AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 25085–25136.
DOI: 10.3934/math.20251111
Received: 05 August 2025
Revised: 15 September 2025
Accepted: 24 September 2025
Published: 03 November 2025

Research article

Frameworks and implementation strategies for sustainable waste-to-energy alternatives: a study using bipolar complex intuitionistic fuzzy-based multi-attribute decision-making

Hariwan Z. Ibrahim[1,*] and Mesfer H. Alqahtani[2]

[1] Department of Mathematics, College of Education, University of Zakho, Kurdistan Region-Iraq
[2] Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk 48322, Saudi Arabia

* **Correspondence:** Email: hariwan_math@yahoo.com.

Abstract: This study presented a comprehensive multi-attribute decision-making framework founded on bipolar complex intuitionistic fuzzy numbers (BCIFNs) to enhance decision reliability under conditions of uncertainty and dual (positive-negative) evaluation. First, new operational laws were formulated for BCIFNs, providing a rigorous mathematical foundation for developing two aggregation mechanisms: the bipolar complex intuitionistic fuzzy weighted average (BCIFWA) and the bipolar complex intuitionistic fuzzy weighted geometric (BCIFWG) operators. These operators were systematically derived to ensure logical consistency, robustness, and suitability for handling uncertain, imprecise, and bipolar information. Building on these operators, a structured multi-attribute decision-making methodology was proposed to evaluate alternatives across multiple conflicting attributes. The approach was applied to a real-world case study on waste-to-energy (WtE) and circular economy solutions, focusing on identifying the most sustainable strategy for integrated waste management. Six WtE alternatives were assessed across environmental, economic, and social dimensions. The analysis revealed that incineration with energy recovery emerged as the most effective solution, offering a balance of high energy recovery, substantial waste volume reduction, and technological reliability. A comparative analysis with existing fuzzy decision-making methods highlighted the superior ranking stability and adaptability of the proposed BCIFWA and BCIFWG operators, particularly in complex evaluation scenarios with bipolar uncertainty. The results demonstrated the framework's flexibility, scalability, and enhanced decision precision, contributing to both the theoretical development of fuzzy multi-attribute decision-making and the practical advancement of sustainable waste management and circular economy strategies.

Keywords: bipolar complex intuitionistic fuzzy set; fuzzy aggregations; decision support systems
Mathematics Subject Classification: 03E72, 90B50

1. Introduction

Decision-making in modern engineering, environmental management, and industrial systems often involves complex uncertainties and multi-faceted evaluations, which cannot be effectively addressed using classical set theory alone. The advent of fuzzy set (FS) theory, first introduced by Zadeh [1], marked a significant shift by allowing partial membership degrees within the interval $[0, 1]$, thus enabling the representation of imprecise and vague information. While conventional FSs offered a foundational tool for uncertainty modeling, their inability to explicitly handle hesitation or dual preferences limited their expressiveness in practical decision-making scenarios. To overcome these limitations, Atanassov [2] proposed the intuitionistic fuzzy set (IFS), which augments the traditional membership function $\mathscr{MR}$ with a nonmembership function $\mathscr{NR}$. This enhancement allows the derivation of an explicit hesitation degree as $1 - (\mathscr{MR} + \mathscr{NR})$, capturing the uncertainty between full membership and nonmembership. IFSs have since been widely applied in expert systems, decision analysis, and multi-criteria evaluations, where hesitation and incomplete information play a critical role. Further advancements in uncertainty modeling emerged with the introduction of complex fuzzy sets (CFSs) by Ramot et al. [3]. In CFSs, the membership value of an element is expressed in complex form, $\mathscr{MR}e^{i\mathscr{Q}}$, where the magnitude $\mathscr{MR}$ reflects the traditional membership degree, and the phase term $\mathscr{Q}$ captures periodic or contextual variations. This formulation is particularly useful in time-dependent, cyclic, or oscillatory systems, such as signal processing, control systems, and dynamic decision-making environments. Building on this foundation, complex intuitionistic fuzzy sets (CIFSs) were introduced by Alkouri and Salleh [4], incorporating both membership and nonmembership functions in the complex domain. CIFSs provide an enriched modeling capacity, capturing phase-dependent hesitation and multidimensional uncertainty, making them suitable for medical diagnostics, pattern recognition, and artificial intelligence (AI)-based decision models. Meanwhile, the concept of bipolarity in FSs has offered another dimension for modeling positive and negative evidence simultaneously. Bipolar fuzzy sets (BFSs), proposed by Zhang [5, 6], introduced dual membership functions to represent supporting (positive) and opposing (negative) tendencies within the same framework. This dual nature has proven effective in multi-agent decision-making, cognitive modeling, and social network analysis. The approach was further refined through bipolar intuitionistic fuzzy sets (BIFSs) [7], which integrate bipolar membership and nonmembership functions, providing a richer depiction of uncertainty by addressing affirmative, opposing, and hesitant evaluations simultaneously.

The evolution of FS theory has given rise to a series of advanced models capable of capturing increasingly complex forms of uncertainty. Among these, bipolar complex fuzzy sets (BCFSs) extend both BFSs and CFSs by integrating dual membership functions with complex-valued representations. This combination enables a richer modeling framework, simultaneously accounting for positive and negative evaluations along with magnitude and phase information, making it highly suitable for environments with dynamic, oscillatory, or time-dependent behaviors. Several researchers have contributed to the theoretical development and practical applications of BCFSs. Alkouri et al. [8] examined the fundamental properties and potential applications of BCFSs, establishing their role in uncertainty handling within time-varying and cyclic systems. Al-Husban et al. [9] further investigated their mathematical characteristics, setting the stage for their use in decision-making and cognitive modeling. Gulistan et al. [10] demonstrated the practical utility of complex bipolar fuzzy sets in transportation decision problems, where positive and negative influences fluctuate dynamically.

Mahmood and Ur Rehman [11] expanded the application scope of BCFSs to similarity measures, showcasing their relevance in pattern recognition, decision analysis, and system optimization. Building upon this foundation, bipolar complex intuitionistic fuzzy sets (BCIFSs) introduce an additional layer of expressiveness by incorporating complex-valued membership and nonmembership functions, allowing decision-makers to capture both affirmative and opposing tendencies with explicit hesitation. Al-Husban [12] first provided the formal definition of BCIFSs, establishing their theoretical properties. Later, Alkouri and Alshboul [13] presented an alternative formulation, emphasizing their ability to represent environmental impact assessment data through the real and imaginary components of bipolar intuitionistic fuzzy information, thereby expanding their application to sustainability and multi-criteria evaluations

In recent years, these fuzzy extensions have found broad applications in multi-attribute decision-making (MADM) and intelligent decision support systems across engineering, environmental, healthcare, and industrial domains. Classical FSs and IFSs have been widely adopted for expert evaluations and multi-expert decision-making. For instance, Liu [14] proposed a fuzzy multi-expert decision algorithm for optimizing university labor education courses, while Liu and Wang [15] developed intuitionistic fuzzy aggregation operators for handling complex multi-attribute problems. Over time, researchers have broadened the scope of fuzzy modeling by embedding it into more sophisticated mathematical structures and specialized environments. For instance, significant progress has been achieved by introducing fuzzy concepts into fuzzy normed spaces [16], ordered semigroups [17], and semirings [18], as well as by extending them to structural representations such as m-polar fuzzy graphs [19]. In addition, recent research has demonstrated the versatility of fuzzy theory through diverse applications. Examples include the development of fractional fuzzy differential equations with applications in plasma physics [20], the fuzzification of rough sets with practical implementations in medical treatment selection [21], and their integration into multi-criteria group decision-making frameworks [22]. Further contributions involve the study of generalized fuzzy-valued convexity and the derivation of related inequalities [23], which expand the theoretical underpinnings of fuzzy analysis. Parallel to these advancements, fuzzy algebraic structures have also received considerable attention. Scholars have provided detailed characterizations of ordered semigroups within a fuzzy context [24] and introduced concepts such as m-polar fuzzy q-ideals in BCI-algebras [25], thereby enriching the algebraic foundations of fuzzy mathematics. On the application side, m-polar fuzzy graphs have proven valuable in addressing real-world optimization problems, such as efficient resource allocation [26] and robotic manufacturing task distribution using inverse graph methodologies [27]. Recent contributions have also addressed advanced decision environments, including renewable energy investment using spherical fuzzy MADM [28], and optimization in technology industries using generalized intuitionistic FSs [29]. The introduction of CFSs and CIFSs has enabled the modeling of phase-dependent and oscillatory uncertainty. Dai [30] presented linguistic complex FSs, and Khan et al. [31] applied complex T-spherical fuzzy aggregation operators in engineering MADM scenarios, complemented by studies on complex probabilistic hesitant fuzzy operators by Yang et al. [32]. BFSs and BIFSs have further enhanced decision-making by simultaneously representing positive and negative evaluations. These models have been effectively employed in unmanned aerial vehicle (UAV) software selection [33], AI-driven telecommunication decision models [34], supplier evaluation in sustainable supply chains [35], and medical diagnosis [36]. Additionally, studies have explored Aczel-Alsina power aggregation operators for complex intuitionistic fuzzy and bipolar fuzzy environments, with applications spanning industrial optimization and even quantum computing [37–39].

