AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25227–25252.
DOI:10.3934/math.20251117
Received: 02 July 2025
Revised: 05 October 2025
Accepted: 22 October 2025
Published: 04 November 2025

Research article

A comprehensive analysis of Riemann-Liouville fractional multiplicative integral inequalities

Mhamed Eddahbi[1,*], Abdelghani Lakhdari[2,3], Badreddine Meftah[4], Lassaad Mchiri[5] and Mohamed Rhaima[6]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Z.C. 11451, Riyadh, Saudi Arabia
[2] Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Umuttepe Campus, Kocaeli 41001, Türkiye
[3] Department CPST, National Higher School of Technology and Engineering, Annaba 23005, Algeria
[4] Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, Guelma 24000, Algeria
[5] Panthéon-Assas University Paris II, 92 Rue d'Assas, 75006 Paris, France
[6] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: meddahbi@ksu.edu.sa

Abstract: In this article, we explored a comprehensive class of quadrature formulas characterized by a bi-parametric expression via the concept of multiplicative (s, P)-convexity. Inspired by prior works in this field, we investigated formulas with varying points (ranging from 1 to 4) and established associated fractional multiplicative inequalities for functions whose multiplicative first-order derivatives exhibit multiplicative (s, P)-convexity.

Keywords: multiplicative calculus; Newton-Cotes-type inequalities; multiplicative (s, P)-convex functions; fractional calculus
Mathematics Subject Classification: 26A51, 26D10, 26D15

1. Introduction

The 17th century witnessed a monumental shift in mathematics with the independent discoveries of differential and integral calculus by Isaac Newton and Gottfried Wilhelm Leibniz. These discoveries laid the foundation for modern calculus, enabling the analysis of rates of change and accumulation of quantities through the operations of differentiation and integration. Differentiation elucidates infinitesimal changes and growth rates, while integration encapsulates the summation of quantities over intervals.

In the latter part of the 20th century, a profound departure from conventional calculus emerged. Grossman and Katz presented an innovative framework [14], challenging the conventional arithmetic operations of addition and subtraction. Their revolutionary approach introduced multiplicative calculus, a distinct system that redefined the essence of calculus. Within this framework, infinitesimal operations are framed in terms of multiplication and division, offering a fresh perspective on mathematical analysis.

Notably, multiplicative calculus finds its application exclusively within the domain of positive functions, defining a distinct realm of operation. As a result, its applicability is more restricted compared to the broader scope of Newtonian calculus. Despite this limitation, multiplicative calculus has unearthed intriguing and profound results across diverse domains [7, 22, 26]. Its inception in the 1970's marked a departure from classical calculus, ushering in a new era of mathematical exploration.

As a profound consequence of introducing multiplicative calculus, a new concepts emerged: the multiplicative derivative and integral. Unlike classical calculus, where differentiation focuses on quantifying the rate of change, the multiplicative derivative, denoted as ξ^*, explores the relative growth of a function. This perspective allows us to measure how a function grows concerning its own values, marking a fundamental shift in our understanding of mathematical analysis.

Definition 1.1 (cf. [6])**.** *The multiplicative derivative ($*$-derivative) of a positive function ξ defined on $\mathbb{R}$, denoted by ξ^*, and evaluated at the point u_0, is defined by*

$$\frac{d^*\xi}{du}(u_0) = \xi^*(u_0) = \lim_{k\to 0}\left(\frac{\xi(u_0+k)}{\xi(u_0)}\right)^{\frac{1}{k}}.$$

The following relationship between the multiplicative and the classical derivatives can be established as follows:

$$\xi^*(u) = \exp\{(\ln \xi(u))'\} = \exp\left\{\frac{\xi'(u)}{\xi(u)}\right\}.$$

From this representation, we can deduce that if ξ has positive values and is differentiable, then ξ^* exists.

In a similar manner, the concept of the multiplicative integral, represented by $\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du}$ was introduced by Bashirov et al. [6], allowing accumulation to be computed through multiplication. This presents an alternative viewpoint on accumulation, diverging from the conventional additive method.

The relationship between the Riemann integral and the multiplicative integral can be summarized as follows:

$$\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du} = \exp\left\{\int_{\varpi_1}^{\varpi_2}\ln(\xi(u))\,du\right\}.$$

From this representation, we can also deduce that if ξ has positive values and the function $\ln(\xi)$ is Riemann integrable on $[\varpi_1, \varpi_2]$, then ξ is multiplicative integrable on $[\varpi_1, \varpi_2]$.

Integral inequalities play a fundamental role within the broader field of inequality theory in mathematics. These inequalities provide relationships between integrals of functions and play a fundamental role in analyzing the properties and behaviors of various mathematical expressions. Integral inequalities come in various forms and are used to establish bounds and estimates for integrals involving different functions.

Convexity is also a fundamental and foundational concept in the development of integral inequalities. It serves a critical role, enabling the formulation and proof of these key inequalities. Convex functions are a cornerstone in this process, allowing for the derivation and understanding of these vital inequalities. Numerous extensions and adaptations of the concept of convexity have been cultivated. Yet, within the realm of multiplicative calculus, the most fitting variant is logarithmic convexity, also referred to as multiplicative convexity, defined succinctly as follows:

A positive function ξ is deemed multiplicatively convex over an interval $\mathscr{D}$ if, for all $\varpi_1, \varpi_2 \in \mathscr{D}$ and $t \in [0,1]$, the subsequent inequality holds:

$$\xi\left(t\varpi_1 + (1-t)\,\varpi_2\right) \leq \left[\xi\left(\varpi_1\right)\right]^t \left[\xi(\varpi_2)\right]^{1-t}.$$

Significant research has been conducted in the domain of multiplicative integral inequalities. Initially, in 2019, Ali et al. established the Hermite-Hadamard inequality concerning multiplicative integrals [3]. Subsequently, researchers delved into error bounds for several quadrature formulas. Specifically, the midpoint- and trapezoid-type inequalities were introduced in [17] through the lens of multiplicative convexity. Ali et al. [1] examined Ostrowski and Simpson-type inequalities in the framework of multiplicatively convex functions. Moreover, another independent investigation elaborated on dual Simpson inequalities [20], and Chasreechai et al. [10] explored Simpson's and Newton's type inequalities for the same class of functions. Together, these studies significantly advance our comprehension of inequalities within the domain of multiplicative calculus. Readers interested in further exploration of multiplicative integral inequalities are encouraged to refer to [21, 29, 31].

In 2016, Abdeljawad and Grossman presented the formulation of multiplicative Riemann-Liouville fractional integrals in the following manner:

Definition 1.2 (cf. [2]). *The left- and right-sided multiplicative Riemann-Liouville fractional integral operators of order $\alpha \in \mathbb{C}$, $Re\,(\alpha) > 0$ are defined, respectively, by:*

$$_{\varpi_1}\mathbf{J}_*^\alpha \xi\,(u) = \exp\left\{\mathbf{I}_{\varpi_1^+}^\alpha \left(\ln\circ\xi\right)(u)\right\}, \quad \varpi_1 < u$$

and

$$_*\mathbf{J}_{\varpi_2}^\alpha \xi\,(u) = \exp\left\{\mathbf{I}_{\varpi_2^-}^\alpha \left(\ln\circ\xi\right)(u)\right\}, \quad u < \varpi_2,$$

where $\mathbf{I}_{\varpi_1^+}^\alpha$ and $\mathbf{I}_{\varpi_2^-}^\alpha$ represent the left- and right-sided Riemann-Liouville operators, defined as follows:

$$\mathbf{I}_{\varpi_1^+}^\alpha \psi\,(v) = \frac{1}{\Gamma(\alpha)} \int_{\varpi_1}^{v} (v-u)^{\alpha-1}\, \psi\,(u)\, du$$

and

$$\mathbf{I}_{\varpi_2^-}^\alpha \psi\,(v) = \frac{1}{\Gamma(\alpha)} \int_{v}^{\varpi_2} (u-v)^{\alpha-1}\, \psi\,(u)\, du.$$

First, Budak et al. established the Hermite-Hadamard inequality pertaining to multiplicative Riemann-Liouville fractional integrals [9]. Expanding upon this, in their research presented in [13], Fu et al. delved into an examination of multiplicative tempered fractional integrals, extending the discoveries outlined by Ali et al. in [3] and Budak et al. in [9]. This signifies a significant advancement in understanding the intricacies of fractional integrals in a multiplicative context. Furthermore, within the realm of fractional multiplicative integral inequalities, Moumen et al. [23] were credited with introducing Simpson inequalities, and Boulares et al. [8] showcased Bullen-type inequalities. Additionally, Peng and Du [24] enriched the domain by investigating fractional multiplicative Maclaurin-type inequalities, while Lakhdari et al. [19] presented multiplicative fractional Newton-type inequalities. However, the two pioneering works conducted in this regard are those presented in [5] and [11], where the authors carried out extensive parametric studies and were able to establish noteworthy estimates related to the three-point Newton-Cotes formulas for multiplicative differentiable convex functions and twice-multiplicative differentiable convex functions, respectively. More recently, Zhou and Du [30] presented a parametrized five-point multiplicative fractional inequality via multiplicative convexity. For a comprehensive exploration of the related findings, one can refer to [12, 16, 27] and the referenced literature therein.

Among the various extensions of the notion of multiplicative convexity is the concept of multiplicative s-convexity, introduced by Xi and Qi in [28].

Definition 1.3 (cf. [28]). *A function $\xi : \mathscr{D} \subset \mathbb{R} \to \mathbb{R}^+$ is considered multiplicatively s-convex in the second sense for some fixed $s \in (0,1]$ if, for all $u_1, u_2 \in \mathscr{D}$ and $t \in [0,1]$, the following inequality holds:*

$$\xi\left(tu_1 + (1-t)u_2\right) \leq \left[\xi\left(u_1\right)\right]^{t^s}\left[\xi(u_2)\right]^{(1-t)^s}.$$

In [15], Kadakal defined the concept of multiplicative P-functions as follows:

Definition 1.4 (cf. [25]). *A function $\xi : \mathscr{D} \subset \mathbb{R} \to \mathbb{R}^+$ is considered multiplicatively P-function if, for all $u_1, u_2 \in \mathscr{D}$ and $t \in [0,1]$, the following inequality holds:*

$$\xi\left(tu_1 + (1-t)u_2\right) \leq \xi\left(u_1\right)\xi(u_2).$$

Peng and Du [25] presented the concept of multiplicative (s,P)-convexity in the following way:

Definition 1.5 (cf. [25]). *A function $\xi : \mathscr{D} \subset \mathbb{R} \to \mathbb{R}^+$ is considered multiplicatively (s,P)-convex for some fixed $s \in (0,1]$ if, for all $u_1, u_2 \in \mathscr{D}$ and $t \in [0,1]$, the following inequality holds:*

$$\xi\left(tu_1 + (1-t)u_2\right) \leq \left[\xi\left(u_1\right)\xi(u_2)\right]^{t^s+(1-t)^s}.$$

Proposition 1.1 (cf. [25]). *It holds that any multiplicative P-function qualifies as a multiplicative (s,P)-convex function. Moreover, if a function $\xi : \mathscr{D} \to (1,+\infty)$ is multiplicative s-convex, then it is necessarily multiplicative (s,P)-convex as well.*

The authors in [25] provided the following version of the Hermite-Hadamard inequality for multiplicative (s,P)-convex functions via Riemann-Liouville fractional multiplicative integrals.

