AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25346–25357.
DOI: 10.3934/math.20251122
Received: 06 September 2025
Revised: 19 October 2025
Accepted: 23 October 2025
Published: 04 November 2025

Research article

A note on some stability results for stochastic delay differential equations

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Abstract: We investigate the stability of stochastic delay differential equations (SDDEs) that describe systems affected by both memory effects and stochastic disturbances. By adopting a comparison principle approach, we derive unified criteria for p-th moment, asymptotic, and exponential stability under conditions that relax the conventional requirement of a negative diffusion operator. The proposed results broaden the applicability of classical methods and are validated through illustrative numerical examples, demonstrating their potential relevance to engineering and applied sciences.

Keywords: stability; stochastic delay differential equations; Halanay inequalities
Mathematics Subject Classification: 34K50, 90B15, 93D20

1. Introduction

Stochastic dynamical systems with delays have become an indispensable framework for modeling real-world processes where memory effects and random perturbations coexist. Their applications span a broad spectrum of science and engineering, including communication networks, population biology, and financial mathematics. In such contexts, delays are inevitable due to factors such as signal transmission, processing lags, or inherent reaction times, while stochastic disturbances typically represent environmental noise or structural uncertainties. The interplay between these two factors often leads to complex behaviors, including oscillations or even instability, thereby motivating extensive studies on stochastic delay differential equations (SDDEs).

Over the past decades, numerous analytical techniques have been developed to investigate the stability properties of SDDEs. Among these, the Razumikhin method has played a central role in establishing asymptotic and exponential stability criteria [3, 7, 8]. Further developments of this method were provided in [10, 13, 14], and several extensions were discussed in [16, 21, 23]. More recent contributions address stochastic and hybrid systems using similar ideas [24, 27, 28]. Additional improvements and related applications can be found in [29]. The Lyapunov functional approach, another powerful tool, has yielded refined conditions for moment stability under various delay structures [2, 6, 9]. Further theoretical improvements based on this approach were reported in [15–17], and additional developments under different delay structures can be found in [22,27]. In addition, fixed-point theorems have been applied to address existence and stability problems [11,12], while differential and integral inequality techniques have been employed to analyze hybrid and functional differential systems [18–20]. Collectively, these approaches have greatly enriched the theoretical foundations of SDDEs and expanded their applicability to a diverse range of scientific and engineering models.

Despite these advances, many classical results rely on restrictive structural assumptions. In particular, several existing criteria [1–3] require the diffusion operator $\mathcal{L}V$ associated with a Lyapunov function to be negative definite or to satisfy inequalities of the form

$$\mathcal{L}V(t,\phi) \leq \xi(t)V(t,\phi(0)) + \chi(t) \sup_{\theta\in[-\tau,0]} V(t+\theta,\phi(\theta)),$$

with $\xi(t) + \chi(t) \leq a \leq 0$ for all $t \geq t_0$ (see [8, 19, 20]). While mathematically elegant, these conditions significantly limit practical applicability. For instance, when $\xi(t) = -0.5t + t\sin t$ and $\chi(t) = 0$, the term $\xi(t) + \chi(t)$ oscillates without a global bound, rendering the classical framework inapplicable. Similar challenges arise whenever the diffusion operator becomes positive over certain time intervals in cases that are not adequately addressed by existing theories.

To address these limitations, this work develops stability criteria for SDDEs under more general conditions. Our approach neither requires the diffusion operator to remain negative at all times nor assumes that $\xi(t) + \chi(t)$ is globally bounded. By leveraging comparison principles, we relax these restrictive assumptions and derive broader stability conditions, thereby extending the scope of systems that can be rigorously analyzed.

The remainder of the paper is organized as follows. Section 2 introduces notations and preliminary definitions. Section 3 presents the main theoretical results. Section 4 demonstrates the effectiveness of the proposed criteria through numerical examples. Section 5 concludes the paper with discussions of potential extensions and future research directions.

2. Preliminaries

Let $\mathbb{R}_a = [a, +\infty)$ and let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, with $|\cdot|$ representing the Euclidean norm. The class $\mathcal{VK}$ consists of continuous, strictly increasing, convex functions, and $\mathcal{CK}$ of continuous, strictly increasing, concave functions, both defined on $\mathbb{R}_0^+$ with $\mu(0) = 0$.