The integration of complex-valued representation into bipolar frameworks has led to BCFSs and BCIFSs, which are increasingly applied to high-stakes and dynamic decision contexts. Mahmood and Ur Rehman [40] introduced BCFS-based Maclaurin symmetric mean operators for MADM, while Zhao et al. [41] developed BCFS Hamy mean operators for industrial decision support. Emam et al. [42] demonstrated hesitant BCFS Hamacher power aggregation operators in AI-driven energy management systems. Healthcare and diagnostic applications have been addressed by Alolaiyan et al. [43], while Mahmood et al. [44] applied the technique for order preference by similarity to ideal solution (TOPSIS) in the BCIFS N-soft set framework. Advanced BCIFS structures have also been extended to graph-based models by Nandhinii and Amsaveni [45], providing tools for dynamic network and multi-agent decision problems. Collectively, these studies demonstrate that the progressive evolution from FSs to IFSs, CFSs, CIFSs, BFSs, BIFSs, BCFSs, and BCIFSs equips decision-makers with increasingly powerful tools for handling uncertainty, hesitation, and dual-sided evaluations across diverse real-world multi-attribute decision-making scenarios.

For clarity and ease of understanding, Table 1 provides a concise comparative overview of the main FS models, highlighting their key features and extensions in membership, bipolarity, and complex representation.

Table 1. Comparative overview of FS models.

Model	Reference	Key Features
FS	Zadeh [1]	Basic fuzzy set with membership values in $[0,1]$; handles imprecise data.
IFS	Atanassov [2]	Adds non-membership and hesitation degree to model uncertainty more effectively.
CFS	Ramot et al. [3]	Complex-valued membership; captures periodic or phase-dependent information.
CIFS	Alkouri and Salleh [4]	Complex extension of IFS, incorporating hesitation in the complex plane.
BFS	Zhang [5,6]	Bipolar extension of FS; models positive and negative membership simultaneously.
BIFS	Ezhilmaran and Sankar [7]	Bipolar extension of IFS; handles positive/negative membership and non-membership.
BCFS	Mahmood and Ur Rehman [11]	Bipolar complex extension of FS; includes complex-valued positive and negative memberships.
BCIFS	Alkouri and Alshboul [13]	Bipolar complex extension of IFS; fully models positive/negative, membership/non-membership in complex plane.

1.1. Motivation

Decision-making in complex real-world environments often requires handling multidimensional uncertainty, conflicting criteria, and both positive and negative evaluations. Traditional decision-making frameworks, based on classical set theory, fail to capture the nuanced interplay of hesitation, dual preferences, and phase-dependent variations. To overcome these limitations, FS theory and its extensions—such as IFSs, CFSs, and BFSs—have been developed to enhance the modeling of imprecision and multi-perspective evaluations. While IFSs capture hesitation, BFSs introduce dual positive-negative assessments, and CFSs incorporate magnitude-phase information, their standalone use cannot fully represent dynamic and contradictory decision environments.

Prior to the widespread use of fuzzy-based approaches, a variety of traditional decision-making methods have long been employed to address complex engineering and sustainability challenges, including waste-to-energy strategy selection. Classical tools such as cost-benefit analysis, and life-cycle assessment, have been useful for quantifying economic and environmental trade-offs, while conventional multi-criteria decision-making techniques—such as the Analytic Hierarchy Process, TOPSIS, and vlseKriterijumska optimizacija i kompromisno resenje (VIKOR)—have provided structured ranking and selection mechanisms. These approaches offer transparency and analytical clarity; however, they generally rely on deterministic data and precise expert judgments. This assumption limits their applicability in real-world contexts where uncertainty, vagueness, and conflicting stakeholder preferences are inherent. As highlighted in recent energy decision-making research, traditional methods often fail to capture dynamic interactions, heterogeneous behaviors, and the uncertain nature of sustainable energy systems [46]. Such limitations have driven the development of FS theory and its numerous extensions, which enable more effective modeling of hesitation, dual (positive-negative) assessments, and complex interdependencies among attributes in practical decision environments.

To unify these strengths, BCIFSs have emerged as a comprehensive framework capable of simultaneously accommodating dual (positive-negative) information, hesitation, and complex-valued uncertainty. This framework is particularly suitable for multi-attribute decision-making problems involving environmental, economic, and social trade-offs, where evaluations often reflect both supportive and opposing perspectives. In the context of sustainable energy and resource management, such as waste-to-energy alternatives within circular economy strategies, BCIF-based MADM methods provide a structured approach to evaluate options like anaerobic digestion, incineration with energy recovery, pyrolysis/gasification, refuse-derived fuel (RDF) production, landfill gas utilization, and integrated recycling hubs. By incorporating both uncertainty and dual-sided assessments, the proposed methodology enhances decision reliability, ranking stability, and the robustness of strategy selection under complex, real-world conditions. Consequently, this study aligns with the journal's focus on sustainable energy system planning and implementation by offering a flexible, computationally implementable decision-support framework that bridges theoretical advances in fuzzy decision-making with practical applications in energy and circular economy contexts.

1.2. Significant contributions

This research offers several key contributions to advancing MADM under uncertainty:

(1) Two novel aggregation operators—the bipolar complex intuitionistic fuzzy weighted average (BCIFWA) and weighted geometric (BCIFWG)—are developed to effectively integrate multi-criteria information while preserving the nuanced bipolar and complex features of the data.
(2) The BCIFS-based MADM framework is applied to assess waste-to-energy alternatives within a circular economy setting, demonstrating improved ranking stability, flexible uncertainty handling,

and enhanced decision precision for sustainable resource management.

(3) Comparative analysis with existing fuzzy MADM methods highlights the proposed framework's superior accuracy, robustness, and ability to capture complex interdependencies in decision data.

(4) Graphical visualizations of the BCIFWA and BCIFWG operators provide deeper insights into their computational behavior and effectiveness in modeling bipolar complex uncertainty in real-world decision scenarios.

(5) By combining bipolarity, intuitionistic fuzziness, and complex fuzzy representations, this work significantly enriches the theoretical foundations of fuzzy MADM and introduces a versatile tool for handling complex, uncertain, and contradictory decision problems.

(6) The study paves the way for further enhancements, including increased computational efficiency for large-scale applications, extensions to dynamic decision contexts, integration with machine learning techniques, and exploration of applications in financial risk analysis, medical diagnosis, and engineering optimization.

In summary, this research establishes BCIFS as a comprehensive and flexible framework that substantially advances MADM methodologies for complex and uncertain decision-making challenges.

1.3. Paper organization

The paper is organized as follows:

- Section 1 provides a comprehensive review of the relevant literature, highlighting existing research gaps that motivate the current study.
- Section 2 covers the necessary theoretical background, including detailed overviews of CIFS, BIFS, and BCIFS, along with their fundamental operations.
- Section 3 formulates new operational laws for bipolar complex intuitionistic fuzzy numbers, establishing the groundwork for developing aggregation techniques.
- Section 4 introduces the novel BCIFWA and BCIFWG aggregation operators and explores their key theoretical properties.
- Section 5 presents a structured MADM framework based on these operators and demonstrates its practical utility through a real-world WtE application.
- Section 6 conducts a comparative analysis between the proposed methods and existing fuzzy decision-making approaches, showcasing improvements in accuracy, robustness, and flexibility.
- Section 7 concludes the study by summarizing the major contributions and outlining potential directions for future research.

2. Preliminaries

This section establishes the foundation by introducing essential core concepts.

