AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25329–25345.
DOI: 10.3934/math.20251121
Received: 11 July 2025
Revised: 16 October 2025
Accepted: 30 October 2025
Published: 04 November 2025

Research article

On global existence and blow up of weak solutions for a wave equation with mixed local and nonlocal propagation

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Abstract: In this paper, we investigate the initial boundary value problem of a wave equation with mixed local and nonlocal propagation. First, we introduce the spatial framework to study the wave equation, which is the intersection of a classical Sobolev space and a fractional Sobolev space. By the Mountain Pass Theorem, we obtain the attainability of the optimal embedding constant from the introduced space to the suitable Lebesgue space. Second, by introducing a family of potential wells in the introduced space, we obtain the existence of a global weak solution through the utilization of potential well theory. At last, by analyzing the properties of the energy functional, we show that any weak solution must blow up at the existence time.

Keywords: potential well; mixed local and nonlocal propagation; wave equation; global solution; blow up
Mathematics Subject Classification: 35R11, 35A15, 45K05

1. Introduction

Because of the so-called memory effect, there exist many phenomena that ordinary calculus is unable to describe. To explain this dependency, fractional ordinary and/or partial differential equations must be used. Ordinary differential equations as well as partial differential equations of fractional orders have gained considerable importance due to their applications in physics, mechanics, chemistry, engineering, and other fields. Recently, there has been significant progress concerning ordinary differential equations and partial differential equations of fractional orders; see, for example, [1–6] and the references therein. Especially in [7], Fu studied the initial boundary value problems for space-fractional wave equations.

Due to mathematical structure and rich physical background, i.e., physical scenarios describing the coexistence of short-range and long-range diffusion within the same medium, problems of mixed local and nonlocal operators have become an important topic in partial differential equations. Local and nonlocal mixed operators can be regarded as the superposition of two different scales of random processes, i.e., classical Brownian motion and Lévy flight. Among them, there is a classic category of local and nonlocal operators:

$$-\Delta + (-\Delta)^s, \ \ s \in (0, 1),$$

where Δ is the Laplace operator, and the nonlocal operator $(-\Delta)^s$ is the fractional Laplace operator. This type of operator has received widespread attention and research in recent years and has shown important applications in many fields. For example, in biology, such operators provide new insights for studying ecological issues such as population dynamics modeling. Dipierro and Valdinoci [8] proposed a model of an evolution equation driven by mixed operators:

$$\frac{\partial U}{\partial t} = \alpha\Delta U - \beta(-\Delta)^s U \ \ \text{in } \Sigma \times (0, +\infty),$$

where

$$\lim_{\vartheta\to 0^+} \frac{U(x+\vartheta v, t) - U(x,t)}{\vartheta} =: \frac{\partial U}{\partial v}(x) = 0 \text{ for every } x \in \partial\Sigma,$$
$$\int_\Sigma \frac{U(x) - U(y)}{|x-y|^{n+2s}} dy = 0 \text{ for every } x \in \mathbb{R}^n \backslash \bar{\Sigma}.$$

It is applied to characterize the spatiotemporal evolution of population density in ecological niches, where the domain of the equation corresponds to the population ecological niche space of the location. In addition, Dipierro, Proietti, and Valdinoci [9] introduced logistic equations using classical diffusion and nonlocal diffusion. They applied spectral analysis methods to systematically analyze favorable and unfavorable scenarios to quantify and then study whether the environmental niche is suitable for the survival of the population. Mixed operators are applied to other fields including plasma physics (see, for example, [10]) and stochastic control (see, for example, [11, 12]).

Regarding the parabolic equations mixed with local and nonlocal diffusions:

$$u_t - \Delta u + (-\Delta)^s u = 0,$$

Barlow, Bass, Chen et al. [13] established the Harnack inequality for bounded nonnegative solutions. Chen and Kumagai et al. [14] proved the Hölder regularity and Harnack estimation of weak solutions by means of probabilistic methods. Garain and Kinnunen [15] obtained the weak Harnack inequality. Lately, Das [16] applied perturbation methods to prove the regularity $C^{1,\alpha}$ for the solutions.

As we know, there are no works on wave equations with mixed local and nonlocal propagations, so in this paper, we study this topic to generalize the results in [7] to wave equations with mixed local and nonlocal propagations, i.e.,

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^s u = |u|^{p-2}u, & (x,t) \in \Sigma \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Sigma, \\ u(x,t) = 0, & x \in \mathbb{R}^N \backslash \Sigma, \ t \in (0, +\infty). \end{cases} \tag{1.1}$$

Throughout this paper, we assume that $\Sigma \subset \mathbb{R}^N$ is a smooth bounded domain, $s \in (0, 1)$, $N > 2s$, and p satisfies $2 < p \leqslant 2_s^* = \frac{2N}{N-2s}$. We call $u \in L^\infty(0, +\infty; Y_0)$ with $u_t \in L^\infty(0, +\infty; L^2(\Sigma))$ a weak solution of (1.1) if

$$\begin{aligned}
&-\int_0^t\int_\Sigma u_\omega(x,\omega)\varphi_\omega(x,\omega)dxd\omega+\int_0^t\int_\Sigma \nabla u(x,\omega)\nabla\varphi(x,\omega)dxd\omega\\
&+\int_0^t\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x,\omega)-u(y,\omega))(\varphi(x,\omega)-\varphi(y,\omega))}{|x-y|^{N+2s}}dxdyd\omega\\
&+\int_\Sigma u_t(x,t)\varphi(x,t)dx-\int_\Sigma u_1(x,0)\varphi(x,0)dx\\
=&\int_0^t\int_\Sigma |u(x,\omega)|^{p-2}u(x,\omega)\varphi(x,\omega)dxd\omega
\end{aligned}$$

and

$$u(x,0)=u_0(x)\in Y_0$$

for any $\varphi\in L^1(0,+\infty;Y_0)$ with $\varphi_t\in L^1(0,+\infty;L^2(\Sigma))$ and any $t\in[0,+\infty)$. See the next section for the definition of Y_0. Throughout this paper, the potential well theory is used here and there. Concerning potential wells, see [17–21] and the references therein.

The organizational structure of this paper is as follows. In Section 2, we provide symbols and some facts about classical Sobolev spaces and fractional Sobolev spaces for the utilization afterwards. In Section 3, we introduce a family of potential wells for studying wave equations with mixed local and non-local propagation. In Section 4, we establish the existence of global weak solutions and the blow-up of weak solutions.