Let $w(t)$ be an n-dimensional standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, P)$, where the filtration $\{\mathcal{F}_t\}$ is right-continuous and $\mathcal{F}_{t_0}$ contains all P-null sets. Denote by $\mathbb{E}$ the expectation with respect to P [5]. Define

$$C := C((-\infty, t_0]; \mathbb{R}^n) = \{\phi : (-\infty, t_0] \to \mathbb{R}^n \mid \phi \text{ is continuous and bounded}\},$$

equipped with the norm $\|\phi\| = \sup_{\theta\in(-\infty,t_0]} |\phi(\theta)| < \infty$. Let $\mathcal{L}^p_{\mathcal{F}_t}((-\infty, t_0]; \mathbb{R}^n)$ denote the set of $\{\mathcal{F}_t\}$-adapted, C-valued random variables $\phi = \{\phi(\theta) : \theta \in (-\infty, t_0]\}$ satisfying

$$\|\phi\|^p_{\mathcal{L}^p} = \sup_{\theta\in(-\infty,t_0]} \mathbb{E}|\phi(\theta)|^p < \infty.$$

For any function $a(t)$ defined on $[t_0, +\infty)$, define its extended form by

$$a^*(t) = \begin{cases} a(t), & t \geq t_0, \\ 0, & t < t_0. \end{cases}$$

Consider the following stochastic delay differential equation (SDDE):

$$\begin{cases} dz(t) = f(t, z(t), z(t - \tau(t)))\, dt + \sigma(t, z(t), z(t - \tau(t)))\, dw(t), & t \geq t_0, \\ z(t) = \phi(t) \in \mathcal{L}^p_{\mathcal{F}_{t_0}}((-\infty, t_0]; \mathbb{R}^n), & t \in (-\infty, t_0], \end{cases} \tag{2.1}$$

where $f : \mathbb{R}_{t_0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_{t_0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are Borel-measurable functions satisfying standard conditions that guarantee the existence and uniqueness of solutions. $\tau(t) : \mathbb{R}_{t_0^+} \to \mathbb{R}_{0^+}$ characterizes the system's time delay, which may be bounded or unbounded. For simplicity, we write $z(t)$ instead of $z(t, t_0, \phi)$. Furthermore, if $f(t, 0, 0) = 0$ and $\sigma(t, 0, 0) = 0$ for all $t \geq t_0$, then the identically zero process $z(t) \equiv 0$ constitutes a trivial solution.

Following [9], we introduce several p-th moment stability concepts.

Definition 2.1. *[9] The zero solution of (2.1) is called p-*th moment stable *(PS) if, for every $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon) > 0$ such that whenever $\|\phi\|_{\mathcal{L}^p} < \delta$, it follows that*

$$\mathbb{E}|z(t, t_0, \phi)|^p < \epsilon, \quad t \geq t_0.$$

Definition 2.2. *[9] The zero solution of (2.1) is called is p-*th moment asymptotically stable *(PAS) if it is PS and there exists $\delta > 0$ such that $\|\phi\|_{\mathcal{L}^p} < \delta$ implies*

$$\lim_{t\to+\infty} \mathbb{E}|z(t, t_0, \phi)|^p = 0.$$

Definition 2.3. *[9] The zero solution of (2.1) is called p-*th moment exponentially stable *(PES) if there exist constants $a > 0$ and $\kappa > 0$ such that*

$$\mathbb{E}|z(t, t_0, \phi)|^p \leq \kappa \|\phi\|^p_{\mathcal{L}^p} e^{-a(t-t_0)}, \quad t \geq t_0.$$

Remark 2.4. *In the special case of $p = 2$, Definitions 2.1–2.3 correspond to 2S, 2AS, and 2ES, which represent mean square stability, mean square asymptotic stability, and mean square exponential stability, respectively.*

Throughout this paper, let $\mathcal{V} \in C^{2,1}[\mathbb{R}_{t_0} \times \mathbb{R}^n; \mathbb{R}_0]$ denote a nonnegative functional $\mathcal{V}(t, z)$ that is continuously differentiable once in t and twice in z. Given a $\mathcal{V}(t, x)$ in this class, define $\mathcal{L}\mathcal{V} : \mathbb{R}_{t_0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{L}\mathcal{V}(t, x, y) = \mathcal{V}_t(t, x) + \mathcal{V}_x^T(t, x) f(t, x, y) + \tfrac{1}{2}\, \mathrm{trace}\Big[\sigma^T(t, x, y)\, \mathcal{V}_{xx}(t, x)\, \sigma(t, x, y)\Big],$$

where $\mathcal{V}_t = \partial\mathcal{V}/\partial t$, $\mathcal{V}_x = (\partial\mathcal{V}/\partial x_1, \ldots, \partial\mathcal{V}/\partial x_n)$, and $\mathcal{V}_{xx} = (\partial^2\mathcal{V}/\partial x_i \partial x_j)_{n\times n}$ [25].

3. Stability analysis

In this section, the comparison principle is employed to derive stability conditions for system (2.1). The first main theorem is presented below.