Theorem 1.1 (cf. [25]). *Assume that the function $\xi : [\varpi_1, \varpi_2] \to \mathbb{R}^+$ is multiplicative (s,P)-convex, then we have*

$$\xi\left(\frac{\varpi_1+\varpi_2}{2}\right) \leq \left[{}_{\varpi_1}\mathbf{J}_*^{\alpha}\xi(\varpi_2) \times {}_*\mathbf{J}_{\varpi_2}^{\alpha}\xi(\varpi_1)\right]^{\frac{2^{1-s}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}} \leq \left[\xi(\varpi_1)\xi(\varpi_2)\right]^{\alpha 2^{2-s}\left(\frac{1}{\alpha+s}+B(\alpha,s+1)\right)},$$

where B denotes the beta function.

Theorem 1.2 (cf. [25]). *Under the same conditions of Theorem 1.1, we have*

$$\xi\left(\tfrac{\varpi_1+\varpi_2}{2}\right) \leq \left[{}_{\left(\frac{\varpi_1+\varpi_2}{2}\right)}\mathbf{J}_*^{\alpha}\xi(\varpi_2) \times {}_*\mathbf{J}^{\alpha}_{\left(\frac{\varpi_1+\varpi_2}{2}\right)}\xi(\varpi_1) \right]^{\frac{2^{\alpha+1-s}\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}} \leq \left[\xi(\varpi_1)\xi(\varpi_2)\right]^{\alpha 4^{1-s}\left(\frac{1}{\alpha+s}+\Theta(\alpha,s)\right)},$$

where

$$\Theta(\alpha, s) = \int_0^1 t^{\alpha-1}(2-t)^s \, dt.$$

In the same paper, the authors conducted a parametrized analysis of fractional multiplicative Newton-type inequalities for $*$-differentiable (s, P)-convex functions.

Theorem 1.3 (cf. [25]). *Let $\xi : [\varpi_1, \varpi_2] \to \mathbb{R}^+$ be $*$-increasing and $*$-differentiable mapping on $[\varpi_1, \varpi_2]$ with $\varpi_1 < \varpi_2$. If ξ^* is multiplicative (s, P)-convex on $[\varpi_1, \varpi_2]$, then for all $\ell_1, \ell_2, \ell_3 \geq 0$, we have*

$$\left| (\xi(\varpi_1)\xi(\varpi_2))^{1+\ell_1-\ell_3} \left(\xi\left(\frac{2\varpi_1+\varpi_2}{3}\right)\xi\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right)^{\ell_3-\ell_1} \left[{}_{\varpi_1}\mathbf{J}_*^{\alpha}\xi(\varpi_2) \times {}_*\mathbf{J}^{\alpha}_{\varpi_2}\xi(\varpi_1)\right]^{-\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^{\alpha}}} \right|_*$$
$$\leq (\xi^*(\varpi_1)\xi^*(\varpi_2))^{2(\sigma_2-\sigma_1)[\mathcal{A}_1(\ell_1,\alpha,s)+\mathcal{A}_2(\ell_2,\alpha,s)+\mathcal{A}_3(\ell_3,\alpha,s)]},$$

with $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ being defined as in Theorem 4.3 from [25].

Although numerous fractional multiplicative integral inequalities have been established via multiplicative convexity, the literature lacks results concerning the class of multiplicative (s, P)-convex functions. In this paper, inspired by the aforementioned works, particularly the study conducted by Peng and Du in [25], we investigate a broader class of 1-, 2-, 3-, and 4-point formulas, characterized for $\sigma \in [0, 1]$ and $\varkappa \in [\varpi_1, \frac{\varpi_1+\varpi_2}{2}]$ by the following bi-parametric expression:

$$\mathcal{L}(\varpi_1, \varkappa, \varpi_2; \xi) = (\xi(\varpi_1)\xi(\varpi_2))^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}} (\xi(\varkappa)\xi(\varpi_1+\varpi_2-\varkappa))^{\frac{(\varpi_1+\varpi_2-2\varkappa)+2(1-\sigma)(\varkappa-\varpi_1)}{2(\varpi_2-\varpi_1)}}. \tag{1.1}$$

By introducing a new general integral identity, we establish numerous inequalities associated with different quadrature formulas. This is done for functions whose multiplicative derivatives in multiplicative absolute value are multiplicatively (s, P)-convex. The results obtained represent a generalization and extension of several previously established works as well as some newly established ones. This work differs from prior contributions that assume multiplicative monotonicity by introducing innovative tools such as the multiplicative absolute value, thereby establishing results for a more general class of functions.

The remainder of the paper is organized as follows: In Section 2, we recall key notions related to fractional calculus and multiplicative calculus. Section 3 is devoted to introducing an auxiliary result in the form of a fractional multiplicative integral identity, which serves as the foundation for establishing the multiparametrized fractional multiplicative integral inequality for functions whose first-order multiplicative derivatives are multiplicative (s, P)-convex. In Section 4, we present some special instances derived by choosing specific values of the parameters σ and $\varkappa$, followed by some applications and a numerical example with graphical representations to confirm the validity of our results. Additional results using Hölder and power mean inequality are provided in Section 5. The study is concluded with a summary in Section 6.

2. Preliminaries

In this section, we begin by recalling some fundamental definitions related to fractional calculus, along with the key properties of multiplicative differentiation and integration.

Definition 2.1 (cf. [18]). *The beta function is defined for any complex numbers x_1, x_2 such that $Re(x_1) > 0$ and $Re(x_2) > 0$ by*

$$B(x_1, x_2) = \int_0^1 u^{x_1-1}(1-u)^{x_2-1}\,du = \frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2)},$$

where $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.2 (cf. [18]). *The incomplete beta function is defined for any complex numbers x_1, x_2 such that $Re(x_1) > 0$ and $Re(x_2) > 0$ by*

$$B_{\varpi}(x_1, x_2) = \int_0^{\varpi} u^{x_1-1}(1-u)^{x_2-1}\,du,\ 0 \leq \varpi < 1.$$

Definition 2.3. *[18] For any complex numbers x_1, x_2, x_3, and z such that $Re(x_3) > Re(x_2) > 0$ and $|z| < 1$, the hypergeometric function is defined as follows:*

$${}_2F_1(x_1, x_2, x_3; z) = \frac{1}{B(x_2, x_3-x_2)}\int_0^1 u^{x_2-1}(1-u)^{x_3-x_2-1}(1-zu)^{-x_1}\,du,$$

where $B(\cdot,\cdot)$ is the beta function.

The multiplicative derivative and integral admit the following properties:

Definition 2.4 (The multiplicative absolute value [6]). *Given $\mathfrak{u} \in \mathbb{R}^+ = (0,\infty)$, the multiplicative absolute value of $\mathfrak{u}$, also called $*$-absolute value, is defined by*

$$|\mathfrak{u}|_* = \begin{cases} \mathfrak{u} & \textit{if} \quad \mathfrak{u} \geq 1 \\ \frac{1}{\mathfrak{u}} & \textit{if} \quad 0 < \mathfrak{u} < 1, \end{cases}$$

which can also be written as $\ln|\mathfrak{u}|_* = |\ln(\mathfrak{u})|$, where $|\cdot|$ is the classical absolute value.

Remark 2.1 (The multiplicative triangle inequality). *The $*$-triangle inequality, which is also known as the multiplicative triangle inequality, is expressed as*

$$|\mathfrak{u}\mathfrak{v}|_* \leq |\mathfrak{u}|_* |\mathfrak{v}|_*,$$

for $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}^+$.

Theorem 2.1 (cf. [6]). *Let ξ and ζ be two $*$-differentiable functions, and λ is an arbitrary constant. Then, the functions $\lambda\xi$, $\xi\zeta$, $\xi+\zeta$, ξ/ζ and ξ^{ζ} are $*$-differentiable and*

- $(\lambda\xi)^{*}(u)=\xi^{*}(u)$,
- $(\xi\zeta)^{*}(u)=\xi^{*}(u)\,\zeta^{*}(u)$,
- $(\xi+\zeta)^{*}(u)=\xi^{*}(u)^{\frac{\xi(u)}{\xi(u)+\zeta(u)}}\,\zeta^{*}(u)^{\frac{\zeta(u)}{\xi(u)+\zeta(u)}}$,
- $\left(\frac{\xi}{\zeta}\right)^{*}(u)=\frac{\xi^{*}(u)}{\zeta^{*}(u)}$,
- $\left(\xi^{\zeta}\right)^{*}(u)=\xi^{*}(u)^{\zeta(u)}\,\xi(u)^{\zeta'(u)}$.

Theorem 2.2 (cf. [6]). *Let ξ be a positive and Riemann integrable on $[\varpi_1,\varpi_2]$, then we have*

- $\int_{\varpi_1}^{\varpi_2}\left((\xi(u))^{k}\right)^{du}=\left(\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du}\right)^{k}$,
- $\int_{\varpi_1}^{\varpi_2}(\xi(u)\,\zeta(u))^{du}=\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du}\int_{\varpi_1}^{\varpi_2}(\zeta(u))^{du}$,
- $\int_{\varpi_1}^{\varpi_2}\left(\frac{\xi(u)}{\zeta(u)}\right)^{du}=\frac{\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du}}{\int_{\varpi_1}^{\varpi_2}(\zeta(u))^{du}}$,
- $\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du}=\int_{\varpi_1}^{m}(\xi(u))^{du}\int_{m}^{\varpi_2}(\xi(u))^{du}$, $\varpi_1<m<\varpi_2$,
- $\int_{\varpi_1}^{\varpi_1}(\xi(u))^{du}=1$ *and* $\int_{\varpi_1}^{\varpi_2}(\xi(u))^{du}=\left(\int_{\varpi_2}^{\varpi_1}(\xi(u))^{du}\right)^{-1}$.

Lemma 2.1 (Multiplicative integration by parts [4]). *Let $\xi:[\varpi_1,\varpi_2]\to\mathbb{R}$ be multiplicative differentiable, let $\zeta:[\varpi_1,\varpi_2]\to\mathbb{R}$, and let $\psi:\mathscr{D}\subset\mathbb{R}\to\mathbb{R}$ be two differentiable functions. Then we have*

$$\int_{\varpi_1}^{\varpi_2}\left(\xi^{*}(\psi(u))^{\psi'(u)\zeta(u)}\right)^{du}=\frac{\xi(\psi(\varpi_2))^{\zeta(\varpi_2)}}{\xi(\psi(\varpi_1))^{\zeta(\varpi_1)}}\times\frac{1}{\int_{\varpi_1}^{\varpi_2}\left(\xi(\psi(u))^{\zeta'(u)}\right)^{du}}.$$

3. Main results

This section presents the main finding of our study. We begin by introducing the following notations:

$$\begin{aligned}\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi)=&\left[{}_{*}\mathbf{J}^{\alpha}_{\varkappa}\xi(\varpi_1)\times{}_{(\varpi_1+\varpi_2-\varkappa)}\mathbf{J}^{\alpha}_{*}\xi(\varpi_2)\right]^{(\varkappa-\varpi_1)^{1-\alpha}}\\&\times\left[{}_{\varkappa}\mathbf{J}^{\alpha}_{*}\xi\left(\tfrac{\varpi_1+\varpi_2}{2}\right)\times{}_{*}\mathbf{J}^{\alpha}_{(\varpi_1+\varpi_2-\varkappa)}\xi\left(\tfrac{\varpi_1+\varpi_2}{2}\right)\right]^{\frac{(\varpi_1+\varpi_2-2\varkappa)^{1-\alpha}}{2^{1-\alpha}}},\end{aligned}\tag{3.1}$$

$$\begin{aligned}\Lambda(\alpha,\sigma,s)=&\int_0^1|t^{\alpha}-\sigma|\,(t^{s}+(1-t)^{s})\,dt=\int_0^1|\sigma-(1-t)^{\alpha}|\,(t^{s}+(1-t)^{s})\,dt\\=&\frac{1-2\sigma^{\frac{s+\alpha+1}{\alpha}}}{s+\alpha+1}+2\sigma\frac{\sigma^{\frac{s+1}{\alpha}}-\left(1-\sigma^{\frac{1}{\alpha}}\right)^{s+1}}{s+1}+B(\alpha+1,s+1)-2B_{\sigma^{\frac{1}{\alpha}}}(\alpha+1,s+1)\end{aligned}\tag{3.2}$$

and

$$\Omega(\alpha,\sigma,p)=\int_0^1 |t^\alpha-\sigma|^p\,dt=\int_0^1 |\sigma-(1-t)^\alpha|^p\,dt$$
$$=\frac{\sigma^{p+\frac{1}{\alpha}}}{\alpha}B\left(\frac{1}{\alpha},p+1\right)+\frac{(1-\sigma)^{p+1}}{\alpha(p+1)}\;{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;1-\sigma\right), \tag{3.3}$$

where ${}_2F_1$ is the hypergeometric function.

