AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25307–25328.
DOI: 10.3934/math.20251120
Received: 29 August 2025
Revised: 22 October 2025
Accepted: 29 October 2025
Published: 04 November 2025

Research article

Closed-form solutions of systems of nonlinear difference equations and their connections to generalized Fibonacci numbers and related sequences

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Abstract: In this work, we conducted a comprehensive analytical investigation of an extended nonlinear difference system, which can be regarded as a generalization of a classical nonlinear difference equation. The proposed extension introduces additional variables, allowing for the examination of richer and more intricate dynamical behaviors, particularly those arising from nonlinear interactions among the components. By applying suitable variable transformation techniques, the system is reduced to a linear form, thereby facilitating the derivation of explicit and closed-form solutions. Furthermore, we analyzed the presence of periodic solutions and established their connection to the generalized Fibonacci sequence and other related number sequences, which are known to be highly relevant in various scientific and engineering contexts. Our findings reveal that the system exhibits a periodic structure with a fundamental period of 12, indicating a repeating pattern in the variable evolution. In addition, we proposed a broader generalization of the system through nonlinear functional transformations, which preserve the underlying mathematical framework while allowing the modeling of even more complex behaviors. Several illustrative examples were provided to demonstrate the applicability and effectiveness of the theoretical results.

Keywords: system of difference equations; nonlinear difference equations; closed-form solutions; generalized Fibonacci sequence; periodic solutions
Mathematics Subject Classification: 39A10, 40A05

1. Introduction

Nonlinear difference equations and their systems are fundamental tools across a wide range of scientific and applied fields, including engineering, physics, economics, and biology. Recent research has explored these models from multiple perspectives. For instance, Althagafi and Ghezal (2025, [1]) analyzed the stability of biological rhythms using three-dimensional difference systems with quadratic terms, highlighting the importance of such models for understanding complex biological dynamics. The same authors also introduced a bilinear dynamical framework to address nonlinear difference systems, demonstrating its capability in capturing periodic and complex behaviors [2]. In a different context, Althagafi (2025, [3]) investigated difference systems with applications in neural dynamics, illustrating the potential of these models to explain neural phenomena and interactions within complex networks. Further contributions have linked theoretical development with applications in finance: Ghezal and Alzeley (2024, [4]) examined the probabilistic properties and estimation techniques of thresholded cyclical stochastic volatility models, while Alzeley and Ghezal (2024, [5]) studied multivariate difference systems with asymmetric stochastic volatility, emphasizing their statistical and economic significance. Earlier works, such as those by Abo-Zeid and Cinar (2013, [6]), examined the general behavior of a rational difference equation, emphasizing stability properties and global dynamics. Building on this, Elsayed (2016, 2017, [7–9]) presented detailed analyses of rational systems and higher-order recursive sequences, focusing on periodic solutions, maximal behaviors, and explicit solution representations. Similarly, the contributions of Gümüş et al. (2020–2023, [10–13]) advanced the study of unsteady and periodic behavior in higher-order and delayed difference equations, offering explicit formulations for forbidden sets and closed-form solutions, thus enriching the theoretical understanding of such systems. In addition, Tran et al. (2021, [14]) investigated the general dynamics of certain systems of second-order difference equations, drawing attention to the universality of their dynamical properties. Collectively, these works highlight the breadth of analytical approaches applied to nonlinear systems and underscore the importance of developing extended models that more accurately capture complex interactions. For instance, the following nonlinear equation has been analyzed:

$$\lambda_m = \frac{\lambda_{m-3}\lambda_{m-4}}{\lambda_{m-1}\left(\pm 1 \pm \lambda_{m-3}\lambda_{m-4}\right)},\ m \in \mathbb{N}, \tag{1.1}$$

where the behavior of the solutions was determined under specific initial conditions λ_{-j}, $j = \{0, 1, 2, 3\}$, as reported in [15]. Another significant contribution in this context, building upon the foundation laid by Eq (1.1), is the nonlinear difference equation introduced by Kara et al. [16], which is given by

$$\lambda_m = \frac{\lambda^r_{m-2}\lambda_{m-3}\lambda_{m-4}}{\lambda_{m-1}\left(\alpha\lambda^r_{m-5} + \beta\lambda_{m-3}\lambda_{m-4}\right)},\ r, m \in \mathbb{N}, \tag{1.2}$$

which was studied under the initial values λ_{-j}, $j = \{1, 2, 3, 4, 5\}$. Building on the model described by Eq (1.2), we propose an extension that incorporates an additional sequence of variables. Let $\{\lambda_m\}$ and $\{\mu_m\}$ be sequences of strictly positive real numbers, i.e., $\lambda_m, \mu_m \in \mathbb{R}^*_+$. The extended system is defined as follows:

$$\lambda_m = \frac{\mu^r_{m-2}\mu_{m-3}\lambda_{m-4}}{\mu_{m-1}\left(\alpha\lambda^r_{m-5} + \beta\mu_{m-3}\lambda_{m-4}\right)},\ \mu_m = \frac{\lambda^r_{m-2}\lambda_{m-3}\mu_{m-4}}{\lambda_{m-1}\left(\alpha\mu^r_{m-5} + \beta\lambda_{m-3}\mu_{m-4}\right)},\ m \in \mathbb{N}, \tag{1.3}$$

where $\alpha, \beta > 0$ are fixed real parameters and $r \in \mathbb{N}$ denotes an exponent (power). It is assumed that the initial conditions for both sequences $\left(\lambda_{-j}, \mu_{-j}\right)$, $j = \{1, 2, 3, 4, 5\}$, are strictly positive, ensuring that the system is well-defined. This extension not only broadens the scope of the original model but also enables the study of more complex dynamic behaviors resulting from nonlinear interactions between the two sequences. The importance of the two-dimensional system lies in its ability to represent the interrelationship between two interconnected series, something that one-dimensional models cannot achieve. This construction enables the simulation of realistic phenomena in which multiple components evolve simultaneously, such as biological, economic, and population processes. This extension aims primarily to elucidate how periodic patterns and analytical solutions arise from the interaction between the two sequences, thus deepening the mathematical understanding of nonlinear behavior and enhancing the theoretical value of the proposed model.

Recent research in complex dynamical systems has focused on multi-agent models and control strategies based on event-triggered mechanisms. Zhang et al. (2025, [17]) proposed a hybrid distributed control mechanism that achieves efficient coordination of multi-agent systems despite external disturbances, while minimizing the number of sensor and controller updates. Another study, Zhang et al. (2024, [18]), examined mechanisms for tracking time-varying configurations using adaptive protocols and novel observers. These studies confirm the importance of event-triggered control strategies in understanding complex dynamical systems, which aligns with the goal of this research: analyzing explicit solutions and nonlinear interactions in extended difference systems. This extension is a theoretical innovation that enables a more flexible and accurate representation of interactions between variables, as it allows the nonlinear system to be transformed into an analytically linear form without losing the nonlinear properties of the functions used. This framework provides a solid foundation for deriving closed-loop and explicit solutions, revealing cyclical patterns and complex behaviors that previous models could not describe, thus enhancing the theoretical value of this paper and contributing to addressing a clear gap in the current literature.

The significance of this study lies in its comprehensive analysis of the possible solutions to this extended system. We begin by examining the existence of periodic solutions and exploring the possibility of transforming nonlinear equations into linear equations using appropriate transformation techniques, which facilitate the derivation of closed-form analytical solutions. We also discuss the relationship between the solutions to this system and the generalized Fibonacci sequence and its related sequences, which play key roles in many scientific applications, such as biological modeling, financial forecasting, and engineering design. Recent studies have highlighted similar applications of nonlinear difference systems, including the stability analysis of biological rhythms using three-dimensional systems of difference equations with squared terms (see Althagafi and Ghezal [1]), the modeling of animal behavioral dynamics through bilinear systems (see Althagafi and Ghezal [2]), and the investigation of neural system dynamics using difference equations (see Althagafi [3]). These works underscore the importance and versatility of nonlinear discrete models in capturing complex patterns across various disciplines. An additional motivation for our study lies in the fact that expressing closed-form solutions of nonlinear systems in terms of well-known integer sequences provides both theoretical and practical advantages. Such representations not only simplify the analysis of periodicity and stability but also reveal deeper structural connections between nonlinear dynamics and classical number sequences. A fundamental example is the classical Fibonacci sequence ($F_0 = 0$, $F_1 = 1$, and $F_m = F_{m-1} + F_{m-2}$, $m \geq 2$), which has inspired numerous generalizations in the study of nonlinear recurrence relations. Recent contributions in the literature highlight this perspective: for instance, bidimensional systems of rational difference equations have been explicitly solved using Mersenne numbers and Mersenne–Lucas numbers (see [19]); higher-order two-dimensional systems have been analyzed via Pell numbers and their generalizations (see [20, 21]); and other solvable models have been linked to Jacobsthal numbers, Jacobsthal–Lucas numbers, and even co-balancing numbers (see [22, 23]). These studies demonstrate that embedding solutions within the framework of known integer sequences provide a unifying language for comparing different nonlinear systems and for uncovering hidden algebraic structures. Following this approach, our work emphasizes how the periodic solutions of the extended system are connected not only to generalized Fibonacci numbers but also potentially to a broader family of integer sequences, thereby enriching both the theoretical foundations and the range of possible applications. Despite the importance of previous studies in advancing the theoretical understanding of nonlinear difference systems, most models are still limited by linear structures or constrained transformations, which limit their ability to represent the complex interactions between nonlinear components. To address this shortcoming, this paper proposes an extended model that transforms the nonlinear system into a simplified, analyzable linear representation, with the ability to incorporate various functions such as logarithmic, exponential, and power functions. The primary contribution of this study is to derive explicit, closed-form solutions that maintain the model's flexibility and enhance its ability to describe periodic and complex behaviors with greater accuracy than previous models. This approach relies on an analytical methodology that reformulates the model to facilitate the study of its evolution in terms of its initial conditions and parameters, enabling an explicit representation of the dynamics and facilitating the transition from theoretical analysis to numerical simulation. Thus, the proposed framework contributes to a deeper understanding of nonlinear interactions and supports the evaluation of the model in describing natural and engineering phenomena.

In this paper, we propose a methodology for obtaining explicit solutions of the system using the closed-form solution approach, a powerful mathematical technique for analyzing dynamical systems, where the solutions are expressed explicitly in terms of system parameters and initial conditions. We also demonstrate how the system can be transformed into a simplified linear form through an appropriate variable transformation, thereby facilitating the derivation of simpler and more analytically tractable solutions. Furthermore, we derive and discuss theoretical results that offer a comprehensive understanding of the system's behavior, focusing on the periodic nature of the solutions and their connection to the generalized Fibonacci sequence and its related numerical sequences. The results indicate that the system exhibits a periodic structure with a fundamental period of 12, revealing a recurring pattern in the evolution of the variables. This understanding of the periodic structure and its associated mathematical relationships provides valuable insights into nonlinear systems and their potential applications, particularly in fields such as biology, neuroscience, and behavioral sciences, as demonstrated by recent contributions [1–3]. To facilitate the reading and comprehension of this paper, the following subsection presents the main symbols and notations used throughout.

1.1. Notation

The primary notations used in our analysis are listed as follows:

(1) **Number sets and functions**

(a) $\mathbb{N}$: the set of natural numbers.