Definition 1. *[4] Considering a universal set $\mathscr{U}$, the set $\mathcal{D}$ is defined as follows:*

$$\mathcal{D} = \{\langle \beta, \mathscr{MR}(\beta), \mathscr{NR}(\beta)\rangle : \beta \in \mathscr{U}\},$$

where $\mathscr{MR} : \mathscr{U} \to \mathscr{MC} : \mathscr{MC} \in \mathcal{D}, |\mathscr{MC}| \leq 1$ and $\mathscr{NR} : \mathscr{U} \to \mathscr{NC} : \mathscr{NC} \in \mathcal{D}, |\mathscr{NC}| \leq 1$ satisfy the conditions:

$$\mathscr{MR}(\beta) = \mathscr{MC} = r_1 + im_1, \quad \mathscr{NR}(\beta) = \mathscr{NC} = r_2 + im_2,$$

with the constraint:

$$0 \leq |\mathscr{MC}| + |\mathscr{NC}| \leq 1.$$

In polar form, these functions are represented as

$$\mathscr{MR}(\beta) = \Phi_{\mathcal{D}}(\beta)e^{i2\pi\mathscr{MI}_{\mathcal{D}}(\beta)}, \text{ and } \mathscr{NR}(\beta) = \Theta_{\mathcal{D}}(\beta)e^{i2\pi\mathscr{NI}_{\mathcal{D}}(\beta)},$$

subject to

$$0 \leq \Phi_{\mathcal{D}}(\beta) + \Theta_{\mathcal{D}}(\beta) \leq 1, \text{ and } 0 \leq \mathscr{MI}_{\mathcal{D}}(\beta) + \mathscr{NI}_{\mathcal{D}}(\beta) \leq 1.$$

Here, $\Phi_{\mathcal{D}}, \Theta_{\mathcal{D}}, \mathscr{MI}_{\mathcal{D}}, \mathscr{NI}_{\mathcal{D}} \in [0,1]$, *with* $i = \sqrt{-1}$. *Then,* $\mathcal{D}$ *is called a CIFS.*

Definition 2. *[7] Considering a universal set* $\mathscr{U}$, *the set* $\mathcal{D}$ *is defined as follows:*

$$\mathcal{D} = \{\langle\beta, \mathscr{MR}^+_{\mathcal{D}}(\beta), \mathscr{NR}^+_{\mathcal{D}}(\beta), \mathscr{MR}^-_{\mathcal{D}}(\beta), \mathscr{NR}^-_{\mathcal{D}}(\beta)\rangle : \beta \in \mathscr{U}\},$$

then $\mathcal{D}$ *is called BIFS if* $0 \leq \mathscr{MR}^+_{\mathcal{D}} + \mathscr{NR}^+_{\mathcal{D}} \leq 1$, *and* $-1 \leq \mathscr{MR}^-_{\mathcal{D}} + \mathscr{NR}^-_{\mathcal{D}} \leq 0$.

Definition 3. *[12] Considering a universal set* $\mathscr{U}$, *the set* $\mathcal{D}$ *is defined as follows:* $\mathcal{D} = \left\{\left\langle\beta, \mathscr{MR}^+_{\mathcal{D}}(\beta)e^{i\mathscr{AM}^+_{\mathcal{D}}(\beta)}, \mathscr{NR}^+_{\mathcal{D}}(\beta)e^{i\mathscr{BN}^+_{\mathcal{D}}(\beta)}, \mathscr{MR}^-_{\mathcal{D}}(\beta)e^{i\mathscr{AM}^-_{\mathcal{D}}(\beta)}, \mathscr{NR}^-_{\mathcal{D}}(\beta)e^{i\mathscr{BN}^-_{\mathcal{D}}(\beta)}\right\rangle : \beta \in \mathscr{U}\right\}$, *then* $\mathcal{D}$ *is called a BCIFS if*

$$0 \leq \mathscr{MR}^+_{\mathcal{D}}(\beta) + \mathscr{NR}^+_{\mathcal{D}}(\beta) \leq 1, \quad 0 < \mathscr{AM}^+_{\mathcal{D}}(\beta) + \mathscr{BN}^+_{\mathcal{D}}(\beta) \leq 2\pi,$$

and

$$-1 \leq \mathscr{MR}^-_{\mathcal{D}}(\beta) + \mathscr{NR}^-_{\mathcal{D}}(\beta) \leq 0, \quad 0 < |\mathscr{AM}^-_{\mathcal{D}}(\beta)| + |\mathscr{BN}^-_{\mathcal{D}}(\beta)| \leq 2\pi,$$

where

$$\mathscr{MR}^+_{\mathcal{D}}(\beta),\ \mathscr{NR}^+_{\mathcal{D}}(\beta) \in [0,1], \quad \mathscr{MR}^-_{\mathcal{D}}(\beta),\ \mathscr{NR}^-_{\mathcal{D}}(\beta) \in [-1,0],$$

and

$$\mathscr{AM}^+_{\mathcal{D}}(\beta),\ \mathscr{BN}^+_{\mathcal{D}}(\beta), -\mathscr{AM}^-_{\mathcal{D}}(\beta),\ -\mathscr{BN}^-_{\mathcal{D}}(\beta) \in (0, 2\pi].$$

Definition 4. *[13] Considering a universal set* $\mathscr{U}$, *the set* $\mathcal{D}$ *is defined as follows:*

$$\mathcal{D} = \{\langle\beta, \mathscr{MR}^+_{\mathcal{D}}(\beta) + i\mathscr{MI}^+_{\mathcal{D}}(\beta), \mathscr{NR}^+_{\mathcal{D}}(\beta) + i\mathscr{NI}^+_{\mathcal{D}}(\beta),$$
$$\mathscr{MR}^-_{\mathcal{D}}(\beta) + i\mathscr{MI}^-_{\mathcal{D}}(\beta), \mathscr{NR}^-_{\mathcal{D}}(\beta) + i\mathscr{NI}^-_{\mathcal{D}}(\beta)\rangle : \beta \in \mathscr{U}\},$$

where

$$\mathscr{MR}^+_{\mathcal{D}}(\beta),\ \mathscr{NR}^+_{\mathcal{D}}(\beta) \in [0,1] \quad \text{(amplitude positive components)},$$
$$\mathscr{MI}^+_{\mathcal{D}}(\beta),\ \mathscr{NI}^+_{\mathcal{D}}(\beta) \in [0,1] \quad \text{(phase positive components)},$$

with

$$0 \leq (\mathscr{MR}^+_{\mathcal{D}}(\beta)) + (\mathscr{NR}^+_{\mathcal{D}}(\beta)) \leq 1, \quad 0 \leq (\mathscr{MI}^+_{\mathcal{D}}(\beta)) + (\mathscr{NI}^+_{\mathcal{D}}(\beta)) \leq 1,$$

and

$$\mathscr{MR}^-_{\mathcal{D}}(\beta),\ \mathscr{NR}^-_{\mathcal{D}}(\beta) \in [-1,0] \quad \text{(amplitude negative components)},$$
$$\mathscr{MI}^-_{\mathcal{D}}(\beta),\ \mathscr{NI}^-_{\mathcal{D}}(\beta) \in [-1,0] \quad \text{(phase negative components)},$$

with

$$0 \le |\mathscr{MR}^{-}_{\mathcal{D}}(\beta)| + |\mathscr{NR}^{-}_{\mathcal{D}}(\beta)| \le 1, \quad 0 \le |\mathscr{MI}^{-}_{\mathcal{D}}(\beta)| + |\mathscr{NI}^{-}_{\mathcal{D}}(\beta)| \le 1.$$

Then, $\mathcal{D}$ is called a BCIFS, and the value of β is computed as

$$\mathcal{D} = \langle \mathscr{MR}^{+}_{\mathcal{D}} + i\mathscr{MI}^{+}_{\mathcal{D}}, \mathscr{NR}^{+}_{\mathcal{D}} + i\mathscr{NI}^{+}_{\mathcal{D}}, \mathscr{MR}^{-}_{\mathcal{D}} + i\mathscr{MI}^{-}_{\mathcal{D}}, \mathscr{NR}^{-}_{\mathcal{D}} + i\mathscr{NI}^{-}_{\mathcal{D}} \rangle,$$

which represents the BCIF number (BCIFN).

Figure 1 depicts the graded space of bipolar complex intuitionistic fuzzy values, emphasizing the interplay between amplitude and phase elements in both positive and negative domains.

