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Research article

Bi-univalent functions connected to Bazilevič and λ-Pseudo functions and their Lucas-Balancing polynomial applications

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Abstract: In this work, we define two families $\mathcal{V}_\Sigma(\mu,\gamma,\lambda;r)$ and $\mathcal{W}_\Sigma(\mu,\gamma,\lambda;r)$ of holomorphic and bi-univalent functions connected with Bazilevič functions and λ-pseudo functions defined by Lucas-Balancing polynomials. We demonstrate the upper bounds for the initial Taylor-Maclaurin coefficients. In addition, the Fekete-Szegö type inequalities are derived for functions in these families. Moreover, we indicate certain special cases and consequences for our results.

Keywords: bi-univalent function; holormorphic function; Bazilevič function; λ-Pseudo functions; upper bounds; Lucas-Balancing polynomials; convolution, Fekete-Szegö problem
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1. Introduction

Indicate by $\mathcal{B}$ the family of holomorphic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$\mathcal{F}(z) = z + \sum_{k=2}^{\infty} d_k z^k. \tag{1.1}$$

The subfamily of $\mathcal{B}$ that consists of functions that are also univalent in $\mathbb{U}$ is represented by $\mathbb{S}$.

Let $\mathcal{F} \in \mathbb{S}$ be said to be a starlike of order $\delta (0 \leqq \delta < 1)$ if

$$\Re\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) > \delta, \qquad (z \in \mathbb{U})$$

and a function $\mathcal{F} \in \mathbb{S}$ is called convex of order $\delta (0 \leqq \delta < 1)$ if

$$\Re\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + 1\right) > \delta, \qquad (z \in \mathbb{U}).$$

The families of functions $\mathcal{S}^*(\delta)$ and $C(\delta)$ are called starlike of order δ and convex of order δ in $\mathbb{U}$, respectively.

A function $\mathcal{F} \in \mathcal{B}$ is called Bazilevič function in $\mathbb{U}$ if (see [17]),

$$\Re\left(\frac{z^{1-\gamma}\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\gamma}}\right) > 0 \qquad (z \in \mathbb{U};\ \gamma \geqq 0).$$

A function $\mathcal{F} \in \mathcal{B}$ is called a λ-pseudo-starlike function in $\mathbb{U}$ if

$$\Re\left\{\frac{z\left(\mathcal{F}'(z)\right)^{\lambda}}{\mathcal{F}(z)}\right\} > 0, \qquad (z \in \mathbb{U}; \lambda \geqq 1).$$

Let $\mathcal{F}$ and $\mathcal{G}$ be two analytic functions in $\mathbb{U}$. The function $\mathcal{F}$ is said to be subordinated to $\mathcal{G}$ if there exists a Schwarz function $w(z)$, i.e., that is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$, such that $\mathcal{F}(z) = \mathcal{G}(w(z))$ for all $z \in \mathbb{U}$. This subordination notion is denoted by

$$\mathcal{F} \prec \mathcal{G} \quad \text{or} \quad \mathcal{F}(z) \prec \mathcal{G}(z).$$

If the function $\mathcal{G}$ is univalent in $\mathbb{U}$, then we have the inclusion equivalence

$$\mathcal{F}(z) \prec \mathcal{G}(z) \Leftrightarrow \mathcal{F}(0) = \mathcal{G}(0) \quad \text{and} \quad \mathcal{F}(\mathbb{U}) \subset \mathcal{G}(\mathbb{U}).$$

Recently, several authors introduced and studied different subfamilies associated with Bazilevič and λ-pseudo functions (see, for example, [2, 4, 19, 20]).

Based on the Koebe one-quarter theorem [5], the image of $\mathbb{U}$ under any univalent function $\mathcal{F} \in \mathcal{B}$ contains a disk of radius $\frac{1}{4}$, and each function $\mathcal{F} \in \mathbb{S}$ has an inverse $\mathcal{F}^{-1}$ that is defined as $\mathcal{F}^{-1}(\mathcal{F}(z)) = z$ and

$$\mathcal{F}(\mathcal{F}^{-1}(w)) = w, \quad \left(|w| < r_0(\mathcal{F}), r_0(\mathcal{F}) \geqq \frac{1}{4}\right),$$

where

$$\mathcal{G}(w) = \mathcal{F}^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots .$$

A function $\mathcal{F} \in \mathcal{B}$ is named bi-univalent function in $\mathbb{U}$ if both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are univalent functions in $\mathbb{U}$: The family of all bi-univalent functions in $\mathbb{U}$ denoted by Σ.