(b) $f_{\alpha,\beta}^{(m,r)}(x,y,z) = \alpha^{2m}\dfrac{x^r}{yz} + \beta\left(\alpha^{2m}-1\right)/(\alpha-1)$.

(c) $g_{\alpha,\beta}^{(m,r)}(x,y,z) = \alpha^{2m}\left(\alpha\frac{x^r}{yz}+\beta\right)+\beta\left(\alpha^{2m}-1\right)/(\alpha-1)$.

(d) $\varphi_{\alpha,\beta}^{(m,r)}(x,y,z) = \frac{x^r}{yz}+\beta(\alpha+1)m$.

(e) $\psi_{\alpha,\beta}^{(m,r)}(x,y,z) = \alpha\frac{x^r}{yz}+\beta((\alpha+1)m+1)$.

(f) $\xi_m(n) = 2m+[n/3]$.

(g) G is a one-to-one continuous function on $\mathbb{R}$.

(2) **Some sequences**

(a) (F_m) denotes the generalized Fibonacci numbers, defined by

$$F_0 = 0,\ F_1 = 1, \quad F_m = F_{m-1}+rF_{m-2},\ m \geq 2.$$

(b) (ϕ_m) denotes the conjugate Fibonacci numbers, defined by

$$\phi_0 = \phi_1 = 1, \quad \phi_m = \phi_{m-1}-r\phi_{m-2},\ m \geq 2.$$

(c) (ψ_m) denotes the auxiliary Fibonacci numbers, defined by

$$\psi_0 = 1,\ \psi_1 = 0, \quad \psi_m = \psi_{m-1}-r\psi_{m-2},\ m \geq 2.$$

(3) **Product sequences**

(a) The product sequence $\left(\Gamma_{m,n}^{(l)}(x,y)\right)$ is defined as:

$$\begin{aligned}\Gamma_{m,n}^{(l)}(x,y) = &\left\{\prod_{j=1-l}^{\xi_m(n+1)}\left(x_{6j-3+l}y_{6j-3+l}\right)^{F_{2(6m+n-j+l+2)}}\right\}\left\{\prod_{j=0}^{\xi_m(n)}\left(x_{6j-1+l}y_{6j-1+l}\right)^{F_{2(6m+n-j+l+2)}}\right\}\\ &\times\left\{\prod_{j=0}^{\xi_m(n-1)}\left(x_{6j+1+l}y_{6j+1+l}\right)^{F_{2(6m+n-j+l+2)}}\right\}\left\{\prod_{j=0}^{\xi_m(n+1+l)}\left(x_{6j-2-l}y_{6j-2-l}\right)^{-F_{2(6m+n-j)+5}}\right\}\\ &\times\left\{\prod_{j=0}^{\xi_m(n+l)}\left(x_{6j-l}y_{6j-l}\right)^{-F_{2(6m+n-j)+5}}\right\}\left\{\prod_{j=0}^{\xi_m(n-1+l)}\left(x_{6j+2-l}y_{6j+2-l}\right)^{-F_{2(6m+n-j)+5}}\right\}.\end{aligned}$$

(b) The product sequence $\left(\widehat{\Gamma}_{m,n}^{(l)}(x,y)\right)$ is defined as:

$$\begin{aligned}\widehat{\Gamma}_{m,n}^{(l)}(x,y) = &\left\{\prod_{j=1}^{\xi_m(n+1+l)}\frac{x_{6j-3}^{\phi_{12m+2n+l+3-6j}}}{y_{6j-3}^{\phi_{12m+2n+l+3-6j}}}\right\}\left\{\prod_{j=0}^{\xi_m(n+1)}\frac{x_{6j-2}^{\phi_{12m+2n+l+2-6j}}}{y_{6j-2}^{\phi_{12m+2n+l+2-6j}}}\right\}\left\{\prod_{j=0}^{\xi_m(n+l)}\frac{x_{6j-1}^{\phi_{12m+2n+l+1-6j}}}{y_{6j-1}^{\phi_{12m+2n+l+1-6j}}}\right\}\\ &\times\left\{\prod_{j=0}^{\xi_m(n)}\frac{x_{6j}^{\phi_{12m+2n+l-6j}}}{y_{6j}^{\phi_{12m+2n+l-6j}}}\right\}\left\{\prod_{j=0}^{\xi_m(n-1+l)}\frac{x_{6j+1}^{\phi_{12m+2n+l-1-6j}}}{y_{6j+1}^{\phi_{12m+2n+l-1-6j}}}\right\}\left\{\prod_{j=0}^{\xi_m(n-1)}\frac{x_{6j+2}^{\phi_{12m+2n+l-2-6j}}}{y_{6j+2}^{\phi_{12m+2n+l-2-6j}}}\right\}.\end{aligned}$$

(4) **Product notation convention**

$$\prod_{j=n}^{m}a_j = a_n a_{n+1}...a_m \text{ with } \prod_{j=n}^{m}a_j = 1 \text{ if } m < n.$$

2. Solving system (1.3) explicitly: A closed-form approach

In this section, we focus on deriving explicit solutions for system (1.3) using a closed-form analytical methodology. This approach provides a systematic and direct means of expressing solutions in terms of the system's parameters and initial conditions. Before deriving the solution, it is crucial to ensure that the system is well-defined, with specific conditions on the variables μ and λ being satisfied. By applying an appropriate variable transformation, the original nonlinear system can be converted into a simplified linear form, thereby enabling more explicit and analytically tractable solutions. This methodology not only clarifies the underlying structure of the system but also facilitates the derivation of closed-form expressions for its solutions.

2.1. Linear transformation and its closed-form solution

Before proceeding to solve system (1.3), it is necessary to ensure that the solutions are well-defined and mathematically regular, according to the following conditions:

$$\mu_{m-1}\left(\alpha\lambda^{r}_{m-5}+\beta\mu_{m-3}\lambda_{m-4}\right)\neq 0 \text{ and } \lambda_{m-1}\left(\alpha\mu^{r}_{m-5}+\beta\lambda_{m-3}\mu_{m-4}\right)\neq 0,\ m\in\mathbb{N}.$$

These conditions are not merely formal restrictions; they constitute essential mathematical requirements ensuring that the system admits meaningful and analyzable solutions. Without these constraints, the solutions could lose mathematical coherence or become inapplicable in practical contexts. Therefore, verifying these conditions is a crucial preliminary step in solving the system, as it establishes a rigorous foundation for constructing accurate and reliable solutions. Once these conditions are satisfied, system (1.3) can be represented as follows:

$$\frac{\mu^{r}_{m-2}}{\lambda_m\mu_{m-1}}=\frac{\alpha\lambda^{r}_{m-5}+\beta\mu_{m-3}\lambda_{m-4}}{\mu_{m-3}\lambda_{m-4}},\ \frac{\lambda^{r}_{m-2}}{\mu_m\lambda_{m-1}}=\frac{\alpha\mu^{r}_{m-5}+\beta\lambda_{m-3}\mu_{m-4}}{\lambda_{m-3}\mu_{m-4}}.$$

Using the following variable transformation:

$$\widehat{\lambda}_m=\frac{\mu^{r}_{m-2}}{\lambda_m\mu_{m-1}},\ \widehat{\mu}_m=\frac{\lambda^{r}_{m-2}}{\mu_m\lambda_{m-1}},$$

we can transform the original system into a more analyzable linear system. This transformation represents a strategic mathematical step aimed at simplifying the complex recursive relationships between the variables λ and μ. After applying this transformation, we obtain the following linear system:

$$\widehat{\lambda}_m=\alpha\widehat{\mu}_{m-3}+\beta,\ \widehat{\mu}_m=\alpha\widehat{\lambda}_{m-3}+\beta. \tag{2.1}$$

This linear system (2.1) represents a simplified form of the original system, in which each variable is expressed in terms of the other with constant coefficients α and β. This simplification opens the way for applying well-established techniques in linear system analysis, thereby making the overall analytical process more efficient and transparent. By analyzing linear system (2.1), we can further reduce it to the following recursive form:

$$\widehat{\lambda}_m=\alpha^{2}\widehat{\lambda}_{m-6}+\beta\left(\alpha+1\right),\ \widehat{\mu}_m=\alpha^{2}\widehat{\mu}_{m-6}+\beta\left(\alpha+1\right). \tag{2.2}$$

This simplified system (2.2) exhibits a direct recursive relationship between the variables λ and μ with larger time steps (from $m-6$ instead of $m-3$). Such a reduction allows us to capture the system's behavior over extended periods, providing a solid analytical foundation for deriving closed-form solutions that accurately describe the dynamical evolution of the system.

In the context of analyzing system (2.2), we present the following lemma, which provides the closed-form solution to one of the main recursive equations in the system. This lemma, originally introduced by Elaydi [24], constitutes a fundamental step in understanding the behavior of complex dynamical systems, as it provides a general analytical expression to recursive equations of this type.

Lemma 2.1. *The closed-form solution of the following difference equation:*

$$u_m = \alpha^2 u_{m-6} + \beta(\alpha+1), \ m \geq 6, \tag{2.3}$$

where u_{-j}, $j \in \{0, ..., 5\}$, *are the initial conditions, can be explicitly expressed as*

$$u_{6m-j} = \begin{cases} \alpha^{2m} u_{-j} + \beta\left(\alpha^{2m}-1\right)/(\alpha-1), \ \alpha \neq 1 \\ u_{-j} + \beta(\alpha+1) m, \ \alpha = 1 \end{cases}, \quad j \in \{0, ..., 5\}, \ m \in \mathbb{N}.$$

Proof. The proof is based on partitioning the sequence (u_m) into six distinct subsequences according to the remainders upon division by 6, and then solving each subsequence independently as a first-order recursive relation. This procedure yields an explicit closed-form expression for each case: When $\alpha \neq 1$ and when $\alpha = 1$. The final result is precisely stated in the lemma above. It is worth noting that this methodology is consistent with well-established analytical approaches in the literature, and a similar treatment can be found in the work of Elaydi [24], which remains a fundamental reference in the theory of linear difference equations. □

The following result represents a key lemma in this analysis, as it provides explicit closed-form solutions to system (2.1) under two distinct cases: When $\alpha \neq 1$ and when $\alpha = 1$.