2. Preliminaries

The Hilbert space $H_0^1(\Sigma)$ is defined by

$$H_0^1(\Sigma)=\left\{u\in L^2(\Sigma):|\nabla u|^2\in L^1(\Sigma),\ u=0\text{ on }\partial\Sigma\right\}$$

endowed with the norm

$$\|u\|_{H_0^1(\Sigma)}=\|\nabla u\|_{L^2(\Sigma)}$$

and inner product

$$\langle u,v\rangle_{H_0^1(\Sigma)}=\int_\Sigma\nabla u\nabla vdx.$$

Denote

$$2^*=+\infty\text{ if }N=1,2;\ \frac{2N}{N-2}\text{ if }N\geqslant 3.$$

Lemma 2.1. *(1) Provided $p\in[1,2^*)$, there holds the compact embedding $H_0^1(\Sigma)\hookrightarrow\hookrightarrow L^p(\Sigma)$.*
(2) There holds the continuous embedding $H_0^1(\Sigma)\hookrightarrow L^{2^*}(\Sigma)$.

See [22] for the facts on the Hilbert space $H_0^1(\Sigma)$.

Let $s\in(0,1)$ and $2s<N$. The fractional Laplace operator is defined as

$$(-\triangle)^s u(x)=\int_{\mathbb{R}^N}\frac{u(x)-u(x+y)-u(x-y)}{|y|^{N+2s}}dy$$

for $x \in \mathbb{R}^N$. This definition is consistent (up to a normalization constant depending on N and s) with the classical definition of a fractional Laplace operator; see [23].

The fractional Sobolev space $H^s(\mathbb{R}^N)$ is the collection of functions $u \in L^2(\mathbb{R}^N)$ satisfying

$$\frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N).$$

$H^s(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + [u]_{H^s(\mathbb{R}^N)},$$

where

$$[u]_{H^s(\mathbb{R}^N)} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

Let X_0 be the subspace of $H^s(\mathbb{R}^N)$ as

$$X_0 = \left\{ u \in H^s(\mathbb{R}^N) : \ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Sigma \right\}.$$

For X_0, we can use

$$\|u\|_{X_0} = [u]_{H^s(\mathbb{R}^N)} = (\iint_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Sigma^c \times \Sigma^c)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy)^{\frac{1}{2}}$$

as norm on X_0. It is immediately apparent that X_0 is a Hilbert space with inner product

$$\langle u, v \rangle_{X_0} = \iint_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Sigma^c \times \Sigma^c)} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy.$$

For fractional Sobolev spaces, there is the following embedding result; see [23] for the details.

Lemma 2.2. *(1) If Σ has a Lipschitz boundary, then there holds the compact embedding $X_0 \to L^p(\mathbb{R}^N)$ for $p \in [1, 2_s^*)$.*

(2) There holds the continuous embedding $X_0 \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$.

Let

$$Y_0 = H_0^1(\Sigma) \bigcap X_0$$

endowed with the norm

$$\|u\|_{Y_0}^2 = \|u\|_{H_0^1(\Sigma)}^2 + \|u\|_{X_0}^2.$$

Let X be a Banach space. In the case $1 \leqslant p < +\infty$,

$$L^p(0, +\infty; X) = \left\{ u : \int_0^{+\infty} \|u\|_X^p dt < +\infty \right\}$$

equipped with the following norm:

$$\|u\|_{L^p(0,+\infty;X)} = \left(\int_0^{+\infty} \|u\|_X^p dt \right)^{\frac{1}{p}}.$$

In the case $p = +\infty$,

$$L^\infty(0, +\infty; X) = \left\{u : \sup_{t\in[0,+\infty)} \|u\|_X < +\infty\right\}$$

equipped with the norm:

$$\|u\|_{L^\infty(0,+\infty;X)} = \sup_{t\in[0,+\infty)} \|u\|_X.$$

$L^p(0, +\infty; X)$ is a Banach space for $1 \leqslant p \leqslant +\infty$

Refer to [24] for results on this kind of spaces. In the following, X is usually taken as $L^2(\Sigma)$ or $L^p(\Sigma)$ or Y_0.

3. Introduction of potential wells

For problem (1.1), define

$$E(t) = \frac{1}{2}\|u_t\|_{L^2(\Sigma)} + \frac{1}{2}\|u\|^2_{Y_0} - \frac{1}{p}\|u\|^p_{L^p(\Sigma)},$$

$$I(u) = \frac{1}{2}\|u\|^2_{Y_0} - \frac{1}{p}\|u\|^p_{L^p(\Sigma)}$$

and

$$W = \{u \in Y_0 : \langle I'(u), u\rangle > 0, I(u) < h\} \bigcup \{0\},$$

where

$$h = \inf_{u\in Y_0, u\neq 0} \sup_{\vartheta\geqslant 0} I(\vartheta u).$$

It is easy to calculate that the Frechét derivative of $I(u)$ is

$$\langle I'(u), \phi\rangle = \int_\Sigma \nabla u \nabla \phi dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy - \int_\Sigma |u|^{p-2} u\phi dx$$

for $\phi \in Y_0$ and especially

$$\langle I'(u), u\rangle = \|u\|^2_{Y_0} - \|u\|^p_{L^p(\Sigma)}.$$

It is also easy to obtain that with respect to ϑ, $I(\vartheta u)$ takes its maximum at $\vartheta^* = \left(\frac{\|u\|^2_{Y_0}}{\|u\|^p_{L^p(\Sigma)}}\right)^{\frac{1}{p-2}}$. Normalize u such that $\vartheta^* = 1$, i.e., $\|u\|^2_{Y_0} = \|u\|^p_{L^p(\Sigma)}$, then there holds

$$h = \inf I(u)$$

subject to $0 \neq u \in Y_0$, $\langle I'(u), u\rangle = 0$.

Proposition 3.1. *There exists a nontrivial solution to the following problem:*

$$\begin{cases} -\Delta u + (-\Delta)^s u = |u|^{p-2} u, & in\ \Sigma, \\ u = 0, & in\ \mathbb{R}^N \backslash \Sigma. \end{cases} \tag{3.1}$$

Proof. First, it is immediately apparent that I is the energy functional for problem (3.1), and $I \in C^1(Y_0, \mathbb{R})$, and a critical point of I is a weak solution of problem (3.1). Next, we apply the Mountain Pass Theorem to prove the proposition in two steps.