Theorem 3.1. *Let* $\mathcal{V} \in C^{2,1}[\mathbb{R}_{t_0} \times \mathbb{R}^n; \mathbb{R}_0]$, $c_1 \in \mathcal{VK}$, $c_2 \in \mathcal{CK}$, $\xi : \mathbb{R}_{t_0} \to \mathbb{R}$, $\chi : \mathbb{R}_{t_0} \to \mathbb{R}_0$, *and* $p > 0$. *Suppose that:*
(**A.1**) $c_1(|z|^p) \le \mathcal{V}(t, z) \le c_2(|z|^p)$, $\forall (t, z) \in \mathbb{R}_{t_0} \times \mathbb{R}^n$;
(**A.2**)

$$\mathbb{E}\mathcal{L}\mathcal{V}(t, z(t), z(t - \tau(t))) \le \xi(t)\, \mathbb{E}\mathcal{V}(t, z(t)) + \chi(t) \sup_{s \in [t-\tau(t), t]} \mathbb{E}\mathcal{V}(s, z(s)), \quad t \ge t_0;$$

(**A.3**)

$$\lim_{t \to +\infty} \int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)]\, dv < +\infty,$$

where $\rho(t) = \exp\Big(\sup_{s \in [t-\tau(t), t]} \int_s^t (-\xi(v))\, dv \Big)$.

Then, system (2.1) is PS.

Proof. Let $M(t) = \mathbb{E}\mathcal{V}(t, z(t))$. By Itô's formula, one has

$$M(t) = M(t_0) + \int_{t_0}^{t} \mathbb{E}\mathcal{L}\mathcal{V}(s, z(s), z(s - \tau(s)))ds, \qquad t \ge t_0. \tag{3.1}$$

From (3.1), for all $t \ge t_0$ and $\triangle t > 0$, we have

$$M(t + \triangle t) = M(t) + \int_{t}^{t+\triangle t} \mathbb{E}\mathcal{L}\mathcal{V}(s, z(s), z(s - \tau(s)))ds.$$

Hence,

$$\begin{aligned} D^+ M(t) &= \limsup_{\triangle t \to 0^+} \frac{M(t + \triangle t) - M(t)}{\triangle t} = \mathbb{E}\,\mathcal{L}\mathcal{V}(t, z(t), z(t - \tau(t))) \\ &\le \xi(t) M(t) + \chi(t) \sup_{s \in [t-\tau(t), t]} M(s), \quad t \ge t_0, \end{aligned}$$

where D^+ denotes the upper right Dini derivative [4] defined by

$$D^+ M(t) = \limsup_{\triangle t \to 0^+} \frac{M(t + \triangle t) - M(t)}{\triangle t}.$$

Constructing the following delay differential equation,

$$\begin{cases} \dot{u}(t) = \xi(t) u(t) + \chi(t) \sup_{s \in [t-\tau(t), t]} u(s), & t \in \mathbb{R}_{t_0}, \\ u(t) = u_0 = c_2(\|\phi\|^p_{\mathcal{L}^p}), & t \le t_0, \end{cases} \tag{3.2}$$

we show that

$$u(t) \le u_0 e^{\int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)]\, dv}, \quad t \in \mathbb{R}_{t_0}, \tag{3.3}$$

in three steps.

First, we verify the nonnegativity of $u(t)$ for all $t \in \mathbb{R}$. Clearly, for $t \leq t_0$, we have $u(t) = u_0 = c_2(\|\phi\|^p_{\mathcal{L}^p}) \geq 0$. For $t > t_0$, suppose, for contradiction, that $u(t) < 0$ at some t, and define

$$t^* = \inf\{t \geq t_0 \mid u(t) < 0\}.$$

Given that $u(t)$ is continuous and $u(t^*) < 0$, it follows that $u(t) \geq 0$ for $t < t^*$. Therefore, the upper right-hand derivative satisfies

$$D^+u(t_0) < 0.$$

However, according to (3.2), we have

$$\dot{u}(t^*) = \xi(t^*)u(t^*) + \chi(t^*) \sup_{s\in[t^*-\tau(t^*),t^*]} u(s) \geq 0,$$

which is in contradiction with the previous inequality. Hence, we deduce that $u(t) \geq 0$ for all $t \in \mathbb{R}$.

Next, we assert that

$$\sup_{s\in[t-\tau(t),t]} u(s) \leq \rho(t)u(t), \quad t \in \mathbb{R}_{t_0}.$$

To see this, we apply the variation of constants formula, which yields

$$u(t) = u_0 e^{\int_{t_0}^t \xi(s)ds} + \int_{t_0}^t e^{\int_s^t \xi(u)du}\chi(s) \sup_{v\in[s-\tau(s),s]} u(v)ds, \quad t \in \mathbb{R}_{t_0}. \tag{3.4}$$

Noting $\chi(t) \geq 0$ and $u(t) \geq 0$, we deduce that

$$\begin{aligned} u(t) &= u_0 e^{\int_{t_0}^t \xi(v)dv} + \int_{t_0}^t e^{\int_s^t \xi(v)dv}\chi(s) \sup_{\theta\in[s-\tau(s),s]} u(\theta)ds \\ &= e^{\int_{t-\tau(t)}^t \xi^*(v)dv}\left(u_0 e^{\int_{t_0}^{t-\tau(t)} \xi^*(v)dv} + \int_{t_0}^{t-\tau(t)} e^{\int_s^{t-\tau(t)} \xi^*(v)dv}\chi(s) \sup_{\theta\in[s-\tau(s),s]} u(\theta)ds\right) \\ &\quad + \int_{t-\tau(t)}^t e^{\int_s^t \xi^*(v)dv}\chi(s) \sup_{\theta\in[s-\tau(s),s]} u(\theta)ds \\ &\geq e^{\int_{t-\tau(t)}^t \xi(v)dv} u(t-\tau(t)). \end{aligned} \tag{3.5}$$