Lemma 2.2. *The closed-form solution of system (2.1) is given, for all m, by:*

$$\text{When } \alpha \neq 1: \begin{array}{llll} \widehat{\lambda}_{6m-3} & = f_{\alpha,\beta}^{(m,r)}(\mu_{-5}, \lambda_{-3}, \mu_{-4}), & \widehat{\mu}_{6m-3} & = f_{\alpha,\beta}^{(m,r)}(\lambda_{-5}, \mu_{-3}, \lambda_{-4}), \\ \widehat{\lambda}_{6m-2} & = f_{\alpha,\beta}^{(m,r)}(\mu_{-4}, \lambda_{-2}, \mu_{-3}), & \widehat{\mu}_{6m-2} & = f_{\alpha,\beta}^{(m,r)}(\lambda_{-4}, \mu_{-2}, \lambda_{-3}), \\ \widehat{\lambda}_{6m-1} & = f_{\alpha,\beta}^{(m,r)}(\mu_{-3}, \lambda_{-1}, \mu_{-2}), & \widehat{\mu}_{6m-1} & = f_{\alpha,\beta}^{(m,r)}(\lambda_{-3}, \mu_{-1}, \lambda_{-2}), \\ \widehat{\lambda}_{6m} & = f_{\alpha,\beta}^{(m,r)}(\mu_{-2}, \lambda_{0}, \mu_{-1}), & \widehat{\mu}_{6m} & = f_{\alpha,\beta}^{(m,r)}(\lambda_{-2}, \mu_{0}, \lambda_{-1}), \\ \widehat{\lambda}_{6m+1} & = g_{\alpha,\beta}^{(m,r)}(\lambda_{-4}, \mu_{-2}, \lambda_{-3}), & \widehat{\mu}_{6m+1} & = g_{\alpha,\beta}^{(m,r)}(\mu_{-4}, \lambda_{-2}, \mu_{-3}), \\ \widehat{\lambda}_{6m+2} & = g_{\alpha,\beta}^{(m,r)}(\lambda_{-3}, \mu_{-1}, \lambda_{-2}), & \widehat{\mu}_{6m+2} & = g_{\alpha,\beta}^{(m,r)}(\mu_{-3}, \lambda_{-1}, \mu_{-2}), \end{array} \tag{2.4}$$

and

$$\text{when } \alpha = 1: \begin{array}{llll} \widehat{\lambda}_{6m-3} & = \varphi_{\alpha,\beta}^{(m,r)}(\mu_{-5}, \lambda_{-3}, \mu_{-4}), & \widehat{\mu}_{6m-3} & = \varphi_{\alpha,\beta}^{(m,r)}(\lambda_{-5}, \mu_{-3}, \lambda_{-4}), \\ \widehat{\lambda}_{6m-2} & = \varphi_{\alpha,\beta}^{(m,r)}(\mu_{-4}, \lambda_{-2}, \mu_{-3}), & \widehat{\mu}_{6m-2} & = \varphi_{\alpha,\beta}^{(m,r)}(\lambda_{-4}, \mu_{-2}, \lambda_{-3}), \\ \widehat{\lambda}_{6m-1} & = \varphi_{\alpha,\beta}^{(m,r)}(\mu_{-3}, \lambda_{-1}, \mu_{-2}), & \widehat{\mu}_{6m-1} & = \varphi_{\alpha,\beta}^{(m,r)}(\lambda_{-3}, \mu_{-1}, \lambda_{-2}), \\ \widehat{\lambda}_{6m} & = \varphi_{\alpha,\beta}^{(m,r)}(\mu_{-2}, \lambda_{0}, \mu_{-1}), & \widehat{\mu}_{6m} & = \varphi_{\alpha,\beta}^{(m,r)}(\lambda_{-2}, \mu_{0}, \lambda_{-1}), \\ \widehat{\lambda}_{6m+1} & = \psi_{\alpha,\beta}^{(m,r)}(\lambda_{-4}, \mu_{-2}, \lambda_{-3}), & \widehat{\mu}_{6m+1} & = \psi_{\alpha,\beta}^{(m,r)}(\mu_{-4}, \lambda_{-2}, \mu_{-3}), \\ \widehat{\lambda}_{6m+2} & = \psi_{\alpha,\beta}^{(m,r)}(\lambda_{-3}, \mu_{-1}, \lambda_{-2}), & \widehat{\mu}_{6m+2} & = \psi_{\alpha,\beta}^{(m,r)}(\mu_{-3}, \lambda_{-1}, \mu_{-2}), \end{array} \tag{2.5}$$

with initial conditions $\left(\lambda_{-j}, \mu_{-j}\right)$, $j = \{1, 2, 3, 4, 5\}$.

Proof. From systems (2.1) and (2.2), the general solution is obtained by solving Eq (2.3) for each variable. By applying Lemma 2.1, we derive a closed-form expression for the recursive equations, distinguishing between the two cases: when $\alpha \neq 1$ and when $\alpha = 1$. For $\alpha \neq 1$, the solution explicitly depends on the initial conditions and exhibits a multiplicative structure involving α^{2m}. Conversely, when $\alpha = 1$, the solution simplifies to a linear expression in m, reflecting a distinct type of dynamical behavior. Furthermore, this reasoning follows the standard methodology established in the theory of linear difference equations, as detailed in the work of Elaydi [24], thereby ensuring that the proof is mathematically rigorous and consistent with classical analytical frameworks. □

Remark 2.1. *Lemma 2.2 provides a comprehensive framework for understanding the behavior of system (2.1) across different parameter regimes. When $\alpha \neq 1$, the solutions exhibit exponential growth or decay, modulated by the factor α^{2m}. When $\alpha = 1$, they display linear growth in m. This dichotomy highlights the crucial role of the parameter α in determining the long-term evolution and stability of the system.*

2.2. Closed-form solutions for system (1.3) and its generalized form

Building on the linear transformations and conditions introduced in the previous subsection, we now proceed to derive the closed-form solutions for system (1.3) and its generalized counterpart. Employing the framework established by Lemma 2.2, which addresses the solutions of iterative linear equations, these techniques can be extended to the more complex nonlinear structure of system (1.3). This extension enables the derivation of explicit expressions for the system's variables. The following theorem formalizes this approach, providing a comprehensive understanding of the system's dynamics through concise expressions that capture the temporal evolution of μ and λ.

Theorem 2.1. *Consider nonlinear system (1.3), where the initial values are given by $\left(\lambda_{-j}, \mu_{-j}\right)$, $j \in \{0, ..., 5\}$. Then, the closed-form solution of the system exhibits a periodic structure with a period of 12. The following general relations hold for all $m \geq -1$:*

$$\begin{aligned}
\lambda^2_{12m+2n+l} = & \left((\mu_{-4+l}\lambda_{-4+l})^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}} \Big/ (\mu_{-3-l}\lambda_{-3-l})^{((1-l)+lr)F_{12m+2n+4}} \right) \\
& \times \Gamma^{(l)}_{m,n}\left(\widehat{\lambda}, \widehat{\mu}\right) \widehat{\Gamma}^{(l)}_{m,n}\left(\widehat{\lambda}, \widehat{\mu}\right) \frac{\lambda_{-3}^{\phi_{12m+2n+l+3}}}{\mu_{-3}^{\phi_{12m+2n+l+3}}} \frac{\lambda_{-4}^{\psi_{12m+2n+l+4}}}{\mu_{-4}^{\psi_{12m+2n+l+4}}}, \\
\mu^2_{12m+2n+l} = & \left((\mu_{-4+l}\lambda_{-4+l})^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}} \Big/ (\mu_{-3-l}\lambda_{-3-l})^{((1-l)+lr)F_{12m+2n+4}} \right) \\
& \times \Gamma^{(l)}_{m,n}\left(\widehat{\mu}, \widehat{\lambda}\right) \widehat{\Gamma}^{(l)}_{m,n}\left(\widehat{\mu}, \widehat{\lambda}\right) \frac{\mu_{-3}^{\phi_{12m+2n+l+3}}}{\lambda_{-3}^{\phi_{12m+2n+l+3}}} \frac{\mu_{-4}^{\psi_{12m+2n+l+4}}}{\lambda_{-4}^{\psi_{12m+2n+l+4}}},
\end{aligned}$$

where $n \in \{0, 1, 2, 3, 4, 5\}$ and $l \in \{0, 1\}$. Moreover, the values $\left(\widehat{\lambda}_{6m+j}, \widehat{\mu}_{6m+j}\right)$, $j \in \{0, ..., 5\}$, are determined according to (2.4) when $\alpha \neq 1$ and according to (2.5) when $\alpha = 1$.

Proof. To derive the closed-form solution for system (1.3), we begin by applying the inverse variable transformation, which constitutes a fundamental step in analyzing the system's dynamic structure. This transformation allows us to reformulate the relationships between successive variables, thereby facilitating the derivation of a general expression for the closed-form solution. The relationship between the variables is defined as follows:

$$\lambda_m = \frac{\mu^r_{m-2}}{\widehat{\lambda}_m \mu_{m-1}}, \ \mu_m = \frac{\lambda^r_{m-2}}{\widehat{\mu}_m \lambda_{m-1}}, \text{ for all } m.$$

Next, we introduce an additional change of variables to obtain a nonlinear system of difference equations with separated variables. For all m, define:

$$x_m = \mu_m \lambda_m, \ y_m = \frac{\lambda_m}{\mu_m}, \ s_m = \widehat{\lambda}_m \widehat{\mu}_m, \ t_m = \frac{\widehat{\lambda}_m}{\widehat{\mu}_m}.$$

Using these new variables, the system can be written as:

$$x_m = \frac{x_{m-2}^r}{s_m x_{m-1}}, \ y_m = \frac{t_m y_{m-1}}{y_{m-2}^r}, \ \text{for all } m. \tag{2.6}$$

From the first equation of system (2.6), we obtain

$$\begin{aligned}
x_{-2} &= \frac{x_{-4}^r}{s_{-2}x_{-3}} &&= \frac{x_{-4}^{rF_1}}{s_{-2}^{F_1} x_{-3}^{F_2}}, \\
x_{-1} &= \frac{x_{-3}^r}{s_{-1}x_{-2}} &&= \frac{x_{-3}^{r+1} s_{-2}}{s_{-1} x_{-4}^r} = \frac{x_{-3}^{F_3} s_{-2}^{F_2}}{s_{-1}^{F_1} x_{-4}^{rF_2}}, \\
x_0 &= \frac{x_{-2}^r}{s_0 x_{-1}} &&= \frac{x_{-4}^{r+r^2} s_{-1}}{s_0 s_{-2}^{r+1} x_{-3}^{2r+1}} = \frac{x_{-4}^{rF_3} s_{-1}^{F_2}}{s_0^{F_1} s_{-2}^{F_3} x_{-3}^{F_4}}, \\
x_1 &= \frac{x_{-1}^r}{s_1 x_0} &&= \frac{x_{-3}^{1+3r+r^2} s_{-2}^{2r+1} s_0}{s_1 s_{-1}^{r+1} x_{-4}^{r+2r^2}} = \frac{x_{-3}^{F_5} s_{-2}^{F_4} s_0^{F_2}}{s_1^{F_1} s_{-1}^{F_3} x_{-4}^{rF_4}}, \\
x_2 &= \frac{x_0^r}{s_2 x_1} &&= \frac{x_{-4}^{\left(1+3r+r^2\right)r} s_{-1}^{2r+1} s_1}{s_2 s_0^{1+r} s_{-2}^{1+3r+r^2} x_{-3}^{1+4r+3r^2}} = \frac{x_{-4}^{rF_5} s_{-1}^{F_4} s_1^{F_2}}{s_2^{F_1} s_0^{F_3} s_{-2}^{F_5} x_{-3}^{F_6}}, \\
x_3 &= \frac{x_1^r}{s_3 x_2} &&= \frac{x_{-3}^{1+5r+6r^2+r^3} s_{-2}^{1+4r+3r^2} s_0^{2r+1} s_2}{s_3 s_1^{1+r} s_{-1}^{1+3r+r^2} x_{-4}^{r\left(1+4r+3r^2\right)}} = \frac{x_{-3}^{F_7} s_{-2}^{F_6} s_0^{F_4} s_2^{F_2}}{s_3^{F_1} s_1^{F_3} s_{-1}^{F_5} x_{-4}^{rF_6}}.
\end{aligned}$$

