (A_2)_{B_r(x_0)}(z)|}{|z|^{p-1} + a(x)|z|^{q-1}} \\ &\le \frac{|A_1(x,z) - (A_1)_{B_r(x_0)}(z)|}{|z|^{p-1}} + \frac{|A_2(x,z) - (A_2)_{B_r(x_0)}(z)|}{|z|^{q-1}} \\ &\le \theta_1(A_1; B_r(x_0))(x) + \theta_2(A_2; B_r(x_0))(x). \end{aligned}$$

Then, by (2.3),

$$\fint_{B_r(x_0)} \Theta(G; B_r(y_0))(x)\, dx \le \fint_{B_r(x_0)} \theta_1(A_1; B_r(x_0))(x)\, dx + \fint_{B_r(x_0)} \theta_2(A_2; B_r(x_0))(x)\, dx \le 2\delta$$

holds, whenever $B_r(x_0) \subset \Omega$ and $0 < r \le R$.

From the arbitrariness of the constant $\delta > 0$, we conclude that (1.1) with the assumptions (1.4)–(1.7) can be derived from the assumptions (2.1)–(2.3).

Remark 2.1. *The paper [27] establishes Calderón-Zygmund type result for double phase problems with coefficients depending on both x and u variables under the assumption* $\frac{q}{p} < 1 + \frac{\alpha}{n}$. *To deal with such coefficients, it is assumed that the solution u is bounded and the coefficient function satisfies the small BMO condition with respect to the variable x and uniform continuity with respect to the variable u. It is worth noting that the case of [27], where the coefficient is independent of the variable u, is a special case covered by [6]. Therefore, as discussed in this section, Theorem 1.2 also explains this specific case from [27].*

2.2. The structure conditions in [7]

The article [7] establishes Calderón-Zygmund type estimates for a class of equations generalizing the double phase problem. Unlike [6], which distinguishes between small BMO coefficients for p and q phases, the equations in [7] do not make such a distinction. To address these equations, the authors in [7] imposed additional assumptions on the range of p and q, specifically in (2.7).

By comparing the equations presented in [7] with the conditions (1.4)–(1.7), we now summarize the relevant assumptions from the article. Specifically, the Eq (1.1) is considered in [7] with the following growth and ellipticity assumptions below:

$$\begin{cases} |A(x,z)||z| + |\partial_z A(x,z)||z|^2 \leq LH(x,z), \\ \nu \dfrac{H(x,z)}{|z|^2}|\xi|^2 \leq \langle \partial_z A(x,z)\xi, \xi \rangle, \end{cases} \tag{2.5}$$

for some $0 < \nu \leq L < \infty$, whenever $x \in \Omega$, $z, \xi \in \mathbb{R}^n$. In addition, the following (δ, R)-vanishing condition is also assumed:

$$\sup_{0<\rho\leq R} \sup_{B_r(x_0)\subset\Omega} \fint_{B_r(x_0)} \theta(A; B_r(x_0))(x)\, dx \leq \delta, \tag{2.6}$$

where

$$\theta(A; B_r(x_0))(x) := \sup_{z\in\mathbb{R}^n\setminus\{0\}} \left| \frac{A(x,z)}{|z|^{p-1} + a(x)|z|^{q-1}} - \left(\frac{A(\cdot,z)}{|z|^{p-1} + a(\cdot)|z|^{q-1}} \right)_{B_r(x_0)} \right|.$$

The last assumption made in [7] is

$$q - p < \frac{\nu}{L}, \tag{2.7}$$

which naturally arises from the structure assumptions (2.5).

We now set

$$G(x,y,z) = \frac{H(x,z)}{H(y,z)} A(y,z) \tag{2.8}$$

for any $x, y \in \Omega$ and $z \in \mathbb{R}^n$. Then, (1.4) directly holds. To proceed further, we use the simple notation $H'(x,t) = \partial_t H(x,t)$ in this section.

First, we show that the function G defined in (2.8) satisfies the growth condition $(1.5)_1$. By basic manipulations, there holds

$$|G(x,y,z)||z| = \frac{H(x,z)}{H(y,z)} |A(y,z)| \leq LH(x,z).$$

In addition, by

$$\partial_z G(x,y,z) = \left(\frac{H'(x,z)H(y,z) - H(x,z)H'(y,z)}{H(y,z)^2}\right) A(y,z) \otimes \frac{z}{|z|} + \frac{H(x,z)}{H(y,z)} \partial_z A(y,z),$$

we have

$$|\partial_z G(x,y,z)||z|^2 \le \frac{H'(x,z)}{H(y,z)}|A(y,z)||z|^2 + \frac{H(x,z)H'(y,z)}{H(y,z)^2}|A(y,z)||z|^2 + \frac{H(x,z)}{H(y,z)}|\partial_z A(y,z)||z|^2.$$

