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Research article

Comprehensive study of a (3+1)-dimensional nonlinear Vakhnenko-Parkes dynamical equation with applications in nonlinear wave propagation in relaxing media

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Abstract: This research conducted a study of the nonlinear (3 + 1)-dimensional Vakhnenko-Parkes (VP) equation since it acts as a vital model for high-frequency wave perturbations in relaxing high-rate active barotropic media. The analytical solutions emerged through the modified auxiliary equation method and the improved F-expansion method which serve as advanced tools for studying nonlinear waves. These methods present diverse solutions that contain solitary waves combined with periodic waves and rational forms. The obtained solutions exhibit their behavior through illustrated 2D and 3D plots showing how waves evolve and how their structures transform with varying parameters. This visual analysis shows how solutions disperse and stay stable thus making them relevant for fluid dynamics investigations and wave propagation studies. The analytical understanding of nonlinear wave models in high-frequency barotropic systems receives new insights through our results which enhance mathematical and physical VP equation descriptions.

Keywords: Vakhnenko-Parkes equation; the improved F-expansion method; the modified auxiliary equation method; soliton solution; nonlinear waves
Mathematics Subject Classification: 35B10, 35C07, 35C08, 35Q35

1. Introduction

Nonlinear partial differential equations (NLPDEs) have been capable of simulating the majority of intricate real-life processes, such as wave propagation, plasma oscillations [1, 2], signal transmission in optical fibers [3, 4] and gas evolution in fluid dynamics [5, 6]. Models containing the NLPDEs have produced the superior predictions and had a wide range of applications in nature [7, 8]. The investigation of many numerical and accurate solution approaches [9, 10] has been crucial in our scientific community [11, 12]. Over the past few decades, several effective analytical techniques have been developed to construct traveling wave solutions that describe diverse physical structures such as solitary waves, shock waves, periodic waves, and breather-type excitations. Examples of these methods include: the Backlünd transformation [13–15], inverse scattering method [16], Hirota's bilinear transformation [17, 18], Darboux transformation [19, 20], logarithmic transformation and symbolic computation [21], Lie-symmetry analysis [22], multiple exp-function method [23], extended algebraic method [24], Jacobi elliptic function expansion [25, 26], modified auxiliary equation [27], generalized Kudryashov method [28], extended tanh-function method, simplest equation method [29], sine-Gordon expansion method [30], sub-ODE method [31], separation of variables method [32], and modified extended direct algebraic method [33]. There are just a few of the practical techniques that have been put forth in recent decades to obtain traveling wave solutions with various physical structures [34]. The (3+1) Vakhnenko-Parkes equation (VPE) is a nonlinear partial differential equation (PDE) that arises in the study of wave propagation in nonlinear media, especially in fluid dynamics [35, 36], nonlinear acoustics, and plasma physics [37]. It originally appeared in the context of describing short pulses in a relaxing medium [38]. We outline below the stages of development of the equation under study in the research:

1. Vakhnenko and Parkes in 1992 [39] presented for the first time their equation, which models the propagation of short waves in a relaxing medium, expressed as follows:

$$\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial t}+\mathcal{U}\frac{\partial}{\partial x}\Big)+\mathcal{U}=0. \tag{1.1}$$

2. Victor et al. in 2008 [40], proposed a model equation to describe the relaxation of high-rate processes in active barotropic media, expressed as follows:

$$\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial t}+\mathcal{U}\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)\Big)\mathcal{U}+\mathcal{U}=0. \tag{1.2}$$

3. In 2021, Wazwaz [41] formulated a (3+1)-dimensional Vakhnenko-Parkes equation as follows:

$$\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\Big)\Big(\frac{\partial}{\partial t}+\mathcal{U}\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\Big)\Big)\mathcal{U}+\mathcal{U}=0. \tag{1.3}$$

When we simplify, we get

$$\mathcal{U}_{xxt}+\mathcal{U}_{xxy}+\mathcal{U}_{xxz}+\mathcal{U}_t\mathcal{U}_x+\mathcal{U}_y\mathcal{U}_x+\mathcal{U}_z\mathcal{U}_x+\mathcal{U}_x=0. \tag{1.4}$$

There are numerous effective techniques to construct the VPE solutions, such as: The simplified Hirota method [41], Riccati equation mapping, exponential rational function and generalized

Kudryashov methods [42], modified F-expansion, extended simple equation and generalized direct algebraic methods [43], modified (G'/G) expansion method [44], multiple exp-function method [45], and Hirota bilinear method [46]. The traveling wave solutions of the (3+1) Vakhnenko-Parkes Eq (1.4) are examined in this work using two approaches: The advanced auxiliary equation [47–52] and the improved F-expansion [53–56] methods. To the best of our knowledge, no previous work has applied these two methods to the present equation. Hence, this study is the first to use these techniques to obtain novel analytical solutions. The paper is structured as follows: An overview of the applied approaches is provided in Section 2. In Section 3, the solutions to the Vakhnenko-Parkes equation are obtained using these techniques. The 2D and 3D graphs of the solutions are presented in Section 4. In Section 5, we go over the solutions' physical meaning. Finally, some concluding remarks are made in Section 6.

2. Summary of the implemented methods

2.1. The improved F-expansion method

Theorem 1. Let us consider the nonlinear partial differential equation

$$F(\mathcal{U},\mathcal{U}_t,\mathcal{U}_x,\mathcal{U}_{xx},\mathcal{U}_{xxx},\dots)=0, \tag{2.1}$$

where $\mathcal{U}=\mathcal{U}(x,y,z,t)$ denotes the traveling wave solution to be found, and F represents a polynomial expression in $\mathcal{U}$ and its partial derivatives.