From this recurrence, we derive the following elegant general formula, valid for all $m \geq -1$:

$$x_{2m+l} = \left(x_{-4+l}^{(1-l)rF_{2m+3}+lF_{2m+5}} \Big/ x_{-3-l}^{((1-l)+lr)F_{2m+4}} \right) \left(\prod_{j=1}^{m+l+1} s_{2j-3-l}^{F_{2(m-j+l+2)}} \Big/ \prod_{j=1}^{m+2} s_{2j-4+l}^{F_{2(m-j)+5}} \right),$$

where $l \in \{0, 1\}$. Similarly, from the second equation of system (2.6), we obtain for all $m \geq -1$:

$$\begin{aligned}
y_{-2} &= t_{-2} y_{-3} y_{-4}^{-r} &&= t_{-2}^{\phi_0} y_{-3}^{\phi_1} y_{-4}^{\psi_2}, \\
y_{-1} &= t_{-1} y_{-2} y_{-3}^{-r} &&= t_{-1} t_{-2} y_{-3}^{-r+1} y_{-4}^{-r} = t_{-1}^{\phi_0} t_{-2}^{\phi_1} y_{-3}^{\phi_2} y_{-4}^{\psi_3}, \\
y_0 &= t_0 y_{-1} y_{-2}^{-r} &&= t_0 t_{-1} t_{-2}^{-r+1} y_{-3}^{-2r+1} y_{-4}^{r^2-r} = t_0^{\phi_0} t_{-1}^{\phi_1} t_{-2}^{\phi_2} y_{-3}^{\phi_3} y_{-4}^{\psi_4}, \\
y_1 &= t_1 y_0 y_{-1}^{-r} &&= t_1 t_0 t_{-1}^{-r+1} t_{-2}^{-2r+1} y_{-3}^{r^2-3r+1} y_{-4}^{2r^2-r} = t_1^{\phi_0} t_0^{\phi_1} t_{-1}^{\phi_2} t_{-2}^{\phi_3} y_{-3}^{\phi_4} y_{-4}^{\psi_5}.
\end{aligned}$$

Hence, for all $m \geq -1$, the general relation is

$$y_m = \left\{ \prod_{j=0}^{m+2} t_{m-j}^{\phi_j} \right\} y_{-3}^{\phi_{m+3}} y_{-4}^{\psi_{m+4}}.$$

Returning now to the original variables λ_m and μ_m, since $\lambda_m^2 = x_m y_m$ and $\mu_m^2 = \dfrac{x_m}{y_m}$, we have for all $m \geq -1$:

$$
\begin{aligned}
\lambda_{2m+l}^2 &= x_{2m+l} y_{2m+l} \\
&= \left((\mu_{-4+l}\lambda_{-4+l})^{(1-l)rF_{2m+3}+lF_{2m+5}} \Big/ (\mu_{-3-l}\lambda_{-3-l})^{((1-l)+lr)F_{2m+4}} \right) \\
&\times \left(\prod_{j=1}^{m+l+1} \left(\widehat{\lambda}_{2j-3-l}\widehat{\mu}_{2j-3-l}\right)^{F_{2(m-j+l+2)}} \Big/ \prod_{j=1}^{m+2} \left(\widehat{\lambda}_{2j-4+l}\widehat{\mu}_{2j-4+l}\right)^{F_{2(m-j)+5}} \right) \\
&\times \left\{ \prod_{j=0}^{2m+l+2} \frac{\widehat{\lambda}_{2m+l-j}^{\phi_j}}{\widehat{\mu}_{2m+l-j}^{\phi_j}} \right\} \frac{\lambda_{-3}^{\phi_{2m+l+3}}}{\mu_{-3}^{\phi_{2m+l+3}}} \frac{\lambda_{-4}^{\psi_{2m+l+4}}}{\mu_{-4}^{\psi_{2m+l+4}}},
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_{2m+l}^2 &= \frac{x_{2m+l}}{y_{2m+l}} \\
&= \left((\mu_{-4+l}\lambda_{-4+l})^{(1-l)rF_{2m+3}+lF_{2m+5}} \Big/ (\mu_{-3-l}\lambda_{-3-l})^{((1-l)+lr)F_{2m+4}} \right) \\
&\times \left(\prod_{j=1}^{m+l+1} \left(\widehat{\lambda}_{2j-3-l}\widehat{\mu}_{2j-3-l}\right)^{F_{2(m-j+l+2)}} \Big/ \prod_{j=1}^{m+2} \left(\widehat{\lambda}_{2j-4+l}\widehat{\mu}_{2j-4+l}\right)^{F_{2(m-j)+5}} \right) \\
&\times \left\{ \prod_{j=0}^{2m+l+2} \frac{\widehat{\mu}_{2m+l-j}^{\phi_j}}{\widehat{\lambda}_{2m+l-j}^{\phi_j}} \right\} \frac{\mu_{-3}^{\phi_{2m+l+3}}}{\lambda_{-3}^{\phi_{2m+l+3}}} \frac{\mu_{-4}^{\psi_{2m+l+4}}}{\lambda_{-4}^{\psi_{2m+l+4}}}.
\end{aligned}
$$

The system exhibits periodic behavior of order 6 for the variables $\left(\widehat{\lambda}_m, \widehat{\mu}_m\right)$, reflecting a recurring pattern in their evolution. By applying the results of Lemma 2.2, we conclude that the closed-form solution of the system follows a periodicity of order 12, meaning that the values repeat every 12 steps. This result provides a deeper understanding of the system's periodic nature, as the same structure patterns recur after every 12 iterations. To prove the validity of this periodic pattern, the principle of mathematical induction on the general relationship between the variables is used. Accordingly, the closed-form solution can be expressed as follows:

$$
\begin{aligned}
\lambda^{2}_{12m+2n+l} = {} & \Big(\left(\mu_{-4+l}\lambda_{-4+l}\right)^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}} \Big/ \left(\mu_{-3-l}\lambda_{-3-l}\right)^{((1-l)+lr)F_{12m+2n+4}} \Big) \\
& \times \left\{ \prod_{j=1-l}^{\xi_m(n+1)} \left(\widehat{\lambda}_{6j-3+l}\widehat{\mu}_{6j-3+l}\right)^{F_{2(6m+n-j+l+2)}} \right\} \left\{ \prod_{j=0}^{\xi_m(n)} \left(\widehat{\lambda}_{6j-1+l}\widehat{\mu}_{6j-1+l}\right)^{F_{2(6m+n-j+l+2)}} \right\} \\
& \times \left\{ \prod_{j=0}^{\xi_m(n-1)} \left(\widehat{\lambda}_{6j+1+l}\widehat{\mu}_{6j+1+l}\right)^{F_{2(6m+n-j+l+2)}} \right\} \left\{ \prod_{j=0}^{\xi_m(n+1+l)} \left(\widehat{\lambda}_{6j-2-l}\widehat{\mu}_{6j-2-l}\right)^{-F_{2(6m+n-j)+5}} \right\} \\
& \times \left\{ \prod_{j=0}^{\xi_m(n+l)} \left(\widehat{\lambda}_{6j-l}\widehat{\mu}_{6j-l}\right)^{-F_{2(6m+n-j)+5}} \right\} \left\{ \prod_{j=0}^{\xi_m(n-1+l)} \left(\widehat{\lambda}_{6j+2-l}\widehat{\mu}_{6j+2-l}\right)^{-F_{2(6m+n-j)+5}} \right\} \\
& \times \left\{ \prod_{j=1}^{\xi_m(n+1+l)} \frac{\widehat{\lambda}_{6j-3}^{\phi_{12m+2n+l+3-6j}}}{\widehat{\mu}_{6j-3}^{\phi_{12m+2n+l+3-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n+1)} \frac{\widehat{\lambda}_{6j-2}^{\phi_{12m+2n+l+2-6j}}}{\widehat{\mu}_{6j-2}^{\phi_{12m+2n+l+2-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n+l)} \frac{\widehat{\lambda}_{6j-1}^{\phi_{12m+2n+l+1-6j}}}{\widehat{\mu}_{6j-1}^{\phi_{12m+2n+l+1-6j}}} \right\} \\
& \times \left\{ \prod_{j=0}^{\xi_m(n)} \frac{\widehat{\lambda}_{6j}^{\phi_{12m+2n+l-6j}}}{\widehat{\mu}_{6j}^{\phi_{12m+2n+l-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n-1+l)} \frac{\widehat{\lambda}_{6j+1}^{\phi_{12m+2n+l-1-6j}}}{\widehat{\mu}_{6j+1}^{\phi_{12m+2n+l-1-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n-1)} \frac{\widehat{\lambda}_{6j+2}^{\phi_{12m+2n+l-2-6j}}}{\widehat{\mu}_{6j+2}^{\phi_{12m+2n+l-2-6j}}} \right\} \\
& \times \frac{\lambda_{-3}^{\phi_{12m+2n+l+3}}}{\mu_{-3}^{\phi_{12m+2n+l+3}}} \frac{\lambda_{-4}^{\psi_{12m+2n+l+4}}}{\mu_{-4}^{\psi_{12m+2n+l+4}}},
\end{aligned}
$$

$$
\begin{aligned}
\mu^{2}_{12m+2n+l} = {} & \Big(\left(\mu_{-4+l}\lambda_{-4+l}\right)^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}} \Big/ \left(\mu_{-3-l}\lambda_{-3-l}\right)^{((1-l)+lr)F_{12m+2n+4}} \Big) \\
& \times \left\{ \prod_{j=1-l}^{\xi_m(n+1)} \left(\widehat{\lambda}_{6j-3+l}\widehat{\mu}_{6j-3+l}\right)^{F_{2(6m+n-j+l+2)}} \right\} \left\{ \prod_{j=0}^{\xi_m(n)} \left(\widehat{\lambda}_{6j-1+l}\widehat{\mu}_{6j-1+l}\right)^{F_{2(6m+n-j+l+2)}} \right\} \\
& \times \left\{ \prod_{j=0}^{\xi_m(n-1)} \left(\widehat{\lambda}_{6j+1+l}\widehat{\mu}_{6j+1+l}\right)^{F_{2(6m+n-j+l+2)}} \right\} \left\{ \prod_{j=0}^{\xi_m(n+1+l)} \left(\widehat{\lambda}_{6j-2-l}\widehat{\mu}_{6j-2-l}\right)^{-F_{2(6m+n-j)+5}} \right\} \\
& \times \left\{ \prod_{j=0}^{\xi_m(n+l)} \left(\widehat{\lambda}_{6j-l}\widehat{\mu}_{6j-l}\right)^{-F_{2(6m+n-j)+5}} \right\} \left\{ \prod_{j=0}^{\xi_m(n-1+l)} \left(\widehat{\lambda}_{6j+2-l}\widehat{\mu}_{6j+2-l}\right)^{-F_{2(6m+n-j)+5}} \right\} \\
& \times \left\{ \prod_{j=1}^{\xi_m(n+1+l)} \frac{\widehat{\mu}_{6j-3}^{\phi_{12m+2n+l+3-6j}}}{\widehat{\lambda}_{6j-3}^{\phi_{12m+2n+l+3-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n+1)} \frac{\widehat{\mu}_{6j-2}^{\phi_{12m+2n+l+2-6j}}}{\widehat{\lambda}_{6j-2}^{\phi_{12m+2n+l+2-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n+l)} \frac{\widehat{\mu}_{6j-1}^{\phi_{12m+2n+l+1-6j}}}{\widehat{\lambda}_{6j-1}^{\phi_{12m+2n+l+1-6j}}} \right\} \\
& \times \left\{ \prod_{j=0}^{\xi_m(n)} \frac{\widehat{\mu}_{6j}^{\phi_{12m+2n+l-6j}}}{\widehat{\lambda}_{6j}^{\phi_{12m+2n+l-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n-1+l)} \frac{\widehat{\mu}_{6j+1}^{\phi_{12m+2n+l-1-6j}}}{\widehat{\lambda}_{6j+1}^{\phi_{12m+2n+l-1-6j}}} \right\} \left\{ \prod_{j=0}^{\xi_m(n-1)} \frac{\widehat{\mu}_{6j+2}^{\phi_{12m+2n+l-2-6j}}}{\widehat{\lambda}_{6j+2}^{\phi_{12m+2n+l-2-6j}}} \right\} \\
& \times \frac{\mu_{-3}^{\phi_{12m+2n+l+3}}}{\lambda_{-3}^{\phi_{12m+2n+l+3}}} \frac{\mu_{-4}^{\psi_{12m+2n+l+4}}}{\lambda_{-4}^{\psi_{12m+2n+l+4}}},
\end{aligned}
$$