3.1. Multiplicative fractional identity

In this subsection, we introduce an auxiliary result in the form of a multi-parameter multiplicative fractional integral identity that is essential for establishing our main results.

Lemma 3.1. *Let $\xi:[\varpi_1,\varpi_2]\to\mathbb{R}^+$ be a $*$-differentiable mapping on $[\varpi_1,\varpi_2]$ with $\varpi_1<\varpi_2$. If ξ^* is multiplicative integrable on $[\varpi_1,\varpi_2]$, then we have the following identity for multiplicative fractional integrals*

$$\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)\left(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}$$
$$=\left(\int_0^1\left(\left(\xi^*\left((1-t)\varpi_1+t\varkappa\right)\right)^{t^\alpha-\sigma}\right)^{dt}\right)^{\frac{(\varkappa-\varpi_1)^2}{(\varpi_2-\varpi_1)}}$$
$$\times\left(\int_0^1\left(\left(\xi^*\left((1-t)\varkappa+t\frac{\varpi_1+\varpi_2}{2}\right)\right)^{-(1-t)^\alpha}\right)^{dt}\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}}$$
$$\times\left(\int_0^1\left(\left(\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2}+t\left(\varpi_1+\varpi_2-\varkappa\right)\right)\right)^{t^\alpha}\right)^{dt}\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}}$$
$$\times\left(\int_0^1\left(\left(\xi^*\left((1-t)\left(\varpi_1+\varpi_2-\varkappa\right)+t\varpi_2\right)\right)^{(\sigma-(1-t)^\alpha)}\right)^{dt}\right)^{\frac{(\varkappa-\varpi_1)^2}{(\varpi_2-\varpi_1)}},$$

where $\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)$ is defined as in (1.1)*, $\sigma\in[0,1]$, and $\varkappa\in[\varpi_1,\frac{\varpi_1+\varpi_2}{2}]$.*

Proof. Let

$$\mathscr{L}_1=\left(\int_0^1\left(\left(\xi^*\left((1-t)\varpi_1+t\varkappa\right)\right)^{t^\alpha-\sigma}\right)^{dt}\right)^{\frac{(\varkappa-\varpi_1)^2}{(\varpi_2-\varpi_1)}},$$

$$\mathscr{L}_2=\left(\int_0^1\left(\left(\xi^*\left((1-t)\varkappa+t\frac{\varpi_1+\varpi_2}{2}\right)\right)^{-(1-t)^\alpha}\right)^{dt}\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}},$$

$$\mathscr{L}_3 = \left(\int_0^1 \left(\left(\xi^* \left((1-t) \frac{\varpi_1+\varpi_2}{2} + t(\varpi_1+\varpi_2-\varkappa) \right) \right)^{t^\alpha} \right)^{dt} \right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}}$$

and

$$\mathscr{L}_4 = \left(\int_0^1 \left((\xi^* ((1-t)(\varpi_1+\varpi_2-\varkappa) + t\varpi_2))^{(\sigma-(1-t)^\alpha)} \right)^{dt} \right)^{\frac{(\varkappa-\varpi_1)^2}{(\varpi_2-\varpi_1)}}.$$

Using Lemma 2.1, from $\mathscr{L}_1$, we have

$$\begin{aligned}
\mathscr{L}_1 &= \left(\int_0^1 \left((\xi^* ((1-t)\varpi_1 + t\varkappa))^{t^\alpha-\sigma} \right)^{dt} \right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}} \\
&= \left(\int_0^1 \left((\xi^* ((1-t)\varpi_1 + t\varkappa))^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}(t^\alpha-\sigma)} \right)^{dt} \right) \\
&= \frac{(\xi(\varkappa))^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}}{(\xi(\varpi_1))^{-\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}} \frac{1}{\int_0^1 \left(\xi((1-t)\varpi_1 + t\varkappa)^{\frac{\alpha(\varkappa-\varpi_1)}{\varpi_2-\varpi_1} t^{\alpha-1}} \right)^{dt}} \\
&= \frac{(\xi(\varpi_1))^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}} (\xi(\varkappa))^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}}{\exp\left\{ \int_0^1 \frac{\alpha(\varkappa-\varpi_1)}{\varpi_2-\varpi_1} t^{\alpha-1} \ln \xi((1-t)\varpi_1 + t\varkappa)\, dt \right\}} \\
&= \frac{(\xi(\varpi_1))^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}} (\xi(\varkappa))^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}}{\exp\left\{ \frac{\Gamma(\alpha+1)(\varkappa-\varpi_1)^{1-\alpha}}{\varpi_2-\varpi_1} \left(\frac{1}{\Gamma(\alpha)} \int_{\varpi_1}^{\varkappa} (u-\varpi_1)^{\alpha-1} \ln \xi(u)\, du \right) \right\}} \\
&= (\xi(\varpi_1))^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}} (\xi(\varkappa))^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}} \left(\left({}_*\mathbf{J}_{\varkappa}^{\alpha} \xi \right)(\varpi_1) \right)^{-\frac{(\varkappa-\varpi_1)^{1-\alpha}\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}.
\end{aligned} \tag{3.4}$$

Likewise,

$$\begin{aligned}
\mathscr{L}_2 &= \left(\int_0^1 \left(\left(\xi^* \left((1-t)\varkappa + t\frac{\varpi_1+\varpi_2}{2} \right) \right)^{-(1-t)^\alpha} \right)^{dt} \right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}} \\
&= \left(\int_0^1 \left(\left(\xi^* \left((1-t)\varkappa + t\frac{\varpi_1+\varpi_2}{2} \right) \right)^{-\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}(1-t)^\alpha} \right)^{dt} \right)
\end{aligned}$$

$$
\begin{aligned}
&=\frac{(\xi(\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}}{\int_0^1\left(\left(\xi\left((1-t)\varkappa+t\frac{\varpi_1+\varpi_2}{2}\right)\right)^{\frac{\alpha(\varpi_1+\varpi_2-2\varkappa)}{2(\varpi_2-\varpi_1)}(1-t)^{\alpha-1}}\right)^{dt}}\\
&=\frac{(\xi(\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}}{\exp\left\{\int_0^1\frac{\alpha(\varpi_1+\varpi_2-2\varkappa)}{2(\varpi_2-\varpi_1)}(1-t)^{\alpha-1}\ln\left(\xi\left((1-t)\varkappa+t\frac{\varpi_1+\varpi_2}{2}\right)\right)dt\right\}}\\
&=\frac{(\xi(\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}}{\exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^{1-\alpha}\Gamma(\alpha+1)}{2^{1-\alpha}(\varpi_2-\varpi_1)}\left(\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\frac{\varpi_1+\varpi_2}{2}}\left(\frac{\varpi_1+\varpi_2}{2}-u\right)^{\alpha-1}\ln(\xi(u))\,du\right)\right\}}\\
&=(\xi(\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}\left(\left({}_{\varkappa}\mathbf{J}_{*}^{\alpha}\xi\right)\left(\frac{\varpi_1+\varpi_2}{2}\right)\right)^{-\frac{(\varpi_1+\varpi_2-2\varkappa)^{1-\alpha}\Gamma(\alpha+1)}{2^{1-\alpha}(\varpi_2-\varpi_1)}},
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\mathscr{L}_3&=\left(\int_0^1\left(\left(\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2}+t(\varpi_1+\varpi_2-\varkappa)\right)\right)^{t^\alpha}\right)^{dt}\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}}\\
&=\int_0^1\left(\left(\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2}+t(\varpi_1+\varpi_2-\varkappa)\right)\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}t^\alpha}\right)^{dt}\\
&=\frac{(\xi(\varpi_1+\varpi_2-\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}}{\int_0^1\left(\left(\xi((1-t)\frac{\varpi_1+\varpi_2}{2}+t(\varpi_1+\varpi_2-\varkappa))\right)^{\frac{\alpha(\varpi_1+\varpi_2-2\varkappa)}{2(\varpi_2-\varpi_1)}t^{\alpha-1}}\right)^{dt}}\\
&=\frac{(\xi(\varpi_1+\varpi_2-\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}}{\exp\left\{\frac{\alpha(\varpi_1+\varpi_2-2\varkappa)}{2(\varpi_2-\varpi_1)}\left(\int_0^1 t^{\alpha-1}\ln\left(\xi\left((1-t)\frac{\varpi_1+\varpi_2}{2}+t(\varpi_1+\varpi_2-\varkappa)\right)\right)dt\right)\right\}}\\
&=\frac{(\xi(\varpi_1+\varpi_2-\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}}{\exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^{1-\alpha}\Gamma(\alpha+1)}{2^{1-\alpha}(\varpi_2-\varpi_1)}\left(\frac{1}{\Gamma(\alpha)}\int_{\frac{\varpi_1+\varpi_2}{2}}^{\varpi_1+\varpi_2-\varkappa}\left(u-\frac{\varpi_1+\varpi_2}{2}\right)^{\alpha-1}\ln(\xi(u))\,du\right)\right\}}\\
&=(\xi(\varpi_1+\varpi_2-\varkappa))^{\frac{\varpi_1+\varpi_2-2\varkappa}{2(\varpi_2-\varpi_1)}}\left(\left({}_{*}\mathbf{J}_{(\varpi_1+\varpi_2-\varkappa)}^{\alpha}\xi\right)\left(\frac{\varpi_1+\varpi_2}{2}\right)\right)^{-\frac{(\varpi_1+\varpi_2-2\varkappa)^{1-\alpha}\Gamma(\alpha+1)}{2^{1-\alpha}(\varpi_2-\varpi_1)}}
\end{aligned}
\tag{3.6}
$$

and

$$
\mathscr{L}_4=\left(\int_0^1\left((\xi^*((1-t)(\varpi_1+\varpi_2-\varkappa)+t\varpi_2))^{(\sigma-(1-t)^\alpha)}\right)^{dt}\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}}
$$