A very large number of works related to the bi-univalent functions have been presented in the papers. In [3], Cotîrlă used the $(p, q)-$derivative operator and investigated the estimates on coefficients and the Fekete-Szegö functional for subclasses of analytic and bi-univalent functions. El-Deeb et al. [6] obtained estimates for the first two Taylor-Maclaurin coefficients associated with q-analogue derivative. In 2024, Krushkal [11] discussed a variational technique for biunivalent functions, which provides a powerful tool for solving the general extremal problems on the classes of biunivalent holomorphic functions, while in 2025 Krushkal [12] considered the connection between biunivalence and the geometry of Teichmuller balls and provided some sufficient conditions for biunivalence of holomorphic functions on the disk. Murugusundaramoorthy et al. [13] presented and examined a family of bi-starlike functions with respect to symmetric conjugate points associated with a Lucas balancing polynomial. We recall some examples of functions in the family Σ, from the work of Srivastava et al. [18],

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Fekete and Szegö (see [7]) disproved of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by 1, which is the basis of the Fekete-Szegö problem $\left|d_3 - \eta d_2^2\right|$ for $\mathcal{F} \in \mathbb{S}$, which has a long history in the field of geometric function theory. Fekete-Szegö inequalities for various function families were obtained by many authors.

Behera and Panda [1] introduced the concept of balancing numbers B_n, $n \geq 0$. In actuality, the Diophantine equation can be calculated by the balancing number n and its balancer τ.

$$1 + 2 + 3 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+\tau).$$

It is known that if n is a balancing number, then $8n^2 + 1$ is a perfect square, and its positive square root is called a Lucas-Balancing number [16]. Recently, Frontczak [8] was concerned with Balancing and Lucas-Balancing polynomials. These polynomials are a natural extension of Balancing and Lucas-Balancing numbers. He derived many interesting properties of the former related to Chebyshev polynomials. In [10], Keskin and Karaatli have studied some properties of balancing numbers and square triangular numbers. Comprehensive information about Lucas-Balancing numbers is accessible for interested readers in [16]. Lucas-Balancing polynomials are expansions of the Lucas-Balancing numbers that occur naturally. For $r \in \mathbb{C}$ and $n \geq 2$, the recurrence relation that follows defines the Lucas-Balancing polynomials [15].

$$C_n(r) = 6rC_{n-1}(r) - C_{n-2}(r),$$

with initial conditions

$$C_0(r) = 1, \qquad C_1(r) = 3r \qquad \textit{and} \qquad C_2(r) = 18r^2 - 1. \tag{1.2}$$

The Lucas-Balancing polynomials' generating function can be written as follows (see [9]):

$$\mathcal{B}(r, z) = \sum_{k=0}^{\infty} C_k(r)z^k = \frac{1 - 3rz}{1 - 6rz + z^2}, \tag{1.3}$$

where $r \in [-1, 1]$ and $z \in \mathbb{U}$.

2. Definitions and preliminaries

Now, by using the Lucas-Balancing polynomials, we provide the following families of holomorphic bi-Bazilevič and λ-pseudo functions.

Definition 2.1. *Let $\mathcal{V}_\Sigma(\mu,\gamma,\lambda;r)$ be the family of functions $\mathcal{F}\in\Sigma$ satisfying the following subordinations:*

$$(1-\mu)\,\frac{z^{1-\gamma}\mathcal{F}'(z)}{(\mathcal{F}(z))^{1-\gamma}}+\mu\,\frac{z(\mathcal{F}'(z))^{\lambda}}{\mathcal{F}(z)}\prec\frac{1-3rz}{1-6rz+z^2}=:\mathcal{B}(r,z)$$

and

$$(1-\mu)\,\frac{w^{1-\gamma}\mathcal{G}'(w)}{(\mathcal{G}(w))^{1-\gamma}}+\mu\,\frac{w(\mathcal{G}'(w))^{\lambda}}{\mathcal{G}(w)}\prec\frac{1-3rw}{1-6rw+w^2}=:\mathcal{B}(r,w),$$

where $0\leqq\mu\leqq1$, $\gamma\geqq0$, $\lambda\geqq1$, $r\in[-1,1]$, and $\mathcal{G}(w)=\mathcal{F}^{-1}(w)$.