Step 1. The transformation utilized to simplify the partial differential equation (PDE) into an ordinary differential equation (ODE) is expressed as

$$\mathcal{U}(x,y,z,t)=\mathcal{U}(\eta),\quad \eta=kx+my+nz-tw, \tag{2.2}$$

where η denotes the traveling wave variable, the parameters k, m, and n are wave numbers that describe the rate of wave variation along the x, y, and z axes, respectively, while w is the angular frequency, indicating the speed of wave oscillations over time t. By substituting (2.2) into (2.1), the partial differential equation is transformed into the following reduced ordinary differential equation:

$$H(\mathcal{U}',\mathcal{U}'',\mathcal{U}''',\mathcal{U}'''',...)=0, \tag{2.3}$$

where H represents a polynomial in $\mathcal{U}(\eta)$ along with its derivatives.

Step 2. We consider the traveling wave solution of (2.3) to be re-presentable as a polynomial in $G(\eta)$, given by:

$$\mathcal{U}(\eta)=\sum_{i=0}^{N} f_i(G(\eta)+\mu)^i+\sum_{i=1}^{N} Q_i(G(\eta)+\mu)^{-i}, \tag{2.4}$$

where either f_N or Q_N may be zero, but both cannot simultaneously be zero at any time. f_i, Q_i ($i=0,1,2,\dots,N$), and μ are arbitrary constants that will be determined.

Step 3. The positive integer N is determined by equating the highest-order derivative term to the highest-order nonlinear term in (2.3).

Step 4. $G(\eta)$ satisfies the following Riccati equation:

$$G'(\eta)=G(\eta)^2+\rho, \tag{2.5}$$

where the prime denotes differentiation with respect to η, and ρ is a real parameter.
Step 5. The three cases of the general solution for the Riccati equation (2.5) are presented below:
Set I: When $\rho < 0$,

$$\begin{aligned} &G(\eta) = -\sqrt{-\rho}\tanh\left(\sqrt{-\rho}\eta\right) \\ &or \\ &G(\eta) = -\sqrt{-\rho}\coth\left(\sqrt{-\rho}\eta\right). \end{aligned} \tag{2.6}$$

Set II: When $\rho > 0$,

$$\begin{aligned} &G(\eta) = \sqrt{\rho}\tan\left(\sqrt{\rho}\eta\right) \\ &or \\ &G(\eta) = -\sqrt{\rho}\cot\left(\sqrt{\rho}\eta\right). \end{aligned} \tag{2.7}$$

Set III: When $\rho = 0$,

$$G(\eta) = \frac{-1}{\eta}. \tag{2.8}$$

Step 6. By substituting (2.4) into (2.3), a polynomial in $G(\eta)$ is obtained. Grouping terms with the same powers and equating their coefficients to zero results in a system of algebraic equations, which can be solved using the Mathematica software.

2.2. The modified auxiliary equation method (MAE)

The main steps of the modified auxiliary equation method are outlined below:
Theorem 2.
Step 1. The solution to equation (2.3) is expressed as:

$$\mathcal{U}(\eta) = \sum_{i=0}^{N} b_i a^{i\mathfrak{f}(\eta)}, \tag{2.9}$$

where $b_i (i = 0, 1, 2, ..., N), b_N \neq 0$.
The function $\mathfrak{f}(\eta)$ is determined as a solution to the following ordinary differential equation:

$$\mathfrak{f}' = \frac{1}{\ln a}(\mu a^{-\mathfrak{f}(\eta)} + \sigma + \lambda a^{\mathfrak{f}(\eta)}), \tag{2.10}$$

where the constants μ, σ, and λ need to be identified.
Step 2. In (2.3), N represents a positive integer determined through the application of the homogeneous balance principle, as discussed earlier.
Step 3. Substituting (2.9) and (2.10) into (2.3), grouping terms with identical powers of $a^{\mathfrak{f}(\eta)}$, and setting their coefficients to zero results in a system of equations. This system is solved using the Mathematica software.
Step 4. (2.10) has multiple sets of solutions:
Set I: $\sigma^2 - 4\lambda\mu < 0$ and $\lambda \neq 0$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\lambda} + \frac{\sqrt{4\mu\lambda - \sigma^2}}{2\lambda}\tan(\frac{\sqrt{4\mu\lambda - \sigma^2}\eta}{2})\\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.11)\\
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\lambda} + \frac{\sqrt{4\mu\lambda - \sigma^2}}{2\lambda}\cot(\frac{\sqrt{4\mu\lambda - \sigma^2}\eta}{2}).
\end{aligned}
$$

Set II: $\sigma^2 - 4\lambda\mu > 0$ and $\lambda \neq 0$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\lambda} - \frac{\sqrt{\sigma^2 - 4\mu\lambda}}{2\lambda}\tanh(\frac{\sqrt{\sigma^2 - 4\mu\lambda}\eta}{2})\\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.12)\\
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\lambda} - \frac{\sqrt{\sigma^2 - 4\mu\lambda}}{2\lambda}\coth(\frac{\sqrt{\sigma^2 - 4\mu\lambda}\eta}{2}).
\end{aligned}
$$

Set III: $\sigma^2 + 4\mu^2 < 0, \lambda \neq 0$, and $\lambda = -\mu$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = \frac{\sigma}{2\mu} - \frac{\sqrt{-\sigma^2 - 4\mu^2}}{2\mu}\tan(\frac{\sqrt{-\sigma^2 - 4\mu^2}\eta}{2})\\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.13)\\
&a^{\mathfrak{f}(\eta)} = \frac{\sigma}{2\mu} - \frac{\sqrt{-\sigma^2 - 4\mu^2}}{2\mu}\cot(\frac{\sqrt{-\sigma^2 - 4\mu^2}\eta}{2}).
\end{aligned}
$$

Set IV: $\sigma^2 + 4\mu^2 > 0, \lambda \neq 0$, and $\lambda = -\mu$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = \frac{\sigma}{2\mu} + \frac{\sqrt{\sigma^2 + 4\mu^2}}{2\mu}\tanh(\frac{\sqrt{\sigma^2 + 4\mu^2}\eta}{2})\\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.14)\\
&a^{\mathfrak{f}(\eta)} = \frac{\sigma}{2\mu} + \frac{\sqrt{\sigma^2 + 4\mu^2}}{2\mu}\coth(\frac{\sqrt{\sigma^2 + 4\mu^2}\eta}{2}).
\end{aligned}
$$