Thus, from (3.5), we have

$$u(t-\tau(t)) \leq e^{\int_{t-\tau(t)}^t (-\xi(v))dv} u(t), \quad t \in \mathbb{R}_{t_0}. \tag{3.6}$$

Consequently, it follows from (3.6) that

$$\sup_{s\in[t-\tau(t),t]} u(s) \leq \rho(t)\, u(t), \quad t \in \mathbb{R}_{t_0}. \tag{3.7}$$

Finally, we prove the claim (3.3). In fact, by (3.7), (3.2) can be rewritten as

$$\begin{cases} \dot{u}(t) \leq [\xi(t) + \chi(t)\rho(t)]u(t), & t \in \mathbb{R}_{t_0}, \\ u(t) = u_0 = c_2(\|\phi\|^p_{\mathcal{L}^p}), & t \leq t_0, \end{cases}$$

which immediately leads to

$$M(t) \leq u(t) \leq u_0 e^{\int_{t_0}^t [\xi(v)+\chi(v)\rho(v)]\, dv}, \quad t \in \mathbb{R}_{t_0}. \tag{3.8}$$

By assumption $(A.3)$, there exists an $M_1 \geq 0$ such that

$$\int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)]\, dv \leq M_1, \quad t \in \mathbb{R}_{t_0}.$$

Exploiting the properties of c_1 and c_2, for any $\varepsilon > 0$, one can choose a sufficiently small $\delta > 0$ such that

$$e^{M_1} c_2(\delta) < c_1(\varepsilon).$$

Consequently, whenever $\|\phi\|_{\mathcal{L}^p}^p < \delta$, it holds that

$$c_1(\mathbb{E}|z(t)|^p) \leq \mathbb{E}[c_1(|z(t)|^p)] \leq M(t) \leq e^{M_1} c_2(\delta) < c_1(\varepsilon), \tag{3.9}$$

which immediately implies $\mathbb{E}|z(t)|^p < \varepsilon$ for all $t \in \mathbb{R}_{t_0}$. This completes the proof.

□

Theorem 3.2. *Let $\mathcal{V} \in C^{2,1}[\mathbb{R}_{t_0} \times \mathbb{R}^n; \mathbb{R}_0]$, $c_1 \in \mathcal{VK}$, $c_2 \in \mathcal{CK}$, $\xi : \mathbb{R}_{t_0} \to \mathbb{R}$, $\chi : \mathbb{R}_{t_0} \to \mathbb{R}_0$, and $p > 0$. Suppose assumptions $(A.1)$, $(A.2)$, and the following hold:*
(A.3′) $\lim_{t\to+\infty} \int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)]\, dv = -\infty$, *where $\rho(t)$ is as defined in Theorem 3.1. Then the solution of system (2.1) is PAS.*

Proof. Assumption $(A.3')$ implies $(A.3)$, hence the solution of (2.1) is PS. From $(A.1)$ and (3.8), one has

$$c_1(\mathbb{E}|z(t)|^p) \leq u_0 \exp\Big(\int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)] dv \Big).$$

Applying the definition of c_1 and $(A.3')$, it follows that

$$\lim_{t\to+\infty} \mathbb{E}|z(t)|^p = 0.$$

This completes the proof. □

Theorem 3.3. *Let $\mathcal{V} \in C^{2,1}[\mathbb{R}_{t_0} \times \mathbb{R}^n; \mathbb{R}_0]$, $c_1 \in \mathcal{VK}$, $c_2 \in \mathcal{CK}$, $\xi : \mathbb{R}_{t_0} \to \mathbb{R}$, $\chi : \mathbb{R}_{t_0} \to \mathbb{R}_0$, $\lambda > 0$, $p > 0$, and constants $0 < \kappa_1 \leq \kappa_2$. Suppose that assumption $(A.2)$ and the following hold:*
(A.1′) $\kappa_1 |z|^p \leq \mathcal{V}(t, z) \leq \kappa_2 |z|^p$ *for all $(t, z) \in \mathbb{R}_{t_0} \times \mathbb{R}^n$;*
(A.3″) $\lim_{t\to+\infty} \int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v) + \lambda] dv = -\infty$,
where $\rho(t)$ is defined as in Theorem 3.1. Then, (2.1) is PES.