$$=\int_0^1\left(\left(\xi^*\left(\left(1-t\right)\left(\varpi_1+\varpi_2-\varkappa\right)+t\varpi_2\right)\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}(\sigma-(1-t)^{\alpha})}\right)^{dt}$$

$$=\frac{\left(\xi\left(\varpi_2\right)\right)^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}\left(\xi\left(\varpi_1+\varpi_2-\varkappa\right)\right)^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}}{\int_0^1\left(\left(\xi\left(\left(1-t\right)\left(\varpi_1+\varpi_2-\varkappa\right)+t\varpi_2\right)\right)^{\frac{\alpha(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}(1-t)^{\alpha-1}}\right)^{dt}}$$

$$=\frac{\left(\xi\left(\varpi_1+\varpi_2-\varkappa\right)\right)^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}\left(\xi\left(\varpi_2\right)\right)^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}}{\exp\left\{\int_0^1\frac{\alpha(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}\left(1-t\right)^{\alpha-1}\ln\left(\xi\left(\left(1-t\right)\left(\varpi_1+\varpi_2-\varkappa\right)+t\varpi_2\right)\right)dt\right\}}$$

$$=\frac{\left(\xi\left(\varpi_1+\varpi_2-\varkappa\right)\right)^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}\left(\xi\left(\varpi_2\right)\right)^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}}{\exp\left\{\frac{(\varkappa-\varpi_1)^{1-\alpha}\Gamma(\alpha+1)}{\varpi_2-\varpi_1}\left(\frac{1}{\Gamma(\alpha)}\int_{\varpi_1+\varpi_2-\varkappa}^{\varpi_2}\left(\varpi_2-u\right)^{\alpha-1}\ln\left(\xi\left(u\right)\right)du\right)\right\}}$$

$$=\left(\xi\left(\varpi_1+\varpi_2-\varkappa\right)\right)^{\frac{(1-\sigma)(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}\left(\xi\left(\varpi_2\right)\right)^{\frac{\sigma(\varkappa-\varpi_1)}{\varpi_2-\varpi_1}}\left(\left({}_{(\varpi_1+\varpi_2-\varkappa)}\mathbf{J}_*^{\alpha}\xi\right)\left(\varpi_2\right)\right)^{-\frac{(\varkappa-\varpi_1)^{1-\alpha}\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}. \tag{3.7}$$

Multiplying (3.4)–(3.7), we get the desired result. Therefore, the statement is proven. □

3.2. Parametrized multiplicative fractional inequalities

Based on the identity introduced above, we are now ready to present our first main result in the form of a multiparameter inequality for $*$-differentiable (s,P)-convex functions via multiplicative Riemann–Liouville fractional integrals.

Theorem 3.1. *Let $\xi:[\varpi_1,\varpi_2]\to\mathbb{R}^+$ be a $*$-differentiable mapping on $[\varpi_1,\varpi_2]$ with $\varpi_1<\varpi_2$. If $|\xi^*|_*$ is multiplicative (s,P)-convex on $[\varpi_1,\varpi_2]$, then for all $\sigma\in[0,1]$ and $\varkappa\in[\varpi_1,\frac{\varpi_1+\varpi_2}{2}]$, we have*

$$\left|\mathcal{L}\left(\varpi_1,\varkappa,\varpi_2;\xi\right)\left(\mathcal{I}\left(\varpi_1,\varkappa,\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(\left|\xi^*\left(\varpi_1\right)\right|_*\left|\xi^*\left(\varpi_2\right)\right|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\Lambda(\alpha,\sigma,s)}\left(\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{2(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)}$$
$$\times\left(\left|\xi^*\left(\varkappa\right)\right|_*\left|\xi^*\left(\varpi_1+\varpi_2-\varkappa\right)\right|_*\right)^{\left[\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\Lambda(\alpha,\sigma,s)+\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)\right]},$$

where $\mathcal{L}$, $\mathcal{I}$, and Λ are defined as (1.1)–(3.2)*, respectively, and B is the beta function.*

Proof. From Lemma 3.1, the $*$-absolute value, and the $*$-triangle inequality, along with properties of the multiplicative integral, we have

$$\left|\mathcal{L}\left(\varpi_1,\varkappa,\varpi_2;\xi\right)\left(\mathcal{I}\left(\varpi_1,\varkappa,\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\exp\left\{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\int_0^1\left|t^{\alpha}-\sigma\right|\left|\ln\left(\xi^*\left(\left(1-t\right)\varpi_1+t\varkappa\right)\right)\right|dt\right\}$$

$$\times \exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\int_0^1 (1-t)^\alpha \left|\ln\left(\xi^*\left((1-t)\varkappa + t\frac{\varpi_1+\varpi_2}{2}\right)\right)\right| dt\right\}$$

$$\times \exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\int_0^1 t^\alpha \left|\ln\left(\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2} + t(\varpi_1+\varpi_2-\varkappa)\right)\right)\right| dt\right\}$$

$$\times \exp\left\{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\int_0^1 |\sigma-(1-t)^\alpha| \left|\ln\left(\xi^*\left((1-t)(\varpi_1+\varpi_2-\varkappa)+t\varpi_2\right)\right)\right| dt\right\}$$

$$= \exp\left\{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\int_0^1 |t^\alpha-\sigma| \ln|\xi^*((1-t)\varpi_1+t\varkappa)|_* \, dt\right\}$$

$$\times \exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\int_0^1 (1-t)^\alpha \ln\left|\xi^*\left((1-t)\varkappa + t\frac{\varpi_1+\varpi_2}{2}\right)\right|_* dt\right\}$$

$$\times \exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\int_0^1 t^\alpha \ln\left|\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2} + t(\varpi_1+\varpi_2-\varkappa)\right)\right|_* dt\right\}$$

$$\times \exp\left\{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\int_0^1 |\sigma-(1-t)^\alpha| \ln|\xi^*((1-t)(\varpi_1+\varpi_2-\varkappa)+t\varpi_2)|_* \, dt\right\}, \tag{3.8}$$

where we have used the fact that $|\ln\xi| = \ln|\xi|_*$.

Using the multiplicative (s,P)-convexity of $|\xi^*|_*$, inequality (3.8) yields

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)\left(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$

$$\leq \exp\left\{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\ln|\xi^*(\varpi_1)|_* + \ln|\xi^*(\varkappa)|_*\right)\int_0^1 |t^\alpha-\sigma|\left(t^s+(1-t)^s\right)dt\right\}$$

$$\times \exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\ln|\xi^*(\varkappa)|_* + \ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)\int_0^1 (1-t)^\alpha\left(t^s+(1-t)^s\right)dt\right\}$$

$$\times \exp\left\{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_* + \ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)\int_0^1 t^\alpha\left(t^s+(1-t)^s\right)dt\right\}$$

$$\times \exp\left\{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_* + \ln|\xi^*(\varpi_2)|_*\right)\int_0^1 |\sigma-(1-t)^\alpha|\left(t^s+(1-t)^s\right)dt\right\}$$

$$= \left(|\xi^*(\varpi_1)|_*\,|\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\Lambda(\alpha,\sigma,s)} \left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{2(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)}$$

$$\times \left(|\xi^*(\varkappa)|_*\,|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\left[\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\Lambda(\alpha,\sigma,s)+\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)\right]},$$

where we have used (3.2), and

$$\int_0^1 t^\alpha \left(t^s + (1-t)^s\right) dt = \int_0^1 (1-t)^\alpha \left(t^s + (1-t)^s\right) dt = \frac{1}{\alpha+s+1} + B(s+1, \alpha+1). \tag{3.9}$$

This completes the proof. □

Corollary 3.1. *If we attempt to take $s = 1$, Theorem 3.1 yields the following bi-parametrized fractional inequalities via multiplicative P-convexity*

$$\begin{aligned}
&\left| \mathcal{L}(\varpi_1, \varkappa, \varpi_2; \xi) \left(\mathcal{I}(\varpi_1, \varkappa, \varpi_2; \xi)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}} \right|_* \\
&\leq \left(|\xi^*(\varpi_1)|_* |\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left[\frac{1-(\alpha+1)\sigma}{\alpha+1} + \frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right]} \left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{2(\varpi_2-\varpi_1)(\alpha+1)}} \\
&\quad \times \left(|\xi^*(\varkappa)|_* |\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\left[\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{1-(\alpha+1)\sigma}{\alpha+1} + \frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right) + \frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)(\alpha+1)}\right]},
\end{aligned}$$

where $\mathcal{L}(\varpi_1, \varkappa, \varpi_2; \xi)$ is defined as in (1.1).

Corollary 3.2. *By choosing $\alpha = 1$, Theorem 3.1 yields the following bi-parametrized inequalities via multiplicative (s, P)-convexity*

$$\begin{aligned}
&\left| \mathcal{L}(\varpi_1, \varkappa, \varpi_2; \xi) \left(\int_{\varpi_1}^{\varpi_2} (\xi(u))^{du} \right)^{-\frac{1}{\varpi_2-\varpi_1}} \right|_* \\
&\leq \left(|\xi^*(\varpi_1)|_* |\xi^*(\varkappa)|_* |\xi^*(\varpi_1+\varpi_2-\varkappa)|_* |\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left[\frac{s+2\sigma^{s+2}+2(1-\sigma)^{s+2}}{(s+1)(s+2)}\right]} \\
&\quad \times \left(|\xi^*(\varkappa)|_* \left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2 |\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)(s+1)}}.
\end{aligned}$$

Moreover, by choosing $s = 1$, we obtain the following bi-parametrized inequalities via P-functions

$$\begin{aligned}
&\left| \mathcal{L}(\varpi_1, \varkappa, \varpi_2; \xi) \left(\int_{\varpi_1}^{\varpi_2} (\xi(u))^{du} \right)^{-\frac{1}{\varpi_2-\varpi_1}} \right|_* \\
&\leq \left(|\xi^*(\varpi_1)|_* |\xi^*(\varkappa)|_* |\xi^*(\varpi_1+\varpi_2-\varkappa)|_* |\xi^*(\varpi_2)|_*\right)^{\frac{\left(1-2\sigma+2\sigma^2\right)(\varkappa-\varpi_1)^2}{2(\varpi_2-\varpi_1)}} \\
&\quad \times \left(|\xi^*(\varkappa)|_* \left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2 |\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{8(\varpi_2-\varpi_1)}}.
\end{aligned}$$

4. Special instances, applications, and numerical validation

This section presents particular cases of the main result, discusses their applications, and includes a numerical example with graphical validation.

4.1. Particular cases

By fixing specific values of the parameters σ and $\varkappa$, the general inequality reduces to several noteworthy particular cases, which we present below.