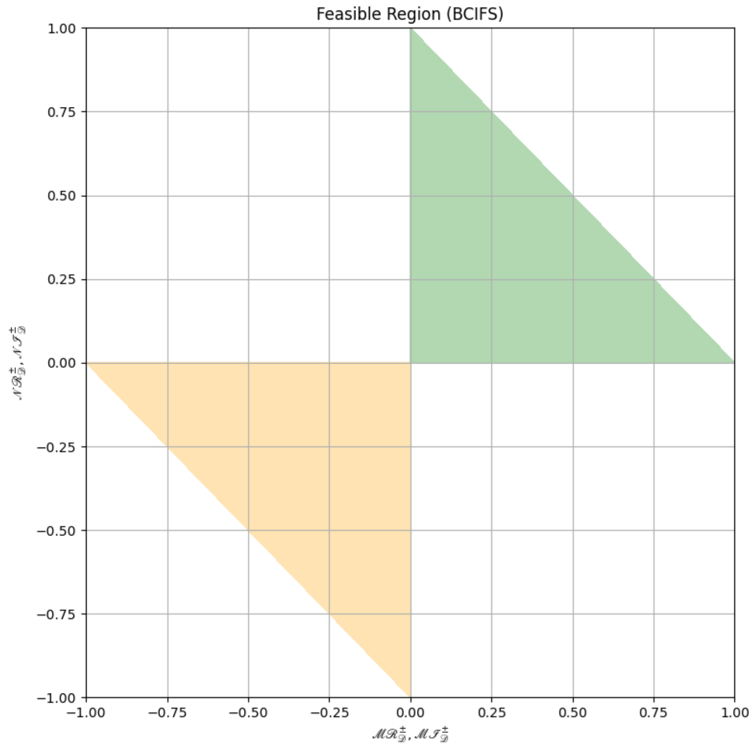

Figure 1. Grades space of BCIF values.

Definition 5. *[13] Let two BCIFNs be given as*

$$\mathcal{D}_1 = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_1} + i\mathscr{MI}^{+}_{\mathcal{D}_1}, \mathscr{NR}^{+}_{\mathcal{D}_1} + i\mathscr{NI}^{+}_{\mathcal{D}_1}, \mathscr{MR}^{-}_{\mathcal{D}_1} + i\mathscr{MI}^{-}_{\mathcal{D}_1}, \mathscr{NR}^{-}_{\mathcal{D}_1} + i\mathscr{NI}^{-}_{\mathcal{D}_1} \right\rangle,$$

and

$$\mathcal{D}_2 = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_2} + i\mathscr{MI}^{+}_{\mathcal{D}_2}, \mathscr{NR}^{+}_{\mathcal{D}_2} + i\mathscr{NI}^{+}_{\mathcal{D}_2}, \mathscr{MR}^{-}_{\mathcal{D}_2} + i\mathscr{MI}^{-}_{\mathcal{D}_2}, \mathscr{NR}^{-}_{\mathcal{D}_2} + i\mathscr{NI}^{-}_{\mathcal{D}_2} \right\rangle.$$

Then

(1) $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if and only if

$$\mathscr{MR}^{+}_{\mathcal{D}_1} \le \mathscr{MR}^{+}_{\mathcal{D}_2}, \quad \mathscr{MR}^{-}_{\mathcal{D}_1} \ge \mathscr{MR}^{-}_{\mathcal{D}_2}, \quad \mathscr{NR}^{+}_{\mathcal{D}_1} \ge \mathscr{NR}^{+}_{\mathcal{D}_2}, \quad \mathscr{NR}^{-}_{\mathcal{D}_1} \le \mathscr{NR}^{-}_{\mathcal{D}_2},$$

and

$$\mathscr{MI}^{+}_{\mathcal{D}_1} \le \mathscr{MI}^{+}_{\mathcal{D}_2}, \quad \mathscr{MI}^{-}_{\mathcal{D}_1} \ge \mathscr{MI}^{-}_{\mathcal{D}_2}, \quad \mathscr{NI}^{+}_{\mathcal{D}_1} \ge \mathscr{NI}^{+}_{\mathcal{D}_2}, \quad \mathscr{NI}^{-}_{\mathcal{D}_1} \le \mathscr{NI}^{-}_{\mathcal{D}_2}.$$

(2) $\mathcal{D}_1 = \mathcal{D}_2$ *if and only if*

$$\mathscr{MR}^+_{\mathcal{D}_1} = \mathscr{MR}^+_{\mathcal{D}_2}, \quad \mathscr{MR}^-_{\mathcal{D}_1} = \mathscr{MR}^-_{\mathcal{D}_2}, \quad \mathscr{NR}^+_{\mathcal{D}_1} = \mathscr{NR}^+_{\mathcal{D}_2}, \quad \mathscr{NR}^-_{\mathcal{D}_1} = \mathscr{NR}^-_{\mathcal{D}_2},$$
$$\mathscr{MI}^+_{\mathcal{D}_1} = \mathscr{MI}^+_{\mathcal{D}_2}, \quad \mathscr{MI}^-_{\mathcal{D}_1} = \mathscr{MI}^-_{\mathcal{D}_2}, \quad \mathscr{NI}^+_{\mathcal{D}_1} = \mathscr{NI}^+_{\mathcal{D}_2}, \quad \mathscr{NI}^-_{\mathcal{D}_1} = \mathscr{NI}^-_{\mathcal{D}_2}.$$

(3) $\mathcal{D}_1{}^c = \langle \mathscr{NR}^+_{\mathcal{D}_1} + i\mathscr{NI}^+_{\mathcal{D}_1}, \mathscr{MR}^+_{\mathcal{D}_1} + i\mathscr{MI}^+_{\mathcal{D}_1}, -|\mathscr{NR}^-_{\mathcal{D}_1}| + i(-|\mathscr{NI}^-_{\mathcal{D}_1}|), -|\mathscr{MR}^-_{\mathcal{D}_1}| + i(-|\mathscr{MI}^-_{\mathcal{D}_1}|)\rangle$, *where*

$$\mathscr{MR}^-_{\mathcal{D}} = -\left|\mathscr{MR}^-_{\mathcal{D}}\right|, \quad \mathscr{NR}^-_{\mathcal{D}} = -\left|\mathscr{NR}^-_{\mathcal{D}}\right|,$$
$$\mathscr{MI}^-_{\mathcal{D}} = -\left|\mathscr{MI}^-_{\mathcal{D}}\right|, \quad \mathscr{NI}^-_{\mathcal{D}} = -\left|\mathscr{NI}^-_{\mathcal{D}}\right|.$$

(4) $\mathcal{D}_1 \cap \mathcal{D}_2 = \begin{array}{l} \langle \min\{\mathscr{MR}^+_{\mathcal{D}_1}, \mathscr{MR}^+_{\mathcal{D}_2}\} + i\min\{\mathscr{MI}^+_{\mathcal{D}_1}, \mathscr{MI}^+_{\mathcal{D}_2}\}, \\ \max\{\mathscr{NR}^+_{\mathcal{D}_1}, \mathscr{NR}^+_{\mathcal{D}_2}\} + i\max\{\mathscr{NI}^+_{\mathcal{D}_1}, \mathscr{NI}^+_{\mathcal{D}_2}\}, \\ \max\{\mathscr{MR}^-_{\mathcal{D}_1}, \mathscr{MR}^-_{\mathcal{D}_2}\} + i\max\{\mathscr{MI}^-_{\mathcal{D}_1}, \mathscr{MI}^-_{\mathcal{D}_2}\}, \\ \min\{\mathscr{NR}^-_{\mathcal{D}_1}, \mathscr{NR}^-_{\mathcal{D}_2}\} + i\min\{\mathscr{NI}^-_{\mathcal{D}_1}, \mathscr{NI}^-_{\mathcal{D}_2}\}\rangle. \end{array}$

(5) $\mathcal{D}_1 \cup \mathcal{D}_2 = \begin{array}{l} \langle \max\{\mathscr{MR}^+_{\mathcal{D}_1}, \mathscr{MR}^+_{\mathcal{D}_2}\} + i\max\{\mathscr{MI}^+_{\mathcal{D}_1}, \mathscr{MI}^+_{\mathcal{D}_2}\}, \\ \min\{\mathscr{NR}^+_{\mathcal{D}_1}, \mathscr{NR}^+_{\mathcal{D}_2}\} + i\min\{\mathscr{NI}^+_{\mathcal{D}_1}, \mathscr{NI}^+_{\mathcal{D}_2}\}, \\ \min\{\mathscr{MR}^-_{\mathcal{D}_1}, \mathscr{MR}^-_{\mathcal{D}_2}\} + i\min\{\mathscr{MI}^-_{\mathcal{D}_1}, \mathscr{MI}^-_{\mathcal{D}_2}\}, \\ \max\{\mathscr{NR}^-_{\mathcal{D}_1}, \mathscr{NR}^-_{\mathcal{D}_2}\} + i\max\{\mathscr{NI}^-_{\mathcal{D}_1}, \mathscr{NI}^-_{\mathcal{D}_2}\}\rangle. \end{array}$

3. Operational framework of bipolar complex intuitionistic fuzzy numbers

This section presents the essential operations and foundational mechanisms for handling bipolar complex intuitionistic fuzzy numbers, which form the basis for subsequent aggregation and decision-making processes.