where $n \in \{0, 1, 2, 3, 4, 5\}$ and $l \in \{0, 1\}$. □

Remark 2.2. *The result presented in Theorem 2.1 unveils a profound relationship between the solutions of the nonlinear system and the generalized Fibonacci sequence (r-Fibonacci sequence). The general solution, expressed through the relations $\lambda_{12m+2n+l}$ and $\mu_{12m+2n+l}$, fundamentally depends on the generalized Fibonacci numbers (F_m). The constant parameter r plays a crucial role in shaping the behavior of this sequence and, consequently, in determining the dynamical evolution of the system. This dependence is clearly reflected in the exponents to which the variables are raised, such as*

$\mu_{-4+l}^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}}$ *and* $\widehat{\lambda}_{6j+2-l}^{-F_{2(6m+n-j)+5}}$, *demonstrating how the mathematical structure of the system is intricately linked to the properties of the r-Fibonacci sequence. In addition, the theorem reveals that the system exhibits a periodic structure with a fundamental period of* 12, *reflecting a recurring behavior that depends on the interplay between the initial values* $\left(\lambda_{-j}, \mu_{-j}\right)$ *and the generalized Fibonacci sequence. This interaction highlights how these mathematical components collectively influence the system's temporal evolution, thereby offering valuable insights into the relationship between nonlinear dynamics and number-theoretic sequences. It is also important to emphasize that the analysis involves not only the generalized Fibonacci sequence* (F_m), *but also its associated conjugate and auxiliary sequences,* (ϕ_m) *and* (ψ_m), *respectively. By incorporating these sequences, the closed-form solutions attain an elegant structure, expressed in terms of products and powers involving the initial values and the Fibonacci-type sequences. Thus, the interplay among the generalized, conjugate, and auxiliary Fibonacci sequences governs both the exponents and multiplicative structure of the solutions and explains the periodicity of order* 12, *which constitutes a fundamental characteristic of the nonlinear system under consideration.*

To better motivate the forthcoming generalization, it is essential to recognize that specific functional transformations can significantly modify the system's dynamics. Such transformations introduce a scaling mechanism that compresses larger values while expanding smaller ones, effectively reshaping the oscillatory patterns of trajectories and, in some cases, stabilizing growth behaviors. Alternatively, they may amplify or dampen fluctuations depending on parameter choices. These observations demonstrate that the proposed generalization is not merely an abstract mathematical extension but a mechanism that directly alters the qualitative properties of the system, paving the way for a broader and more comprehensive formulation. Accordingly, the generalized system is defined as

$$\begin{aligned} \lambda_m &= G^{-1}\left(\frac{(G(\mu_{m-2}))^r G(\mu_{m-3}) G(\lambda_{m-4})}{G(\mu_{m-1})(\alpha(G(\lambda_{m-5}))^r + \beta G(\mu_{m-3}) G(\lambda_{m-4}))}\right), \\ \mu_m &= G^{-1}\left(\frac{(G(\lambda_{m-2}))^r G(\lambda_{m-3}) G(\mu_{m-4})}{G(\lambda_{m-1})(\alpha(G(\mu_{m-5}))^r + \beta G(\lambda_{m-3}) G(\mu_{m-4}))}\right), \end{aligned} \tag{2.7}$$

where $r, m \in \mathbb{N}$. The system is supplemented with strictly positive initial conditions $\left(\lambda_{-j}, \mu_{-j}\right)$, $j = \{1, 2, 3, 4, 5\}$, ensuring well-posedness of the recursive relations. Building upon Theorem 2.1, this formulation can be extended to system (2.7), as established in the following corollary. The generalized framework broadens the scope of nonlinear systems with periodic structures, where the solutions are expressed through intricate mathematical relations that combine the generalized Fibonacci sequence with functional mappings possessing special properties.

Corollary 2.1. *Consider system (2.7). The solutions are well-defined whenever the denominators are nonzero, i.e.,*

$$\begin{aligned} G(\mu_{m-1})(\alpha(G(\lambda_{m-5}))^r + \beta G(\mu_{m-3}) G(\lambda_{m-4})) \neq 0, \\ G(\lambda_{m-1})(\alpha(G(\mu_{m-5}))^r + \beta G(\lambda_{m-3}) G(\mu_{m-4})) \neq 0. \end{aligned}$$

Under these conditions, the closed-form solution of the system exhibits a periodic structure of order 12. *Moreover, the following general relations hold for all m:*

$$\begin{aligned}
\lambda_{12m+2n+l} = \quad & G^{-1}\Big(sqrt\Big(\Big((G(\mu_{-4+l})\,G(\lambda_{-4+l}))^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}}\\
& \times (G(\mu_{-3-l})\,G(\lambda_{-3-l}))^{((l-1)-lr)F_{12m+2n+4}}\Big)\Gamma^{(l)}_{m,n}\Big(\widehat{\Lambda},\widehat{\Phi}\Big)\widehat{\Gamma}^{(l)}_{m,n}\Big(\widehat{\Lambda},\widehat{\Phi}\Big)\\
& \times \frac{(G(\lambda_{-3}))^{\phi_{12m+2n+l+3}}}{(G(\mu_{-3}))^{\phi_{12m+2n+l+3}}}\frac{(G(\lambda_{-4}))^{\psi_{12m+2n+l+4}}}{(G(\mu_{-4}))^{\psi_{12m+2n+l+4}}}\Big)\Big),
\end{aligned}$$

$$\begin{aligned}
\mu_{12m+2n+l} = \quad & G^{-1}\Big(sqrt\Big(\Big((G(\mu_{-4+l})\,G(\lambda_{-4+l}))^{(1-l)rF_{2(6m+n)+3}+lF_{2(6m+n)+5}}\\
& \times (G(\mu_{-3-l})\,G(\lambda_{-3-l}))^{((l-1)-lr)F_{12m+2n+4}}\Big)\Gamma^{(l)}_{m,n}\Big(\widehat{\Phi},\widehat{\Lambda}\Big)\widehat{\Gamma}^{(l)}_{m,n}\Big(\widehat{\Phi},\widehat{\Lambda}\Big)\\
& \times \frac{(G(\mu_{-3}))^{\phi_{12m+2n+l+3}}}{(G(\lambda_{-3}))^{\phi_{12m+2n+l+3}}}\frac{(G(\mu_{-4}))^{\psi_{12m+2n+l+4}}}{(G(\lambda_{-4}))^{\psi_{12m+2n+l+4}}}\Big)\Big),
\end{aligned}$$

where $n \in \{0,1,2,3,4,5\}$, $l \in \{0,1\}$, $\widehat{\Lambda}_m = \dfrac{(G(\mu_{m-2}))^r}{G(\lambda_m)\,G(\mu_{m-1})}$, *and* $\widehat{\Phi}_m = \dfrac{(G(\lambda_{m-2}))^r}{G(\mu_m)\,G(\lambda_{m-1})}$. *The quantities* $\Big(\widehat{\Lambda}_{6m+j},\widehat{\Phi}_{6m+j}\Big)$, $j \in \{-3,-2-1,0,1,2\}$, *are determined according to the following relations:*

When $\alpha \neq 1$:
$$\begin{aligned}
\widehat{\Lambda}_{6m-3} &= f^{(m,r)}_{\alpha,\beta}(G(\mu_{-5}),G(\lambda_{-3}),G(\mu_{-4})), & \widehat{\Phi}_{6m-3} &= f^{(m,r)}_{\alpha,\beta}(G(\lambda_{-5}),G(\mu_{-3}),G(\lambda_{-4})),\\
\widehat{\Lambda}_{6m-2} &= f^{(m,r)}_{\alpha,\beta}(G(\mu_{-4}),G(\lambda_{-2}),G(\mu_{-3})), & \widehat{\Phi}_{6m-2} &= f^{(m,r)}_{\alpha,\beta}(G(\lambda_{-4}),G(\mu_{-2}),G(\lambda_{-3})),\\
\widehat{\Lambda}_{6m-1} &= f^{(m,r)}_{\alpha,\beta}(G(\mu_{-3}),G(\lambda_{-1}),G(\mu_{-2})), & \widehat{\Phi}_{6m-1} &= f^{(m,r)}_{\alpha,\beta}(G(\lambda_{-3}),G(\mu_{-1}),G(\lambda_{-2})),\\
\widehat{\Lambda}_{6m} &= f^{(m,r)}_{\alpha,\beta}(G(\mu_{-2}),G(\lambda_{0}),G(\mu_{-1})), & \widehat{\Phi}_{6m} &= f^{(m,r)}_{\alpha,\beta}(G(\lambda_{-2}),G(\mu_{0}),G(\lambda_{-1})),\\
\widehat{\Lambda}_{6m+1} &= g^{(m,r)}_{\alpha,\beta}(G(\lambda_{-4}),G(\mu_{-2}),G(\lambda_{-3})), & \widehat{\Phi}_{6m+1} &= g^{(m,r)}_{\alpha,\beta}(G(\mu_{-4}),G(\lambda_{-2}),G(\mu_{-3})),\\
\widehat{\Lambda}_{6m+2} &= g^{(m,r)}_{\alpha,\beta}(G(\lambda_{-3}),G(\mu_{-1}),G(\lambda_{-2})), & \widehat{\Phi}_{6m+2} &= g^{(m,r)}_{\alpha,\beta}(G(\mu_{-3}),G(\lambda_{-1}),G(\mu_{-2})),
\end{aligned}$$

and

when $\alpha = 1$:
$$\begin{aligned}
\widehat{\Lambda}_{6m-3} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\mu_{-5}),G(\lambda_{-3}),G(\mu_{-4})), & \widehat{\Phi}_{6m-3} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\lambda_{-5}),G(\mu_{-3}),G(\lambda_{-4})),\\
\widehat{\Lambda}_{6m-2} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\mu_{-4}),G(\lambda_{-2}),G(\mu_{-3})), & \widehat{\Phi}_{6m-2} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\lambda_{-4}),G(\mu_{-2}),G(\lambda_{-3})),\\
\widehat{\Lambda}_{6m-1} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\mu_{-3}),G(\lambda_{-1}),G(\mu_{-2})), & \widehat{\Phi}_{6m-1} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\lambda_{-3}),G(\mu_{-1}),G(\lambda_{-2})),\\
\widehat{\Lambda}_{6m} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\mu_{-2}),G(\lambda_{0}),G(\mu_{-1})), & \widehat{\Phi}_{6m} &= \varphi^{(m,r)}_{\alpha,\beta}(G(\lambda_{-2}),G(\mu_{0}),G(\lambda_{-1})),\\
\widehat{\Lambda}_{6m+1} &= \psi^{(m,r)}_{\alpha,\beta}(G(\lambda_{-4}),G(\mu_{-2}),G(\lambda_{-3})), & \widehat{\Phi}_{6m+1} &= \psi^{(m,r)}_{\alpha,\beta}(G(\mu_{-4}),G(\lambda_{-2}),G(\mu_{-3})),\\
\widehat{\Lambda}_{6m+2} &= \psi^{(m,r)}_{\alpha,\beta}(G(\lambda_{-3}),G(\mu_{-1}),G(\lambda_{-2})), & \widehat{\Phi}_{6m+2} &= \psi^{(m,r)}_{\alpha,\beta}(G(\mu_{-3}),G(\lambda_{-1}),G(\mu_{-2})).
\end{aligned}$$