Step 1. The functional I satisfies the (PS) condition, i.e., if $\|u_n\|_{Y_0} \leqslant C$ for some constant C and $I'(u) \to 0$, then $\{u_n\}$ admits a convergent subsequence.

By the (PS) condition, we have

$$\begin{aligned} c\left(1 + \|u_n\|_{Y_0}\right) &\geqslant I(u_n) - \left\langle I'(u_n), \frac{1}{p}u_n \right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_{Y_0}^2, \end{aligned}$$

from which we conclude that $\{u_n\}$ is bounded in Y_0 as $p > 2$. As Y_0 is reflexive, $\{u_n\}$ admits a subsequence, which is still denoted by $\{u_n\}$, weakly convergent to $u \in Y_0$. By Lemmas 2.1 and 2.2, we have

$$u_n \to u \text{ in } L^p(\Sigma).$$

In view of

$$\begin{aligned} &\langle I'(u_n) - I'(u), u_n - u\rangle \\ &= \int_\Sigma |\nabla u_n - \nabla u|^2 \, dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|}{|x-y|^{n+2s}} dxdy \\ &\quad - \int_\Sigma \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\, dx, \end{aligned}$$

we have

$$\begin{aligned} &\int_\Sigma |\nabla u_n - \nabla u|^2 \, dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|}{|x-y|^{n+2s}} dxdy \\ &= \langle I'(u_n) - I'(u), u_n - u\rangle + \int_\Sigma \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\, dx. \end{aligned}$$

By (PS) condition and $u_n \to u$ weakly in Y_0, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \text{ as } n \to +\infty.$$

As $u_n \to u$ in $L^p(\Sigma)$, by Hölder inequality we have

$$\int_\Sigma \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\, dx \to 0 \text{ as } n \to +\infty.$$

Now we conclude that

$$\int_\Sigma |\nabla u_n - \nabla u|^2 \, dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|}{|x-y|^{n+2s}} dxdy \to 0$$

as $n \to +\infty$, which implies that $u_n \to u$ in Y_0.

Step 2. We verify that the Mountain Pass geometry structure is satisfied.

By Lemmas 2.1 and 2.2 once more, we know

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\Sigma}|u|^2dx + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy - \frac{1}{p}\int_{\Sigma}|u(x)|^p dx\\ &\geqslant \frac{1}{2}\left(\|u\|_{Y_0}^2 - C\|u\|_{Y_0}^p\right)\\ &> 0\\ &= I(0)\end{aligned}$$

if $\|u\|_{Y_0} = \left(\frac{1}{2C}\right)^{\frac{1}{p-2}}$.

Take $x_0 \in \Sigma$ such that $B_{2R}(x_0) \subset \Sigma$. Take $\phi \in C_0^\infty(B_{2R}(x_0))$ with $0 \leqslant \phi(x) \leqslant 1$, $|\nabla\phi(x)| \leqslant \frac{2}{R}$ and $\phi(x) \equiv 1$ in $B_R(x_0)$. Then

$$\begin{aligned}I(\kappa\phi) =&\frac{1}{2}\kappa^2\left(\int_{B_{2R}(x_0)}|\nabla\phi(x)|^2dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|}{|x-y|^{n+2s}}dxdy\right)\\ &- \frac{\kappa^p}{p}\int_{B_{2R}(x_0)}|\phi(x)|^p dx\\ \leqslant&\frac{1}{2}\kappa^2\left(\int_{B_{2R}(x_0)}|\nabla\phi(x)|^2dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|}{|x-y|^{n+2s}}dxdy\right)\\ &- \frac{\kappa^p}{p}\,\text{meas}\, B_R(x_0)\\ <&0,\end{aligned}$$

if κ is sufficiently large. Therefore, we can get some $\kappa > 0$ such that $I(\kappa\phi) < 0$ and $\|\kappa\phi\|_{Y_0} > \left(\frac{1}{2C}\right)^{\frac{1}{p-2}}$.

At last, by the Mountain Pass Theorem we conclude that there exists a nontrivial solution to problem (3.1). Now the proof is finished.

For all $\mu > 0$, $v = \mu^{\frac{1}{2-p}}u$ satisfies

$$\begin{cases}-\Delta v + (-\Delta)^s v = \mu|v|^{p-2}v, & \text{in } \Sigma,\\ v = 0, & \text{in } \mathbb{R}^N\backslash\Sigma.\end{cases}$$

We define

$$S_p = \inf_{0\neq v\in Y_0}\frac{\|v\|_{Y_0}^2}{\|v\|_{L^p(\Sigma)}^2}.$$

For the variational problem above, the Euler equation is

$$-\Delta u + (-\Delta)^s u = \mu|u|^{p-2}u,$$

where μ is a Lagrange Multiplier. Thus, we can have

$$S_p = \frac{\|u\|_{Y_0}^2}{\|u\|_{L^p(\Sigma)}^2}.$$

On the other hand,

$$\|u\|_{Y_0}^2 = \|u\|_{L^p(\Sigma)}^p$$

and further

$$h = \frac{p-2}{2p}\|u\|^p_{L^p(\Sigma)}.$$

It is immediately apparent that

$$S_p = \|u\|^{p-2}_{L^p(\Sigma)} = \left(\frac{2p}{p-2}h\right)^{\frac{p-2}{p}}$$

and

$$h = \frac{p-2}{2p}S_p^{\frac{2p}{p-2}}. \tag{3.2}$$

Next, for problem (1.1) and $\lambda \in (0, 1)$, define

$$I_\lambda(u) = \frac{\lambda}{2}\|u\|^2_{Y_0} - \frac{1}{p}\|u\|^p_{L^p(\Sigma)}$$

and

$$h(\lambda) = (1-\lambda)(p\lambda)^{\frac{2}{p-2}}\left(\frac{S_p}{2}\right)^{\frac{p}{p-2}}. \tag{3.3}$$

Proposition 3.2. *Provided that $I(u) \leqslant h(\lambda)$, there holds the following results:*

(i) $I_\lambda(u) > 0$ implies

$$0 < \|u\|^2_{Y_0} < \left(\frac{p\lambda}{2}\right)^{\frac{2}{p-2}} S_p^{\frac{p}{p-2}} \tag{3.4}$$

and vice versa. Here, S_p is the optimal embedding constant from Y_0 to $L^p(\Sigma)$.