Using the inequality $tH'(x,t)/H(x,t) \le q$ for any $t > 0$, we estimate each term on the right-hand side by

$$\begin{cases} \dfrac{H'(x,z)}{H(y,z)}|A(y,z)||z|^2 \le L|z||H'(x,z)| \le LqH(x,z), \\ \dfrac{H(x,z)H'(y,z)}{H(y,z)^2}|A(y,z)||z|^2 \le L\dfrac{H(x,z)H'(y,z)|z|}{H(y,z)} \le LqH(x,z), \\ \dfrac{H(x,z)}{H(y,z)}|\partial_z A(y,z)||z|^2 \le LH(x,z). \end{cases}$$

Combining the above estimates, we obtain

$$|\partial_z G(x,y,z)||z|^2 \le L(2q+1)H(x,z).$$

We now investigate if G defined in (2.8) fulfills the condition $(1.5)_2$. By definition of $H(\cdot)$,

$$H'(x,t)H(y,t) - H(x,t)H'(y,t) = (q-p)(a(x)-a(y))t^{p+q-1}. \tag{2.9}$$

To proceed further, we need to check that

$$\frac{(q-p)|a(x)-a(y)|t^{p+q-1}}{H(y,t)^2} \le (q-p)\frac{H(x,t)}{tH(y,t)} \tag{2.10}$$

holds for each $t > 0$, by considering two cases $0 \le 2a(y) \le a(x)$ and $0 \le a(x) \le 2a(y)$. When $0 \le 2a(y) \le a(x)$ holds, we find

$$0 \le \frac{(q-p)(a(x)-a(y))t^{p+q-1}}{H(y,t)^2} \le (q-p)\frac{a(x)t^{p+q-1}}{H(y,t)^2} \le (q-p)\frac{H(x,t)}{tH(y,t)}.$$

On the other hand, if $0 \le a(x) \le 2a(y)$, equivalently the inequality $|a(x)-a(y)| \le a(y)$, holds, then

$$\frac{(q-p)|a(x)-a(y)|t^{p+q-1}}{H(y,t)^2} \le \frac{(q-p)a(y)t^{p+q-1}}{a(y)t^q H(y,t)} \le (q-p)\frac{H(x,t)}{tH(y,t)}.$$

Combining the above inequalities, we have (2.10).

Then, $(2.5)_2$, (2.9), and (2.10) yield

$$\begin{aligned} \langle \partial_z G(x,y,z)\xi, \xi\rangle &\ge \frac{H(x,z)}{H(y,z)}\langle \partial_z A(y,z)\xi,\xi\rangle - \frac{|H'(x,z)H(y,z) - H(x,z)H'(y,z)|}{H(y,z)^2}|A(y,z)||\xi|^2 \\ &\ge \nu\frac{H(x,z)}{|z|^2}|\xi|^2 - L(q-p)\frac{H(x,z)}{|z|^2}|\xi|^2 \end{aligned}$$

$$\geq (\nu - L(q-p))\frac{H(x,z)}{|z|^2}|\xi|^2.$$

Recalling (2.7), we have (1.5) with new ellipticity and growth constants.

Now, we check the (δ, R)-vanishing condition (1.7). For any $B_r(x_0) \subset \Omega$ with $0 < r \leq R$, we have

$$\frac{|G(x,x,z) - (G)_{B_r(x_0),y}(x,z)|}{|x|^{p-1} + a(x)|z|^{q-1}} = \left| \frac{A(x,z)}{|z|^{p-1} + a(x)|z|^{q-1}} - \left(\frac{A(\cdot,z)}{|z|^{p-1} + a(\cdot)|z|^{q-1}} \right)_{B_r(x_0)} \right| \leq \theta(A; B_r(x_0))(x).$$

Therefore, (2.6) implies (1.7) with the same $\delta > 0$.

3. Preliminaries

Throughout this paper, we write

$$\mathcal{M} = \|H(x,F)\|_{L^1(\Omega)}.$$

Moreover, we denote by $c \geq 1$, generic constants depending only on $\mathtt{data}$, which may vary from line to line. If the constant depends on other parameters, we will explicitly denote the dependence by using parentheses. We use the notation $p' = p/(p-1)$ for the Hölder conjugate exponent of p for any $p > 1$. For any $x \in \Omega$ and $r > 0$, let $B_r(x)$ be the open ball centered at x with radius r. We may simply write B_r when the center is clear from the context.