Corollary 4.1. *By setting $\varkappa = \frac{\varpi_1+\varpi_2}{2}$, Theorem 3.1 yields the following fractional three-point inequality for $*$-differentiable (s, P)-convex functions*

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{\sigma}{2}}\left(\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right)^{1-\sigma}\left(\mathcal{I}\left(\varpi_1,\frac{\varpi_1+\varpi_2}{2},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{4}\Lambda(\alpha,\sigma,s)},$$

where Λ is defined as in (3.2)*, and*

$$\mathcal{L}\left(\omega_1,\frac{\omega_1+\omega_2}{2},\omega_2;\xi\right)=\left[{}_*\mathbf{J}^{\alpha}_{\left(\frac{\omega_1+\omega_2}{2}\right)}\xi(\omega_1)\times{}_{\left(\frac{\omega_1+\omega_2}{2}\right)}\mathbf{J}^{\alpha}_*\xi(\omega_2)\right]^{\frac{(\omega_2-\omega_1)^{1-\alpha}}{2^{1-\alpha}}}.$$

Moreover, if we take $s = 1$, we get the following fractional three-point inequality for $*$-differentiable P functions

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{\sigma}{2}}\left(\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right)^{1-\sigma}\left(\mathcal{I}\left(\varpi_1,\frac{\varpi_1+\varpi_2}{2},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{4}\left[\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right]}.$$

Corollary 4.2. *By taking $\sigma = 0$, Corollary 4.1 yields the following fractional midpoint-type inequality for $*$-differentiable multiplicative (s, P)-convex functions*

$$\left|\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\left(\mathcal{I}\left(\varpi_1,\frac{\varpi_1+\varpi_2}{2},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{4}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)}.$$

Moreover, if we attempt to take $s = 1$, then we get the following fractional midpoint-type inequality for $*$-differentiable P-functions

$$\left|\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\left(\mathcal{I}\left(\varpi_1,\frac{\varpi_1+\varpi_2}{2},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{4(\alpha+1)}}.$$

Corollary 4.3. *By taking $\sigma = 1$, Corollary 4.1 yields the following fractional trapezium-type inequality for $*$-differentiable multiplicative (s, P)-convex functions*

$$\left|\sqrt{\xi(\varpi_1)\xi(\varpi_2)}\left(\mathcal{I}\left(\varpi_1,\frac{\varpi_1+\varpi_2}{2},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$

$$\leq \left(|\xi^*(\varpi_1)|_* \left| \xi^* \left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right|_*^2 |\xi^*(\varpi_2)|_* \right)^{\frac{\varpi_2-\varpi_1}{4}\left(\frac{s+1+2\alpha}{(s+1)(\alpha+s+1)} - B(\alpha+1,s+1) \right)}.$$

Moreover, if we attempt to take $s = 1$, then we get the following fractional trapezium-type inequality for $*$-differentiable P-functions

$$\left| \sqrt{\xi(\varpi_1)\xi(\varpi_2)} \left(\mathcal{I}\left(\varpi_1, \tfrac{\varpi_1+\varpi_2}{2}, \varpi_2; \xi \right) \right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}} \right|_*$$
$$\leq \left(|\xi^*(\varpi_1)|_* \left| \xi^* \left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right|_*^2 |\xi^*(\varpi_2)|_* \right)^{\frac{\alpha(\varpi_2-\varpi_1)}{4(\alpha+1)}}.$$

Corollary 4.4. *By taking $\sigma = \frac{1}{3}$, Corollary 4.1 yields the following fractional Simpson-type inequality for $*$-differentiable multiplicative (s, P)-convex functions*

$$\left| (\xi(\varpi_1)\xi(\varpi_2))^{\frac{1}{6}} \left(\xi\left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right)^{\frac{2}{3}} \left(\mathcal{I}\left(\varpi_1, \tfrac{\varpi_1+\varpi_2}{2}, \varpi_2; \xi \right) \right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}} \right|_*$$
$$\leq \left(|\xi^*(\varpi_1)|_* \left| \xi^* \left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right|_*^2 |\xi^*(\varpi_2)|_* \right)^{\frac{\varpi_2-\varpi_1}{4}\Lambda\left(\alpha,\frac{1}{3},s\right)},$$

where $\Lambda(\alpha, \frac{1}{3}, s)$ is given by :

$$\Lambda\left(\alpha, \tfrac{1}{3}, s\right) = \frac{1 - 2\left(\frac{1}{3}\right)^{\frac{\alpha+s+1}{\alpha}}}{\alpha+s+1} + 2\frac{\left(\frac{1}{3}\right)^{\frac{s+1}{\alpha}} - \left(1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)^{s+1}}{3(s+1)} + B(\alpha+1, s+1) - 2B_{\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}(s+1, \alpha+1).$$

Moreover, if we attempt to take $s = 1$, then we get the following fractional Simpson-type inequality for $*$-differentiable P-functions

$$\left| (\xi(\varpi_1)\xi(\varpi_2))^{\frac{1}{6}} \left(\xi\left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right)^{\frac{2}{3}} \left(\mathcal{I}\left(\varpi_1, \tfrac{\varpi_1+\varpi_2}{2}, \varpi_2; \xi \right) \right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}} \right|_*$$
$$\leq \left(|\xi^*(\varpi_1)|_* \left| \xi^* \left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right|_*^2 |\xi^*(\varpi_2)|_* \right)^{\frac{\varpi_2-\varpi_1}{4}\left(\frac{2-\alpha}{3(\alpha+1)} + \frac{2\alpha}{3(\alpha+1)}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} \right)}.$$

Corollary 4.5. *By taking $\sigma = \frac{1}{2}$, Corollary 4.1 yields the following fractional Bullen-type inequality for $*$-differentiable multiplicative (s, P)-convex functions*

$$\left| (\xi(\varpi_1)\xi(\varpi_2))^{\frac{1}{4}} \left(\xi\left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right)^{\frac{1}{2}} \left(\mathcal{I}\left(\varpi_1, \tfrac{\varpi_1+\varpi_2}{2}, \varpi_2; \xi \right) \right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}} \right|_*$$
$$\leq \left(|\xi^*(\varpi_1)|_* \left| \xi^* \left(\tfrac{\varpi_1+\varpi_2}{2} \right) \right|_*^2 |\xi^*(\varpi_2)|_* \right)^{\frac{\varpi_2-\varpi_1}{4}\Lambda\left(\alpha,\frac{1}{2},s\right)},$$

where $\Lambda(\alpha, \frac{1}{2}, s)$ is given by :

$$\Lambda\left(\alpha, \frac{1}{2}, s\right) = \frac{1 - 2\left(\frac{1}{2}\right)^{\frac{\alpha+s+1}{\alpha}}}{\alpha+s+1} + \frac{\left(\frac{1}{2}\right)^{\frac{s+1}{\alpha}} - \left(1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}}{s+1} + B(\alpha+1, s+1) - 2B_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(\alpha+1, s+1).$$

Moreover, if we attempt to take $s = 1$, then we get the following fractional Bullen-type inequality for $*$-differentiable P-functions

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{1}{4}}\left(\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right)^{\frac{1}{2}}\left(\mathcal{I}\left(\varpi_1,\frac{\varpi_1+\varpi_2}{2},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{8(\alpha+1)}\left(1-\alpha+\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)}.$$

Corollary 4.6. *In Theorem 3.1, if we take $\varkappa = \frac{2\varpi_1+\varpi_2}{3}$, then we get the following fractional Newton-type inequality for $*$-differentiable (s,P)-convex functions*

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{\sigma}{3}}\left(\xi\left(\frac{2\varpi_1+\varpi_2}{3}\right)\xi\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right)^{\frac{3-2\sigma}{6}}\left(\mathcal{I}\left(\varpi_1,\frac{2\varpi_1+\varpi_2}{3},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{9}\Lambda(\alpha,\sigma,s)}\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^{\frac{\varpi_2-\varpi_1}{18}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)}$$
$$\times\left(\left|\xi^*\left(\frac{2\varpi_1+\varpi_2}{3}\right)\right|_*\left|\xi^*\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right|_*\right)^{\frac{\varpi_2-\varpi_1}{36}\left(4\Lambda(\alpha,\sigma,s)+\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)},$$

where $\mathcal{I}(\varpi_1,\frac{2\varpi_1+\varpi_2}{3},\varpi_2;\xi)$ is given by

$$\mathcal{I}\left(\varpi_1,\frac{2\varpi_1+\varpi_2}{3},\varpi_2;\xi\right)=\left[{}_*\mathbf{J}^{\alpha}_{\left(\frac{2\varpi_1+\varpi_2}{3}\right)}\xi(\varpi_1)\times{}_{\left(\frac{\varpi_1+2\varpi_2}{3}\right)}\mathbf{J}^{\alpha}_*\xi(\varpi_2)\right]^{\frac{(\varpi_2-\varpi_1)^{1-\alpha}}{3^{1-\alpha}}}$$
$$\times\left[{}_{\left(\frac{2\varpi_1+\varpi_2}{3}\right)}\mathbf{J}^{\alpha}_*\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\times{}_*\mathbf{J}^{\alpha}_{\left(\frac{\varpi_1+2\varpi_2}{3}\right)}\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right]^{\frac{(\varpi_2-\varpi_1)^{1-\alpha}}{6^{1-\alpha}}}.$$

Moreover, by setting $s = 1$, we get the following fractional Newton-type inequality for $*$-differentiable P- functions

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{\sigma}{3}}\left(\xi\left(\frac{2\varpi_1+\varpi_2}{3}\right)\xi\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right)^{\frac{3-2\sigma}{6}}\left(\mathcal{I}\left(\varpi_1,\frac{2\varpi_1+\varpi_2}{3},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{9}\left[\frac{1-2\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}+\sigma\left(2\sigma^{\frac{1}{\alpha}}-1\right)+\frac{1}{(\alpha+1)(\alpha+2)}-2\left(\frac{\sigma^{\frac{\alpha+1}{\alpha}}}{\alpha+1}-\frac{\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}\right)\right]}\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^{\frac{\varpi_2-\varpi_1}{18(\alpha+1)}}$$
$$\times\left(\left|\xi^*\left(\frac{2\varpi_1+\varpi_2}{3}\right)\right|_*\left|\xi^*\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right|_*\right)^{\frac{\varpi_2-\varpi_1}{36}\left(\frac{4-8\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}+4\sigma\left(2\sigma^{\frac{1}{\alpha}}-1\right)+\frac{\alpha+6}{(\alpha+1)(\alpha+2)}-8\left(\frac{\sigma^{\frac{\alpha+1}{\alpha}}}{\alpha+1}-\frac{\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}\right)\right)}.$$

Corollary 4.7. *In Corollary 4.1, if we take $\sigma = \frac{3}{8}$, then we get the following fractional Simpson $3/8$ inequality for $*$-differentiable (s,P)-convex functions*

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{1}{8}}\left(\xi\left(\frac{2\varpi_1+\varpi_2}{3}\right)\xi\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right)^{\frac{3}{8}}\left(\mathcal{I}\left(\varpi_1,\frac{2\varpi_1+\varpi_2}{3},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq\left(|\xi^*(\varpi_1)|_*|\xi^*(\varpi_2)|_*\right)^{\frac{\varpi_2-\varpi_1}{9}\Lambda\left(\alpha,\frac{3}{8},s\right)}\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^{\frac{\varpi_2-\varpi_1}{18}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)}$$

$$\times\left(\left|\xi^{*}\left(\frac{2\varpi_1+\varpi_2}{3}\right)\right|_{*}\left|\xi^{*}\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right|_{*}\right)^{\frac{\varpi_2-\varpi_1}{36}\left[4\Lambda\left(\alpha,\frac{3}{8},s\right)+\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right]},$$

where $\Lambda\left(\alpha,\frac{3}{8},s\right)$ *is given by*

$$\Lambda\left(\alpha,\frac{3}{8},s\right)=\frac{1-2\left(\frac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}}}{\alpha+s+1}+3\frac{\left(\frac{3}{8}\right)^{\frac{s+1}{\alpha}}-\left(1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}}{4(s+1)}+B(\alpha+1,s+1)-2B_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1).$$