(ii) $I_\lambda(u) < 0$ implies

$$\|u\|^2_{Y_0} > \left(\frac{p\lambda}{2}\right)^{\frac{2}{p-2}} S_p^{\frac{p}{p-2}} \tag{3.5}$$

and vice versa.

Proof. (i) If (3.4) holds, there holds

$$\begin{aligned}\|u\|^{2p}_{L^p(\Sigma)} &\leqslant \left(\frac{1}{S_p^p}\right)\|u\|^{2p}_{Y_0}\\ &= \left(\frac{1}{S_p^p}\right)\|u\|^{2p-4}_{Y_0}\|u\|^4_{Y_0}\\ &< \left(\frac{p\lambda}{2}\right)^2\|u\|^4_{Y_0},\end{aligned}$$

furthermore, $I_\lambda(u) > 0$. In another respect, if $I_\lambda(u) > 0$, then $\|u\|_{Y_0} > 0$ and

$$I(u) = \frac{1-\lambda}{2}\|u\|^2_{Y_0} + I_\lambda(u) \leqslant h(\lambda),$$

from which we get

$$\begin{aligned}\|u\|^2_{Y_0} &< \frac{2}{1-\lambda}h(\lambda)\\ &= \left(\frac{p\lambda}{2}\right)^{\frac{2}{p-2}} S_p^{\frac{p}{p-2}}.\end{aligned}$$

(ii) If $I_\lambda(u) < 0$, then

$$\frac{p\lambda}{2}\|u\|_{Y_0}^2 < \|u\|_{L^p(\Sigma)}^p \leqslant \left(\frac{1}{S_p}\right)^{\frac{p}{2}} \|u\|_{Y_0}^p,$$

from which we get (3.5). On the other hand, from (3.4) and (3.5), it is easy to obtain $I_\lambda(u) < 0$.

From Proposition 3.2, it is immediate to conclude that under the condition $I(u) = h(\lambda)$, $I_\lambda(u) = 0$ implies

$$\|u\|_{Y_0}^2 = \left(\frac{p\lambda}{2}\right)^{-\frac{2}{p-2}} S_p^{\frac{p}{p-2}}$$

and vice versa.

It is immediate to obtain the following lemma:

Proposition 3.3. *For the function $h(\lambda)$ on the interval $[0,1]$, there holds the following results:*

(i) $h(0) = h(1) = 0$;

(ii) $h(\lambda)$ takes the maximum $h(\lambda_0) = d$ at $\lambda_0 = \frac{2}{p}$;

(iii) $h(\lambda)$ is increasing on $[0,\lambda_0]$ and decreasing on $[\lambda_0, 1]$;

(iv) Let $e \in (0,h)$ be given. There exist exactly two solutions, $\lambda_1 \in (0,\lambda_0)$ and $\lambda_2 \in (\lambda_0, 1)$, to the equation $h(\lambda) = e$.

Theorem 3.1. *$h(\lambda) = \inf I(u)$ subject to $0 \neq u \in Y_0$, $I_\lambda(u) = 0$.*

Proof. Suppose that $I_\lambda(u) = 0$ with $u \neq 0$; thus,

$$I(u) = \frac{1-\lambda}{2}\|u\|_{Y_0}^2 + I_\lambda(u) = \frac{1-\lambda}{2}\|u\|_{Y_0}^2$$

and

$$\frac{p}{2}\lambda\|u\|_{Y_0}^2 = \|u\|_{L^p(\Sigma)}^p \leqslant \left(\frac{1}{S_p}\|u\|_{Y_0}^2\right)^{\frac{p}{2}},$$

which implies

$$I(u) \geqslant h(\lambda),$$

and furthermore, we finish the proof according to the definition of S_p.

Corollary 3.1. *$h = h(\lambda_0) = \inf I(u)$ subject to $0 \neq u \in Y_0$, $\langle I'(u), u\rangle = 0$.*

As the necessary and sufficient condition for $I_{\lambda_0}(u) = 0$ is $\langle I'(u), u\rangle = 0$, it is immediate to conclude the corollary by Theorem 3.1.

Next define:

$$W_\lambda = \{u \in Y_0 : I_\lambda(u) > 0, I(u) < h(\lambda)\} \bigcup \{0\},\ \lambda \in (0,1).$$

Certainly, W_λ, $\lambda \in (0,1)$, is a family of potential wells.

In addition, define

$$V_\lambda = \{u \in Y_0 : I_\lambda(u) < 0, I(u) < h(\lambda)\}, 0 < \lambda < 1,$$

$$V = \{u \in Y_0 : \langle I'(u), u\rangle < 0, I(u) < d\},$$

$$B_\lambda = \left\{ u \in Y_0 : \|u\|_{Y_0}^2 < \left(\frac{p\lambda}{2}\right)^{\frac{2}{p-2}} S_p^{\frac{p}{p-2}} \right\}.$$

It is immediate that $W_{\lambda_0} = W$ and $V_{\lambda_0} = V$.

As $I(u) \leqslant \frac{1}{2}\|u\|_{Y_0}^2$, for any given $\lambda \in (0,1)$

$$0 < \|u\|_{Y_0}^2 < (1-\lambda)\left(\frac{p\lambda}{2}\right)^{\frac{2}{p-2}} S_p^{\frac{p}{p-2}}$$

implies $I(u) < h(\lambda)$ and $I_\lambda(u) > 0$, which further implies

$$B_{\overline{\lambda}} \subset W_\lambda$$

provided $\overline{\lambda}$ satisfies

$$\overline{\lambda} = (1-\lambda)^{\frac{p-2}{2}} \lambda.$$

By this fact and Lemma 3.2, we can easily get:

Theorem 3.2. *$B_{\overline{\lambda}} \subset W_\lambda \subset B_\lambda$, $V_\lambda \subset (\overline{B_\lambda})^c$.*

Corollary 3.2. *$B_{\overline{\lambda_0}} \subset W \subset B_{\lambda_0}$, $V \subset (\overline{B_{\lambda_0}})^c$, where*

$$B_{\lambda_0} = \{u \in Y_0 : \|u\|_{Y_0}^2 < S_p^{\frac{p}{p-2}}\},$$

and

$$\overline{\lambda} = \frac{2}{p}\left(\frac{p-2}{p}\right)^{\frac{p-2}{2}}.$$

From Proposition 3.3, we can have

Proposition 3.4. *(i) If $0 < \lambda' < \lambda'' \leqslant \lambda_0$, then $W_{\lambda'} \subset W_{\lambda''}$.*

(ii) If $\lambda_0 \leqslant \lambda' < \lambda'' < 1$, then $V_{\lambda''} \subset V_{\lambda'}$.