Set V: $\sigma^2 - 4\mu^2 < 0$ and $\lambda = \mu$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\mu} + \frac{\sqrt{-\sigma^2 + 4\mu^2}}{2\mu}\tan(\frac{\sqrt{-\sigma^2 + 4\mu^2}\eta}{2})\\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.15)\\
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\mu} + \frac{\sqrt{-\sigma^2 + 4\mu^2}}{2\mu}\cot(\frac{\sqrt{-\sigma^2 + 4\mu^2}\eta}{2}).
\end{aligned}
$$

Set VI: $\sigma^2 - 4\mu^2 > 0$ and $\lambda = \mu$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\mu} - \frac{\sqrt{\sigma^2 - 4\mu^2}}{2\mu} \tanh(\frac{\sqrt{\sigma^2 - 4\mu^2}\eta}{2}) \\
&or \\
&a^{\mathfrak{f}(\eta)} = \frac{-\sigma}{2\mu} - \frac{\sqrt{\sigma^2 - 4\mu^2}}{2\mu} \coth(\frac{\sqrt{\sigma^2 - 4\mu^2}\eta}{2}).
\end{aligned} \tag{2.16}
$$

Set VII: $\sigma^2 = 4\lambda\mu$,

$$
a^{\mathfrak{f}(\eta)} = -\frac{2 + \sigma\eta}{2\lambda\eta}. \tag{2.17}
$$

Set VIII: $\lambda\mu < 0, \sigma = 0$, and $\lambda \neq 0$,

$$
\begin{aligned}
&a^{\mathfrak{f}(\eta)} = -\sqrt{\frac{-\mu}{\lambda}} \tanh(\sqrt{-\mu\lambda}\eta) \\
&or \\
&a^{\mathfrak{f}(\eta)} = -\sqrt{\frac{-\mu}{\lambda}} \coth(\sqrt{-\mu\lambda}\eta).
\end{aligned} \tag{2.18}
$$

Set IX: $\sigma = 0$ and $\mu = -\lambda$,

$$
a^{\mathfrak{f}(\eta)} = \frac{1 + e^{-2\lambda\eta}}{-1 + e^{-2\lambda\eta}}. \tag{2.19}
$$

Set X: $\mu = \lambda = 0$,

$$
a^{\mathfrak{f}(\eta)} = \cosh(\sigma\eta) + \sinh(\sigma\eta). \tag{2.20}
$$

Set XI: $\mu = \sigma = h$ and $\lambda = 0$,

$$
a^{\mathfrak{f}(\eta)} = e^{h\eta} - 1. \tag{2.21}
$$

Set XII: $\lambda = \sigma = h$ and $\mu = 0$,

$$
a^{\mathfrak{f}(\eta)} = \frac{e^{h\eta}}{1 - e^{h\eta}}. \tag{2.22}
$$

Set XIII: $\sigma = \lambda + \mu$,

$$
a^{\mathfrak{f}(\eta)} = -\frac{1 - \mu e^{(\mu-\lambda)\eta}}{1 - \lambda e^{(\mu-\lambda)\eta}}. \tag{2.23}
$$

Set XIV: $\sigma = -(\lambda + \mu)$,

$$
a^{\mathfrak{f}(\eta)} = \frac{\mu - e^{(\mu-\lambda)\eta}}{\lambda - e^{(\mu-\lambda)\eta}}. \tag{2.24}
$$

Set XV: $\mu = 0$,

$$a^{\mathfrak{f}(\eta)} = \frac{\sigma e^{\sigma\eta}}{1 - \lambda e^{\sigma\eta}}. \tag{2.25}$$

Set XVI: $\lambda = \mu = \sigma \neq 0$,

$$a^{\mathfrak{f}(\eta)} = \sqrt{3}\tan(\frac{\sqrt{3}}{2}\mu\eta) - 1. \tag{2.26}$$

Set XVII: $\lambda = \sigma = 0$,

$$a^{\mathfrak{f}(\eta)} = \mu\eta. \tag{2.27}$$

Set XVIII: $\mu = \sigma = 0$,

$$a^{\mathfrak{f}(\eta)} = \frac{-1}{\lambda\eta}. \tag{2.28}$$

Set XIX: $\lambda = \mu$ and $\sigma = 0$,

$$a^{\mathfrak{f}(\eta)} = \tan(\mu\eta). \tag{2.29}$$

Set XX: $\lambda = 0$,

$$a^{\mathfrak{f}(\eta)} = e^{\sigma\eta} - \frac{\mu}{\sigma}. \tag{2.30}$$

3. Implementation of solution techniques

We start by transforming the partial differential equation into an ordinary differential equation. Substituting (2.2) into (1.4) results in:

$$k(m + n - w)\mathcal{U}^{(3)}(\eta) + (m + n - w)\mathcal{U}'(\eta)^2 + \mathcal{U}'(\eta) = 0. \tag{3.1}$$

Balancing $\mathcal{U}'(\eta)^2$ with $\mathcal{U}^{(3)}(\eta)$ in (3.1), we get $N + 3 = 2N + 2$, and then $N = 1$.

3.1. Utilizing the improved F-expansion method for solutions

Equation (3.1) has a solution given by:

$$\mathcal{U}(\eta) = f_0 + f_1(G(\eta) + \mu) + \frac{Q}{G(\eta) + \mu}. \tag{3.2}$$