Moreover, if we attempt to take $s=1$*, then we get the following fractional Simpson* $3/8$ *inequality for* $*$*-differentiable P-functions*

$$\left|(\xi(\varpi_1)\xi(\varpi_2))^{\frac{1}{8}}\left(\xi\left(\frac{2\varpi_1+\varpi_2}{3}\right)\xi\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right)^{\frac{3}{8}}\left(\mathcal{I}\left(\varpi_1,\frac{2\varpi_1+\varpi_2}{3},\varpi_2;\xi\right)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_{*}$$

$$\leq\left(|\xi^{*}(\varpi_1)|_{*}|\xi^{*}(\varpi_2)|_{*}\right)^{\frac{\varpi_2-\varpi_1}{9}\left(\frac{1-3\alpha}{4(\alpha+1)}+\frac{3\alpha}{4(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)}\left|\xi^{*}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_{*}^{\frac{\varpi_2-\varpi_1}{18(\alpha+2)}}$$

$$\times\left(\left|\xi^{*}\left(\frac{2\varpi_1+\varpi_2}{3}\right)\right|_{*}\left|\xi^{*}\left(\frac{\varpi_1+2\varpi_2}{3}\right)\right|_{*}\right)^{\frac{\varpi_2-\varpi_1}{36}\left(4\left(\frac{1-3\alpha}{4(\alpha+1)}+\frac{3\alpha}{4(\alpha+1)}\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)+\frac{1}{\alpha+2}\right)}.$$

4.2. Applications to special means

The theoretical framework developed above leads to several direct applications, which we now illustrate. For arbitrary real numbers ϖ_1,ϖ_2, let us consider the following means:

- The arithmetic mean: $A(\varpi_1,\varpi_2)=\frac{\varpi_1+\varpi_2}{2}$.
- The weighted arithmetic mean $A(n,m,\omega_1,\omega_2)=\frac{n\omega_1+m\omega_2}{n+m}$.
- The logarithmic mean: $L(\varpi_1,\varpi_2)=\frac{\varpi_2-\varpi_1}{\ln\varpi_2-\ln\varpi_1}$, $\varpi_1,\varpi_2>0$ and $a\neq b$.
- The p-Logarithmic mean: $L_p(\varpi_1,\varpi_2)=\left(\frac{(\varpi_2)^{p+1}-(\varpi_1)^{p+1}}{(p+1)(\varpi_2-\varpi_1)}\right)^{\frac{1}{p}}$, $\varpi_1,\varpi_2>0,\varpi_1\neq\varpi_2$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 4.1. *Let* $\omega_1,\omega_2\in\mathbb{R}$ *with* $0<\omega_1<\omega_2$ *, then we have*

$$\left|\exp\left\{\frac{2A\left(\omega_1^2,\omega_2^2\right)+A^2(3,1,\omega_1,\omega_2)+A^2(1,3,\omega_1,\omega_2)}{4}-L_2^2(\omega_1,\omega_2)\right\}\right|^{*}\leq\exp\left\{\frac{\omega_2^2-\omega_1^2}{8}\right\}.$$

Proof. The assertion follows from the second inequality of Corollary 3.2 with $\sigma=1$ and $\varkappa=\frac{3\omega_1+\omega_2}{4}$ applied to the function $\xi(u)=\exp\left\{u^2\right\}$, where $\xi^{*}(u)=\exp\{2u\}$ and $\left(\int_{\omega_1}^{\omega_2}(\xi(u))^{du}\right)^{\frac{1}{\omega_1-\omega_2}}=\exp\left\{-L_2^2(\omega_1,\omega_2)\right\}$. □

Proposition 4.2. *Let* $\omega_1,\omega_2\in\mathbb{R}$ *with* $0<\omega_1<\omega_2$ *, then we have*

$$\left|\exp\left\{A^{-1}(\omega_1,\omega_2)-L^{-1}(\omega_1,\omega_2)\right\}\right|^{*}\leq\exp\left\{-\left(\frac{1}{\omega_1^2}+\frac{8}{(\omega_1+\omega_2)^2}+\frac{1}{\omega_2^2}\right)\frac{\omega_2-\omega_1}{8}\right\}$$

Proof. The assertion follows from Corollary 4.1 with $\sigma=0$ and $\alpha=1$, applied to the function $\xi(u)=\exp\left\{\frac{1}{u}\right\}$, where $\xi^{*}(u)=\exp\left\{-\frac{1}{u^2}\right\}$ and $\left(\int_{\omega_1}^{\omega_2}(\xi(u))^{du}\right)^{\frac{1}{\omega_1-\omega_2}}=\exp\left\{-L^{-1}(\omega_1,\omega_2)\right\}$. □

4.3. Numerical example and graphical validation

Now, we provide an example with graphical representation to confirm the accuracy of our results. It is noteworthy that in all figures, the red color corresponds to the right-hand side, and the blue color corresponds to the left-hand side of the respective inequalities.

Example 4.1. *Let us consider the function ξ defined on* $[0,1]$ *as* $\xi(u) = e^{u^{s+1}}$ *for some fixed* $s \in (0,1]$. *The multiplicative derivative is* $\xi^*(u) = |\xi^*(u)|_* = e^{(s+1)u^s}$ *which is multiplicative* (s,P)*-convex over the interval* $[0,1]$. *From Theorem 3.1, we have*

$$\left|\frac{\exp\left\{\sigma\varkappa + \left(\varkappa^{s+1} + (1-\varkappa)^{s+1}\right)\frac{1-2\sigma\varkappa}{2}\right\}}{\exp\left\{\varkappa^{1-\alpha}\left(\int\limits_0^{\varkappa} u^{\alpha+s}du + \int\limits_{1-\varkappa}^{1} u^{s+1}(1-u)^{\alpha-1}du\right) + \frac{(1-2\varkappa)^{1-\alpha}}{2^{1-\alpha}}\left(\int\limits_{\varkappa}^{\frac{1}{2}} u^{s+1}\left(\frac{1}{2}-u\right)^{\alpha-1}du + \int\limits_{\frac{1}{2}}^{1-\varkappa} u^{s+1}\left(u-\frac{1}{2}\right)^{\alpha-1}du\right)\right\}}\right|_*$$
$$\leq \exp\left\{(s+1)\varkappa^2\Lambda(\alpha,\sigma,s) + \frac{(s+1)(1-2\varkappa)^2}{2^{s+1}}\left(\frac{1}{\alpha+s+1} + B(s+1,\alpha+1)\right)\right\}$$
$$\times \exp\left\{(s+1)\left[\varkappa^s + (1-\varkappa)^s\right]\left[\varkappa^2\Lambda(\alpha,\sigma,s) + \frac{(1-2\varkappa)^2}{4}\left(\frac{1}{\alpha+s+1} + B(s+1,\alpha+1)\right)\right]\right\},$$

where $\Lambda(\alpha,\sigma,s)$ *is defined as* (3.2)*, and B is the beta function.*

Furthermore, to facilitate the representation of this result, which depends on four parameters, we will set $s = 1$. *Thus, we obtain the following result:*

$$\left|\frac{\exp\left\{\sigma\varkappa + (2\varkappa^2 - 2\varkappa + 1)\frac{1-2\sigma\varkappa}{2}\right\}}{\exp\left\{\alpha\left[\varkappa^{1-\alpha}\left(\frac{\varkappa^\alpha}{\alpha} - \frac{2\varkappa^{\alpha+1}}{\alpha+1} + \frac{2\varkappa^{\alpha+2}}{\alpha+2}\right) + (1-2\varkappa)^{1-\alpha}\left(\frac{2^{\alpha-2}\left(\frac{1}{2}-\varkappa\right)^\alpha}{\alpha} + \frac{2^\alpha\left(\frac{1}{2}-\varkappa\right)^{\alpha+2}}{\alpha+2}\right)\right]\right\}}\right|_* \leq \exp\left\{4\varkappa^2\left[\frac{1-(\alpha+1)\sigma}{\alpha+1} + \frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right] + \frac{(1-2\varkappa)^2}{2(\alpha+1)}\right\}. \tag{4.1}$$

Since the left-hand side involves an absolute value that depends on the values of σ, α, and $\varkappa$, and in order to be able to graphically represent the results, we write inequality (4.1) *in the following two forms:*

$$\frac{\exp\left\{\sigma\varkappa + (2\varkappa^2 - 2\varkappa + 1)\frac{1-2\sigma\varkappa}{2}\right\}}{\exp\left\{\alpha\left[\varkappa^{1-\alpha}\left(\frac{\varkappa^\alpha}{\alpha} - \frac{2\varkappa^{\alpha+1}}{\alpha+1} + \frac{2\varkappa^{\alpha+2}}{\alpha+2}\right) + (1-2\varkappa)^{1-\alpha}\left(\frac{2^{\alpha-2}\left(\frac{1}{2}-\varkappa\right)^\alpha}{\alpha} + \frac{2^\alpha\left(\frac{1}{2}-\varkappa\right)^{\alpha+2}}{\alpha+2}\right)\right]\right\}} \leq \exp\left\{4\varkappa^2\left[\frac{1-2\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2} + \sigma\left(2\sigma^{\frac{1}{\alpha}} - 1\right) + \frac{1}{(\alpha+1)(\alpha+2)} - 2\left(\frac{\sigma^{\frac{\alpha+1}{\alpha}}}{\alpha+1} - \frac{\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}\right)\right] + \frac{(1-2\varkappa)^2}{2(\alpha+1)}\right\} \tag{4.2}$$

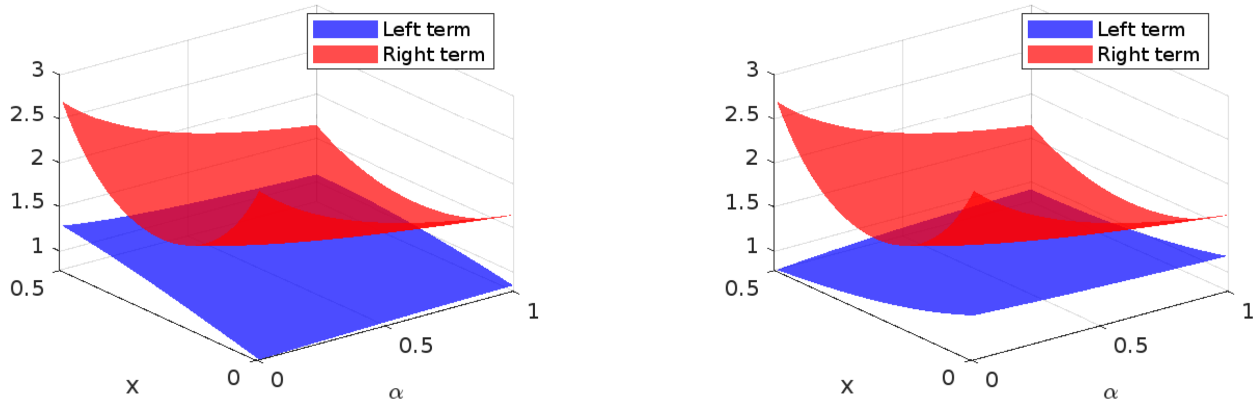
and

$$\frac{\exp\left\{\alpha\left[\varkappa^{1-\alpha}\left(\frac{\varkappa^\alpha}{\alpha} - \frac{2\varkappa^{\alpha+1}}{\alpha+1} + \frac{2\varkappa^{\alpha+2}}{\alpha+2}\right) + (1-2\varkappa)^{1-\alpha}\left(\frac{2^{\alpha-2}\left(\frac{1}{2}-\varkappa\right)^\alpha}{\alpha} + \frac{2^\alpha\left(\frac{1}{2}-\varkappa\right)^{\alpha+2}}{\alpha+2}\right)\right]\right\}}{\exp\left\{\sigma\varkappa + (2\varkappa^2 - 2\varkappa + 1)\frac{1-2\sigma\varkappa}{2}\right\}} \tag{4.3}$$

$$\leq \exp\left\{4\varkappa^2\left[\frac{1-2\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}+\sigma\left(2\sigma^{\frac{1}{\alpha}}-1\right)+\frac{1}{(\alpha+1)(\alpha+2)}-2\left(\frac{\sigma^{\frac{\alpha+1}{\alpha}}}{\alpha+1}-\frac{\sigma^{\frac{\alpha+2}{\alpha}}}{\alpha+2}\right)\right]+\frac{(1-2\varkappa)^2}{2(\alpha+1)}\right\}.$$

Given that this result depends on three parameters, we will consider two specific cases by setting $\sigma = 0$ and then $\varkappa = \frac{1}{2}$, and plot the corresponding results.