Proposition 3.5. *For some $u \in Y_0$, suppose that $0 < I(u) < h$ and $\lambda_1 < \lambda_2$ satisfy $h(\lambda) = I(u)$. Then $I_\lambda(u)$ keeps the sign for $\lambda \in (\lambda_1, \lambda_2)$.*

Proof. First, $I(u) > 0$ implies $u \neq 0$. As $\lambda_1 < \lambda_2$ are the two solutions of the equation $h(\lambda) = I(u)$, provided $I_\lambda(u)$ changes sign, then there exists some $\lambda^* \in (\lambda_1, \lambda_2)$ satisfying $I_{\lambda^*}(u) = 0$. Now applying Proposition 3.3 and Theorem 3.1, it is easy to get

$$I(u) \geqslant h(\lambda^*) > h(\lambda_1) = h(\lambda_2),$$

which is a contradiction because $I(u) = h(\lambda_1) = h(\lambda_2)$.

4. Global existence and blow-up of weak solutions

In this section we study the global existence and blow-up of weak solutions for the problem (1.1).

Theorem 4.1. *Suppose that* $u_0 \in Y_0$, $u_1 \in L^2(\Sigma)$. *If* $0 < E(0) < h$, $\lambda_1 < \lambda_2$ *satisfy* $h(\lambda) = E(0)$, *and* $I_{\lambda_2}(u_0) > 0$ *or* $u_0 = 0$, *then there exists a global weak solution* u *for the problem* (1.1). *Moreover,* $u \in W_\lambda$ *for* $\lambda \in (\lambda_1, \lambda_2)$ *and* $t \in [0, +\infty)$.

Proof. First, we manage to get approximate solutions by the Galerkin method. Choose a sequence of functions $\{e_j\} \subset C_0^\infty(\Sigma)$ that forms an orthonormal basis with respect to the Hilbert space of $L^2(\Sigma)$. Based on the theory of ordinary differential equations (see, for example, [25]), we can construct approximate solutions of the problem (1.1).

$$u_m(x,t) = \sum_{j=1}^{m} h_{jm}(t)e_j(x), m = 1, 2, \cdots,$$

satisfying

$$\begin{aligned}(u_{mtt}, e_j)_{L^2(\Sigma)} + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_m(x) - u_m(y))(e_j(x) - e_j(y))}{|x-y|^{N+2s}} dxdy \\ = \int_\Sigma |u_m|^{p-2}u_m e_j dx,\end{aligned} \tag{4.1}$$

$$u_m(0) = \sum_{j=1}^{m} a_{jm}e_j(x) \to u_0(x) \text{ in } Y_0, \tag{4.2}$$

$$u_{mt}(0) = \sum_{j=1}^{m} b_{jm}e_j(x) \to u_1(x) \text{ in } L^2(\Sigma). \tag{4.3}$$

Next, multiply both sides of (4.1) by $h'_{jm}(t)$, take the sum with respect to j from 1 to m, and integrate from 0 to t, we obtain

$$\begin{aligned} E_m(t) &= \tfrac{1}{2}\|u_{mt}\|^2_{L^2(\Sigma)} + \tfrac{1}{2}\|u_m\|^2_{Y_0} - \tfrac{1}{p}\|u_m\|^p_{L^p(\Sigma)} \\ &= \tfrac{1}{2}\|u_{mt}(0)\|^2_{L^2(\Sigma)} + \tfrac{1}{2}\|u_m(0)\|^2_{Y_0} - \tfrac{1}{p}\|u_m(0)\|^p_{L^p(\Sigma)} \\ &= E_m(0). \end{aligned} \tag{4.4}$$

By $I_{\lambda_2}(u_0) > 0$, we have $\|u_0\|_{Y_0} \neq 0$, and further, similar to the argument of Proposition 3.5, we get $I_\lambda(u_0) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$. Because $I(u_0) \leqslant E(0) = h(\lambda_1)$, there holds $u_0 \in W_\lambda$ for $\lambda \in (\lambda_1, \lambda_2)$.

If $u_0 = 0$, then $u_0 \in W_\lambda$ for $\lambda \in (0, 1)$.

For any fixed $\lambda \in (\lambda_1, \lambda_2)$ and for m large enough, $I_{\lambda_2}(u_0) > 0$ implies $I_\lambda(u_m(0)) > 0$ and $E_m(0) < h(\lambda)$, while $u_0 = 0$ implies $u_m(0) \in B_{\overline{\lambda}}$ where $\overline{\lambda}$ comes from Theorem 3.2; all in all, we obtain $u_m(0) \in W_\lambda$ for all cases.

In the following, we prove by contradiction that for $t \in (0, +\infty)$ and m large enough, $u_m(t) \in W_\lambda$. If not, there exists $t_0 \in (0, +\infty)$ such that $u_m(t_0) \in \partial W_\lambda$, i.e., there holds either $I_\lambda(u_m(t_0)) = 0$ and $\|u_m(t_0)\|_{Y_0} \neq 0$ or $I(u_m(t_0)) = h(\lambda)$. By (4.4), for $t \in (0, +\infty)$ we have

$$I(u_m(t)) \leqslant E_m(0) < h(\lambda)$$

and further $I(u_m(t_0)) = h(\lambda)$ is false. In the case that $I_\lambda(u_m(t_0)) = 0$ and $\|u_m(t_0)\|_{Y_0} \neq 0$, in view of Theorem 3.1 there holds $I(u_m(t_0)) \geqslant h(\lambda)$, which is false as well.