We introduce the auxiliary vector field $V_s : \mathbb{R}^n \to \mathbb{R}^n$ for any $s > 1$ by

$$V_s(z) = |z|^{(s-2)/2} z, \quad z \in \mathbb{R}^n.$$

This vector field is commonly employed to handle the monotonicity property of the structure vector $A(\cdot)$. For any $z_1, z_2 \in \mathbb{R}^n$, the following inequalities hold:

$$|z_1 - z_2|^s \leq \begin{cases} c\varepsilon^{(s-2)/s}|V_s(z_1) - V_s(z_2)|^2 + \varepsilon(|z_1| + |z_2|)^s & \text{if } s \in (1,2), \\ c|V_s(z_1) - V_s(z_2)|^2 & \text{if } s \geq 2. \end{cases} \tag{3.1}$$

Here, the first inequality holds for any $\varepsilon \in (0,1)$. By a standard manipulation, under the ellipticity and growth assumptions in (1.5), there exists a constant $c = c(n, p, \nu, L)$ such that

$$|V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2 \leq c\langle G(x,y,z_1) - G(x,y,z_2), z_1 - z_2\rangle. \tag{3.2}$$

Combining (3.1) and (3.2), for any $\varepsilon \in (0,1)$,

$$H(x, |z_1 - z_2|) \leq \varepsilon^{-q'}\langle G(x,y,z_1) - G(x,y,z_2), z_1 - z_2\rangle + \varepsilon H(x, |z_1| + |z_2|). \tag{3.3}$$

We now turn our attention to the function spaces required for our analysis. We begin by defining the Musielak-Orlicz space $L^{H(\cdot)}(\Omega)$ to introduce the Musielak-Orlicz-Sobolev space $W^{1,H(\cdot)}(\Omega)$ for the function $H : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ given in (1.6). The space $L^{H(\cdot)}(\Omega)$ consists of all measurable functions $v : \Omega \to \mathbb{R}$ such that

$$\int_\Omega H(x,v)\,dx < \infty,$$

where the (Luxemburg) norm is given by

$$\|v\|_{L^{H(\cdot)}(\Omega)} := \inf \left\{ t > 0 \,\middle|\, \int_\Omega H\left(x, \frac{|v|}{t}\right) dx \le 1 \right\}.$$

Then, the Musielak-Orlicz-Sobolev space $W^{1,H(\cdot)}(\Omega)$ is defined as

$$W^{1,H(\cdot)}(\Omega) := \left\{ v \in W^{1,1}(\Omega) \mid v \in L^{H(\cdot)}(\Omega) \text{ and } H(\cdot, Dv) \in L^1(\Omega) \right\}.$$

Under the assumption (2.7), $W^{1,H(\cdot)}(\Omega)$ is a Banach space with the norm given by $\|v\|_{W^{1,H(\cdot)}(\Omega)} := \|v\|_{L^{H(\cdot)}(\Omega)} + \|Dv\|_{L^{H(\cdot)}(\Omega)}$. Moreover, we can define $W_0^{1,H(\cdot)}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,H(\cdot)}(\Omega)}$. For more details on the space $W^{1,H(\cdot)}(\Omega)$, especially focusing on the density of $W^{1,\infty}(\Omega)$ in $W^{1,H(\cdot)}(\Omega)$, we refer to [15, 19, 31].

With the above preliminaries at hand, we are now ready to prove the main result.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 largely follows the strategy outlined in [11, Section 4]. We will therefore focus on the steps that require careful handling of the structure condition (1.5) involving the function G. For the remaining details, we refer the reader to [11], as the completion of the proof follows the same reasoning developed therein.

Let us consider any ball $B_r \subset\subset \Omega$ with radius $r \le R$, where $R \le 1$ will be chosen small enough later in this section. Let E_λ^s be the super-level set given by

$$E_\lambda^s := \{x \in B_s : H(x, Du(x)) > \lambda\}, \qquad r/2 \le s \le r, \quad \lambda > 0, \tag{4.1}$$

where B_s is the ball with radius $s > 0$ and the same center as B_r. Then, for almost every $x_0 \in E_\lambda^s$ and any $s \in [r/2, r]$,

$$\lim_{\rho \to 0} \fint_{B_\rho(x_0)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx > \lambda,$$

where $\delta \in (0, 1)$ will be chosen later.

In order to carry out the subsequent calculations, we introduce two radii r_1, r_2 satisfying $r/2 \le r_1 < r_2 \le r$. Let us denote

$$\lambda_0 := \frac{20^n r_2^n}{(r_2 - r_1)^n} \fint_{B_{r_2}} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx.$$

Then, for any $\rho \in [(r_2 - r_1)/20, r_2 - r_1]$, there holds

$$\fint_{B_\rho(x_0)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \le \lambda_0.$$

Note that the radii r_1 and r_2 are needed to apply [11, Lemma 4.1] later in this section.