Substituting (3.2) into (3.1) and equating the coefficients of corresponding powers of $G^i(\eta)$ to zero, we derive the following system:

$$
\begin{aligned}
&-2f_1\mu^2\rho^2\left(Q-k\mu^2\right)(m+n-w)+f_1\mu^4\rho+f_1^2\mu^4\rho^2(m+n-w)+\rho Q\times\\
&\left(\rho(Q-6k\rho)(m+n-w)+\mu^2(-2k\rho(m+n-w)-1)\right)=0,\\
&-4f_1\mu\rho^2\left(Q-k\mu^2\right)(m+n-w)+4f_1k\mu^3\rho^2(m+n-w)+4f_1\mu^3\rho+4f_1^2\mu^3\rho^2\times\\
&(m+n-w)+8k\mu m\rho^2Q+8k\mu n\rho^2Q-8k\mu\rho^2Qw-2\mu\rho Q=0,\\
&-2f_1\rho^2\left(Q-k\mu^2\right)(m+n-w)-2f_1\mu^2\rho\left(Q-k\mu^2\right)(m+n-w)+2f_1\mu^2\rho\times\\
&(m+n-w)\left(k\left(3\mu^2+\rho\right)-Q\right)+8f_1k\mu^2\rho^2(m+n-w)+f_1\mu^4+6f_1\mu^2\rho+2f_1^2\mu^4\rho\times\\
&(m+n-w)+6f_1^2\mu^2\rho^2(m+n-w)-8k\mu^2m\rho Q-8km\rho^2Q-8k\mu^2n\rho Q-8kn\rho^2Q\\
&+8k\mu^2\rho Qw+8k\rho^2Qw+2m\rho Q^2+2n\rho Q^2-2\rho Q^2w-\mu^2Q-\rho Q=0,\\
&-4f_1\mu\rho\left(Q-k\mu^2\right)(m+n-w)+4f_1\mu\rho(m+n-w)\left(k\left(3\mu^2+\rho\right)-Q\right)+16f_1k\mu^3\rho\times\\
&(m+n-w)+4f_1k\mu\rho^2(m+n-w)+4f_1\mu^3+4f_1\mu\rho+8f_1^2\mu^3\rho(m+n-w)\\
&+4f_1^2\mu\rho^2(m+n-w)+8k\mu m\rho Q+8k\mu n\rho Q-8k\mu\rho Qw-2\mu Q=0,\\
&2f_1\mu^2(m+n-w)\left(k\left(3\mu^2+\rho\right)-Q\right)-2f_1\rho\left(Q-k\mu^2\right)(m+n-w)+2f_1\rho\times\\
&(m+n-w)\left(k\left(3\mu^2+\rho\right)-Q\right)+38f_1k\mu^2\rho(m+n-w)+6f_1\mu^2+f_1^2\mu^4(m+n-w)\\
&+12f_1^2\mu^2\rho(m+n-w)+f_1^2\rho^2(m+n-w)+f_1\rho-6k\mu^2mQ-2km\rho Q-6k\mu^2nQ\\
&-2kn\rho Q+6k\mu^2Qw+2k\rho Qw+mQ^2+nQ^2-Q^2w-Q=0,\\
&4f_1\mu(m+n-w)\left(k\left(3\mu^2+\rho\right)-Q\right)+12f_1k\mu^3(m+n-w)+28f_1k\mu\rho(m+n-w)\\
&+4f_1\mu+4f_1^2\mu^3(m+n-w)+8f_1^2\mu\rho(m+n-w)=0,\\
&2f_1(m+n-w)\left(k\left(3\mu^2+\rho\right)-Q\right)+30f_1k\mu^2(m+n-w)+6f_1k\rho(m+n-w)\\
&+6f_1^2\mu^2(m+n-w)+2f_1^2\rho(m+n-w)+f_1=0,\\
&24f_1k\mu(m+n-w)+4f_1^2\mu(m+n-w)=0,\\
&6f_1k(m+n-w)+f_1^2(m+n-w)=0.
\end{aligned}
\tag{3.3}
$$

We obtain five families of solutions:

Family I:

$$w=\frac{2f_1m\rho+2f_1n\rho+3}{2f_1\rho},Q=0,k=-\frac{f_1}{6}. \tag{3.4}$$

Family II:

$$w=\frac{4km\rho+4kn\rho-1}{4k\rho},f_1=0,Q=6\left(k\mu^2+k\rho\right). \tag{3.5}$$

Family III:

$$w=\frac{2f_1m\rho+2f_1n\rho+3}{2f_1\rho},Q=0,\mu=0,k=-\frac{f_1}{6}. \tag{3.6}$$

Family IV:

$$w=\frac{8f_1m\rho+8f_1n\rho+3}{8f_1\rho},Q=f_1(-\rho),\mu=0,k=-\frac{f_1}{6}. \tag{3.7}$$

Family V:

$$w=\frac{2mQ+2nQ-3}{2Q},f_1=0,\mu=0,k=\frac{Q}{6\rho}. \tag{3.8}$$

These are the solitary wave solutions:
Set I: When $\rho < 0$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = f_0 + f_1\left(\mu - \sqrt{-\rho}\tanh\left(\eta\sqrt{-\rho}\right)\right) + \frac{Q}{\mu - \sqrt{-\rho}\tanh\left(\eta\sqrt{-\rho}\right)} \\
&or \\
&\mathcal{U}(x,y,z,t) = f_0 + f_1\left(\mu - \sqrt{-\rho}\coth\left(\eta\sqrt{-\rho}\right)\right) + \frac{Q}{\mu - \sqrt{-\rho}\coth\left(\eta\sqrt{-\rho}\right)}.
\end{aligned} \tag{3.9}$$

Set II: When $\rho > 0$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = f_0 + f_1\,(\sqrt{\rho}\tan(\eta\sqrt{\rho}) + \mu) + \frac{Q}{\sqrt{\rho}\tan\left(\eta\sqrt{\rho}\right) + \mu} \\
&or \\
&\mathcal{U}(x,y,z,t) = f_0 + f_1\,(-\sqrt{\rho}\cot(\eta\sqrt{\rho}) + \mu) + \frac{Q}{-\sqrt{\rho}\cot\left(\eta\sqrt{\rho}\right) + \mu}.
\end{aligned} \tag{3.10}$$

Set III: When $\rho = 0$,

$$\mathcal{U}(x,y,z,t) = f_0 + f_1\left(\mu - \frac{1}{\eta}\right) + \frac{Q}{\mu - \frac{1}{\eta}}. \tag{3.11}$$