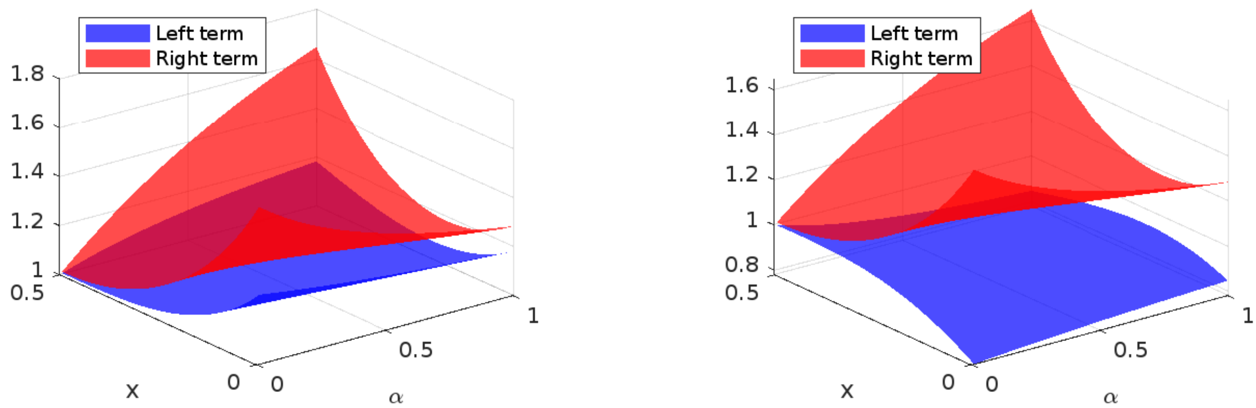
Case 1. *We begin by setting $\sigma = 0$, yielding the companion of Ostrowski's inequality for multiplicative P-functions. This result is visually depicted in Figure 1, where the surface corresponding to the right term is colored red, while that corresponding to the left term is colored blue.*



(a) Inequality (4.2) **(b)** Inequality (4.3)

Figure 1. Case 1: $\alpha \in (0, 1]$, $\varkappa \in [0, \frac{1}{2}]$, $s = 1$ and $\sigma = 0$.

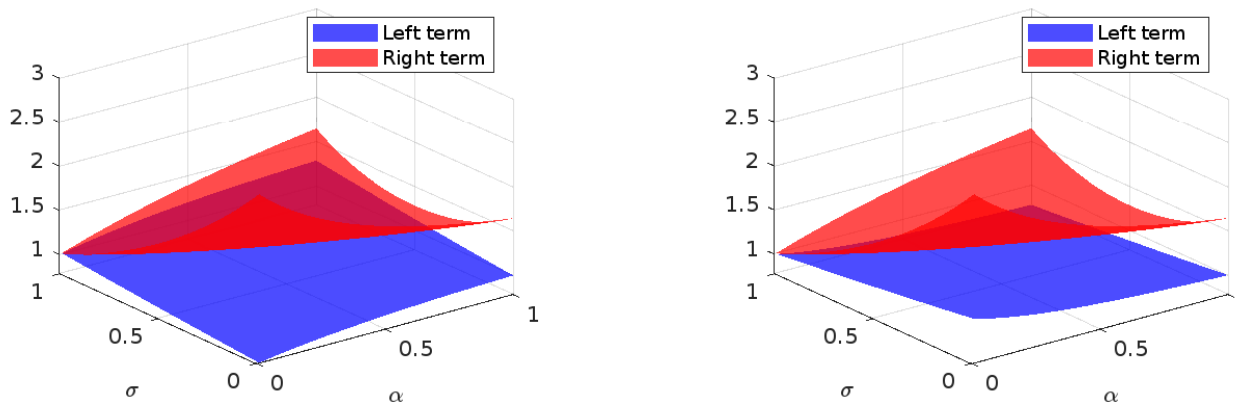
Case 2. *Now, if we attempt to take $\sigma = 1$, it yields the four-point Newton-type inequality for multiplicative P-functions, including Simpson's 3/8 and corrected Simpson's 3/8 inequalities. This result is visually depicted in Figure 2.*



(a) Inequality (4.2) **(b)** Inequality (4.3)

Figure 2. Case 2: $\alpha \in (0, 1]$, $\varkappa \in [0, \frac{1}{2}]$, $s = 1$ and $\sigma = 1$.

Case 3. *Now, we consider the parameters α and σ to be variables while fixing $\varkappa = \frac{1}{2}$. Consequently, we derive the three-point Newton-Cotes-type inequalities via multiplicative P-functions, including* *midpoint, trapezium, Bullen, and Simpson's inequalities, among others. The result is visually presented in Figure 3.*



(a) Inequality (4.2) **(b)** Inequality (4.3)

Figure 3. Case 3: $\alpha \in (0,1]$, $\sigma \in [0,1]$, $s = 1$ and $\varkappa = \frac{1}{2}$.

From the provided figures, we observe that the left-hand side is always greater than the right-hand side for all the cases studied. This validates the accuracy of our results.

5. Additional results

In this section, we provide additional results for functions whose first-order multiplicative derivatives satisfy that $(\ln|\xi^*|)^q$ is (s,P)-convex on $[\varpi_1,\varpi_2]$, where $q>1$.

Theorem 5.1. *Let $\xi : [\varpi_1,\varpi_2] \to \mathbb{R}^+$ be a $*$-differentiable function on $[\varpi_1,\varpi_2]$ with $\varpi_1 < \varpi_2$. If $(\ln|\xi^*|_*)^q$ is (s,P)-convex on $[\varpi_1,\varpi_2]$, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for all $\sigma\in[0,1]$ and $\varkappa \in [\varpi_1, \frac{\varpi_1+\varpi_2}{2}]$, we have*

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)\left(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq \left(|\xi^*(\varpi_1)|_*\,|\xi^*(\varkappa)|_*\,|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\,|\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}(\Omega(\alpha,\sigma,p))^{\frac{1}{p}}}$$
$$\times\left(|\xi^*(\varkappa)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}},$$

where $\mathcal{L}$, $\mathcal{I}$ and Ω are defined as (1.1)*,* (3.1)*, and* (3.3)*, respectively.*

Proof. By using Hölder inequality, inequality 3.8 gives

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)\left(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$=\exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\int_0^1|t^\alpha-\sigma|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\ln|\xi^*((1-t)\varpi_1+t\varkappa)|_*\right)^q dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\int_0^1 (1-t)^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\ln\left|\xi^*\left((1-t)\varkappa + t\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^q dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\ln\left|\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2} + t(\varpi_1+\varpi_2-\varkappa)\right)\right|_*\right)^q dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\int_0^1 |\sigma-(1-t)^\alpha|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (\ln|\xi^*((1-t)(\varpi_1+\varpi_2-\varkappa)+t\varpi_2)|_*)^q\,dt\right)^{\frac{1}{q}}\right). \tag{5.1}$$

Using the (s,P)-convexity of $(\ln|\xi^*|_*)^q$, inequality (5.1) yields

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)\left(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi)\right)^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$

$$\leq \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}(\Omega(\alpha,\sigma,p))^{\frac{1}{p}}\left[(\ln|\xi^*(\varpi_1)|_*)^q+(\ln|\xi^*(\varkappa)|_*)^q\right]^{\frac{1}{q}}\left(\int_0^1((1-t)^s+t^s)\,dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[(\ln|\xi^*(\varkappa)|_*)^q+\left(\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^q\right]^{\frac{1}{q}}\left(\int_0^1((1-t)^s+t^s)\,dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^q+(\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*)^q\right]^{\frac{1}{q}}\left(\int_0^1((1-t)^s+t^s)\,dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}(\Omega(\alpha,\sigma,p))^{\frac{1}{p}}\left[(\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*)^q+(\ln|\xi^*(\varpi_2)|_*)^q\right]^{\frac{1}{q}}\left(\int_0^1((1-t)^s+t^s)\,dt\right)^{\frac{1}{q}}\right)$$

$$\leq \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}(\Omega(\alpha,\sigma,p))^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\ln|\xi^*(\varpi_1)|_*+\ln|\xi^*(\varpi_2)|_*+\ln|\xi^*(\varkappa)|_*+\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right]\right)$$

$$\times \exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left(\ln|\xi^*(\varkappa)|_*+\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2+\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)\right)$$

$$=\left(|\xi^*(\varpi_1)|_*|\xi^*(\varkappa)|_*|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*|\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}(\Omega(\alpha,\sigma,p))^{\frac{1}{p}}}$$

$$\times\left(|\xi^*(\varkappa)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}}.$$

where we have used the fact that $A^q+B^q\leq(A+B)^q$ for $A\geq 0, B\geq 0$ with $q\geq 1$, with $\Omega(\alpha,\sigma,p)$ being defined as in (3.3). This completes the proof. □

Theorem 5.2. *Let $\xi:[\varpi_1,\varpi_2]\to\mathbb{R}^+$ be a $*$-differentiable function on $[\varpi_1,\varpi_2]$ with $\varpi_1<\varpi_2$. If $(\ln|\xi^*|_*)^q$ is (s,P)-convex on $[\varpi_1,\varpi_2]$, where $q>1$, then for all $\sigma\in[0,1]$ and $\varkappa\in[\varpi_1,\frac{\varpi_1+\varpi_2}{2}]$, we* *have*

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi))^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq \left(|\xi^*(\varpi_1)|_*|\xi^*(\varkappa)|_*|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*|\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}(\Lambda(\alpha,\sigma,s))^{\frac{1}{q}}}$$
$$\times\left(|\xi^*(\varkappa)|_*\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)^{\frac{1}{q}}},$$

where $\mathcal{L}$, $\mathcal{I}$, *and* Λ_1 *are defined as* (1.1)–(3.2), *respectively, and B is the beta function.*

Proof. By using power mean inequality, inequality (3.8) gives

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi))^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\int_0^1|t^\alpha-\sigma|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t^\alpha-\sigma|(\ln|\xi^*((1-t)\varpi_1+t\varkappa)|_*)^q\,dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\int_0^1(1-t)^\alpha\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^\alpha\left(\ln\left|\xi^*\left((1-t)\varkappa+t\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^q dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha\left(\ln\left|\xi^*\left((1-t)\frac{\varpi_1+\varpi_2}{2}+t(\varpi_1+\varpi_2-\varkappa)\right)\right|_*\right)^q dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\int_0^1|\sigma-(1-t)^\alpha|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\sigma-(1-t)^\alpha|(\ln|\xi^*((1-t)(\varpi_1+\varpi_2-\varkappa)+t\varpi_2)|_*)^q\,dt\right)^{\frac{1}{q}}\right).\tag{5.2}$$