Definition 6. *Let* $\mathcal{D} = \langle \mathscr{MR}^+_{\mathcal{D}} + i\mathscr{MI}^+_{\mathcal{D}}, \mathscr{NR}^+_{\mathcal{D}} + i\mathscr{NI}^+_{\mathcal{D}}, \mathscr{MR}^-_{\mathcal{D}} + i\mathscr{MI}^-_{\mathcal{D}}, \mathscr{NR}^-_{\mathcal{D}} + i\mathscr{NI}^-_{\mathcal{D}}\rangle$, $\mathcal{D}_1 = \langle \mathscr{MR}^+_{\mathcal{D}_1} + i\mathscr{MI}^+_{\mathcal{D}_1}, \mathscr{NR}^+_{\mathcal{D}_1} + i\mathscr{NI}^+_{\mathcal{D}_1}, \mathscr{MR}^-_{\mathcal{D}_1} + i\mathscr{MI}^-_{\mathcal{D}_1}, \mathscr{NR}^-_{\mathcal{D}_1} + i\mathscr{NI}^-_{\mathcal{D}_1}\rangle$, *and* $\mathcal{D}_2 = \langle \mathscr{MR}^+_{\mathcal{D}_2} + i\mathscr{MI}^+_{\mathcal{D}_2}, \mathscr{NR}^+_{\mathcal{D}_2} + i\mathscr{NI}^+_{\mathcal{D}_2}, \mathscr{MR}^-_{\mathcal{D}_2} + i\mathscr{MI}^-_{\mathcal{D}_2}, \mathscr{NR}^-_{\mathcal{D}_2} + i\mathscr{NI}^-_{\mathcal{D}_2}\rangle$ *be three BCIFNs, and* $\mathscr{S} > 0$. *Then,*

$$\begin{aligned} \mathcal{D}_1 \oplus \mathcal{D}_2 = \Big\langle & \left(\mathscr{MR}^+_{\mathcal{D}_1} + \mathscr{MR}^+_{\mathcal{D}_2} - \mathscr{MR}^+_{\mathcal{D}_1}\mathscr{MR}^+_{\mathcal{D}_2}\right) + i\left(\mathscr{MI}^+_{\mathcal{D}_1} + \mathscr{MI}^+_{\mathcal{D}_2} - \mathscr{MI}^+_{\mathcal{D}_1}\mathscr{MI}^+_{\mathcal{D}_2}\right), \\ & \mathscr{NR}^+_{\mathcal{D}_1}\mathscr{NR}^+_{\mathcal{D}_2} + i\left(\mathscr{NI}^+_{\mathcal{D}_1}\mathscr{NI}^+_{\mathcal{D}_2}\right), -\left(\mathscr{MR}^-_{\mathcal{D}_1}\mathscr{MR}^-_{\mathcal{D}_2}\right) + i\left(-\mathscr{MI}^-_{\mathcal{D}_1}\mathscr{MI}^-_{\mathcal{D}_2}\right), \\ & -\left(\left|\mathscr{NR}^-_{\mathcal{D}_1}\right| + \left|\mathscr{NR}^-_{\mathcal{D}_2}\right| - \left|\mathscr{NR}^-_{\mathcal{D}_1}\right|\left|\mathscr{NR}^-_{\mathcal{D}_2}\right|\right) + i\left(-\left(\left|\mathscr{NI}^-_{\mathcal{D}_1}\right| + \left|\mathscr{NI}^-_{\mathcal{D}_2}\right| - \left|\mathscr{NI}^-_{\mathcal{D}_1}\right|\left|\mathscr{NI}^-_{\mathcal{D}_2}\right|\right)\right)\Big\rangle \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathcal{D}_1 \otimes \mathcal{D}_2 = \Big\langle & \left(\mathscr{MR}^+_{\mathcal{D}_1}\mathscr{MR}^+_{\mathcal{D}_2}\right) + i\left(\mathscr{MI}^+_{\mathcal{D}_1}\mathscr{MI}^+_{\mathcal{D}_2}\right), \left(\mathscr{NR}^+_{\mathcal{D}_1} + \mathscr{NR}^+_{\mathcal{D}_2} - \mathscr{NR}^+_{\mathcal{D}_1}\mathscr{NR}^+_{\mathcal{D}_2}\right) \\ & + i\left(\mathscr{NI}^+_{\mathcal{D}_1} + \mathscr{NI}^+_{\mathcal{D}_2} - \mathscr{NI}^+_{\mathcal{D}_1}\mathscr{NI}^+_{\mathcal{D}_2}\right), -\left(\left|\mathscr{MR}^-_{\mathcal{D}_1}\right| + \left|\mathscr{MR}^-_{\mathcal{D}_2}\right| - \left|\mathscr{MR}^-_{\mathcal{D}_1}\right|\left|\mathscr{MR}^-_{\mathcal{D}_2}\right|\right) \\ & + i\left(-\left(\left|\mathscr{MI}^-_{\mathcal{D}_1}\right| + \left|\mathscr{MI}^-_{\mathcal{D}_2}\right| - \left|\mathscr{MI}^-_{\mathcal{D}_1}\right|\left|\mathscr{MI}^-_{\mathcal{D}_2}\right|\right)\right), -\left(\mathscr{NR}^-_{\mathcal{D}_1}\mathscr{NR}^-_{\mathcal{D}_2}\right) + i\left(-\mathscr{NI}^-_{\mathcal{D}_1}\mathscr{NI}^-_{\mathcal{D}_2}\right)\Big\rangle, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \mathscr{S}\mathcal{D} = \Big\langle & \left(1 - (1 - \mathscr{MR}^+_{\mathcal{D}})^{\mathscr{S}}\right) + i\left(1 - (1 - \mathscr{MI}^+_{\mathcal{D}})^{\mathscr{S}}\right), (\mathscr{NR}^+_{\mathcal{D}})^{\mathscr{S}} + i(\mathscr{NI}^+_{\mathcal{D}})^{\mathscr{S}}, \\ & -\left|\mathscr{MR}^-_{\mathcal{D}}\right|^{\mathscr{S}} + i\left(-\left|\mathscr{MI}^-_{\mathcal{D}}\right|^{\mathscr{S}}\right), -\left(1 - \left(1 - \left|\mathscr{NR}^-_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right) + i\left(-(1 - \left(1 - \left|\mathscr{NI}^-_{\mathcal{D}}\right|\right)^{\mathscr{S}})\right)\Big\rangle, \end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathcal{D}^{\mathscr{S}} = \Big\langle & (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}})^{\mathscr{S}} + i\,(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}})^{\mathscr{S}}, \left(1-(1-\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}})^{\mathscr{S}}\right) + i\left(1-(1-\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}})^{\mathscr{S}}\right), \\ & -\left(1-\left(1-\left|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right) + i\left(-(1-\left(1-\left|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}})\right), -\left|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}}\right|^{\mathscr{S}} + i\left(-\left|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}}\right|^{\mathscr{S}}\right)\Big\rangle.\end{aligned} \tag{3.4}$$

Example 1. Two BCIFNs are defined as follows:

$$\mathcal{D}_1 = \langle .36 + i(.48), .48 + i(.52), -.51 + i(-.36), -.48 + i(-.52)\rangle,$$

and

$$\mathcal{D}_2 = \langle .33 + i(.36), .51 + i(.48), -.36 + i(-.52), -.52 + i(-.36)\rangle.$$