3.2. Utilizing the modified auxiliary equation method for solutions

Equation (2.9) has a solution given by:

$$\mathcal{U}(\eta) = b_0 + b_1 a^{f(\eta)}. \tag{3.12}$$

By substituting (3.12) into (3.1), grouping terms with the same powers, and equating their coefficients to zero, we derive the following system of equations:

$$\begin{aligned}
&2b_1k\lambda\mu^2m + b_1k\mu m\sigma^2 + 2b_1k\lambda\mu^2n + b_1k\mu n\sigma^2 - 2b_1k\lambda\mu^2w - b_1k\mu\sigma^2w + b_1\mu \\
&\quad + b_1^2\mu^2m + b_1^2\mu^2n - b_1^2\mu^2w = 0, \\
&8b_1k\lambda\mu m\sigma + b_1km\sigma^3 + 8b_1k\lambda\mu n\sigma + b_1kn\sigma^3 - 8b_1k\lambda\mu\sigma w - b_1k\sigma^3w + 2b_1^2\mu m\sigma \\
&\quad + 2b_1^2\mu n\sigma + b_1\sigma - 2b_1^2\mu\sigma w = 0, \\
&8b_1k\lambda^2\mu m + 7b_1k\lambda m\sigma^2 + 8b_1k\lambda^2\mu n + 7b_1k\lambda n\sigma^2 - 8b_1k\lambda^2\mu w - 7b_1k\lambda\sigma^2w \\
&\quad + b_1\lambda + 2b_1^2\lambda\mu m + b_1^2m\sigma^2 + 2b_1^2\lambda\mu n + b_1^2n\sigma^2 - 2b_1^2\lambda\mu w - b_1^2\sigma^2w = 0, \\
&12b_1k\lambda^2m\sigma + 12b_1k\lambda^2n\sigma - 12b_1k\lambda^2\sigma w + 2b_1^2\lambda m\sigma + 2b_1^2\lambda n\sigma - 2b_1^2\lambda\sigma w = 0, \\
&6b_1k\lambda^3m + 6b_1k\lambda^3n - 6b_1k\lambda^3w + b_1^2\lambda^2m + b_1^2\lambda^2n - b_1^2\lambda^2w = 0.
\end{aligned} \tag{3.13}$$

We obtain the next solution:

$$w = \frac{4k\lambda\mu m - km\sigma^2 + 4k\lambda\mu n - kn\sigma^2 - 1}{k\,(4\lambda\mu - \sigma^2)}, \quad b_1 = -6k\lambda. \tag{3.14}$$

Set I: $\sigma^2 - 4\lambda\mu < 0$ and $\lambda \neq 0$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sqrt{4\lambda\mu - \sigma^2}\tan\left(\frac{1}{2}\eta\sqrt{4\lambda\mu - \sigma^2}\right)}{2\lambda} - \frac{\sigma}{2\lambda}\right) \\
&or \\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sqrt{4\lambda\mu - \sigma^2}\cot\left(\frac{1}{2}\eta\sqrt{4\lambda\mu - \sigma^2}\right)}{2\lambda} - \frac{\sigma}{2\lambda}\right).
\end{aligned} \tag{3.15}$$

Set II: $\sigma^2 - 4\lambda\mu > 0$ and $\lambda \neq 0$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(-\frac{\sqrt{\sigma^2 - 4\lambda\mu}\tanh\left(\frac{1}{2}\eta\sqrt{\sigma^2 - 4\lambda\mu}\right)}{2\lambda} - \frac{\sigma}{2\lambda}\right) \\
&or \\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(-\frac{\sqrt{\sigma^2 - 4\lambda\mu}\coth\left(\frac{1}{2}\eta\sqrt{\sigma^2 - 4\lambda\mu}\right)}{2\lambda} - \frac{\sigma}{2\lambda}\right).
\end{aligned} \tag{3.16}$$

Set III: $\sigma^2 + 4\mu^2 < 0$, $\lambda \neq 0$, and $\lambda = -\mu$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sigma}{2\mu} - \frac{\sqrt{-4\mu^2 - \sigma^2}\tan\left(\frac{1}{2}\eta\sqrt{-4\mu^2 - \sigma^2}\right)}{2\mu}\right) \\
&or \\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sigma}{2\mu} - \frac{\sqrt{-4\mu^2 - \sigma^2}\cot\left(\frac{1}{2}\eta\sqrt{-4\mu^2 - \sigma^2}\right)}{2\mu}\right).
\end{aligned} \tag{3.17}$$

Set IV: $\sigma^2 + 4\mu^2 > 0$, $\lambda \neq 0$, and $\lambda = -\mu$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sqrt{4\mu^2 + \sigma^2}\tanh\left(\frac{1}{2}\eta\sqrt{4\mu^2 + \sigma^2}\right)}{2\mu} + \frac{\sigma}{2\mu}\right) \\
&or \\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sqrt{4\mu^2 + \sigma^2}\coth\left(\frac{1}{2}\eta\sqrt{4\mu^2 + \sigma^2}\right)}{2\mu} + \frac{\sigma}{2\mu}\right).
\end{aligned} \tag{3.18}$$

Set V: $\sigma^2 - 4\mu^2 < 0$ and $\lambda = \mu$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sqrt{4\mu^2-\sigma^2}\tan\left(\frac{1}{2}\eta\sqrt{4\mu^2-\sigma^2}\right)}{2\mu} - \frac{\sigma}{2\mu}\right)\\
&or\\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(\frac{\sqrt{4\mu^2-\sigma^2}\cot\left(\frac{1}{2}\eta\sqrt{4\mu^2-\sigma^2}\right)}{2\mu} - \frac{\sigma}{2\mu}\right).
\end{aligned} \tag{3.19}$$

Set VI: $\sigma^2 - 4\mu^2 > 0$ and $\lambda = \mu$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(-\frac{\sqrt{\sigma^2-4\mu^2}\tanh\left(\frac{1}{2}\eta\sqrt{\sigma^2-4\mu^2}\right)}{2\mu} - \frac{\sigma}{2\mu}\right)\\
&or\\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(-\frac{\sqrt{\sigma^2-4\mu^2}\coth\left(\frac{1}{2}\eta\sqrt{\sigma^2-4\mu^2}\right)}{2\mu} - \frac{\sigma}{2\mu}\right).
\end{aligned} \tag{3.20}$$