Proof. From $(A.1')$ and (3.8), one has

$$M(t) = \mathbb{E}\mathcal{V}(t, z(t)) \leq \kappa_2 \|\phi\|_{\mathcal{L}^p}^p \exp\left(\int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)]\, dv \right). \tag{3.10}$$

Hence,

$$\mathbb{E}|z(t)|^p \leq \frac{\kappa_2}{\kappa_1} \|\phi\|_{\mathcal{L}^p}^p \exp\left(\int_{t_0}^{t} [\xi(v) + \chi(v)\rho(v)]\, dv \right), \quad t \geq t_0.$$

By $(A.3'')$, there exists $T_* \geq t_0$ such that

$$\int_{t_0}^{t} [\xi(u) + \chi(u)\rho(u)]\, du \leq -\lambda(t - t_0), \quad t > T_*,$$

which implies

$$\mathbb{E}|z(t)|^p \leq \frac{\kappa_2}{\kappa_1} \|\phi\|^p_{\mathcal{L}^p} \exp(-\lambda(t - t_0)), \quad t > T_*.$$

For $t \in [t_0, T_*]$, it follows from the previous bound that

$$\mathbb{E}|z(t)|^p \leq \frac{\kappa_2}{\kappa_1} \|\phi\|^p_{\mathcal{L}^p} \exp\left(\sup_{s \in [t_0, T_*]} \int_{t_0}^{s} [\xi(v) + \chi(v)\rho(v)]\, dv\right) \exp(\lambda(T_* - t_0)) \exp(-\lambda(t - t_0)).$$

Combining these estimates, there exists a constant

$$K = \frac{\kappa_2}{\kappa_1} \exp\left(\sup_{s \in [t_0, T_*]} \int_{t_0}^{s} [\xi(v) + \chi(v)\rho(v)]\, dv + \lambda(T_* - t_0)\right)$$

such that

$$\mathbb{E}|z(t)|^p \leq K \|\phi\|^p_{\mathcal{L}^p} \exp(-\lambda(t - t_0)), \quad t \geq t_0.$$

This completes the proof. □

4. Numerical examples

This section presents three numerical examples to demonstrate the practical applicability of the theoretical results.

Example 4.1. *Consider the following SDDE*

$$\begin{cases} dz(t) = \Big(-\dfrac{1}{2(t+1)} - 1 + 0.5 \sin t \Big) z(t)\, dt + \Big(z(t) + \dfrac{z(0.5t)}{3e\sqrt{t+1}} \Big) dw(t), \quad t \geq 0, \\ z(0) = 10, \end{cases} \tag{4.1}$$

where $w(t)$ is the standard Wiener process [5].

Choosing $\mathcal{V}(t, z) = |z|^2$, we obtain

$$\mathbb{E}\mathcal{L}\mathcal{V}(t, z(t)) \leq \Big(-\frac{1}{t+1} + \sin t \Big) \mathbb{E}\mathcal{V}(t, z(t)) + \frac{2}{9e^2(t+1)} \sup_{s \in [0.5t, t]} \mathbb{E}\mathcal{V}(s, z(s)).$$

Define $\xi(t) = -\dfrac{1}{t+1} + \sin t$, $\chi(t) = \dfrac{2}{9e^2(t+1)}$, and note that $\rho(t) = e^{\int_{t-\tau(t)}^{t} (-\xi^*(s)) ds} \leq 2e^2$. Since

$$\lim_{t \to +\infty} \int_0^t [\xi(v) + \chi(v)\rho(v)]\, dv = -\infty,$$

the zero solution of system (4.1) is 2S and 2AS. Figure 1 illustrates the numerical simulation of $z(t)$.

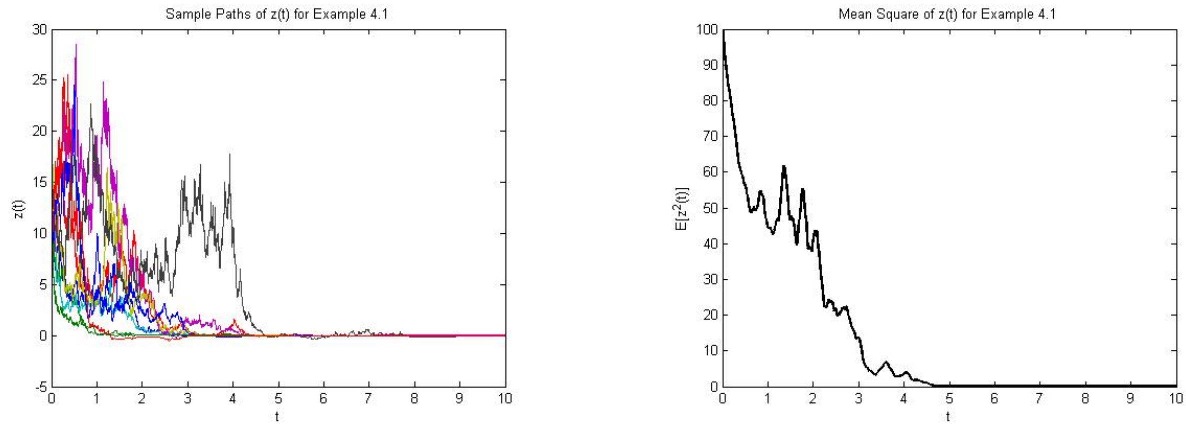

Figure 1. Trajectories of $z(t)$ and its mean square $E[z^2(t)]$ for Example 4.1.