Now by Proposition 3.2, for $t \in (0, +\infty)$ and m large enough, we have

$$\|u_m(t)\|_{Y_0}^2 < \left(\frac{p\lambda}{2}\right)^{\frac{2}{p-2}} S_p^{\frac{p}{p-2}},$$

$$\|u_m(t)\|_{L^p(\Sigma)}^2 \leqslant \frac{1}{S_p}\|u_m(t)\|_{Y_0}^2 \leqslant \left(\frac{p\lambda S_p}{2}\right)^{\frac{2}{p-2}}$$

and

$$\|u_{mt}(t)\|_{L^2(\Sigma)}^2 < 2h(\lambda).$$

As $L^\infty(0, +\infty; L^p(\Sigma))$ and $L^\infty(0, +\infty; Y_0)$ are dual of separable Banach spaces, bounded sets in the two spaces are weak$*$-compactness. Denote the limit of $\{u_m\}$ by u. Then we arrive at the conclusion that the problem (1.1) admits a global weak solution u with $u(t) \in W_\lambda$ for $t \in [0, +\infty)$. According to the arbitrariness of λ, for $\lambda \in (\lambda_1, \lambda_2)$ and $t \in [0, +\infty)$, there holds $u(t) \in W_\lambda$. Now the proof is finished.

Corollary 4.1. *If we replace the assumption $I_{\lambda_2}(u_0) > 0$ or $u_0 = 0$ in Theorem 4.1 by $\langle I'(u_0), u_0\rangle > 0$ or $u_0 = 0$, i.e., $u_0(x) \in W$, then the result of Theorem 4.1 still holds.*

Because we can infer $I_\lambda(u_0) > 0$ and $I_{\lambda_2}(u_0) \geqslant I_{\lambda_0}(u_0)$ from $\langle I'(u_0), u_0\rangle > 0$, it is immediate to obtain this corollary.

Next study problem (1.1) in the critical case: $\langle I'(u_0), u_0\rangle \geqslant 0$ and $E(0) = h$.

Theorem 4.2. *Suppose that $u_0 \in Y_0$, $u_1 \in L^2(\Sigma)$. If $E(0) = h$ and $\langle I'(u_0), u_0\rangle \geqslant 0$, then for problem* (1.1) *there exists a global weak solution u with $u(t) \in \overline{W}$ for $t \in [0, +\infty)$.*

Proof. First, consider the following two cases:

(i) $\|u(0)\|_{Y_0} \neq 0$.

Let $\theta_m = 1 - \frac{1}{m}$ and $u_{0m}(x) = \theta_m u_0(x)$, $m = 1, 2, \cdots$. Consider problem (1.1) under the following initial condition:

$$u(x, 0) = u_{0m}(x), \quad u_t(x, 0) = u_1(x). \tag{4.5}$$

As $\langle I'(u_0), u_0\rangle \geqslant 0$, we have

$$\begin{aligned}\langle I'(u_{0m}), u_{0m}\rangle &= \theta_m^2\|u_0\|_{Y_0}^2 - \theta_m^p\|u_0\|_{L^p(\Sigma)}^p \\ &= \theta_m^2\langle I'(u_0), u_0\rangle > 0 + (\theta_m^2 - \theta_m^p)\|u_0\|_{L^p(\Sigma)}^p \\ &> 0,\end{aligned}$$

and

$$\begin{aligned}I(u_{0m}) &= \frac{1}{2}\|u_{0m}\|_{Y_0}^2 - \frac{1}{p}\|u_{0m}\|_{L^p(\Sigma)}^p \\ &= (\frac{1}{2} - \frac{1}{p})\|u_{0m}\|_{Y_0}^2 + \frac{1}{p}\langle I'(u_{0m}), u_{0m}\rangle \\ &> 0,\end{aligned}$$

and

$$I(u_{0m}) = I(\theta_m u_0) < I(u_0),$$

and

$$\begin{aligned} 0 < E_m(0) &= \tfrac{1}{2}\|u_1\|^2_{L^2(\Sigma)} + I(u_{0m}) \\ &< \tfrac{1}{2}\|u_1\|^2_{L^2(\Sigma)} + I(u_0) \\ &= E(0) < h. \end{aligned}$$

(ii) $u(0) = 0$.

In this case $I(u_0) = 0$ and $E(0) = \frac{1}{2}\|u_1\|^2_{L^2(\Sigma)} = h$. Let $\theta_m = 1 - \frac{1}{m}$ and $u_{1m}(x) = \theta_m u_1(x)$, $m = 1, 2, \cdots$. Study problem (1.1) under the following initial condition:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_{1m}(x). \tag{4.6}$$

As $u(0) = 0$, we have

$$\begin{aligned} 0 < E_m(0) &= \tfrac{1}{2}\|u_{1m}\|^2_{L^2(\Sigma)} + I(u_0) \\ &= \tfrac{\theta_m^2}{2}\|u_1\|^2_{L^2(\Sigma)} \\ &< E(0) = h. \end{aligned}$$

In both case (i) and case (ii), for $m = 1, 2, \cdots$, by Theorem 4.1, there exists a global weak solution u_m with $u_m(t) \in \overline{W}$ for $t \in [0, +\infty)$ to problem (1.1) under initial condition (4.5) or (4.6).

For $t \in [0, +\infty)$, $u_m(t) \in \overline{W}$ implies

$$\begin{aligned} I(u_m(t)) &\geqslant (\tfrac{1}{2} - \tfrac{1}{p})\|u_m\|^2_{Y_0} + \tfrac{1}{p}\langle I'(u_m), u_m\rangle \\ &\geqslant \tfrac{p-1}{2(p+1)} C(N, s)\|u_m\|^2_{Y_0} \end{aligned}$$

and further

$$\|u_m(t)\|^2_{Y_0} \leq \frac{2p}{p-2} h,$$

$$\|u_m(t)\|^2_{L^p(\Sigma)} \leqslant \frac{1}{S_p}\|u_m(t)\|^2_{Y_0} \leqslant \frac{1}{S_p}\left(\frac{2p}{p-1} h\right)$$

and

$$\|u_{mt}(t)\|^2_{L^2(\Sigma)} < 2h.$$

Similar to Theorem 4.1, $\{u_m\} \subset L^\infty(0, +\infty; Y_0)$ and $\{u_{mt}\} \subset L^\infty(0, +\infty; L^2(\Sigma))$ are weak$*$-compactness, and we get the conclusion that for problem (1.1) there exists a global weak solution u, which is the limit of u_m, with $u(t) \in \overline{W}$ for $t \in [0, +\infty)$. Now we finish the proof.