Set VII: $\sigma^2 = 4\lambda\mu$,

$$\mathcal{U}(x,y,z,t) = b_0 - b_1\Big(\frac{\eta\sigma+2}{2\eta\lambda}\Big). \tag{3.21}$$

Set VIII: $\lambda\mu < 0, \sigma = 0$, and $\lambda \neq 0$,

$$\begin{aligned}
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(-\sqrt{-\frac{\mu}{\lambda}}\tanh\left(\eta\sqrt{-\lambda\mu}\right)\right)\\
&or\\
&\mathcal{U}(x,y,z,t) = b_0 + b_1\left(-\sqrt{-\frac{\mu}{\lambda}}\coth\left(\eta\sqrt{-\lambda\mu}\right)\right).
\end{aligned} \tag{3.22}$$

Set IX: $\sigma = 0$ and $\mu = -\lambda$,

$$\mathcal{U}(x,y,z,t) = b_0 + b_1\Big(\frac{1+e^{-2\lambda\eta}}{-1+e^{-2\lambda\eta}}\Big). \tag{3.23}$$

Set X: $\mu = \lambda = 0$,

$$\mathcal{U}(x,y,z,t) = b_0 + b_1(\sinh(\eta\sigma) + \cosh(\eta\sigma)). \tag{3.24}$$

Set XI: $\mu = \sigma = h$ and $\lambda = 0$,

$$\mathcal{U}(x,y,z,t) = b_0 + b_1(e^{\eta h} - 1). \tag{3.25}$$

Set XII: $\lambda = \sigma = h$ and $\mu = 0$,

$$\mathcal{U}(x,y,z,t) = b_0 + b_1\Big(\frac{e^{h\eta}}{1-e^{h\eta}}\Big). \tag{3.26}$$

Set XIII: $\sigma = \lambda + \mu$,

$$\mathcal{U}(x, y, z, t) = b_0 - b_1\left(\frac{1 - \mu e^{(\mu-\lambda)\eta}}{1 - \lambda e^{(\mu-\lambda)\eta}}\right). \tag{3.27}$$

Set XIV: $\sigma = -(\lambda + \mu)$,

$$\mathcal{U}(x, y, z, t) = b_0 + b_1\left(\frac{\mu - e^{(\mu-\lambda)\eta}}{\lambda - e^{(\mu-\lambda)\eta}}\right). \tag{3.28}$$

Set XV: $\mu = 0$,

$$\mathcal{U}(x, y, z, t) = b_0 + b_1\left(\frac{\sigma e^{\sigma\eta}}{1 - \lambda e^{\sigma\eta}}\right). \tag{3.29}$$

Set XVI: $\lambda = \mu = \sigma \neq 0$,

$$\mathcal{U}(x, y, z, t) = b_0 + b_1\left(\sqrt{3}\tan\left(\frac{1}{2}\sqrt{3}\eta\mu\right) - 1\right). \tag{3.30}$$

Set XVII: $\lambda = \sigma = 0$,

$$\mathcal{U}(x, y, z, t) = b_0 + b_1(\eta\mu). \tag{3.31}$$

Set XVIII: $\mu = \sigma = 0$,

$$\mathcal{U}(x, y, z, t) = b_0 - b_1\left(\frac{1}{\eta\lambda}\right). \tag{3.32}$$

Set XIX: $\lambda = \mu$ and $\sigma = 0$,

$$\mathcal{U}(x, y, z, t) = b_0 + b_1\tan(\eta\mu). \tag{3.33}$$

Set XX: $\lambda = 0$,

$$\mathcal{U}(x, y, z, t) = b_0 + b_1\left(e^{\sigma\eta} - \frac{\mu}{\sigma}\right). \tag{3.34}$$

3.3. Symbolic computation verification

To confirm the validity of the obtained analytical solutions, symbolic computation was employed using Mathematica 14.1. The derived solutions were substituted back into the governing nonlinear equation, and the residuals were simplified symbolically to zero. The accuracy and consistency of the suggested methods are guaranteed by this process. Thus, symbolic computation offers a potent verification technique that enhances the dependability of the analytic derivations and complements them [13, 14, 19].

4. Graphical illustrations

To illustrate how the solution behaves and the way its parameters interact, this section provides graphical representations. An efficient tool for illustrating these interactions is a graph. This section employs 2D and 3D plots to visually represent the solution $\mathcal{U}(x, y, z, t)$.

• Applying the improved F-expansion method

Figure 1 illustrates the solution of (3.5) combined with (3.9) at: $\rho = -1, \mu = 2, f_0 = 1, f_1 = 0, k = 1, m = 1, n = 1$. Figure 2 shows the result obtained by applying (3.8) combined with (3.9) at: $\rho = -1, f_0 = 1, f_1 = 0, Q = 4, m = 1, n = 1, \mu = 0$.

• Applying the modified auxiliary equation method

In Figure 3 the graph corresponds to (3.16) at: $\lambda = 1, \mu = 0.1, \sigma = 3, k = 1, m = 1, n = 1$. Figure 4 depicts the solution of (3.18) at $\mu = 1, \lambda = -1, \sigma = 1.5, k = 1, m = 1, n = 1$. Figure 5 presents the graph of (3.22) at $\lambda = -0.6, \mu = 1, \sigma = 0, k = 1, m = 1, n = 1, b_0 = 0$. Figure 6 presents the graph of (3.23) at $\lambda = 1, m\text{u} = -\lambda, k = 1, m = 1, n = -1, \sigma = 0, b_0 = 0$. Finally, Figure 7 presents the solution of (3.33) at: $\lambda = \mu = 0.7, \sigma = 0, k = 1, m = 1, n = -1, b_0 = 0.5$.