Remark 2.1. *Put $\mu=\gamma=0$ in Definition 2.1; the family $\mathcal{V}_\Sigma(\mu,\gamma,\lambda;r)$ reduce to the family ${}_{\mathcal{LB}}\mathcal{S}^*_\Sigma(\mathcal{B}(r,z))$, which was studied recently by Öztürk and Aktaş (see [14]).*

Definition 2.2. *Let $\mathcal{W}_\Sigma(\mu,\gamma,\lambda;r)$ be the family of functions $\mathcal{F}\in\Sigma$ satisfying the following subordinations:*

$$(1-\mu)\left(1+\frac{z^{2-\gamma}\mathcal{F}''(z)}{(z\mathcal{F}'(z))^{1-\gamma}}\right)+\mu\frac{((z\mathcal{F}'(z))')^{\lambda}}{\mathcal{F}'(z)}\prec\frac{1-3rz}{1-6rz+z^2}=:\mathcal{B}(r,z)$$

and

$$(1-\mu)\left(1+\frac{w^{2-\gamma}\mathcal{G}''(w)}{(w\mathcal{G}'(w))^{1-\gamma}}\right)+\mu\frac{((w\mathcal{G}'(w))')^{\lambda}}{\mathcal{G}'(w)}\prec\frac{1-3rw}{1-6rw+w^2}=:\mathcal{B}(r,w),$$

where $0\leqq\mu\leqq1$, $\gamma\geqq0$, $\lambda\geqq1$, $r\in[-1,1]$, and $\mathcal{G}(w)=\mathcal{F}^{-1}(w)$.

Remark 2.2. *If we take $\mu=\gamma=0$ in Definition 2.2, the family $\mathcal{W}_\Sigma(\mu,\gamma,\lambda;r)$ reduce to the family ${}_{\mathcal{LB}}\mathcal{C}_\Sigma(\mathcal{B}(r,z))$, which was introduced recently by Öztürk and Aktaş (see [14]).*

3. Main results

Theorem 3.1. *Let $\mathcal{F}$ given by* (1.1) *be in the family $\mathcal{V}_\Sigma(\mu,\gamma,\lambda;r)$ $(0\leqq\mu\leqq1,\ \gamma\geqq0,\ \lambda\geqq1)$. Then*

$$|d_2|\leqq\frac{3|r|\sqrt{6|r|}}{\sqrt{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}}$$

and

$$|d_3|\leqq\frac{3|r|}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}+\frac{9r^2}{\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2},$$

where

$$\Omega(\mu,\gamma,\lambda)=(1-\mu)(\gamma+1)\left[4\mu(4\lambda-\gamma-3)+3\gamma+2\right]+2\mu\,(2\lambda-1)\,(3\lambda-2)\,. \tag{3.1}$$

Proof. Let $f \in \mathcal{V}_\Sigma(\mu, \gamma, \lambda; r)$ and $f^{-1} = g$. There are two holomorphic functions $\Phi, \Psi : \mathbb{U} \longrightarrow \mathbb{U}$, fulfill the following conditions:

$$(1-\mu)\,\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu\,\frac{z(f'(z))^{\lambda}}{f(z)} = \mathcal{B}(r, \Phi(z)), \quad z \in \mathbb{U} \tag{3.2}$$

and

$$(1-\mu)\,\frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu\,\frac{w(g'(w))^{\lambda}}{g(w)} = \mathcal{B}(r, \Psi(w)), \quad w \in \mathbb{U}, \tag{3.3}$$

where

$$\Phi(z) = x_1 z + x_2 z^2 + x_3 z^3 + \cdots \quad , z \in \mathbb{U} \tag{3.4}$$

and

$$\Psi(z) = y_1 w + y_2 w^2 + y_3 w^3 + \cdots \quad , w \in \mathbb{U} \tag{3.5}$$

are Schwarz functions such that

$$\Phi(0) = \Psi(0) = 0 \qquad \text{and} \qquad |\Phi(z)| < 1, \ |\Psi(w)| < 1 \quad (z, w \in \mathbb{U}).$$