From now on, for $\lambda > \lambda_0$ we will find estimate on the set $E_\lambda^{r_1}$. Then, for almost every $x_0 \in E_\lambda^{r_1}$, there is ρ_{x_0} such that

$$\fint_{B_{\rho_{x_0}}(x_0)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx = \lambda \tag{4.2}$$

and

$$\fint_{B_{\rho_{x_0}}(x_0)} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx < \lambda \quad \text{for every } \rho \in (\rho_{x_0}, r_2 - r_1]. \tag{4.3}$$

Hence, we can apply Vitaili's covering lemma to find a measure zero set N and a family of disjoint balls $\{B_{\rho_{x_i}}(x_i)\}$ such that

$$E_\lambda^{r_1} \setminus N \subset \bigcup_i B_{5\rho_{x_i}}(x_i).$$

Moreover, for each i, (4.2) and (4.3) hold when replacing x_0 and ρ_{x_0} with x_i and ρ_{x_i}. For simplicity, we denote $\rho_i = 5\rho_{x_i}$ and $B_i = B_{\rho_i}(x_i)$. Observe that $\rho_i \le (r_2 - r_1)/4 \le r/8$ by (4.3) and the choice of $\lambda > \lambda_0$.

4.1. Comparison maps

To proceed further, we consider a sequence of weak solutions for each i. The first comparison map $v_{h,i} \in u + W_0^{1,H(\cdot)}(4B_i)$ is the weak solution to the homogeneous equation

$$\operatorname{div} A(x, Dv_{h,i}) = 0 \quad \text{in } 4B_i. \tag{4.4}$$

We now define

$$\bar{A}(x, z) := (G)_{3B_i, y}(x, z) = \fint_{3B_i} G(x, y, z)\, dy.$$

Then, the second comparison map, $v_{f,i} \in v_{h,i} + W_0^{1,H(\cdot)}(3B_i)$, is the weak solution to

$$\operatorname{div} \bar{A}(x, Dv_{f,i}) = 0 \quad \text{in } 3B_i. \tag{4.5}$$

Finally, we introduce our last comparison map, $w_i \in v_{f,i} + W_0^{1,p}(2B_i)$, as the weak solution to the frozen equation

$$\operatorname{div} \bar{A}(x_{m,i}, Dw_i) = 0 \quad \text{in } 2B_i, \tag{4.6}$$

where $x_{m,i}$ is the point satisfying $a(x_{m,i}) = \sup_{x \in 2B_i} a(x)$.

Let us point out that to guarantee the existence of the solution w_i with Dirichlet boundary data $v_{f,i}$, it is not enough to have $v_{f,i} \in W^{1,H(\cdot)}(3B_i)$. The conditional reverse Hölder type result [13, Theorem 5] allows us to address this issue. Indeed, by recalling (1.4) and (1.5), we see that $\bar{A}$ satisfies the assumptions [13, (1.6), (1.8)]. This allows us to apply [13, Theorem 5] to $v_{f,i}$, which ensures that $v_{f,i} \in W_{loc}^{1,q}(3B_i)$. We now state this result without its proof.

Lemma 4.1. *Let $v_{f,i} \in W^{1,p}(3B_i)$ be a weak solution to* (4.5). *Assume*

$$\sup_{x \in 3B_i} a(x) \le K[a]_{C^\alpha} \rho_i^\alpha,$$

for some $K \ge 1$. Then, for every $\bar{q} < np/(n - 2\alpha)$ (= ∞ when $\alpha = 1$ and $n = 2$), there is a constant $c = c(\mathtt{data}, \|Dv_{f,i}\|_{L^p(3B_i)}, K, \bar{q}) > 0$ which is a nondecreasing function of $\|Dv_{f,i}\|_{L^p(3B_i)}$, and satisfying

$$\left(\fint_{2B_i} |Dv_{f,i}|^{\bar{q}}\, dx \right)^{1/\bar{q}} \le c \left(\fint_{3B_i} H(x, Dv_{f,i})\, dx \right)^{1/p}. \tag{4.7}$$

4.2. Comparison estimates

The first comparison lemma we present is about u and $v_{h,i}$.

Lemma 4.2. *Let $u \in W^{1,H(\cdot)}(\Omega)$ be the weak solution to* (1.1) *and $v_{h,i} \in u + W_0^{1,H(\cdot)}(4B_i)$ the weak solution to* (4.4)*. Then, for any $\varepsilon \in (0,1)$, we have*

$$\fint_{4B_i} H(x, Du - Dv_{h,i})\,dx \le \varepsilon \fint_{4B_i} H(x, |Du| + |Dv_{h,i}|)\,dx + c\varepsilon^{-q'} \fint_{4B_i} H(x, F)\,dx, \tag{4.8}$$

where c depends only on $\mathtt{data}$.