At the end of this paper, consider the blow-up of weak solutions for problem (1.1).

Theorem 4.3. *Suppose that $u_0 \in Y_0$, $u_1 \in L^2(\Sigma)$. If $(u_0, u_1)_{L^2(\Sigma)} > 0$ and $E(0) \leqslant 0$ or $0 < E(0) < h$ with $\langle I'(u_0), u_0\rangle < 0$, then for problem* (1.1)*, any weak solution u must blow up at the existence time T, i.e.,*

$$\lim_{t \to T} \|u\|_{L^2(\Sigma)} = +\infty.$$

Proof. Denote

$$\mathcal{E}(t) = \|u(t))\|^2_{L^2(\Sigma)}.$$

In view of Theorem 2.1 in [23], we conclude that $u \in C\left(0, +\infty; L^2(\Sigma)\right)$ and

$$2\int_0^t (u(\omega), u_\omega(\omega))_{L^2(\Sigma)}\, d\omega = \|u(t)\|^2_{L^2(\Sigma)} - \|u_0\|^2_{L^2(\Sigma)},$$

from which, we have

$$\dot{\mathcal{E}} = 2\,(u, u_t)_{L^2(\Sigma)}$$

and further by [20], $\dot{\mathcal{E}}$ is Lipschitz continuous and $\ddot{\mathcal{E}}$ exists a.e., so we can get

$$\ddot{\mathcal{E}} = 2\left(\|u_t\|^2_{L^2(\Sigma)} - \|u\|^2_{Y_0(\Sigma)} + \|u\|^p_{L^p(\Sigma)}\right).$$

Taking $\varphi = u_t\phi$ where ϕ can be any function belonging to $C_0^\infty([0, +\infty))$, for problem (1.1) we obtain

$$\frac{d}{dt}E(t) = 0, \text{ i.e. } E(t) = E(0). \tag{4.7}$$

In the case that $E(0) < 0$ and $(u_0, u_1)_{L^2(\Sigma)} > 0$, by

$$\frac{1}{2}\|u_1\|^2_{L^2(\Sigma)} + \frac{1-\lambda}{2}\|u_0\|^2_{Y_0} + I_{\lambda_0}(u_0) = E(0),$$

we have $I_{\lambda_0}(u_0) < 0$ and $\langle I'(u_0), u_0\rangle < 0$.

In the case that $E(0) = 0$ and $(u_0, u_1)_{L^2(\Sigma)} > 0$, from

$$0 < (u_0, u_1)_{L^2(\Sigma)} \leqslant \|u_0\|_{L^2(\Sigma)}\|u_1\|_{L^2(\Sigma)} \leqslant C\|u_0\|_{Y_0}\|u_1\|_{L^2(\Sigma)},$$

we have $\|u_0\|_{Y_0} > 0$, $\|u_1\|_{L^2(\Sigma)} > 0$. Furthermore by (4.7), we get $I_\lambda(u_0) < 0$ and $\langle I'(u_0), u_0\rangle < 0$.

Next, we prove by contradiction that for $t \in (0, +\infty)$, $\langle I'(u), u\rangle < 0$. If not, there exists $t_0 > 0$ so that $\langle I'(u(t)), u(t)\rangle < 0$ for $t < t_0$ and $\langle I'(u(t_0), u(t_0)\rangle = 0$. Furthermore, $I(u(t_0)) \geqslant h$, which contradicts the fact that $\langle I'(u), u\rangle \leqslant E(t) = E(0) < h$.

In any case, we conclude that for $t \in (0, +\infty)$, $\ddot{\mathcal{E}}(t) > 0$ while $\dot{\mathcal{E}}(0) = 2(u_0, u_1)_{L^2(\Sigma)} > 0$; therefore, for $t \in (0, +\infty)$, $\dot{\mathcal{E}}(t) > 0$, and moreover $\mathcal{E}(t)$ is increasing on $[0, +\infty)$.

(4.7) implies that

$$\|u\|^p_{L^p(\Sigma)} \geqslant \frac{p}{2}\left(\|u\|^2_{Y_0} + \|u_t\|^2_{L^2(\Sigma)}\right) - pE(0).$$

Then

$$\begin{aligned}\ddot{\mathcal{E}} &\geqslant (p+2)\|u_t\|^2_{L^2(\Sigma)} + (p-2)\|u\|^2_{Y_0} - 2pE(0)\\ &\geqslant (p+2)\|u_t\|^2_{L^2(\Sigma)} + \frac{p-2}{S_2}\mathcal{E} - 2(pE(0),\end{aligned}$$

where S_2 is the optimal embedding constant from Y_0 to $L^2(\Sigma)$. As $\mathcal{E}$ is convex, at fixed $t_0 \in (0, +\infty)$ there holds

$$\mathcal{E}(t) \geqslant \mathcal{E}(t_0) + \dot{\mathcal{E}}(t_0)(t - t_0),$$

thus there exists t_1 large enough such that

$$\frac{p-2}{S_2}\mathcal{E} - 2pE(0) > 0 \text{ if } t > t_1.$$

Then for $t > t_1$, there holds that

$$\ddot{F} \geqslant p\|u_t\|^2_{L^2(\Sigma)}$$

and then

$$\begin{aligned}\ddot{F} &= -\frac{p-2}{4\mathcal{E}^{\frac{p+6}{4}}}(\mathcal{E}\ddot{\mathcal{E}} - \frac{p+2}{4}\dot{\mathcal{E}}^2) \\ &\leqslant -\frac{(p-2)(p+2)}{4\mathcal{E}^{\frac{p+6}{4}}}(\mathcal{E}\|u_t\|^2_{L^2(\Sigma)} - (\int_\Sigma uu_t)^2) \\ &\leqslant 0\end{aligned}$$

where $F = \mathcal{E}^{\frac{2-p}{4}}$. Therefore, for t large enough, $\mathcal{E}^{\frac{2-p}{4}}$ is concave. Furthermore, we conclude that the existence time T is finite because $\lim_{t\to T}\mathcal{E}^{\frac{1-p}{4}} = 0$, i.e., $\lim_{t\to T}\|u\|_{L^2(\Sigma)} = \infty$. Now we finish the proof.