On the other hand, it is known that the conditions $|\Phi(z)| < 1$ and $|\Psi(w)| < 1$ imply

$$|x_i| < 1 \qquad \text{and} \qquad |y_i| < 1 \quad \text{for all} \quad i \in \mathbb{N}.$$

It follows from (3.2)–(3.5) that

$$(1-\mu)\,\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \mu\,\frac{z(f'(z))^{\lambda}}{f(z)} = C_0(r) + C_1(r)x_1 z + \left[C_1(r)x_2 + C_2(r)x_1^2\right] z^2 + \cdots \tag{3.6}$$

and

$$(1-\mu)\,\frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \mu\,\frac{w(g'(w))^{\lambda}}{g(w)} = C_0(r) + C_1(r)y_1 w + \left[C_1(r)y_2 + C_2(r)y_1^2\right] w^2 + \cdots \tag{3.7}$$

Equating the coefficients in (3.6) and (3.7) yields

$$[(1-\mu)(\gamma+1) + \mu(2\lambda-1)]\, a_2 = C_1(r)x_1, \tag{3.8}$$

$$[(1-\mu)(\gamma+2) + \mu(3\lambda-1)]\, a_3 + \left[\frac{1}{2}\,(1-\mu)(\gamma+2)(\gamma-1) + \mu\,(2\lambda(\lambda-2)+1)\right] a_2^2$$

$$= C_1(r)x_2 + C_2(r)x_1^2, \tag{3.9}$$

$$-\,[(1-\mu)(\gamma+1) + \mu(2\lambda-1)]\, a_2 = C_1(r)y_1, \tag{3.10}$$

and

$$[(1-\mu)(\gamma+2) + \mu(3\lambda-1)]\,(2a_2^2 - a_3) + \left[\frac{1}{2}\,(1-\mu)(\gamma+2)(\gamma-1) + \mu\,(2\lambda(\lambda-2)+1)\right] a_2^2$$

$$= C_1(r)y_2 + C_2(r)y_1^2. \tag{3.11}$$

From (3.8) and (3.10), we have

$$x_1 = -y_1 \tag{3.12}$$

and

$$2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2 a_2^2 = (C_1(r))^2\,(x_1^2+y_1^2). \tag{3.13}$$

By summing up Eq (3.9) to (3.11), we obtain

$$\left[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\,(2\lambda-1)\right] a_2^2 = C_1(r)(x_2+y_2)+C_2(r)(x_1^2+y_1^2). \tag{3.14}$$

Substituting from (3.13) the value of $x_1^2+y_1^2$ in Eq (3.14), we deduce that

$$a_2^2 = \frac{(C_1(r))^3\,(x_2+y_2)}{\left[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\,(2\lambda-1)\right](C_1(r))^2 - 2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2 C_2(r)}. \tag{3.15}$$

Applying (1.2) in (3.15), we obtain

$$a_2^2 = \frac{27r^3\,(x_2+y_2)}{2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2 - 9\Omega(\mu,\gamma,\lambda)r^2}, \tag{3.16}$$

where $\Omega(\mu,\gamma,\lambda)$ is given by (3.1). Now, using the well-known triangular inequality and taking the square root of (3.16), we conclude that

$$|a_2| \leqq \frac{3|r|\,\sqrt{6|r|}}{\sqrt{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2 - 9\Omega(\mu,\gamma,\lambda)r^2\right|}}.$$

In order to find the bound on $|a_3|$, we subtract Eq (3.11) from (3.9) and consider Eq (3.12); we have

$$2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right](a_3-a_2^2) = C_1(r)(x_2-y_2), \tag{3.17}$$

then, by substituting the value of a_2^2 from (3.13) into (3.17), we obtain

$$a_3 = \frac{C_1(r)(x_2-y_2)}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]} + \frac{(C_1(r))^2\,(x_1^2+y_1^2)}{2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2}. \tag{3.18}$$

Considering Eq (1.2) in (3.18), yield

$$a_3 = \frac{3r(x_2-y_2)}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]} + \frac{9r^2(x_1^2+y_1^2)}{2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2}. \tag{3.19}$$

By making use of the triangular inequality of (3.19) and a straightforward calculation, we find that

$$|a_3| \leqq \frac{3|r|}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)} + \frac{9r^2}{\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2}.$$

□

When $\mu=\gamma=0$, the analogous results presented by Öztürk and Aktaş [14] were obtained by using Theorem 3.1.