Remark 4.2. *Note that the delay function $\tau(t) = 0.5t$ is unbounded. Although*

$$\int_0^t [\xi(v) + \chi(v)\rho(v)]\, dv \to -\infty \quad \text{as } t \to \infty,$$

there does not exist a constant $\lambda > 0$ such that

$$\int_0^t [\xi(v) + \chi(v)\rho(v) + \lambda]\, dv \to -\infty \quad \text{as } t \to \infty.$$

Therefore, condition (A.3”) for exponential stability is not satisfied. Consequently, the zero solution of system (4.1) is $2S$ and $2AS$, but not $2ES$.

Example 4.3. *Consider the following SDDE:*

$$\begin{cases} dz(t) = ((-0.5 + \sin t)z(t) + 0.02z(t - 0.5\pi))dt + 0.2z(t)\, dw(t), & t \geq 0, \\ z(t) = 10, & t \in [-0.5\pi, 0]. \end{cases} \tag{4.2}$$

With $\mathcal{V}(t, z) = |z|^2$, we have

$$\mathbb{E}\mathcal{L}\mathcal{V}(t, z(t)) \leq (-0.94 + 2\sin t)\mathbb{E}\mathcal{V}(t, z(t)) + 0.02 \sup_{s \in [t-0.5\pi, t]} \mathbb{E}\mathcal{V}(s, z(s)).$$

Let $\xi(t) = -0.94 + 2\sin t$, $\chi(t) = 0.02$, and note that $\rho(t) = e^{\int_{t-\tau(t)}^{t}(-\xi^*(s))ds} \leq e^{0.47\pi+2}$. Then,

$$\lim_{t \to +\infty} \int_0^t [\xi(v) + \chi(v)\rho(v) + 0.2]\, du = -\infty,$$

implying that the zero solution of system (4.2) is 2S, 2AS, and 2ES. Figure 2 presents the numerical results.

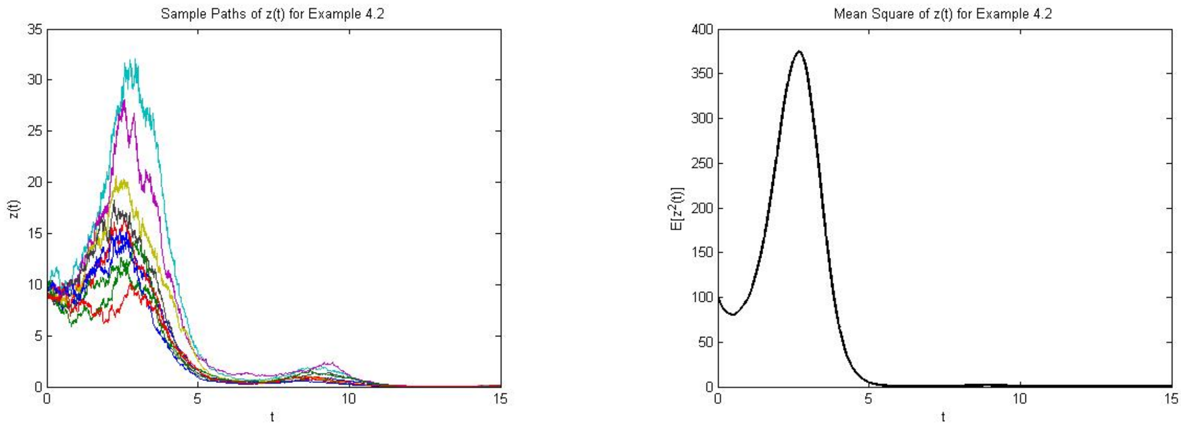

Figure 2. Trajectories of $z(t)$ and its mean square $E[z^2(t)]$ for Example 4.2.

Remark 4.4. *The function $\xi(t)$ is positive on some intervals, indicating that the diffusion operator is not strictly negative. Hence, existing results in the literature [8, 19, 20] do not apply to this case. Related discussions can be found in [22, 26, 27], and further generalizations were reported in [28, 29].*

Example 4.5. *Consider the following SDDE:*

$$\begin{cases} dz(t) = (-0.5t + t\sin t)z(t)\,dt + e^{-4t-2}z(t-\pi)\,dw(t), & t \ge 0, \\ z(t) = 10, & t \in [-\pi, 0]. \end{cases} \tag{4.3}$$

For $\mathcal{V}(t,z) = |z|^2$, it follows that

$$\mathbb{E}\mathcal{L}\mathcal{V}(t, z(t)) \le (-t + 2t\sin t)\mathbb{E}\mathcal{V}(t, z(t)) + e^{-8t-4} \sup_{s\in[t-\pi,t]} \mathbb{E}\mathcal{V}(s, z(s)).$$

Here, $\xi(t) = -t + 2t\sin t$, $\chi(t) = e^{-8t-4}$, and note that $\rho(t) = e^{\int_{t-\tau(t)}^{t}(-\xi^*(s))ds} \le e^{-8t-4}$. Since

$$\lim_{t\to+\infty} \int_0^t [\xi(v) + \chi(v)\rho(v) + 2]\,dv = -\infty,$$

the zero solution of system (4.3) is 2S, 2AS, and 2ES. Figure 3 shows the numerical simulation.