Proof. The proof relies on a variable transformation defined by $\overline{\lambda}_m = G(\lambda_m)$ and $\overline{\mu}_m = G(\mu_m)$. This transformation is fundamental because it reformulates system (2.7) into a structure that reveals its direct correspondence with the original system (1.3):

$$\overline{\lambda}_m = \frac{\overline{\mu}^r_{m-2}\overline{\mu}_{m-3}\overline{\lambda}_{m-4}}{\overline{\mu}_{m-1}\Big(\alpha\overline{\lambda}^r_{m-5}+\beta\overline{\mu}_{m-3}\overline{\lambda}_{m-4}\Big)},\ \overline{\mu}_m = \frac{\overline{\lambda}^r_{m-2}\overline{\lambda}_{m-3}\overline{\mu}_{m-4}}{\overline{\lambda}_{m-1}\Big(\alpha\overline{\mu}^r_{m-5}+\beta\overline{\lambda}_{m-3}\overline{\mu}_{m-4}\Big)},\ m \in \mathbb{N}.$$

Through this transformation, system (2.7) becomes structurally equivalent to (1.3), confirming that the generalized system extends the original while preserving its core mathematical properties. Because G is assumed to be one-to-one and continuous, the transformation maintains the coherence and well-posedness of the solutions. □

Remark 2.3. *The function G is assumed to be a one-to-one continuous function on* $\mathbb{R}$*, introducing additional nonlinear elements into the model. This guarantees that variable transformations remain consistent and that the system admits well-defined analytical solutions. When G is chosen as the identity function, i.e.,* $G(x) = x$*, the generalized system (2.7) naturally reduces to the original system (1.3), confirming the consistency of the framework. However, the scope of the generalization* *becomes evident when explicit forms of G are considered. For instance, if* $G(x) = \log(x)$*, multiplicative relations in the original system become additive, facilitating the study of growth rates and log-periodicity. On the other hand, when* $G(x) = x^s$*, the system exhibits power-law scaling effects, either amplifying or attenuating nonlinear effects depending on s. These examples demonstrate that the choice of G directly shapes the system's dynamical behavior, influencing its periodicity, stability, and scaling properties. Hence, system (2.7) represents not merely a formal extension, but a flexible framework capable of capturing a broad range of nonlinear phenomena.*

Remark 2.4. *In this paper, mathematical conditions such as positive initial values and non-zero denominators are assumed to ensure that the proposed system is both well-defined and analytically tractable. These assumptions are not merely formalities; they constitute essential requirements for the solutions to possess coherent mathematical meaning and for closed-form formulas to be rigorously derived and analyzed. While such assumptions may appear restrictive in the context of practical applications, they are standard in theoretical investigations of nonlinear difference equations. For instance, in the studies of [15, 16], analogous nonlinear systems were examined under the same initial value conditions, enabling precise conclusions about solution behavior to be drawn. Accordingly, adopting these assumptions in the present paper provides a solid mathematical framework that guarantees the theoretical consistency of the solutions. Relaxing or adapting these assumptions to accommodate specific application scenarios remains an important and promising direction for future research.*

Remark 2.5. *From an applied perspective, these assumptions are justified by the fact that most realistic models in biology, economics, and engineering involve inherently positive variables, such as population sizes, prices, or concentrations, rendering nonnegativity both natural and essential. If violated, the system could lose physical interpretability or yield undefined behaviors (e.g., division by zero). Therefore, these conditions are critical for ensuring mathematical soundness and practical relevance. Nevertheless, future investigations could explore relaxed assumptions, such as systems with negative initial values or vanishing denominators, which may reveal new oscillatory regimes or collapse phenomena relevant to applied sciences.*

3. Numerical examples and dynamical behavior of the systems

This section presents a set of numerical experiments designed to analyze the dynamical behavior of nonlinear recursive systems. Three distinct configurations are investigated: A baseline model, a logarithmic transformation, and a power function. Each system is initialized with prescribed positive values and evolves recursively according to the control parameters α and β. The graphical results reveal the trajectories and periodic structures of the sequences, providing insights into their stability properties, oscillatory characteristics, and long-term dynamics.

Example 3.1. *Consider system (1.3) defined by the sequence* (λ_m, μ_m)*, where each successive term is generated recursively from preceding values through nonlinear relations governed by the coefficients* $\alpha = 0.05$ *and* $\beta = 0.03$*, with the exponent* $r = 1$*. The system is initialized with the positive values listed in Table 1:*

Table 1. Initial conditions of the system.

j	1	2	3	4	5
λ_{-j}	0.1	0.2	0.15	0.18	0.12
μ_{-j}	0.2	0.1	0.15	0.12	0.18

To validate the closed-form solution established in Theorem 2.1, we explicitly compute the first few terms of the sequences for the chosen parameters. Substituting the initial values from Table 1 into the derived formula yields, for $m = n = 0$ *and* $r = 1$*:*

$$\begin{aligned}\lambda_l^2 = &\left((\mu_{-4+l}\lambda_{-4+l})^{(1-l)F_3+lF_5} \Big/ (\mu_{-3-l}\lambda_{-3-l})^{F_4} \right)\\ &\times \Gamma_{0,0}^{(l)}\left(\widehat{\lambda},\widehat{\mu}\right)\widehat{\Gamma}_{0,0}^{(l)}\left(\widehat{\lambda},\widehat{\mu}\right)\frac{\lambda_{-3}^{\phi_{l+3}}}{\mu_{-3}^{\phi_{l+3}}}\frac{\lambda_{-4}^{\psi_{l+4}}}{\mu_{-4}^{\psi_{l+4}}},\\ \mu_l^2 = &\left((\mu_{-4+l}\lambda_{-4+l})^{(1-l)F_3+lF_5} \Big/ (\mu_{-3-l}\lambda_{-3-l})^{F_4} \right)\\ &\times \Gamma_{0,0}^{(l)}\left(\widehat{\mu},\widehat{\lambda}\right)\widehat{\Gamma}_{0,0}^{(l)}\left(\widehat{\mu},\widehat{\lambda}\right)\frac{\mu_{-3}^{\phi_{l+3}}}{\lambda_{-3}^{\phi_{l+3}}}\frac{\mu_{-4}^{\psi_{l+4}}}{\lambda_{-4}^{\psi_{l+4}}},\end{aligned}$$

where

$$\begin{aligned}\Gamma_{0,0}^{(l)}\left(\widehat{\lambda},\widehat{\mu}\right) = \Gamma_{0,0}^{(l)}\left(\widehat{\mu},\widehat{\lambda}\right) = &\left\{\prod_{j=1-l}^{\xi_0(1)}\left(\widehat{\lambda}_{6j-3+l}\widehat{\mu}_{6j-3+l}\right)^{F_{2(-j+l+2)}}\right\}\left\{\prod_{j=0}^{\xi_0(0)}\left(\widehat{\lambda}_{6j-1+l}\widehat{\mu}_{6j-1+l}\right)^{F_{2(-j+l+2)}}\right\}\\ &\times\left\{\prod_{j=0}^{\xi_0(-1)}\left(\widehat{\lambda}_{6j+1+l}\widehat{\mu}_{6j+1+l}\right)^{F_{2(-j+l+2)}}\right\}\left\{\prod_{j=0}^{\xi_0(1+l)}\left(\widehat{\lambda}_{6j-2-l}\widehat{\mu}_{6j-2-l}\right)^{-F_{-2j+5}}\right\}\\ &\times\left\{\prod_{j=0}^{\xi_0(l)}\left(\widehat{\lambda}_{6j-l}\widehat{\mu}_{6j-l}\right)^{-F_{-2j+5}}\right\}\left\{\prod_{j=0}^{\xi_0(l-1)}\left(\widehat{\lambda}_{6j+2-l}\widehat{\mu}_{6j+2-l}\right)^{-F_{-2j+5}}\right\},\end{aligned}$$

$$\begin{aligned}\widehat{\Gamma}_{0,0}^{(l)}\left(\widehat{\lambda},\widehat{\mu}\right) = 1/\widehat{\Gamma}_{0,0}^{(l)}\left(\widehat{\mu},\widehat{\lambda}\right) = &\left\{\prod_{j=1}^{\xi_0(1+l)}\frac{\widehat{\lambda}_{6j-3}^{\phi_{l+3-6j}}}{\widehat{\mu}_{6j-3}^{\phi_{l+3-6j}}}\right\}\left\{\prod_{j=0}^{\xi_0(1)}\frac{\widehat{\lambda}_{6j-2}^{\phi_{l+2-6j}}}{\widehat{\mu}_{6j-2}^{\phi_{l+2-6j}}}\right\}\left\{\prod_{j=0}^{\xi_0(l)}\frac{\widehat{\lambda}_{6j-1}^{\phi_{l+1-6j}}}{\widehat{\mu}_{6j-1}^{\phi_{l+1-6j}}}\right\}\\ &\times\left\{\prod_{j=0}^{\xi_0(0)}\frac{\widehat{\lambda}_{6j}^{\phi_{l-6j}}}{\widehat{\mu}_{6j}^{\phi_{l-6j}}}\right\}\left\{\prod_{j=0}^{\xi_0(l-1)}\frac{\widehat{\lambda}_{6j+1}^{\phi_{l-1-6j}}}{\widehat{\mu}_{6j+1}^{\phi_{l-1-6j}}}\right\}\left\{\prod_{j=0}^{\xi_0(0-1)}\frac{\widehat{\lambda}_{6j+2}^{\phi_{l-2-6j}}}{\widehat{\mu}_{6j+2}^{\phi_{l-2-6j}}}\right\}.\end{aligned}$$