Corollary 3.1. *[14] Let $\mathcal{F}$ given by (1.1) is in the family ${}_{\mathcal{LB}}\mathcal{S}^*_{\Sigma}(\mathcal{B}(r,z))$. Then*

$$|d_2| \leqq \frac{3|r|\sqrt{3|r|}}{\sqrt{\left|1-9r^2\right|}}$$

and

$$|d_3| \leqq 3|r|\left(3|r|+\frac{1}{2}\right).$$

Theorem 3.2. *Assume that $\mathcal{F}$ given by (1.1) is in the family $\mathcal{W}_{\Sigma}(\mu,\gamma,\lambda;r)$ $(0 \leqq \mu \leqq 1,\ \gamma \geqq 0,\ \lambda \geqq 1)$. Then*

$$|d_2| \leqq \frac{3|r|\sqrt{6|r|}}{\sqrt{\left|8\left(2\mu(\lambda-1)+1\right)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2\left(\lambda-1\right)^2-6\right]r^2\right|}}$$

and

$$|d_3| \leqq \frac{|r|}{3\mu(\lambda-1)+2}+\frac{9r^2}{4\left(2\mu(\lambda-1)+1\right)^2}.$$

Proof. Let $g=f^{-1}$ and $f\in\mathcal{W}_{\Sigma}(\mu,\gamma,\lambda;r)$. There are the functions $\Phi,\Psi:\mathbb{U}\longrightarrow\mathbb{U}$ which are holomorphic, such that

$$(1-\mu)\left(1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}\right)+\mu\frac{\left((zf'(z))'\right)^{\lambda}}{f'(z)}=\mathcal{B}(r,\Phi(z)),\quad z\in\mathbb{U} \tag{3.20}$$

and

$$(1-\mu)\left(1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}\right)+\mu\frac{\left((wg'(w))'\right)^{\lambda}}{g'(w)}=\mathcal{B}(r,\Psi(w)),\quad w\in\mathbb{U}, \tag{3.21}$$

where $\Phi(z)$ and $\Psi(z)$ have the forms (3.4) and (3.5). From (3.20) and (3.21), we deduce that

$$(1-\mu)\left(1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}\right)+\mu\frac{\left((zf'(z))'\right)^{\lambda}}{f'(z)}=C_0(r)+C_1(r)x_1z+\left[C_1(r)x_2+C_2(r)x_1^2\right]z^2+\cdots \tag{3.22}$$

and

$$(1-\mu)\left(1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}\right)+\mu\frac{\left((wg'(w))'\right)^{\lambda}}{g'(w)}=C_0(r)+C_1(r)y_1w+\left[C_1(r)y_2+C_2(r)y_1^2\right]w^2+\cdots. \tag{3.23}$$

Equating the coefficients in (3.22) and (3.23), yields

$$2\left(2\mu(\lambda-1)+1\right)a_2=C_1(r)x_1, \tag{3.24}$$

$$3\left(3\mu(\lambda-1)+2\right)a_3+4\left[2\lambda\mu(\lambda-2)+\mu(2-\gamma)+\gamma-1\right]a_2^2=C_1(r)x_2+C_2(r)x_1^2, \tag{3.25}$$

$$-2\left(2\mu(\lambda-1)+1\right)a_2=C_1(r)y_1 \tag{3.26}$$

and

$$3\left(3\mu(\lambda-1)+2\right)\left(2a_2^2-a_3\right)+4\left[2\lambda\mu(\lambda-2)+\mu(2-\gamma)+\gamma-1\right]a_2^2=C_1(r)y_2+C_2(r)y_1^2. \tag{3.27}$$

From (3.24) and (3.26), we have

$$x_1 = -y_1 \tag{3.28}$$

and

$$8\,(2\mu(\lambda-1)+1)^2\,a_2^2 = (C_1(r))^2\,(x_1^2+y_1^2). \tag{3.29}$$

If we add (3.25) to (3.27), we obtain

$$2\,[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]\,a_2^2 = C_1(r)(x_2+y_2)+C_2(r)(x_1^2+y_1^2). \tag{3.30}$$