For $\mathscr{S} = 5$, the resulting values are

(1)

$$\begin{aligned}& \mathcal{D}_1 \oplus \mathcal{D}_2 \\ = & \Big\langle\Big(\Big(\mathscr{MR}^{+}_{\mathcal{D}_1}\Big)+\Big(\mathscr{MR}^{+}_{\mathcal{D}_2}\Big)-\Big(\mathscr{MR}^{+}_{\mathcal{D}_1}\Big)\Big(\mathscr{MR}^{+}_{\mathcal{D}_2}\Big)\Big)+i\Big(\Big(\mathscr{MI}^{+}_{\mathcal{D}_1}\Big)+\Big(\mathscr{MI}^{+}_{\mathcal{D}_2}\Big)-\Big(\mathscr{MI}^{+}_{\mathcal{D}_1}\Big)\Big(\mathscr{MI}^{+}_{\mathcal{D}_2}\Big)\Big), \\ & \mathscr{NR}^{+}_{\mathcal{D}_1}\mathscr{NR}^{+}_{\mathcal{D}_2} + i\left(\mathscr{NI}^{+}_{\mathcal{D}_1}\mathscr{NI}^{+}_{\mathcal{D}_2}\right), -\left(\mathscr{MR}^{-}_{\mathcal{D}_1}\mathscr{MR}^{-}_{\mathcal{D}_2}\right) + i\left(-\left(\mathscr{MI}^{-}_{\mathcal{D}_1}\mathscr{MI}^{-}_{\mathcal{D}_2}\right)\right), \\ & -\left(\left|\mathscr{NR}^{-}_{\mathcal{D}_1}\right|+\left|\mathscr{NR}^{-}_{\mathcal{D}_2}\right|-\left|\mathscr{NR}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NR}^{-}_{\mathcal{D}_2}\right|\right)+i\left(-\left(\left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right|+\left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right|-\left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right|\right)\right)\Big\rangle \\ = & \langle((.36) + (.33) - (.36)(.33)) + i((.48) + (.36) - (.48)(.36)), (.48)(.51) + i((.52)(.48)), \\ & -((-.51)(-.36)) + i(-((-.36)(-.52))), -(|-.48| + |-.52| - |-.48||-.52|) \\ & + i(-(|-.52| + |-.36| - |-.52||-.36|))\rangle \\ \approx & \langle .5712 + .6672i, .2448 + .2496i, -.1836 + (-.1872)i, -.7504 + (-.6928)i\rangle.\end{aligned} \tag{3.5}$$

(2)

$$\begin{aligned}& \mathcal{D}_1 \otimes \mathcal{D}_2 \\ = & \Big\langle\left(\mathscr{MR}^{+}_{\mathcal{D}_1}\mathscr{MR}^{+}_{\mathcal{D}_2}\right) + i\left(\mathscr{MI}^{+}_{\mathcal{D}_1}\mathscr{MI}^{+}_{\mathcal{D}_2}\right), \Big(\Big(\mathscr{NR}^{+}_{\mathcal{D}_1}\Big) + \Big(\mathscr{NR}^{+}_{\mathcal{D}_2}\Big) - \Big(\mathscr{NR}^{+}_{\mathcal{D}_1}\Big)\Big(\mathscr{NR}^{+}_{\mathcal{D}_2}\Big)\Big) \\ & + i\Big(\Big(\mathscr{NI}^{+}_{\mathcal{D}_1}\Big)+\Big(\mathscr{NI}^{+}_{\mathcal{D}_2}\Big)-\Big(\mathscr{NI}^{+}_{\mathcal{D}_1}\Big)\Big(\mathscr{NI}^{+}_{\mathcal{D}_2}\Big)\Big), -\left(\left|\mathscr{MR}^{-}_{\mathcal{D}_1}\right|+\left|\mathscr{MR}^{-}_{\mathcal{D}_2}\right|-\left|\mathscr{MR}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MR}^{-}_{\mathcal{D}_2}\right|\right) \\ & + i\left(-\left(\left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right|+\left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right|-\left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right|\right)\right), -\left(\mathscr{NR}^{-}_{\mathcal{D}_1}\mathscr{NR}^{-}_{\mathcal{D}_2}\right)+i\left(-\left(\mathscr{NI}^{-}_{\mathcal{D}_1}\mathscr{NI}^{-}_{\mathcal{D}_2}\right)\right)\Big\rangle \\ = & \langle((.36)(.33))+i((.48)(.36)), ((.48)+(.51)-((.48)(.51)))+i((.52)+(.48)-((.52)(.48))), \\ & -(|-.51| + |-.36| - |-.51||-.36|) + i(-(|-.36| + |-.52| - |-.36||-.52|)), \\ & -((-.48)(-.52)) + i(-((-.52)(-.36)))\rangle \\ \approx & \langle .1188 + .1728i, .7452 + .7504i, -.6864 + (-.6928)i, -.2496 + (-.1872)i\rangle.\end{aligned} \tag{3.6}$$

(3)

$$\begin{aligned}5\mathcal{D}_1 = & \Big\langle(1 - (1 - (.36))^5) + i(1 - (1 - (.48))^5), (.48)^5 + i(.52)^5, -|-.51|^5 + i(-|-.36|^5), \\ & -(1 - (1 - |-.48|)^5) + i(-(1 - (1 - |-.52|)^5))\Big\rangle \\ \approx & \langle .8926 + .9620i, .0255 + .0380i, -.0345 + (-.0060)i, -.9620 + (-.9745)i\rangle.\end{aligned} \tag{3.7}$$

(4)

$$\mathcal{D}_1^5 = \Big\langle (.36)^5 + i(.48)^5, (1-(1-(.48))^5) + i(1-(1-(.52))^5), -(1-(1-|-.51|)^5) + i(-(1-(1-|-.36|)^5)), -|-.48|^5 + i(-|-.52|^5) \Big\rangle \approx \langle .0060 + .0255i, .9620 + .9745i, -.9718 + (-.8926)i, -.0255 + (-.0380)i \rangle. \tag{3.8}$$

Theorem 1. *Let two BCIFNs $\mathcal{D}_1$ and $\mathcal{D}_2$ be expressed as*

$$\mathcal{D}_1 = \Big\langle \mathcal{MR}^+_{\mathcal{D}_1} + i\mathcal{MI}^+_{\mathcal{D}_1}, \mathcal{NR}^+_{\mathcal{D}_1} + i\mathcal{NI}^+_{\mathcal{D}_1}, \mathcal{MR}^-_{\mathcal{D}_1} + i\mathcal{MI}^-_{\mathcal{D}_1}, \mathcal{NR}^-_{\mathcal{D}_1} + i\mathcal{NI}^-_{\mathcal{D}_1} \Big\rangle,$$

and

$$\mathcal{D}_2 = \Big\langle \mathcal{MR}^+_{\mathcal{D}_2} + i\mathcal{MI}^+_{\mathcal{D}_2}, \mathcal{NR}^+_{\mathcal{D}_2} + i\mathcal{NI}^+_{\mathcal{D}_2}, \mathcal{MR}^-_{\mathcal{D}_2} + i\mathcal{MI}^-_{\mathcal{D}_2}, \mathcal{NR}^-_{\mathcal{D}_2} + i\mathcal{NI}^-_{\mathcal{D}_2} \Big\rangle.$$

Consequently, $\mathcal{D}_1 \oplus \mathcal{D}_2$ and $\mathcal{D}_1 \otimes \mathcal{D}_2$ remain valid BCIFNs.

Proof. For the BCIFNs $\mathcal{D}_1$ and $\mathcal{D}_2$, the following relations are satisfied

$$\begin{aligned}
&0 \le \left(\mathcal{MR}^+_{\mathcal{D}_1}\right) \le 1, \quad 0 \le \left(\mathcal{NR}^+_{\mathcal{D}_1}\right) \le 1, \quad 0 \le \left(\mathcal{MR}^+_{\mathcal{D}_1}\right) + \left(\mathcal{NR}^+_{\mathcal{D}_1}\right) \le 1,\\
&0 \le \left|\mathcal{MR}^-_{\mathcal{D}_1}\right| \le 1, \quad 0 \le \left|\mathcal{NR}^-_{\mathcal{D}_1}\right| \le 1, \quad 0 \le \left|\mathcal{MR}^-_{\mathcal{D}_1}\right| + \left|\mathcal{NR}^-_{\mathcal{D}_1}\right| \le 1,\\
&0 \le \left(\mathcal{MR}^+_{\mathcal{D}_2}\right) \le 1, \quad 0 \le \left(\mathcal{NR}^+_{\mathcal{D}_2}\right) \le 1, \quad 0 \le \left(\mathcal{MR}^+_{\mathcal{D}_2}\right) + \left(\mathcal{NR}^+_{\mathcal{D}_2}\right) \le 1,\\
&0 \le \left|\mathcal{MR}^-_{\mathcal{D}_2}\right| \le 1, \quad 0 \le \left|\mathcal{NR}^-_{\mathcal{D}_2}\right| \le 1, \quad 0 \le \left|\mathcal{MR}^-_{\mathcal{D}_2}\right| + \left|\mathcal{NR}^-_{\mathcal{D}_2}\right| \le 1.
\end{aligned}$$