Proof. By testing the difference $u - v_{h,i} \in W_0^{1,H(\cdot)}(4B_i)$ to (1.1) and (4.4), and utilizing (3.3), (1.4), and (1.9), we obtain

$$\begin{aligned}
&\fint_{4B_i} H(x, Du - Dv_{h,i})\,dx \\
&\le c\varepsilon^{q'} \fint_{4B_i} \langle A(x, Du) - A(x, Dv_{h,i}), D(u - v_{h,i})\rangle\,dx + \frac{\varepsilon}{2} \fint_{4B_i} H(x, |Du| + |Dv_{h,i}|)\,dx \\
&= c\varepsilon^{q'} \fint_{4B_i} \langle B(x, F), D(u - v_{h,i})\rangle\,dx + \frac{\varepsilon}{2} \fint_{4B_i} H(x, |Du| + |Dv_{h,i}|)\,dx \\
&\le c\varepsilon^{q'} \fint_{4B_i} H(x, F)\,dx + \varepsilon \fint_{4B_i} H(x, |Du| + |Dv_{h,i}|)\,dx,
\end{aligned}$$

for any $\varepsilon \in (0,1)$. This completes the proof. □

As a next step, we establish a higher integrability result for the weak solution $v_{h,i}$ to (4.4), which will be used in the subsequent comparison estimate.

Lemma 4.3. *Let $v_{h,i} \in u + W_0^{1,H(\cdot)}(\Omega)$ be the weak solution to* (4.4)*. Then, there exist constants $\tau = \tau(\mathtt{data}) \in (0,1)$ and $c = c(\mathtt{data}, \mathcal{M})$ such that*

$$\fint_{3B_i} H(x, Dv_{h,i})^{1+\tau}\,dx \le c \left(\fint_{4B_i} H(x, Dv_{h,i})\,dx \right)^{1+\tau},$$

where c is nondecreasing with respect to $\mathcal{M}$.

Proof. This lemma can be proved by slightly modifying the proof of [6, Lemma 4.1]. However, for the interested readers, we outline the proof. For any $B_{2\rho}(x_0) \subset 4B_i$, we choose a cutoff function $\eta \in C_c^\infty(B_{2\rho}(x_0))$ such that $\eta \equiv 1$ on $B_\rho(x_0)$ and $|D\eta| \le 4/\rho$. By testing $v_{h,i}\eta^q$ to (4.4), a standard manipulation gives the Caccioppoli type estimate

$$\fint_{B_\rho(x_0)} H(x, Dv_{h,i})\,dx \le c \fint_{B_{2\rho}(x_0)} \left(\left| \frac{v_{h,i} - (v_{h,i})_{B_{2\rho}(x_0)}}{\rho} \right|^p + a(x) \left| \frac{v_{h,i} - (v_{h,i})_{B_{2\rho}(x_0)}}{\rho} \right|^q \right) dx.$$

We distinguish two cases to apply Poincaré's inequality. When $\inf_{B_{2\rho}(x_0)} a(x) > (4\rho)^\alpha [a]_{C^\alpha}$, then $\sup_{B_{2\rho}(x_0)} a(x) \le 2 \inf_{B_{2\rho}(x_0)} a(x)$. Hence, the Sobolev-Poincaré inequality gives

$$\fint_{B_\rho(x_0)} H(x, Dv_{h,i})\,dx \le c \left(\fint_{B_{2\rho}(x_0)} |Dv_{h,i}|^{p_*}\,dx \right)^{p/p_*} + c \inf_{B_{2\rho}(x_0)} a(x) \left(\fint_{B_{2\rho}(x_0)} |Dv_{h,i}|^{q_*}\,dx \right)^{q/q_*}$$

$$\leq c\left[\fint_{B_{2\rho}(x_0)} (|Dv_{h,i}|^p + a(x)|Dv_{h,i}|^q)^{q_*/q}\,dx\right]^{q/q_*},$$

where $p_* = np/(n+p)$ and $q_* = nq/(n+q)$.