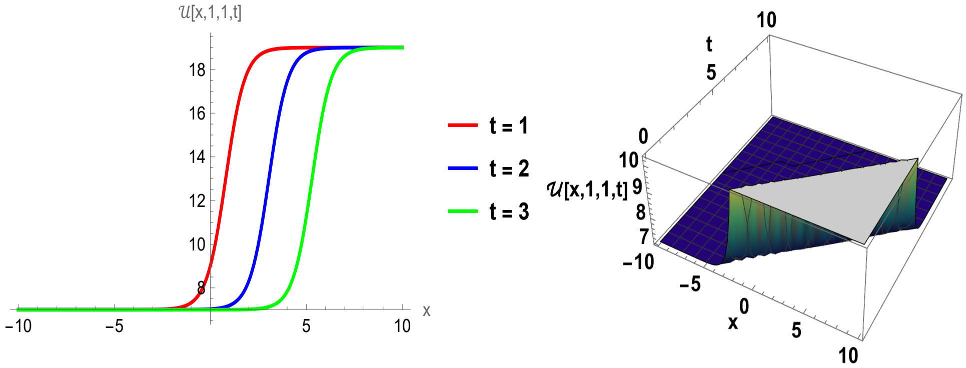

Figure 1. Graph of (3.5) with (3.9) using the improved F-expansion method at $\rho = -1, \mu = 2, f_0 = 1, f_1 = 0, k = 1, m = 1, n = 1$.

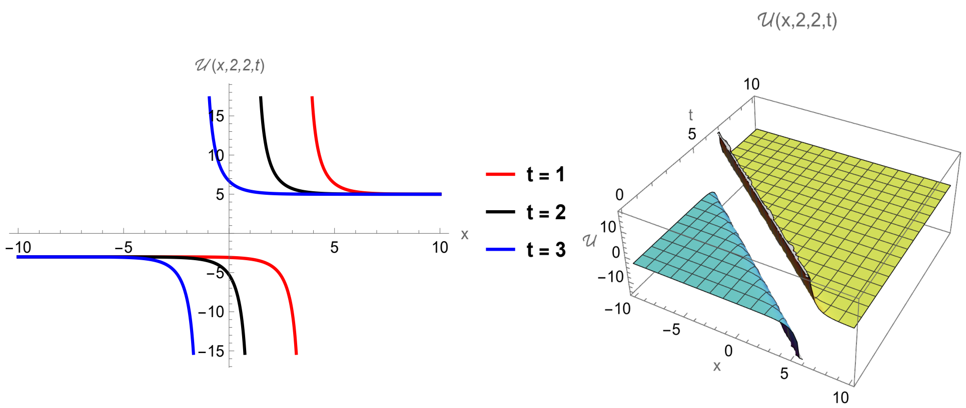

Figure 2. Graph of (3.8) with (3.9) using the improved F-expansion method at $\rho = -1, f_0 = 1, f_1 = 0, Q = 4, m = 1, n = 1, \mu = 0$.

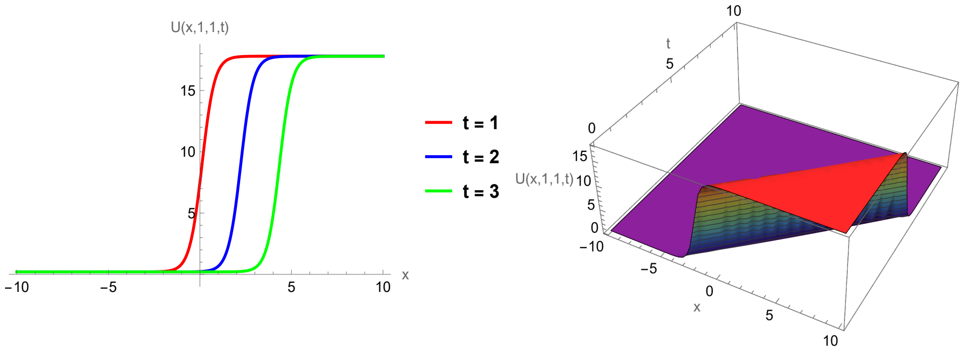

Figure 3. Graph of (3.16) using the modified auxiliary equation method at $\lambda = 1, \mu = 0.1$, $\sigma = 3, k = 1, m = 1, n = 1$.

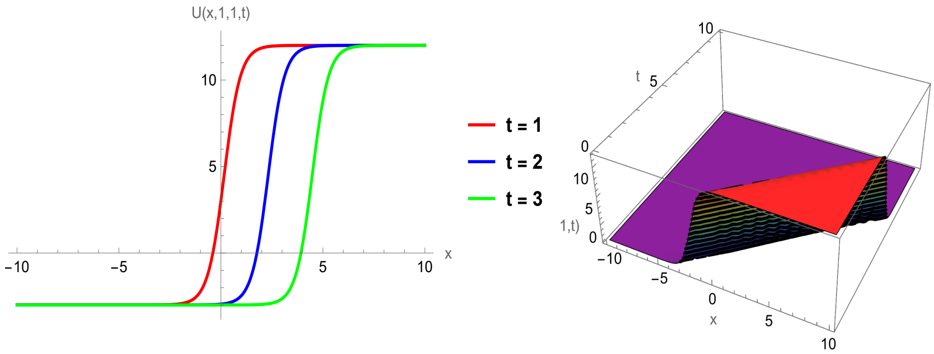

Figure 4. Graph of (3.18) using the modified auxiliary equation method at $\mu = 1, \lambda = -1$, $\sigma = 1.5, k = 1, m = 1, n = 1$.

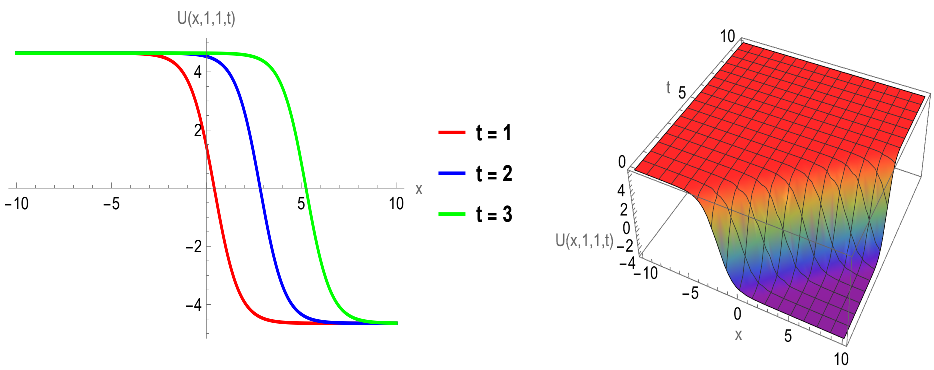

Figure 5. Graph of (3.22) using the modified auxiliary equation method at $\lambda = -0.6, \mu = 1$, $\sigma = 0, k = 1, m = 1, n = 1, b_0 = 0$.