5. Conclusions

In this paper, we study the initial boundary value problem of a wave equation with mixed local and nonlocal propagation. First, we introduce the space Y_0 as the spatial framework, which is the intersection of a classical Sobolev space and a fractional Sobolev space. By means of critical point theory, we show the attainability of the optimal embedding constant from Y_0 to $L^p(\Sigma)$. Second, by introducing a family of potential wells in Y_0, we get the existence of a global weak solution through the utilization of potential well theory. At last, by analyzing the properties of the energy functional in $L^2(\Sigma)$, we conclude that any weak solution must blow up at the existence time.

We can also study the following equation:

$$u_{tt} - \Delta u + (-\Delta)^s u = f(u),$$

where $f(u)$ satisfies the following conditions:

(f1) $f \in C^1(-\infty, +\infty)$ and $f(0) = f'(0) = 0$.

(f2) On $(-\infty, +\infty)$, $f(u)$ is monotone.

(f3) On $(-\infty, 0)$, $f(u)$ is concave, while on $(0, +\infty)$, $f(u)$ is convex.

(f4) $pF(u) \leq uf(u)$ and $|uf(u)| \leq \gamma|F(u)|$, where $F(u) = \int_0^u f(s)ds$ and $2 < p \leq \gamma < \infty$ in the case $N = 1, 2$, and $2 < p \leq \gamma \leq \frac{2N}{N-2s}$ in the case $N \geq 3$.

Conditions (f1)–(f4) were first introduced in [20]. All the results obtained in this paper for problem (1.1) are still valid for the problem of the equation above, equipped with the initial condition in problem (1.1).

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is funded by the Fundamental Research Funds in Heilongjiang Provincial Universities under Grant 145409327.

Conflict of interest

The author declares no conflicts of interest.

References

1. R. Herrmann, *Fractional calculus: An introduction for physicists*, Singapore: World Scientific, 2011. https://doi.org/10.1142/8934
2. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779
3. Y. Luchko, A. Punzi, Modeling anomalous heat transport in geothermal reservoirs via fractional diffusion equations, *Int. J. Geomath.,* **1** (2011), 257–276. http://dx.doi.org/10.1007/s13137-010-0012-8
4. Y. Luchko, Fractional wave equation and damped waves, *J. Math. Phys.,* **54** (2013), 031505. http://dx.doi.org/10.1063/1.4794076
5. F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos Soliton. Fract.,* **7** (1996), 1461–1477. https://doi.org/10.1016/0960-0779(95)00125-5
6. R. Metzler, J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A,* **37** (2004), 161–208. http://dx.doi.org/10.1088/0305-4470/37/31/R01
7. Y. Q. Fu, On potential wells and vacuum isolating of solutions for space-fractional wave equations, *Adv. Differ. Equ. Contr.,* **18** (2017), 149–176. http://dx.doi.org/10.17654/DE018030149
8. S. Dipierro, E. Valdinoci, Description of an ecological niche for a mixed local/nonlocal dispersal: An evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy process, *Physica A,* **575** (2021), 20. https://doi.org/10.1016/j.physa.2021.126052
9. S. Dipierro, L. Proietti, E. Valdinoci, (Non)local logistic equations with Neumann conditions, *Ann. I. H. Poincaré C.,* **40** (2023), 1093–1166. http://dx.doi.org/10.4171/AIHPC/57
10. L. Caffarelli, E. Valdinoci, *A priori bounds for solutions of a nonlocal evolution PDE*, In: Analysis and Numerics of Partial Differential Equations, Heidelberg: Springer, 2013, 141–163. https://doi.org/10.1007/978-88-470-2592-9
11. F. Dell'Oro, V. Pata, Second order linear evolution equations with general dissipation, *Appl. Math. Opt.,* **83** (2021), 1877–1917. http://dx.doi.org/10.1007/s00245-019-09613-x
12. I. H. Biswas, E. R. Jakobsen, K. H. Karlsen, Difference-quadrature schemes for nonlinear degenerate parabolic integro-PDE, *SIAM J. Numer. Anal.,* **48** (2010), 1110–1135. http://dx.doi.org/10.1137/090761501

13. M. Barlow, R. Bass, Z. Q. Chen, M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, *T. Am. Math. Soc.,* **361** (2009), 1963–1999. http://dx.doi.org/10.1090/s0002-9947-08-04544-3

14. Z. Chen, T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps, *Rev. Mat. Iberoam.,* **26** (2010), 551–589. http://dx.doi.org/10.4171/RMI/609

15. P. Garain, J. Kinnunen, Weak Harnack inequality for a mixed local and nonlocal parabolic equation, *J. Differ. Equ.,* **360** (2023), 373–406. http://dx.doi.org/10.1016/j.jde.2023.02.049

16. S. Das, Gradient Hölder regularity in mixed local and nonlocal linear parabolic problem, *J. Math. Anal. Appl.,* **535** (2024), 39. http://dx.doi.org/ 10.1016/j.jmaa.2024.128140

17. H. A. Levine, R. A. Smith, A potential well theory for the wave equation with a nonlinear boundary condition, *J. Reine Angew. Math.,* **374** (1987), 1–23. http://dx.doi.org/10.1515/crll.1987.374.1

18. Y. C. Liu, On potential wells and vacuum isolating of solutions for semilinear wave equations, *J. Differ. Equ.,* **192** (2003), 155–169. http://dx.doi.org/ 10.1016/S0022-0396(02)00020-7

19. T. Matsuyama, R. Ikehata, On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms, *J. Math. Anal. Appl.,* **204** (1996), 729–753. http://dx.doi.org/ 10.1006/jmaa.1996.0464

20. L. E. Payne, D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.,* **22** (1975), 273–303. http://dx.doi.org/10.1007/BF02761595

21. D. H. Sattinger, On global solution of nonlinear hyperbolic equations, *Arch. Ration. Mech. Anal.,* **30** (1968), 148–172. http://dx.doi.org/ 10.1007/BF00250942

22. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, 2 Eds., Amsterdam: Springer, 2003.

23. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.,* **136** (2012), 521–573. http://dx.doi.org/10.1016/j.bulsci.2011.12.004

24. R. Temam, *Navier-Stokes equations: Theory and numerical analysis*, New York: Elsevier Science, 1984. http://dx.doi.org/10.1090/chel/343

25. E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, New York: McGraw-Hill, 1955.