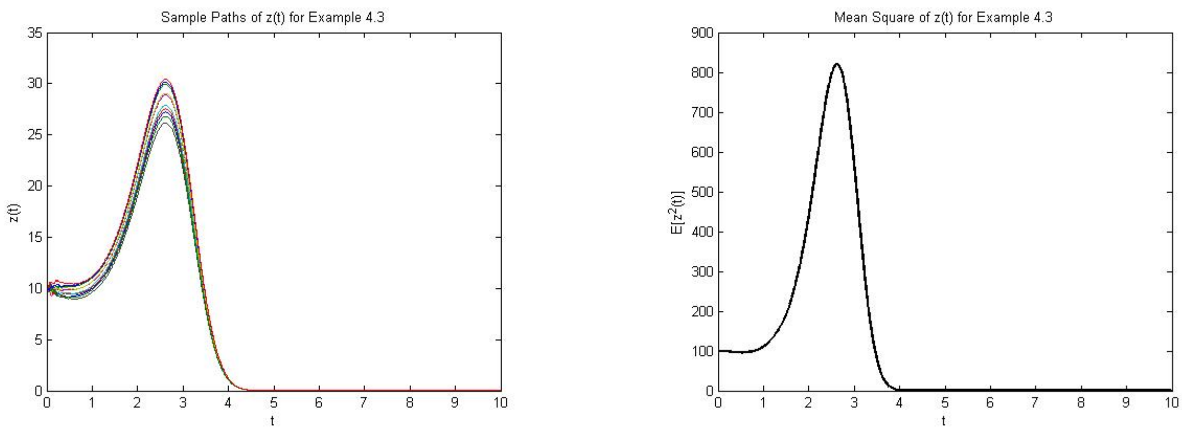

Figure 3. Trajectories of $z(t)$ and its mean square $E[z^2(t)]$ for Example 4.3.

Remark 4.6. *The combined function $\xi(t) + \chi(t)$ is unbounded; Thus, previously established results in the literature [8, 19, 20] do not cover this situation. Similar limitations have been noted in [22, 27], and more recent extensions can be found in [28, 29].*

5. Conclusions

This paper has presented new stability results for SDDEs using a comparison-based analysis. Unlike much of the existing literature, the proposed framework relaxes the sign-definiteness and boundedness restrictions on the Lyapunov diffusion operator, thereby accommodating systems with oscillatory or unbounded coefficients. The approach provides a unified framework for p-th moment, asymptotic, and exponential stability, and its effectiveness is validated through numerical examples. Future research may extend these ideas to models with distributed delays, impulsive dynamics, or non-Gaussian disturbances, with the goal of bridging rigorous stochastic theory and practical applications in control, biology, and finance.

Author contributions

Yao Lu: Writing-original draft, preparation; Dehao Ruan: Validation, software development; Quanxin Zhu: Project administration, supervision. All authors have read and agreed to the submitted version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Natural Science Foundation of Hunan Province, China (2025JJ60047).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. Z. Cai, L. Huang, Generalized Lyapunov approach for functional differential inclusions, *Automatica*, **113** (2020), 108740. https://doi.org/10.1016/j.automatica.2019.108740
2. T. Caraballo, J. Real, L. Shaikhet, Method of Lyapunov functionals construction in stability of delay evolution equations, *J. Math. Anal. Appl.*, **334** (2007), 1130–1145. https://doi.org/10.1016/j.jmaa.2007.01.038

3. Z. Fan, M. Song, M. Liu, The a-th moment stability for the stochastic pantograph equation, *J. Comput. Appl. Math.*, **233** (2009), 109–120. https://doi.org/10.1016/j.cam.2009.04.024

4. A. F. Filippov, Differential equations with discontinuous righthand sides, *Matematicheskii Sbornik*, **93** (1960), 909–128.

5. A. V. Gikhman, A. V. Skorokhod, *Stochastic differential equations*, In: The Theory of Stochastic Processes III, Grundlehren der mathematischen Wissenschaften, New York: Springer, **232** (1972). https://doi.org10.1007/978-1-4615-8065-2_2

6. L. Huang, X. Mao, Delay-dependent exponential stability of neutral stochastic delay systems, *IEEE T. Automat. Contr.*, **54** (2009), 147–152. https://doi.org/10.1109/TAC.2008.2007178

7. S. Janković, J. Randjelović, M. Jovanović, Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations, *J. Math. Anal. Appl.*, **355** (2009), 811–820. https://doi.org/10.1016/j.jmaa.2009.02.011