The second case is when $\inf_{B_{2\rho}(x_0)} a(x) \leq (4\rho)^\alpha [a]_{C^\alpha}$. Then,

$$\begin{aligned}\fint_{B_\rho(x_0)} H(x, Dv_{h,i})\,dx &\leq c\left(\fint_{B_{2\rho}(x_0)} |Dv_{h,i}|^{p_*}\,dx\right)^{p/p_*} + c\rho^\alpha\left(\fint_{B_{2\rho}(x_0)} |Dv_{h,i}|^{q_*}\,dx\right)^{q/q_*}\\ &\leq c\left[1 + \rho^{\alpha-n(q-p)/p}\left(\int_{B_{2\rho}(x_0)} |Dv_{h,i}|^p\,dx\right)^{(q-p)/p}\right]\left(\fint_{B_{2\rho}(x_0)} |Dv_{h,i}|^{q_*}\,dx\right)^{p/q_*}\\ &\leq c(\mathcal{M})\left(\fint_{B_{2\rho}(x_0)} H(x, Dv_{h,i})^{q_*/p}\,dx\right)^{p/q_*}.\end{aligned}$$

Combining the estimates in both cases and using the fact that $q_*/q < q_*/p < 1$, we apply Gehring's lemma to conclude the desired conclusion. □

We now present a second comparison lemma between $v_{h,i}$ and $v_{f,i}$.

Lemma 4.4. *Let $v_{h,i} \in W^{1,H(\cdot)}(4B_i)$ and $v_{f,i} \in v_{h,i} + W_0^{1,H(\cdot)}(3B_i)$ be the weak solutions to* (4.4) *and* (4.5), *respectively. Then, for any $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\mathtt{data}, \mathcal{M}, \varepsilon) \in (0,1)$ such that if* (1.7) *holds with $\delta \leq \delta_1$, then*

$$\fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\,dx \leq \varepsilon \fint_{4B_i} H(x, Dv_{h,i})\,dx.$$

Proof. We test $v_{h,i} - v_{f,i}$ against (4.4) and (4.5). Then, for any $\kappa \in (0,1)$, we use (3.3) to find

$$\begin{aligned}&\fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\,dx\\ &\leq c\kappa^{-q'} \fint_{3B_i} \langle \bar{A}(x, Dv_{h,i}) - \bar{A}(x, Dv_{f,i}), D(v_{h,i} - v_{f,i})\rangle\,dx + \kappa \fint_{3B_i} H(x, |Dv_{h,i}| + |Dv_{f,i}|)\,dx\\ &= c\kappa^{-q'} \fint_{3B_i} \langle \bar{A}(x, Dv_{h,i}) - A(x, Dv_{h,i}), D(v_{h,i} - v_{f,i})\rangle\,dx + \kappa \fint_{3B_i} H(x, |Dv_{h,i}| + |Dv_{f,i}|)\,dx\\ &=: c\kappa^{-q'} I + c\kappa \fint_{3B_i} H(x, Dv_{h,i})\,dx + c_0\kappa \fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\,dx.\end{aligned}$$

Here, c_0 depends only on $\mathtt{data}$.

Now, we estimate I as follows:

$$\begin{aligned}|I| &\overset{(1.4)}{\leq} \fint_{3B_i} \frac{|G(x, x, Dv_{h,i}) - (G)_{3B_i}(x, Dv_{h,i})|}{|Dv_{h,i}|^{p-1} + a(x)|Dv_{h,i}|^{q-1}} (|Dv_{h,i}|^{p-1} + a(x)|Dv_{h,i}|^{q-1})|Dv_{h,i} - Dv_{f,i}|\,dx\\ &\overset{(1.7)}{\leq} \fint_{3B_i} \Theta(G; 3B_i)(x)(|Dv_{h,i}|^{p-1} + a(x)|Dv_{h,i}|^{q-1})|Dv_{h,i} - Dv_{f,i}|\,dx\\ &\leq c(\kappa)\left(\fint_{3B_i} [\Theta(G; 3B_i)(x)]^{p'}|Dv_{h,i}|^p\,dx + \fint_{3B_i} [\Theta(G; 3B_i)(x)]^{q'} a(x)|Dv_{h,i}|^q\,dx\right)\end{aligned}$$

$$
\begin{aligned}
&+ \kappa^{q'+1} \fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\, dx \\
&\overset{(1.8)}{\leq} c(\kappa) \left(\fint_{3B_i} \Theta(G; 3B_i)(x)(|Dv_{h,i}|^p + a(x)|Dv_{h,i}|^q)\, dx \right) + \kappa^{q'+1} \fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\, dx \\
&\leq c(\kappa) \left(\fint_{3B_i} [\Theta(G; 3B_i)(x)]^{\frac{1+\tau}{\tau}}\, dx \right)^{\frac{\tau}{1+\tau}} \left(\fint_{3B_i} H(x, Dv_{h,i})^{1+\tau}\, dx \right)^{\frac{1}{1+\tau}} \\
&\quad + \kappa^{q'+1} \fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\, dx \\
&\overset{(1.7)}{\leq} c(\kappa, \mathcal{M}) \delta^{\frac{\tau}{1+\tau}} \left(\fint_{4B_i} H(x, Dv_{h,i})\, dx \right) + \kappa^{q'+1} \fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\, dx.
\end{aligned}
$$

In the last inequality, we also used Lemma 4.3.