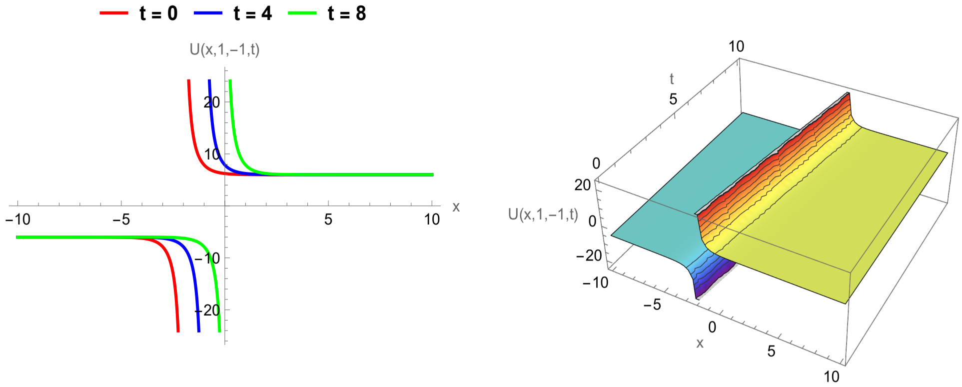

Figure 6. Graph of (3.23) using the modified auxiliary equation method at $\lambda = 1, m\mathrm{u} = -\lambda$, $k = 1, m = 1, n = -1, \sigma = 0, b_0 = 0$.

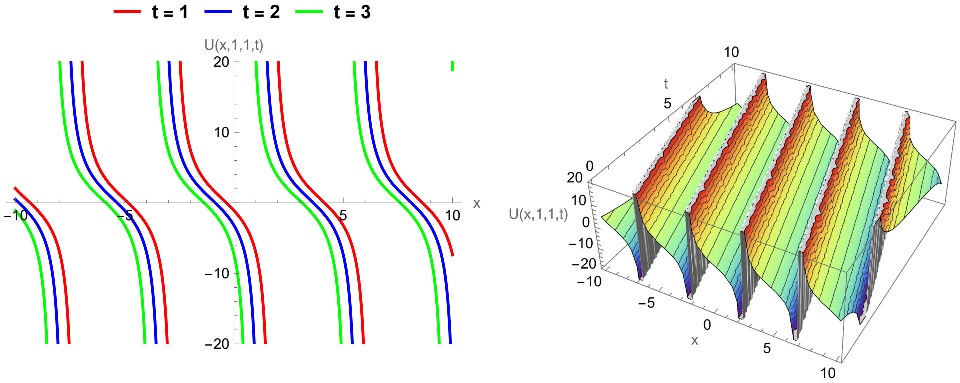

Figure 7. Graph of (3.33) using the modified auxiliary equation method at $\lambda = \mu = 0.7$, $\sigma = 0, k = 1, m = 1, n = -1, b_0 = 0.5$.

5. Discussion

In Figures 1, 3, 4, and 5: The solutions used the hyperbolic tangent and the hyperbolic cotangent functions presenting a kink and anti-kink soliton solution, where the wave propagates toward the right as time increases. Figures 1, 3, and 4 present a positive wave front, while Figure 5 presents a negative wave front. In Figures 2 and 6: The solution used the hyperbolic tangent function presenting a cusp-like singular soliton solution. The upper and lower parts of the curves are mirror-like, showing positive and negative branches of the same wave structure. The wave moves toward the left in Figure 2, while the wave front shifts to the right with time. In Figure 7, the solution used the tangent function presenting a periodic singular wave. The wave front shifts to the left with time.

6. Conclusions

In this article, we investigated the nonlinear (3+1)-dimensional Vakhnenko-Parkes (VP) equation, a key model for understanding high-frequency wave disturbances in active barotropic media. It arises in the study of nonlinear wave propagation in relaxing barotropic and viscoelastic media. It describes the evolution of high-frequency, weakly dispersive waves in systems where nonlinearity and relaxation effects are simultaneously significant. Physically, it models nonlinear acoustic or pressure waves in fluids, solids, and plasmas, and serves as a generalized framework for understanding multi-directional wave dynamics beyond the one-dimensional case. Applying the modified auxiliary equation method and the improved F-expansion method, we acquired analytic solutions including periodic, solitary, and rational forms. The modified auxiliary equation method is suitable for solving nonlinear evolution problems, as this approach is flexible and able to produce a wide classes of solutions, but its implementation is algebraically intensive. The improved F-expansion method has the ability to find a wider range of analytic solutions for nonlinear evolution equations, including novel solutions, in a simpler and more straightforward way than other methods. The presented 2D and 3D graphics provided deeper insight. This helped us to trace wave behavior. Our findings contribute to a more comprehensive understanding of nonlinear wave phenomena in high-frequency barotropic systems. The obtained analytical solutions were graphically compared with the wave profiles reported in related studies [42, 46]. The graphs exhibit similar shapes and propagation characteristics, confirming the physical consistency of the derived solutions. A direct numerical comparison was not attempted, since the mathematical forms of the analytical and numerical results differed. This study not only adds to the theoretical framework of the VP equation but also opens the door to future investigations into wave propagation in high-frequency media.

Author contributions

M. S. Mehanna: Writing original draft, software, visualization, investigation, formal analysis; Ibtehal Alazman: Project administration, validation; review and editing; Aly R. Seadawy: Supervision, methodology, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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