For $l = 0$*, the corresponding expressions are derived as follows:*

$$\begin{aligned}\lambda_0^2 = &\frac{(\mu_{-4}\lambda_{-4})^{F_3}}{(\mu_{-3}\lambda_{-3})^{F_4}}\Gamma_{0,0}^{(0)}\left(\widehat{\lambda},\widehat{\mu}\right)\widehat{\Gamma}_{0,0}^{(0)}\left(\widehat{\lambda},\widehat{\mu}\right)\frac{\lambda_{-3}^{\phi_3}}{\mu_{-3}^{\phi_3}}\frac{\lambda_{-4}^{\psi_4}}{\mu_{-4}^{\psi_4}},\\ \mu_0^2 = &\frac{(\mu_{-4}\lambda_{-4})^{F_3}}{(\mu_{-3}\lambda_{-3})^{F_4}}\Gamma_{0,0}^{(0)}\left(\widehat{\mu},\widehat{\lambda}\right)\widehat{\Gamma}_{0,0}^{(0)}\left(\widehat{\mu},\widehat{\lambda}\right)\frac{\mu_{-3}^{\phi_3}}{\lambda_{-3}^{\phi_3}}\frac{\mu_{-4}^{\psi_4}}{\lambda_{-4}^{\psi_4}},\end{aligned}$$

where

$$\begin{aligned}\Gamma_{0,0}^{(0)}\left(\widehat{\lambda},\widehat{\mu}\right) = \Gamma_{0,0}^{(0)}\left(\widehat{\mu},\widehat{\lambda}\right) \quad &= \left(\widehat{\lambda}_{-1}\widehat{\mu}_{-1}\right)^{F_4}\left(\widehat{\lambda}_{-2}\widehat{\mu}_{-2}\widehat{\lambda}_0\widehat{\mu}_0\right)^{-F_5},\\ \widehat{\Gamma}_{0,0}^{(0)}\left(\widehat{\lambda},\widehat{\mu}\right) = 1/\widehat{\Gamma}_{0,0}^{(0)}\left(\widehat{\mu},\widehat{\lambda}\right) \quad &= \frac{\widehat{\lambda}_{-2}^{\phi_2}}{\widehat{\mu}_{-2}^{\phi_2}}\frac{\widehat{\lambda}_{-1}^{\phi_1}}{\widehat{\mu}_{-1}^{\phi_1}}\frac{\widehat{\lambda}_0^{\phi_0}}{\widehat{\mu}_0^{\phi_0}},\end{aligned}$$

with

$$\begin{aligned}\widehat{\lambda}_{-2} &= \frac{\mu_{-4}}{\lambda_{-2}\mu_{-3}} + \frac{\beta}{1-\alpha}, \widehat{\mu}_{-2} = \alpha\frac{\lambda_{-4}}{\mu_{-2}\lambda_{-3}} + \beta + \frac{\beta}{1-\alpha},\\ \widehat{\lambda}_{-1} &= \frac{\mu_{-3}}{\lambda_{-1}\mu_{-2}} + \frac{\beta}{1-\alpha}, \widehat{\mu}_{-1} = \alpha\frac{\lambda_{-3}}{\mu_{-1}\lambda_{-2}} + \beta + \frac{\beta}{1-\alpha},\\ \widehat{\lambda}_{0} &= \frac{\mu_{-2}}{\lambda_{0}\mu_{-1}} + \frac{\beta}{1-\alpha}, \widehat{\mu}_{0} = \alpha\frac{\lambda_{-2}}{\mu_{0}\lambda_{-1}} + \beta + \frac{\beta}{1-\alpha}.\end{aligned}$$

For $l = 1$, the corresponding expressions are derived as follows:

$$\begin{aligned}\lambda_1^2 &= \frac{(\mu_{-3}\lambda_{-3})^{F_5}}{(\mu_{-4}\lambda_{-4})^{F_4}}\Gamma_{0,0}^{(1)}\left(\widehat{\lambda},\widehat{\mu}\right)\widehat{\Gamma}_{0,0}^{(1)}\left(\widehat{\lambda},\widehat{\mu}\right)\frac{\lambda_{-3}^{\phi_4}}{\mu_{-3}^{\phi_4}}\frac{\lambda_{-4}^{\psi_5}}{\mu_{-4}^{\psi_5}},\\ \mu_1^2 &= \frac{(\mu_{-3}\lambda_{-3})^{F_5}}{(\mu_{-4}\lambda_{-4})^{F_4}}\Gamma_{0,0}^{(1)}\left(\widehat{\mu},\widehat{\lambda}\right)\widehat{\Gamma}_{0,0}^{(1)}\left(\widehat{\mu},\widehat{\lambda}\right)\frac{\mu_{-3}^{\phi_4}}{\lambda_{-3}^{\phi_4}}\frac{\mu_{-4}^{\psi_5}}{\lambda_{-4}^{\psi_5}},\end{aligned}$$

where

$$\begin{aligned}\Gamma_{0,0}^{(1)}\left(\widehat{\lambda},\widehat{\mu}\right) = \Gamma_{0,0}^{(1)}\left(\widehat{\mu},\widehat{\lambda}\right) &= \frac{\left(\widehat{\lambda}_{-2}\widehat{\mu}_{-2}\widehat{\lambda}_{0}\widehat{\mu}_{0}\right)^{F_6}}{\left(\widehat{\lambda}_{-3}\widehat{\mu}_{-3}\widehat{\lambda}_{-1}\widehat{\mu}_{-1}\widehat{\lambda}_{1}\widehat{\mu}_{1}\right)^{F_5}},\\ \widehat{\Gamma}_{0,0}^{(1)}\left(\widehat{\lambda},\widehat{\mu}\right) = 1/\widehat{\Gamma}_{0,0}^{(1)}\left(\widehat{\mu},\widehat{\lambda}\right) &= \frac{\widehat{\lambda}_{-2}^{\phi_3}}{\widehat{\mu}_{-2}^{\phi_3}}\frac{\widehat{\lambda}_{-1}^{\phi_2}}{\widehat{\mu}_{-1}^{\phi_2}}\frac{\widehat{\lambda}_{0}^{\phi_1}}{\widehat{\mu}_{0}^{\phi_1}}\frac{\widehat{\lambda}_{1}^{\phi_0}}{\widehat{\mu}_{1}^{\phi_0}},\end{aligned}$$

with

$$\begin{aligned}\widehat{\lambda}_{-3} &= \frac{\mu_{-5}}{\lambda_{-3}\mu_{-4}} + \frac{\beta}{1-\alpha}, \widehat{\mu}_{-3} = \alpha\frac{\lambda_{-5}}{\mu_{-3}\lambda_{-4}} + \beta + \frac{\beta}{1-\alpha},\\ \widehat{\lambda}_{1} &= \frac{\lambda_{-4}}{\mu_{-2}\lambda_{-3}} + \frac{\beta}{1-\alpha}, \widehat{\mu}_{1} = \alpha\frac{\mu_{-4}}{\lambda_{-2}\mu_{-3}} + \beta + \frac{\beta}{1-\alpha}.\end{aligned}$$

By performing the calculations under the specified initial conditions, we obtain the following numerical values:

The results in Table 2 confirm that the closed-form expressions closely match the numerically computed values. The minor deviations observed are solely due to floating-point rounding errors. Consequently, this example provides a concrete verification of the analytical formulas.

Table 2. Numerical comparison between the iterative evaluation of system (1.3) and the corresponding closed-form solution.

	System (1.3)	The closed form
λ_0	1.9824	1.9832
μ_0	3.7736	3.7727
λ_1	0.0841	0.0834
μ_1	0.2193	0.2201

Based on the initial conditions in Table 1, the trajectories are plotted in Figures 1 and 2.

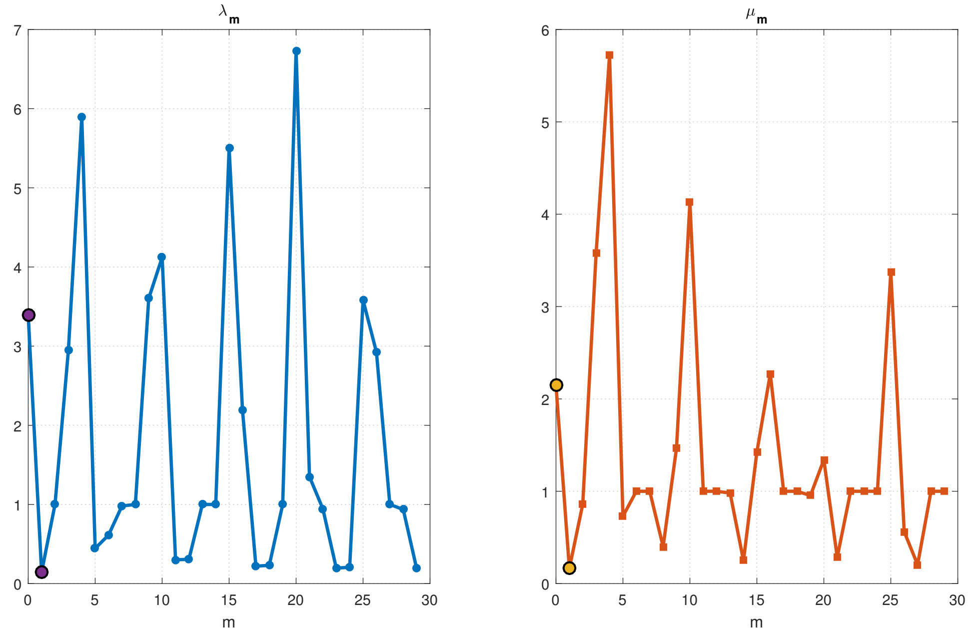

Figure 1. Trajectories of Example (3.1)'s system.

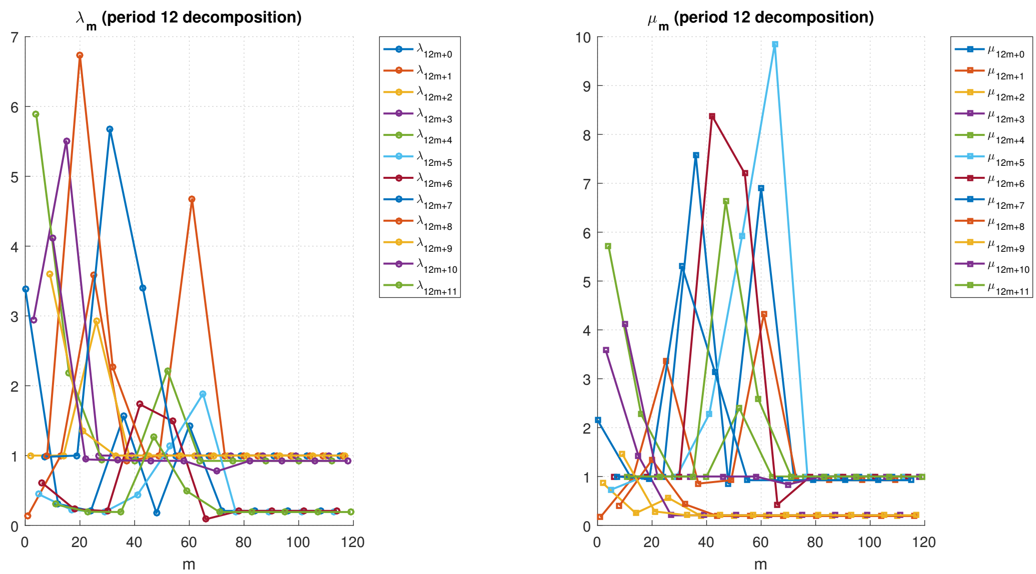

Figure 2. Period-12 decomposition of Example (3.1)'s system.