Furthermore, note that

$$\begin{aligned}
&0 \le \mathcal{MI}^+_{\mathcal{D}_1} \le 1, \quad 0 \le \mathcal{MI}^+_{\mathcal{D}_2} \le 1, \quad 0 \le \mathcal{NI}^+_{\mathcal{D}_1} \le 1, \quad 0 \le \mathcal{NI}^+_{\mathcal{D}_2} \le 1,\\
&0 \le (\mathcal{MI}^+_{\mathcal{D}_1}) + (\mathcal{NI}^+_{\mathcal{D}_1}) \le 1, \quad 0 \le (\mathcal{MI}^+_{\mathcal{D}_2}) + (\mathcal{NI}^+_{\mathcal{D}_2}) \le 1,\\
&0 \le |\mathcal{NI}^-_{\mathcal{D}_1}| \le 1, \quad 0 \le |\mathcal{NI}^-_{\mathcal{D}_2}| \le 1, \quad 0 \le |\mathcal{MI}^-_{\mathcal{D}_1}| \le 1, \quad 0 \le |\mathcal{MI}^-_{\mathcal{D}_2}| \le 1,\\
&0 \le |\mathcal{MI}^-_{\mathcal{D}_1}| + |\mathcal{NI}^-_{\mathcal{D}_1}| \le 1, \quad 0 \le |\mathcal{MI}^-_{\mathcal{D}_2}| + |\mathcal{NI}^-_{\mathcal{D}_2}| \le 1.
\end{aligned}$$

Then, the following inequalities are consequently derived:

$$\begin{aligned}
&\left(\mathcal{MR}^+_{\mathcal{D}_1}\right) \ge \left(\mathcal{MR}^+_{\mathcal{D}_1}\right)\left(\mathcal{MR}^+_{\mathcal{D}_2}\right), \quad \left(\mathcal{MR}^+_{\mathcal{D}_2}\right) \ge \left(\mathcal{MR}^+_{\mathcal{D}_1}\right)\left(\mathcal{MR}^+_{\mathcal{D}_2}\right),\\
&1 \ge \left(\mathcal{MR}^+_{\mathcal{D}_1}\right)\left(\mathcal{MR}^+_{\mathcal{D}_2}\right) \ge 0, \quad \left(\mathcal{NR}^+_{\mathcal{D}_1}\right) \ge \left(\mathcal{NR}^+_{\mathcal{D}_1}\right)\left(\mathcal{NR}^+_{\mathcal{D}_2}\right),\\
&\left(\mathcal{NR}^+_{\mathcal{D}_2}\right) \ge \left(\mathcal{NR}^+_{\mathcal{D}_1}\right)\left(\mathcal{NR}^+_{\mathcal{D}_2}\right), \quad 1 \ge \left(\mathcal{NR}^+_{\mathcal{D}_1}\right)\left(\mathcal{NR}^+_{\mathcal{D}_2}\right) \ge .
\end{aligned}$$

From these relationships, it follows that:

$$\left(\mathcal{MR}^+_{\mathcal{D}_1}\right) + \left(\mathcal{MR}^+_{\mathcal{D}_2}\right) - \left(\mathcal{MR}^+_{\mathcal{D}_1}\right)\left(\mathcal{MR}^+_{\mathcal{D}_2}\right) \ge 0,$$

which consequently yields

$$\left(\left(\mathcal{MR}^+_{\mathcal{D}_1}\right) + \left(\mathcal{MR}^+_{\mathcal{D}_2}\right) - \left(\mathcal{MR}^+_{\mathcal{D}_1}\right)\left(\mathcal{MR}^+_{\mathcal{D}_2}\right)\right) \ge .$$

In a similar manner, the following holds

$$\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\geq 0,$$

which implies

$$\left(\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)\geq .$$

Given that

$$\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\leq 1 \quad \text{and} \quad 0\leq 1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right),$$

we obtain the inequality

$$\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\left(1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\right)\leq 1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right).$$

From this, it follows that

$$\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\leq 1.$$

Using the same logic, a corresponding bound is established

$$\left(\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)\leq 1.$$

Additionally, it is clear that

$$0\leq\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\leq 1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right),\quad 0\leq\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\leq 1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right).$$

Employing these bounds, the following inequality is established

$$\begin{aligned}&\left(\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\\ \leq&\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)+\left(1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\right)\left(1-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)=1.\end{aligned}$$

Thus, we conclude that

$$0\leq\left(\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)\leq 1,$$

$$0\leq\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\leq 1,$$

and

$$0\leq\left(\left(\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\leq 1.$$

In a similar manner, the following inequalities are derived:

(1)

$$\begin{aligned}&0\leq\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\leq 1,\\ &0\leq\left(\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)\leq 1,\\ &0\leq\left(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}\right)+\left(\left(\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)+\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)-\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\right)\right)\right)\leq 1.\end{aligned}$$

(2)

$$
\begin{aligned}
&-1 \leq -\mathscr{MR}^{-}_{\mathcal{D}_1}\mathscr{MR}^{-}_{\mathcal{D}_2} \leq 0,\\
&-1 \leq -\left(\left|\mathscr{NR}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{NR}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{NR}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NR}^{-}_{\mathcal{D}_2}\right|\right) \leq 0,\\
&\;\;0 \leq \left|-\mathscr{MR}^{-}_{\mathcal{D}_1}\mathscr{MR}^{-}_{\mathcal{D}_2}\right| + \left|-\left(\left|\mathscr{NR}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{NR}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{NR}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NR}^{-}_{\mathcal{D}_2}\right|\right)\right| \leq 1.
\end{aligned}
$$

(3)

$$
\begin{aligned}
&-1 \leq -\left(\left|\mathscr{MR}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{MR}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{MR}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MR}^{-}_{\mathcal{D}_2}\right|\right) \leq 0,\\
&-1 \leq -\mathscr{NR}^{-}_{\mathcal{D}_1}\mathscr{NR}^{-}_{\mathcal{D}_2} \leq 0,\\
&\;\;0 \leq \left|-\left(\left|\mathscr{MR}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{MR}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{MR}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MR}^{-}_{\mathcal{D}_2}\right|\right)\right| + \left|-\mathscr{NR}^{-}_{\mathcal{D}_1}\mathscr{NR}^{-}_{\mathcal{D}_2}\right| \leq 1.
\end{aligned}
$$

By applying the same approach, the following results are obtained:

- (i)

$$
0 \leq \left(\left(\mathscr{MI}^{+}_{\mathcal{D}_1}\right) + \left(\mathscr{MI}^{+}_{\mathcal{D}_2}\right) - \left(\mathscr{MI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{MI}^{+}_{\mathcal{D}_2}\right)\right) \leq 1, \quad 0 \leq \left(\mathscr{NI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{NI}^{+}_{\mathcal{D}_2}\right) \leq 1,
$$

  and

$$
0 \leq \left(\left(\left(\mathscr{MI}^{+}_{\mathcal{D}_1}\right) + \left(\mathscr{MI}^{+}_{\mathcal{D}_2}\right) - \left(\mathscr{MI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{MI}^{+}_{\mathcal{D}_2}\right)\right)\right) + \left(\left(\mathscr{NI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{NI}^{+}_{\mathcal{D}_2}\right)\right) \leq 1.
$$

- (ii)

$$
0 \leq \left|-\left(\left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right|\right)\right| \leq 1, \quad 0 \leq \left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right| \leq 1,
$$

  and

$$
0 \leq \left|-\left(\left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right|\right)\right| + \left(\left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right|\right) \leq 1.
$$

- (iii)

$$
0 \leq \left|-\left(\left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right|\right)\right| \leq 1, \quad 0 \leq \left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right| \leq 1,
$$

  and

$$
0 \leq \left|-\left(\left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right| + \left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right| - \left|\mathscr{MI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{MI}^{-}_{\mathcal{D}_2}\right|\right)\right| + \left(\left|\mathscr{NI}^{-}_{\mathcal{D}_1}\right|\left|\mathscr{NI}^{-}_{\mathcal{D}_2}\right|\right) \leq 1.
$$