Substituting from (3.29) the value of $x_1^2+y_1^2$ in the relation (3.30), we deduce that

$$a_2^2 = \frac{(C_1(r))^3\,(x_2+y_2)}{2\,[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)]\,(C_1(r))^2-8\,(2\mu(\lambda-1)+1)^2\,C_2(r)}. \tag{3.31}$$

Applying (1.2) in (3.31), we get

$$a_2^2 = \frac{27r^3\,(x_2+y_2)}{8\,(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2\,(\lambda-1)^2-6\right]r^2}. \tag{3.32}$$

Now, using the well-known triangular inequality and taking square root of (3.32), we conclude that

$$|a_2| \leqq \frac{3|r|\,\sqrt{6|r|}}{\sqrt{\left|8\,(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2\,(\lambda-1)^2-6\right]r^2\right|}}.$$

In order to find the bound on $|a_3|$, from the relation (3.25) we subtract (3.27) and consider Eq (3.28); we will obtain $x_1^2 = y_1^2$ and hence

$$6\,(3\mu(\lambda-1)+2)\,(a_3-a_2^2) = C_1(r)(x_2-y_2), \tag{3.33}$$

then by substituting from (3.29) the value of a_2^2 into (3.33), we get

$$a_3 = \frac{C_1(r)(x_2-y_2)}{6\,(3\mu(\lambda-1)+2)} + \frac{(C_1(r))^2\,(x_1^2+y_1^2)}{8\,(2\mu(\lambda-1)+1)^2}. \tag{3.34}$$

Considering Eq (1.2) in (3.34), yield

$$a_3 = \frac{r(x_2-y_2)}{2\,(3\mu(\lambda-1)+2)} + \frac{9r^2(x_1^2+y_1^2)}{8\,(2\mu(\lambda-1)+1)^2}. \tag{3.35}$$

By making use of the triangular inequality of (3.35) and a straightforward calculation, we find that

$$|a_3| \leqq \frac{|r|}{3\mu(\lambda-1)+2} + \frac{9r^2}{4\,(2\mu(\lambda-1)+1)^2}.$$

□

If we take $\mu = \gamma = 0$ in Theorem 3.2, the results are reduced to the corresponding results of Öztürk and Aktaş (see [14]).

Corollary 3.2. *[14] Let $\mathcal{F}$ given by* (1.1) *is in the family ${}_{\mathcal{LB}}C_{\Sigma}(\mathcal{B}(r,z))$. Then*

$$|d_2| \leqq \frac{3|r|\sqrt{3|r|}}{\sqrt{2\left|2+27r^2\right|}}$$

and

$$|d_3| \leqq \frac{|r|}{2}\left(\frac{9}{2}|r|+1\right).$$

We provide, in the next theorems, the Fekete-Szegö problem for the function families $\mathcal{V}_{\Sigma}(\mu,\gamma,\lambda;r)$ and $\mathcal{W}_{\Sigma}(\mu,\gamma,\lambda;r)$.

Theorem 3.3. *For $0 \leqq \mu \leqq 1$, $\gamma \geqq 0$, $\lambda \geqq 1$, and $\eta \in \mathbb{R}$, let $\mathcal{F} \in \mathcal{V}_{\Sigma}(\mu,\gamma,\lambda;r)$ be of the form* (1.1). *Then*

$$\left|d_3-\eta d_2^2\right| \leqq \begin{cases} \frac{3|r|}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}; \\ |\eta-1| \leqq \frac{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}{18r^2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}, \\ \frac{54|r|^3|\eta-1|}{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}; \\ |\eta-1| \geqq \frac{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}{18r^2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}. \end{cases}$$