Therefore, we obtain

$$
\begin{aligned}
\fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\, dx &\leq c(\kappa, \mathcal{M}) \delta^{\frac{\tau}{1+\tau}} \fint_{4B_i} H(x, Dv_{h,i})\, dx \\
&\quad + c\kappa \fint_{3B_i} H(x, Dv_{h,i})\, dx + c\kappa \fint_{3B_i} H(x, Dv_{h,i} - Dv_{f,i})\, dx,
\end{aligned}
$$

where the first constant on the right-hand side, $c(\kappa, \mathcal{M})$, is decreasing with respect to κ, and it diverges to ∞ as $\kappa \to 0$. Since the constant c_0 depends only on $\mathtt{data}$, we choose κ small enough to absorb the last integral on the right-hand side into the left-hand side. Finally, we choose $\delta_1 = \delta(\mathtt{data}, \mathcal{M}, \varepsilon) \in (0, 1)$ small enough to complete the proof. □

At this stage, we observe that the vector field $\bar{A}$ satisfies all the assumptions in [13, (1.8)]. As the arguments from Steps 1–7 in [13, Section 6] remain valid for our case, we can directly apply their comparison estimate between (4.5) and (4.6). We summarize this result in the following lemma.

Lemma 4.5. *Let $w_i \in v_{f,i} + W_0^{1,H(\cdot)}(2B_i)$ be the weak solution to* (4.6)*. Then, for any $K \geq 4$, there exist constants $\tilde{c} = \tilde{c}(\mathtt{data})$ and $c_* = c_*(\mathtt{data}, \mathcal{M}, K) \geq 1$ such that*

$$
\fint_{2B_i} H(x, Dw_i - Dv_{f,i})\, dx \leq \left[\frac{\tilde{c}}{K} + c_* \rho_i^{\sigma} \right] \fint_{3B_i} H(x, Dv_{f,i})\, dx, \tag{4.9}
$$

where

$$
\sigma = \alpha - \frac{n(q-p)}{p(1+\tau)} > \alpha - n\left(\frac{q}{p} - 1\right) \geq 0
$$

with the small constant $\tau \in (0, 1)$ appearing in Lemma 4.3.

To replace the integral involving $Dv_{f,i}$ with an integral involving Du, we utilize the quasi-minimality property of the weak solutions. Based on the growth and ellipticity assumptions $(1.5)_{1,2}$, the weak solution $v_{h,i}$ is a quasi-minimizer for the functional

$$
v \mapsto \fint_{4B_i} H(x, Dv)\, dx, \qquad \text{for any } v \in u + W_0^{1,p}(4B_i),
$$

and similarly, the weak solution $v_{f,i}$ is a quasi-minimizer of the functional

$$v \mapsto \fint_{3B_i} H(x, Dv)\, dx, \qquad \text{for any } v \in v_{h,i} + W_0^{1,p}(3B_i).$$

Therefore, we have

$$\fint_{3B_i} H(x, Dv_{f,i})\, dx \le c \fint_{3B_i} H(x, Dv_{h,i})\, dx \le c \fint_{4B_i} H(x, Dv_{h,i})\, dx \le c \fint_{4B_i} H(x, Du)\, dx, \tag{4.10}$$

where c depends only on $L/\nu, p, q, n$.

The following lemma presents another useful result concerning the Lipschitz regularity of the weak solution to (4.6).

Lemma 4.6. *Let $w_i \in v_{f,i} + W_0^{1,H(\cdot)}(2B_i)$ be the weak solution to* (4.6)*. Then, $Dw_i \in L^\infty(2B_i)$ with the estimate*

$$\sup_{x \in 2B_i} H(x, Dw_i) \le c \fint_{4B_i} H(x, Du)\, dx,$$

where c depends only on `data`*.*

Proof. In light of [21, Lemma 5.1], we have

$$\sup_{x \in 2B_i} H(x, Dw_i) \le \sup_{x \in 2B_i} H(x_{m,i}, Dw_i) \le c \fint_{2B_i} H(x_{m,i}, Dw_i)\, dx \le c \fint_{2B_i} H(x_{m,i}, Dv_{f,i})\, dx.$$

If $\inf_{x \in 2B_i} a(x) > 10 r^\alpha [a]_{C^\alpha}$, it follows that

$$\begin{aligned} \sup_{x \in 2B_i} H(x, Dw_i) &\le c \fint_{2B_i} H(x_{m,i}, Dv_{f,i})\, dx \\ &\le c \fint_{2B_i} \left[H(x, Dv_{f,i}) + r^\alpha [a]_{C^\alpha} |Dv_{f,i}|^q \right] dx \le c \fint_{3B_i} H(x, Dv_{f,i})\, dx. \end{aligned} \tag{4.11}$$