8. K. Liu, X. Xia, On the exponential stability in mean square of neutral stochastic functional differential equations, *Syst. Control Lett.*, **37** (1999), 207–215. https://doi.org/10.1016/S0167-6911(99)00021-3

9. K. Liu, *Stability of infinite dimensional stochastic differential equations with applications*, New York: Chapman and Hall/CRC, 2005. https://doi.org/10.1201/9781420034820

10. J. Luo, A note on exponential stability in p-th mean of solutions of stochastic delay differential equations, *J. Comput. Appl. Math.*, **198** (2007), 143–148. https://doi.org/10.1016/j.cam.2005.11.019

11. J. Luo, Fixed points and stability of neutral stochastic delay differential equations, *J. Math. Anal. Appl.*, **334** (2007), 431–440. https://doi.org/10.1016/j.jmaa.2006.12.058

12. J. Luo, Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays, *J. Math. Anal. Appl.*, **342** (2008), 753–760. https://doi.org/10.1016/j.jmaa.2007.11.019

13. Q. Luo, X. Mao, Y. Shen, New criteria on exponential stability of neutral stochastic differential delay equations, *Syst. Control Lett.*, **55** (2006), 826–834. https://doi.org/10.1016/j.sysconle.2006.04.005

14. X. Mao, Razumikhin-type theorems on exponential stability of stochastic functional differential equations, *Stoch. Proc. Appl.*, **65** (1996), 233–250. https://doi.org/10.1016/S0304-4149(96)00109-3

15. X. Mao, *Stochastic differential equations and applications*, 2 Eds., Elsevier, 2007.

16. G. Rajchakit, R. Sriraman, Robust passivity and stability analysis of uncertain complex-valued impulsive neural networks with time-varying delays, *Neural Process. Lett.*, **53** (2021), 581–606. https://doi.org/10.1007/s11063-020-10401-w

17. R. Rakkiyappan, P. Balasubramaniam, S. Lakshmanan, Robust stability results for uncertain stochastic neural networks with discrete interval and distributed time-varying delays, *Phys. Lett. A*, **372** (2008), 5290–5298. https://doi.org/10.1016/j.physleta.2008.06.011

18. D. Ruan, L. Xu, J. Luo, Stability of hybrid stochastic functional differential equations, *Appl. Math. Comput.*, **346** (2019), 832–841. https://doi.org/10.1016/j.amc.2018.03.064

19. D. Ruan, X. Guo, Y. Shi, Generalized Halanay inequalities for stability and dissipativity of stochastic functional differential equations, *Syst. Control Lett.*, **173** (2023), 105469. https://doi.org/10.1016/j.sysconle.2023.105469
20. D. Ruan, Y. Lu, Q. Zhu, A note on generalized exponential stability of impulsive stochastic functional differential equations, *Appl. Math. Lett.*, **156** (2024), 109117. https://doi.org/10.1016/j.aml.2024.109117
21. D. Ruan, Q. Zhu, Y. Lu, Generalized exponential stability of stochastic delayed systems with delayed impulses and infinite delays, *Syst. Control Lett.*, **204** (2025), 106869. https://doi.org/10.1016/j.sysconle.2025.106194
22. L. Shaikhet, *Lyapunov functionals and stability of stochastic functional differential equations*, Switzerland: Springer, 2013. https://doi.org/10.1007/978-3-319-00101-2
23. F. Wu, S. Hu, Razumikhin-type theorems on general decay stability and robustness for stochastic functional differential equations, *Int. J. Robust Nonlin.*, **22** (2012), 763–777. https://doi.org/10.1002/rnc.1726
24. F. Wu, G. Yin, L. Wang, Razumikhin-type theorems on moment exponential stability of functional differential equations involving two-time scale Markovian switching, *Math. Control Relat. F.*, **5** (2015), 697–719. https://doi.org/10.3934/mcrf.2015.5.697
25. X. Yao, S. You, W. Mao, X Mao, On the decay rate for a stochastic delay differential equation with an unbounded delay, *Appl. Math. Lett.*, **166** (2025), 109541. https://doi.org/10.1016/j.aml.2025.109541
26. Z. Zhang, H. Chen, H. Zhu, Generalized Halanay inequality and its application to delay differential inclusions, *Automatica*, **166** (2024), 111704.
27. F. Zhu, Z. Han, J. Zhang, Stability analysis of stochastic differential equations with Markovian switching, *Syst. Control Lett.*, **61** (2012), 1209–1214. https://doi.org/10.1016/j.sysconle.2012.08.013
28. X. Zong, F. Wu, C. Huang, Exponential mean square stability of the theta approximations for neutral stochastic differential delay equations, *J. Comput. Appl. Math.*, **286** (2015), 172–185. https://doi.org/10.1016/j.cam.2015.03.016
29. X. Zong, G. Yin, L. Wang, T. Li, J. Zhang, Stability of stochastic functional differential systems using degenerate Lyapunov functionals and applications, *Automatica*, **91** (2018), 197–207. https://doi.org/10.1016/j.automatica.2018.01.038