Proof. It follows from (3.15) and (3.17) that

$$\begin{aligned} a_3-\eta a_2^2 &= \frac{C_1(r)(x_2-y_2)}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}+(1-\eta)\,a_2^2 \\ &= \frac{C_1(r)(x_2-y_2)}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]} \\ &\quad+\frac{(C_1(r))^3\,(x_2+y_2)\,(1-\eta)}{\left[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\,(2\lambda-1)\right](C_1(r))^2-2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2C_2(r)} \\ &= C_1(r)\left[\left(\psi(\eta,r)+\frac{1}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}\right)x_2\right. \\ &\quad\left.+\left(\psi(\eta,r)-\frac{1}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}\right)y_2\right], \end{aligned}$$

where

$$\psi(\eta,r)=\frac{(C_1(r))^2\,(1-\eta)}{\left[(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda\,(2\lambda-1)\right](C_1(r))^2-2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2C_2(r)}.$$

Taking modulus and using triangle inequality with (1.2) in the last equation, we find that

$$\left|a_3-\eta a_2^2\right| \leqq \begin{cases} \frac{3|r|}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}, & 0 \leqq |\psi(\eta,r)| \leqq \frac{1}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}, \\ 6|r|\,|\psi(\eta,r)|, & |\psi(\eta,r)| \geqq \frac{1}{2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}. \end{cases}$$

After some computations, we obtain

$$\left|a_3-\eta a_2^2\right| \leqq \begin{cases} \frac{3|r|}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}; \\ |\eta-1| \leqq \frac{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}{18r^2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}, \\ \frac{54|r|^3|\eta-1|}{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}; \\ |\eta-1| \geqq \frac{\left|2\left[(1-\mu)(\gamma+1)+\mu(2\lambda-1)\right]^2-9\Omega(\mu,\gamma,\lambda)r^2\right|}{18r^2\left[(1-\mu)(\gamma+2)+\mu(3\lambda-1)\right]}. \end{cases}$$

□

For $\mu=\gamma=0$, the results of Öztürk and Aktaş [14] are provided by Theorem 3.3.

Corollary 3.3. *[14] For $\eta \in \mathbb{R}$, let $\mathcal{F} \in {}_{\mathcal{LB}}\mathcal{S}_\Sigma^*(\mathcal{B}(r,z))$ be of the form* (1.1). *Then*

$$\left|d_3-\eta d_2^2\right| \leqq \begin{cases} \frac{3}{2}|r|; & |\eta-1| \leqq \frac{\left|1-9r^2\right|}{18r^2}, \\ \frac{27|r|^3|\eta-1|}{\left|1-9r^2\right|}; & |\eta-1| \geqq \frac{\left|1-9r^2\right|}{18r^2}. \end{cases}$$

If we put $\eta=1$ in Theorem 3.3, we get the next result:

Corollary 3.4. *If $\mathcal{F} \in \mathcal{V}_\Sigma(\mu,\gamma,\lambda;r)$ be of the form* (1.1)*, then we have that*

$$\left|d_3-d_2^2\right| \leqq \frac{3|r|}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}.$$

Theorem 3.4. *For $0 \leqq \mu \leqq 1$, $\gamma \geqq 0$, $\lambda \geqq 1$, and $\eta \in \mathbb{R}$, let $\mathcal{F} \in \mathcal{W}_\Sigma(\mu,\gamma,\lambda;r)$ be of the form* (1.1). *Then*

$$\left|d_3-\eta d_2^2\right| \leqq \begin{cases} \frac{|r|}{3\mu(\lambda-1)+2}; \\ |\eta-1| \leqq \frac{\left|8(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2(\lambda-1)^2-6\right]r^2\right|}{27r^2(3\mu(\lambda-1)+2)}, \\ \frac{27|r|^3|\eta-1|}{\left|8(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2(\lambda-1)^2-6\right]r^2\right|}; \\ |\eta-1| \geqq \frac{\left|8(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2(\lambda-1)^2-6\right]r^2\right|}{27r^2(3\mu(\lambda-1)+2)}. \end{cases}$$