On the other hand, if $\inf_{x \in 2B_i} a(x) \le 10 r^\alpha [a]_{C^\alpha}$, then by Lemma 4.1, we obtain

$$\begin{aligned} \sup_{x \in 2B_i} H(x, Dw_i) &\le c \fint_{2B_i} H(x_{m,i}, Dv_{f,i})\, dx \\ &\le c \fint_{2B_i} \left(|Dv_{f,i}|^p + r^\alpha [a]_{C^\alpha} |Dv_{f,i}|^q \right) dx \\ &\le c \fint_{3B_i} |Dv_{f,i}|^p\, dx + c r^{\alpha - n(q-p)/p} \left(\int_{3B_i} |Dv_{f,i}|^p\, dx \right)^{(q-p)/p} \fint_{3B_i} |Dv_{f,i}|^p\, dx \\ &\le c \fint_{3B_i} H(x, Dv_{f,i})\, dx. \end{aligned} \tag{4.12}$$

In the last line, we have used the quasi-minimizing property of $v_{f,i}$ given in (4.10).

Combining (4.11) and (4.12), and using (4.10) again, the desired estimate is obtained. □

Finally, combining Lemmas 4.2, 4.4, 4.5, and 4.6, and the estimate (4.10), we obtain the following.

Lemma 4.7. *For any constant* $\lambda \geq 1$ *and any* $\varepsilon \in (0,1)$*, there exists a positive constant* $\delta_1 = \delta_1(\mathtt{data}, \mathcal{M}, \varepsilon) \in (0,1)$ *such that if*

$$\fint_{4B_i} \left[H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \leq \lambda$$

and A satisfies (1.4)–(1.7) *for some* $R \in (0,1)$ *and* $\delta \in (0, \delta_1]$*, then for any* $K \geq 4$*, we have*

$$\fint_{B_i} H(x, Du - Dw_i)\, dx \leq c_1 \left[\varepsilon + \varepsilon^{-q'} \delta + \frac{1}{K} + c_* R^{\sigma} \right] \lambda,$$

for some $c_1 > 0$ *depending only on* $\mathtt{data}$*, where* $c_* = c_*(\mathtt{data}, \mathcal{M}, K)$ *is the one given in Lemma 4.5. Moreover, there exists* $c_l = c_l(\mathtt{data})$ *such that*

$$\sup_{x \in 2B_i} H(x, Dw_i) \leq c_l \lambda.$$

We now use the notation

$$S(\varepsilon, R, \delta, K) := c_1 \left[\varepsilon + \varepsilon^{-q'} \delta + \frac{1}{K} + c_* R^{\sigma} \right],$$

where c_1 and c_* are the constants introduced in Lemma 4.5. Then, by Lemma 4.7 with (4.3) and the fact $\rho_i \leq R/8$, for any $\varepsilon > 0$ chosen later, there is $\delta_1 = \delta_1(\mathtt{data}, \mathcal{M}, \varepsilon)$ such that for any $\delta \in (0, \delta_1)$ and B_i, there holds

$$\fint_{2B_i} H(x, Du - Dw_i)\, dx \leq S(\varepsilon, R, \delta, K) \lambda. \tag{4.13}$$

Moreover, there is $c_l = c_l(\mathtt{data})$ such that

$$\sup_{x \in B_i} H(x, Dw_i) \leq c_l \lambda. \tag{4.14}$$

Note that the estimates (4.13) and (4.14) are essentially the same as in [11, (4.47) and (4.51)]. The parameter M in their paper corresponds to $1/\delta$ in ours. Then, the remaining part of the proof of Theorem 1.2 is a standard procedure for deriving Calderón-Zgymund type estimates, which is a repetition of Steps 10–11 in [11, Section 4]. In these steps, with the same choice of the constants $\varepsilon, R, \delta, K$ as in [11, Section 4], one concludes the desired result.

5. Conclusions

In this paper, we successfully established the Calderón-Zygmund estimate for the double phase problem defined by Eq (1.1), under the assumptions (1.4), (1.5), (1.9), and the (δ, R)-vanishing condition for small $\delta > 0$. The principal strength of this result lies in the introduction of a generalized structural condition, which effectively unifies various structural assumptions presented in existing literature into a single framework. This unified approach provides a strong foundation for future analysis and is expected to be extended to establish C^1 regularity results under the Dini mean oscillation condition for the leading coefficients.

Author contributions

Pilsoo Shin: Conceptualization, investigation, methodology, validation, writing–original draft, writing–review and editing; Yeonghun Youn: Formal analysis, investigation, methodology, validation, writing–original draft, writing–review and editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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