Figure 1 illustrates the trajectories of the sequences generated by the system in Example 3.1. The values display noticeable initial fluctuations before any long-term patterns emerge. The behavior remains bounded, indicating the presence of recurring dynamics that may correspond to periodic or quasi-periodic cycles. Figure 2 presents the decomposition of the same system into a 12-step periodic structure. The trajectories reveal that the values begin to repeat consistently after twelve iterations, *confirming the existence of a stable cycle. This decomposition highlights the recurring patterns and dominant frequencies that characterize the system's behavior.*

Example 3.2. *Consider system (2.7), defined by the sequence* (λ_m, μ_m)*, where each successive term is generated recursively from preceding values through nonlinear relations governed by the transformation* $G(x) = \log(x)$ *and its inverse* $G^{-1}(y) = \exp(y)$*. The system is controlled by the coefficients* $\alpha = 0.005$, $\beta = 0.003$*, with the exponent fixed at* $r = 1$*. The initial positive conditions are provided in Table 3 below:*

Table 3. Initial conditions of the system.

j	1	2	3	4	5
λ_{-j}	0.8	1.2	0.9	1.1	1.05
μ_{-j}	1.0	0.9	1.1	0.95	1.05

Using these initial values, the trajectories of the sequences are obtained, as displayed in Figures 3 and 4.

Figure 3 illustrates the trajectories generated by the system in Example 3.2, which incorporates a logarithmic transformation. The sequences display small oscillations around an approximately constant mean value. Compared to Example 3.1, the amplitude of oscillations is much smaller, reflecting the stabilizing effect of the logarithmic transformation, which compresses variations and renders the behavior more regular and predictable. Figure 4 presents the cyclic decomposition of the logarithmic system. The curves indicate that the system settles into a stable 12-step cycle, but with a reduced oscillation range relative to the first example. This demonstrates how the logarithmic transformation mitigates long-term fluctuations while preserving the system's inherent cyclic nature.

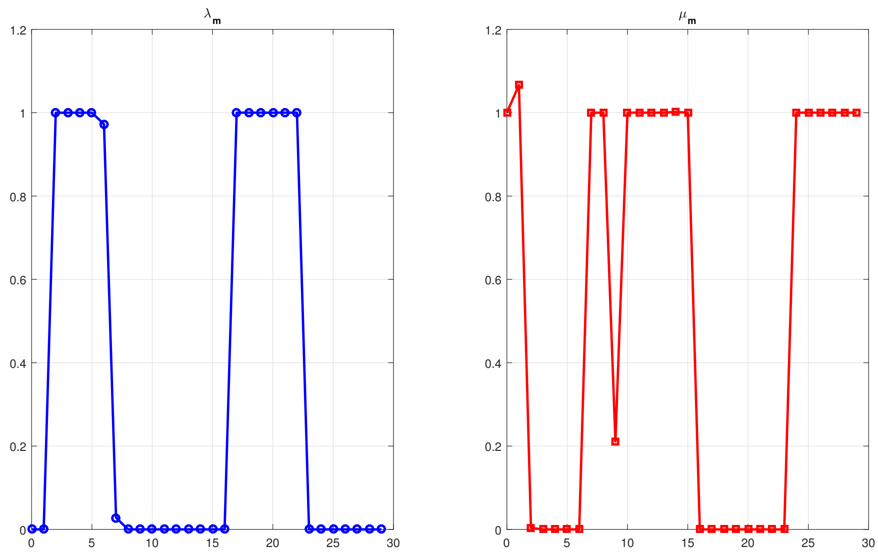

Figure 3. Trajectories of Example (3.2)'s system.

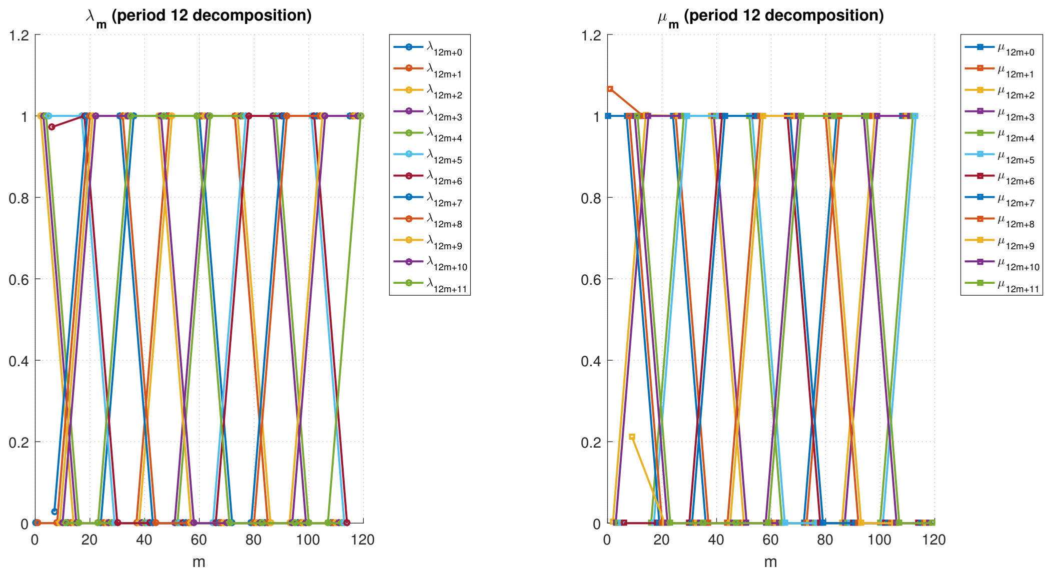

Figure 4. Period-12 decomposition of Example (3.2)'s system.

Example 3.3. *Consider system (2.7), defined by the sequence* (λ_m, μ_m)*, where each successive term is generated recursively from preceding values through nonlinear relations governed by the transformation* $G(x) = x^s$ *and its inverse* $G^{-1}(y) = y^{(1/s)}$*. The system is controlled by the coefficients* $\alpha = 0.005$, $\beta = 0.003$, *with the exponents fixed at* $r = 1$ *and* $s = 2$*. The initial positive conditions are provided in Table 2. Based on these values, the trajectories of the sequences are obtained, as displayed in Figures 5 and 6.*

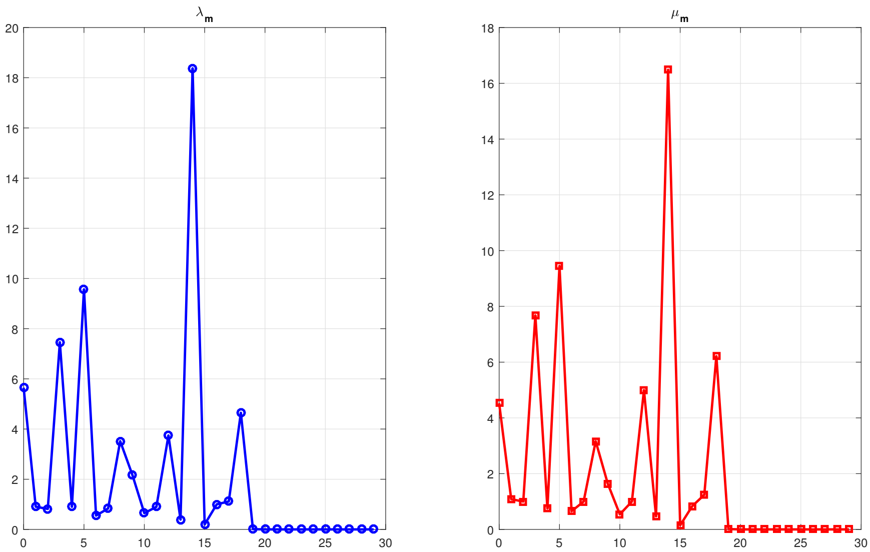

Figure 5. Trajectories of Example (3.3)'s system.

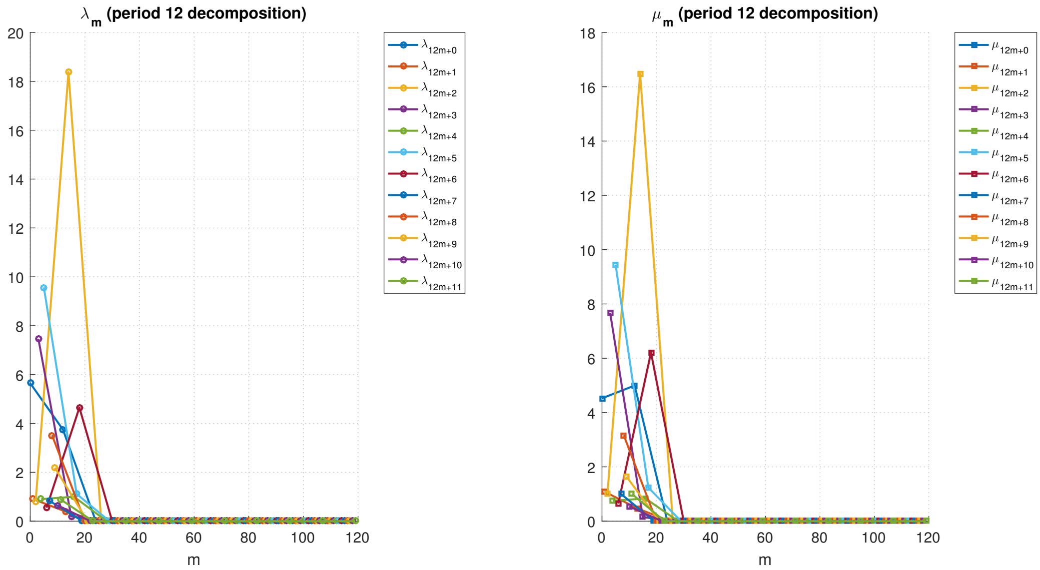

Figure 6. Period-12 decomposition of Example (3.3)'s system.

Figure 5 depicts the trajectories generated by the power transformation in Example 3.3. The values exhibit pronounced oscillations with significantly larger amplitudes compared to the previous two examples. This suggests that the transformation amplifies variations, rendering the system less stable and more sensitive to initial conditions and parameter choices. Figure 6 presents the periodic decomposition of the same system, where a 12-step repeating structure is observed, albeit with relatively large and irregular amplitudes. This indicates that although the system retains a cyclic pattern, it is dominated by strong oscillations. These results clearly demonstrate how nonlinear transformations can induce profound differences in the dynamical behavior of systems, even under identical conditions.

4. Conclusions

In this paper, we have investigated and analyzed an extended system of nonlinear difference equations, with a particular focus on deriving closed-form and periodic solutions through systematic variable transformations. By converting the original nonlinear framework into an equivalent linear form, we successfully obtained explicit and analytically tractable solutions. The findings reveal that the system exhibits a periodic structure with a period of 12, establishing a strong connection between the system's behavior and the generalized Fibonacci sequence. Moreover, by introducing nonlinear transformation functions, we generalized the framework to encompass more complex systems while retaining the core mathematical properties of the original formulation. The choice of transformation function was shown to significantly influence the stability and dynamical characteristics of the solutions, thereby enhancing the model's flexibility and potential for real-world applications.

The main contribution of this paper is the provision of a comprehensive theoretical framework for understanding complex nonlinear systems, while developing effective analytical techniques for solving these systems. The results obtained provide valuable insights into the dynamics of nonlinear systems and their applications in diverse fields such as biological modeling, engineering, and the physical sciences.

In the future, this research could be extended to include the study of more complex nonlinear systems, applying transformation techniques to higher-dimensional systems. The practical applications of these results could also be explored in areas such as predicting dynamic behavior in biological systems or improving the design of engineering systems.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [KFU253865].

Conflict of interest

The author declares no conflicts of interest in this paper.

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