Proof. It follows from (3.31) and (3.33) that

$$\begin{aligned} a_3-\eta a_2^2 = & \frac{C_1(r)(x_2-y_2)}{6\,(3\mu(\lambda-1)+2)} + (1-\eta)\,a_2^2 \\ = & \frac{C_1(r)(x_2-y_2)}{6\,(3\mu(\lambda-1)+2)} \end{aligned}$$

$$+\frac{(C_1(r))^3(x_2+y_2)(1-\eta)}{2\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right](C_1(r))^2-8\left(2\mu(\lambda-1)+1\right)^2C_2(r)}$$
$$=\frac{C_1(r)}{2}\left[\left(\phi(\eta,r)+\frac{1}{3\left(3\mu(\lambda-1)+2\right)}\right)x_2+\left(\phi(\eta,r)-\frac{1}{3\left(3\mu(\lambda-1)+2\right)}\right)y_2\right],$$

where

$$\phi(\eta,r)=\frac{(C_1(r))^2(1-\eta)}{\left[2+4\gamma+9\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)\right](C_1(r))^2-4\left(2\mu(\lambda-1)+1\right)^2C_2(r)}.$$

Taking modulus and using triangle inequality with (1.2) in the last equation, we find that

$$\left|a_3-\eta a_2^2\right|\leqq\begin{cases}\frac{|r|}{3\mu(\lambda-1)+2}, & 0\leqq|\phi(\eta,r)|\leqq\frac{1}{3(3\mu(\lambda-1)+2)},\\ \\ 3|r|\,|\phi(\eta,r)|, & |\phi(\eta,r)|\geqq\frac{1}{3(3\mu(\lambda-1)+2)}.\end{cases}$$

After some computations, we obtain

$$\left|a_3-\eta a_2^2\right|\leqq\begin{cases}\frac{|r|}{3\mu(\lambda-1)+2};\\ \\ |\eta-1|\leqq\frac{\left|8(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2(\lambda-1)^2-6\right]r^2\right|}{27r^2(3\mu(\lambda-1)+2)},\\ \\ \frac{27|r|^3|\eta-1|}{\left|8(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2(\lambda-1)^2-6\right]r^2\right|};\\ \\ |\eta-1|\geqq\frac{\left|8(2\mu(\lambda-1)+1)^2+18\left[4\gamma-23\mu(\lambda-1)+8\lambda\mu(\lambda-2)+4\mu(2-\gamma)-32\mu^2(\lambda-1)^2-6\right]r^2\right|}{27r^2(3\mu(\lambda-1)+2)}.\end{cases}$$

□

When $\mu=\gamma=0$, Theorem 3.4 leads to the known result of Öztürk and Aktaş (see [14]).

Corollary 3.5. *[14] For $\eta\in\mathbb{R}$, let $\mathcal{F}\in{}_{\mathcal{LB}}\mathcal{C}_\Sigma(\mathcal{B}(r,z))$ be of the form* (1.1)*. Then*

$$\left|d_3-\eta d_2^2\right|\leqq\begin{cases}\frac{|r|}{2}; & |\eta-1|\leqq\frac{\left|2-27r^2\right|}{27r^2},\\ \\ \frac{27|r|^3|\eta-1|}{\left|4-54r^2\right|}; & |\eta-1|\geqq\frac{\left|2-27r^2\right|}{27r^2}.\end{cases}$$

Putting in Theorem 3.4 $\eta=1$, we obtain the following result:

Corollary 3.6. *If $\mathcal{F}\in\mathcal{W}_\Sigma(\mu,\gamma,\lambda;r)$ is of the form* (1.1)*, then*

$$\left|d_3-d_2^2\right|\leqq\frac{|r|}{3\mu(\lambda-1)+2}.$$

4. Conclusions

The primary objective was to use the Lucas-Balancing polynomials and create a certain family $\mathcal{V}_\Sigma(\mu, \gamma, \lambda; r)$ and $\mathcal{W}_\Sigma(\mu, \gamma, \lambda; r)$ of bi-univalent functions associating the Bazilevičfunctions and λ-pseudo functions. We generated Taylor-Maclaurin coefficient inequalities for functions belonging to these families and viewed the famous Fekete-Szegö problem. As future research directions, the contents of the paper on Lucas-Balancing polynomials could inspire further research related to other families.

Author contributions

Abbas Kareem Wanas: Conceptualization, methodology, software, formal analysis, visualization, investigation, data curation, writing–original draft; H. M. Srivastava: Conceptualization, methodology, software, formal analysis, visualization, supervision, writing–original draft, writing–review and editing; Adriana Cătaş: Validation, resources, methodology, investigation, project administration, data curation, writing–review and editing, funding acquisition; Sheza M. El-Deeb: Validation, resources, methodology, investigation, project administration, data curation, writing–original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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