Using the (s,P)-convexity of $(\ln|\xi^*|_*)^q$, inequality (5.2) gives

$$\left|\mathcal{L}(\varpi_1,\varkappa,\varpi_2;\xi)(\mathcal{I}(\varpi_1,\varkappa,\varpi_2;\xi))^{-\frac{\Gamma(\alpha+1)}{\varpi_2-\varpi_1}}\right|_*$$
$$\leq \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\left[(\ln|\xi^*(\varpi_1)|_*)^q+(\ln|\xi^*(\varkappa|_*))^q\right]^{\frac{1}{q}}\left(\int_0^1(|t^\alpha-\sigma|((1-t)^s+t^s))\,dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[(\ln|\xi^*(\varkappa)|_*)^q+\left(\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^q\right]^{\frac{1}{q}}\left(\int_0^1(1-t)^\alpha((1-t)^s+t^s)\,dt\right)^{\frac{1}{q}}\right)$$
$$\times\exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*\right)^q+(\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*)^q\right]^{\frac{1}{q}}\left(\int_0^1 t^\alpha((1-t)^s+t^s)\,dt\right)^{\frac{1}{q}}\right)$$

$$\times \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\right.$$

$$\left.\times\left[(\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*)^q+(\ln|\xi^*(\varpi_2)|_*)^q\right]^{\frac{1}{q}}\left(\int_0^1|\sigma-(1-t)^\alpha|\left((1-t)^s+t^s\right)dt\right)^{\frac{1}{q}}\right)$$

$$\leq \exp\left(\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}(\Lambda(\alpha,\sigma,s))^{\frac{1}{q}}\right.$$

$$\times\left[\ln|\xi^*(\varpi_1)|_*+\ln|\xi^*(\varkappa)|_*\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*+\ln|\xi^*(\varpi_2)|_*\right]\Big)$$

$$\exp\left(\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)^{\frac{1}{q}}\right.$$

$$\left.\times\left[\ln|\xi^*(\varkappa)|_*+\ln\left|\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|_*^2+\ln|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*\right]\right)$$

$$=\left(|\xi^*(\varpi_1)|_*|\xi^*(\varkappa)|_*|\xi^*(\varpi_1+\varpi_2-\varkappa)|_*|\xi^*(\varpi_2)|_*\right)^{\frac{(\varkappa-\varpi_1)^2}{\varpi_2-\varpi_1}\left(\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}(\Lambda(\alpha,\sigma,s))^{\frac{1}{q}}}$$

$$\times\left(\xi^*(\varkappa)\left(\xi^*\left(\frac{\varpi_1+\varpi_2}{2}\right)\right)^2\xi^*(\varpi_1+\varpi_2-\varkappa)\right)^{\frac{(\varpi_1+\varpi_2-2\varkappa)^2}{4(\varpi_2-\varpi_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}+B(s+1,\alpha+1)\right)^{\frac{1}{q}}},$$

where we have used (3.2), (3.9), and

$$\int_0^1|\sigma-(1-t)^\alpha|\,dt=\int_0^1|\sigma-t^\alpha|\,dt=\frac{1-(\alpha+1)\sigma}{\alpha+1}+\frac{2\alpha}{\alpha+1}\sigma^{\frac{\alpha+1}{\alpha}},$$

and the fact that $A^q+B^q\leq(A+B)^q$ for $A\geq 0, B\geq 0$ with $q\geq 1$.

This completes the proof. □

6. Conclusions

In conclusion, this study delved into an extensive analysis of quadrature formulas employing a bi-parametric approach. By examining a diverse range of formulas utilizing 1 to 4 points, and considering functions with multiplicative (s,P)-convex $*$-derivatives in $*$-absolute value, we derived a set of associated inequalities. The exploration of a new integral identity significantly enhanced our understanding of these formulas. The obtained results not only provide fresh findings but also illuminate potential directions for future research in the domain of fractional multiplicative calculus.

Author contributions

Abdelghani Lakhdari and Badreddine Meftah: Writing original draft, Writing–review & editing, Methodology; Lassaad Mchiri and Mohamed Rhaima: Visualization, Conceptualization, Investigation, Formal analysis; Mhamed Eddahbi: Writing–review & editing, Supervision, Methodology, Formal analysis.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first-named author extends his appreciation to the Ongoing Research Funding Program, (ORF-2025-1075), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. M. A. Ali, H. Budak, M. Z. Sarikaya, Z. Zhang, Ostrowski and Simpson type inequalities for multiplicative integrals, *Proyecciones (Antofagasta Online)*, **40** (2021), 743–763. https://doi.org/10.22199/issn.0717-6279-4136
2. T. Abdeljawad, M. Grossman, On geometric fractional calculus, *J. Semigroup Theory Appl.*, **2016** (2016), 2.
3. M. A. Ali, M. Abbas, Z. Zhang, I. B. Sial, R. Arif, On integral inequalities for product and quotient of two multiplicatively convex functions, *Asian Research Journal of Mathematics*, **12** (2019), 1–11. https://doi.org/10.9734/arjom/2019/v12i330084
4. M. A. Ali, Z. Zhang, H. Budak, M. Z. Sarıkaya, On Hermite-Hadamard type inequalities for interval-valued multiplicative integrals, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **69** (2020), 1428–1448. https://doi.org/10.31801/cfsuasmas.754842
5. M. B. Almatrafi, W. Saleh, A. Lakhdari, F. Jarad, B. Meftah, On the multiparameterized fractional multiplicative integral inequalities, *J. Inequal. Appl.*, **2024** (2024), 52. https://doi.org/10.1186/s13660-024-03127-z
6. A. E. Bashirov, E. M. Kurpınar, A. Özyapıcı, Multiplicative calculus and its applications, *J. Math. Anal. Appl.*, **337** (2008), 36–48. https://doi.org/10.1016/j.jmaa.2007.03.081
7. A. E. Bashirov, E. Mısırlı, Y. Tandoğdu, A. Özyapıcı, On modeling with multiplicative differential equations, *Appl. Math. J. Chin. Univ.*, **26** (2011), 425–438. https://doi.org/10.1007/s11766-011-2767-6
8. H. Boulares, B. Meftah, A. Moumen, R. Shafqat, H. Saber, T. Alraqad, et al., Fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions, *Symmetry*, **15** (2023), 451. https://doi.org/10.3390/sym15020451
9. H. Budak, K. Özçelik, On Hermite-Hadamard type inequalities for multiplicative fractional integrals, *Miskolc Math. Notes*, **21** (2020), 91–99. https://doi.org/10.18514/MMN.2020.3129
10. S. Chasreechai, M. A. Ali, S. Naowarat, T. Sitthiwirattham, K. Nonlaopon, On some Simpson's and Newton's type of inequalities in multiplicative calculus with applications, *AIMS Math.*, **8** (2023), 3885–3896. https://doi.org/10.3934/math.2023193

11. T. Du, Y. Long, The multi-parameterized integral inequalities for multiplicative Riemann-Liouville fractional integrals, *J. Math. Anal. Appl.*, **541** (2025), 128692. https://doi.org/10.1016/j.jmaa.2024.128692

12. T. Du, Y. Peng, Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals, *J. Comput. Appl. Math.*, **440** (2024), 115582. https://doi.org/10.1016/j.cam.2023.115582

13. H. Fu, Y. Peng, T. Du, Some inequalities for multiplicative tempered fractional integrals involving the λ-incomplete gamma functions, *AIMS Math.*, **6** (2021), 7456–7478. https://doi.org/10.3934/math.2021436

14. M. Grossman, R, Katz, *Non-Newtonian calculus: a self-contained, elementary exposition of the authors' investigations*, Newton Institute, 1978.

15. H. Kadakal, Multiplicatively p-functions and some new inequalities, *New Trends in Mathematical Sciences*, **6** (2018), 111–118. https://doi.org/10.20852/ntmsci.2018.321

16. D. Khan, S. I. Butt, Y. Seol, Analysis on multiplicatively (p, m)-superquadratic functions and related fractional inequalities with applications, *Fractals*, **33** (2025), 2450129. https://doi.org/10.1142/S0218348X24501299

17. S. Khan, H. Budak, On midpoint and trapezoid type inequalities for multiplicative integrals, *Mathematica*, **64** (2022), 95–108. https://doi.org/10.24193/mathcluj.2022.1.11

18. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

19. A. Lakhdari, D. C. Benchettah, B. Meftah, Fractional multiplicative newton-type inequalities for multiplicative s-convex positive functions with application, *J. Comput. Appl. Math.*, **465** (2025), 116600. https://doi.org/10.1016/j.cam.2025.116600

20. B. Meftah, A. Lakhdari, Dual Simpson type inequalities for multiplicatively convex functions, *Filomat*, **37** (2023), 7673–7683. https://doi.org/10.2298/FIL2322673M

21. B. Meftah, A. Lakhdari, S. Wedad, C. D. Benchettah, Companion of Ostrowski inequality for multiplicatively convex functions, *Sahand Commun. Math. Anal.*, **21** (2024), 289–304. https://doi.org/10.22130/scma.2023.2002136.1324

22. E. Misirli, A. Ozyapici, Exponential approximations on multiplicative calculus, *Proceedings of the Jangjeon Mathematical Society*, **12** (2009), 227–236. https://doi.org/10.17777/pjms.2009.12.2.009

23. A. Moumen, H. Boulares, B. Meftah, R. Shafqat, T. Alraqad, E. E. Ali, et al., Multiplicatively Simpson type inequalities via fractional integral, *Symmetry*, **15** (2023), 460. https://doi.org/10.3390/sym15020460

24. Y. Peng, T. Du, Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively p-functions, *Filomat*, **37** (2023), 9497–9509. https://doi.org/10.2298/FIL2328497P

25. Y. Peng, T. Du, On multiplicative (s, p)-convexity and related fractional inequalities within multiplicative calculus, *Fractals*, **32** (2024), 2450048. https://doi.org/10.1142/S0218348X24500488

26. M. Riza, A. Özyapici, E. Misirli, Multiplicative finite difference methods, *Quart. Appl. Math.*, **67** (2009), 745–754.
27. Z. Tan, T. Du, Fractional midpoint-type inequalities in multiplicative calculus, *IAENG International Journal of Applied Mathematics*, **55** (2025), 2550–2563.
28. B.-Y. Xi, F. Qi, Some integral inequalities of Hermite-Hadamard type for s-logarithmically convex functions, *Acta Mathematica Scientis, English Series*, **35A** (2015), 515–526.
29. J. Xie, M. A. Ali, T. Sitthiwirattham, Some new midpoint and trapezoidal type inequalities in multiplicative calculus with applications, *Filomat*, **37** (2023), 6665–6675. https://doi.org/10.2298/FIL2320665X
30. Z. Zhou, T. Du, Error estimation for a broad class of fractional multiplicative quadrature formulas, *Exp. Math.*, in press. https://doi.org/10.1080/10586458.2025.2519603
31. W. Zhu, B. Meftah, H. Xu, F. Jarad, A. Lakhdari, On parameterized inequalities for fractional multiplicative integrals, *Demonstr. Math.*, **57** (2024), 20230155. https://doi.org/10.1515/dema-2023-0155