- (iv)

$$
0 \leq \left(\left(\mathscr{NI}^{+}_{\mathcal{D}_1}\right) + \left(\mathscr{NI}^{+}_{\mathcal{D}_2}\right) - \left(\mathscr{NI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{NI}^{+}_{\mathcal{D}_2}\right)\right) \leq 1, \quad 0 \leq \left(\mathscr{MI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{MI}^{+}_{\mathcal{D}_2}\right) \leq 1,
$$

  and

$$
0 \leq \left(\left(\left(\mathscr{NI}^{+}_{\mathcal{D}_1}\right) + \left(\mathscr{NI}^{+}_{\mathcal{D}_2}\right) - \left(\mathscr{NI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{NI}^{+}_{\mathcal{D}_2}\right)\right)\right) + \left(\left(\mathscr{MI}^{+}_{\mathcal{D}_1}\right)\left(\mathscr{MI}^{+}_{\mathcal{D}_2}\right)\right) \leq 1.
$$

Hence, $\mathcal{D}_1 \oplus \mathcal{D}_2$ and $\mathcal{D}_1 \otimes \mathcal{D}_2$ fulfill the criteria to be considered BCIFNs. □

Theorem 2. *Let*

$$\mathcal{D} = \left\langle \mathscr{MR}^{+}_{\mathcal{D}} + i\,\mathscr{MI}^{+}_{\mathcal{D}},\quad \mathscr{MR}^{-}_{\mathcal{D}} + i\,\mathscr{MI}^{-}_{\mathcal{D}},\quad \mathscr{NR}^{+}_{\mathcal{D}} + i\,\mathscr{NI}^{+}_{\mathcal{D}},\quad \mathscr{NR}^{-}_{\mathcal{D}} + i\,\mathscr{NI}^{-}_{\mathcal{D}} \right\rangle$$

be a BCIFN, and $\mathscr{S} > 0$. *Then, both* $\mathscr{S}\mathcal{D}$ *and* $\mathcal{D}^{\mathscr{S}}$ *are also BCIFNs.*

Proof. Begin by observing the following inequalities:

$$0 \le (\mathscr{MR}^{+}_{\mathcal{D}}) \le 1,\quad 0 \le (\mathscr{NR}^{+}_{\mathcal{D}}) \le 1,\quad 0 \le (\mathscr{MR}^{+}_{\mathcal{D}}) + (\mathscr{NR}^{+}_{\mathcal{D}}) \le 1,$$

$$0 \le \left|\mathscr{MR}^{-}_{\mathcal{D}}\right| \le 1,\quad 0 \le \left|\mathscr{NR}^{-}_{\mathcal{D}}\right| \le 1,\quad 0 \le \left|\mathscr{MR}^{-}_{\mathcal{D}}\right| + \left|\mathscr{NR}^{-}_{\mathcal{D}}\right| \le 1.$$

This leads to the following bounds

$$0 \le (\mathscr{NR}^{+}_{\mathcal{D}}) \le 1 - (\mathscr{MR}^{+}_{\mathcal{D}}),\quad 0 \le \left|\mathscr{MR}^{-}_{\mathcal{D}}\right| \le 1 - \left|\mathscr{NR}^{-}_{\mathcal{D}}\right|.$$

Hence, it can be concluded that

$$0 \le (1 - (\mathscr{MR}^{+}_{\mathcal{D}}))^{\mathscr{S}},\quad 0 \le \left(1 - \left|\mathscr{NR}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}.$$

Subsequently, the following relations are applied:

$$1 - (1 - (\mathscr{MR}^{+}_{\mathcal{D}}))^{\mathscr{S}} \le 1,\quad 1 - \left(1 - \left|\mathscr{NR}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}} \le 1.$$

This leads to the bounds

$$0 \le \left(1 - (1 - (\mathscr{MR}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right) \le 1,\quad 0 \le \left(1 - \left(1 - \left|\mathscr{NR}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right) \le 1.$$

We also observe that

$$0 \le (\mathscr{NR}^{+}_{\mathcal{D}})^{\mathscr{S}} \le 1,\quad -1 \le -\left|\mathscr{MR}^{-}_{\mathcal{D}}\right|^{\mathscr{S}} \le .$$

From this, we deduce

$$0 \le \left(\left(1 - (1 - (\mathscr{MR}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right)\right) + \left((\mathscr{NR}^{+}_{\mathcal{D}})^{\mathscr{S}}\right) \le 1,$$

and

$$0 \le \left|-\left|\mathscr{MR}^{-}_{\mathcal{D}}\right|^{\mathscr{S}}\right| + \left(1 - \left(1 - \left|\mathscr{NR}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right) \le 1.$$

Next, analogous relations are considered for the remaining terms:

(1)

$$0 \le \left(1 - (1 - (\mathscr{MI}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right) \le 1,\quad 0 \le (\mathscr{NI}^{+}_{\mathcal{D}})^{\mathscr{S}} \le 1,$$

and

$$0 \le \left(\left(1 - (1 - (\mathscr{MI}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right)\right) + \left((\mathscr{NI}^{+}_{\mathcal{D}})^{\mathscr{S}}\right) \le 1.$$

(2)

$$0 \le \left|\mathscr{MI}^{-}_{\mathcal{D}}\right|^{\mathscr{S}} \le 1,\quad -1 \le -\left(1 - \left(1 - \left|\mathscr{NI}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right) \le 0,$$

and

$$0 \le \left|-\left|\mathscr{MI}^{-}_{\mathcal{D}}\right|^{\mathscr{S}}\right| + \left|-\left(1 - \left(1 - \left|\mathscr{NI}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right)\right| \le 1.$$

In a similar manner, the following relations are obtained:

$$0 \le \left((\mathscr{MR}^{+}_{\mathcal{D}})^{\mathscr{S}}\right) + \left(\left(1-(1-(\mathscr{NR}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right)\right) \le 1,$$

and

$$0 \le \left|-\left(1-\left(1-\left|\mathscr{MR}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right)\right| + \left|-\left|\mathscr{NR}^{-}_{\mathcal{D}}\right|^{\mathscr{S}}\right| \le 1.$$

In conclusion, based on the above, it follows that:

(1)

$$0 \le (\mathscr{MI}^{+}_{\mathcal{D}})^{\mathscr{S}} \le 1, \quad 0 \le \left(1-(1-(\mathscr{NI}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right) \le 1,$$

and

$$0 \le \left((\mathscr{MI}^{+}_{\mathcal{D}})^{\mathscr{S}}\right) + \left(\left(1-(1-(\mathscr{NI}^{+}_{\mathcal{D}}))^{\mathscr{S}}\right)\right) \le 1.$$

(2)

$$-1 \le -\left(1-\left(1-\left|\mathscr{MI}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right) \le 0, \quad 0 \le \left|\mathscr{NI}^{-}_{\mathcal{D}}\right|^{\mathscr{S}} \le 1,$$

and

$$0 \le \left|-\left(1-\left(1-\left|\mathscr{MI}^{-}_{\mathcal{D}}\right|\right)^{\mathscr{S}}\right)\right| + \left|-\left|\mathscr{NI}^{-}_{\mathcal{D}}\right|^{\mathscr{S}}\right| \le 1.$$

Therefore, both $\mathscr{S}\mathcal{D}$ and $\mathcal{D}^{\mathscr{S}}$ qualify as BCIFNs. □

Theorem 3. *Let*

$$\mathcal{D} = \langle \mathscr{MR}^{+}_{\mathcal{D}} + i\mathscr{MI}^{+}_{\mathcal{D}}, \mathscr{NR}^{+}_{\mathcal{D}} + i\mathscr{NI}^{+}_{\mathcal{D}}, \mathscr{MR}^{-}_{\mathcal{D}} + i\mathscr{MI}^{-}_{\mathcal{D}}, \mathscr{NR}^{-}_{\mathcal{D}} + i\mathscr{NI}^{-}_{\mathcal{D}} \rangle,$$

$$\mathcal{D}_1 = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_1} + i\mathscr{MI}^{+}_{\mathcal{D}_1}, \mathscr{NR}^{+}_{\mathcal{D}_1} + i\mathscr{NI}^{+}_{\mathcal{D}_1}, \mathscr{MR}^{-}_{\mathcal{D}_1} + i\mathscr{MI}^{-}_{\mathcal{D}_1}, \mathscr{NR}^{-}_{\mathcal{D}_1} + i\mathscr{NI}^{-}_{\mathcal{D}_1} \right\rangle,$$

and

$$\mathcal{D}_2 = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_2} + i\mathscr{MI}^{+}_{\mathcal{D}_2}, \mathscr{NR}^{+}_{\mathcal{D}_2} + i\mathscr{NI}^{+}_{\mathcal{D}_2}, \mathscr{MR}^{-}_{\mathcal{D}_2} + i\mathscr{MI}^{-}_{\mathcal{D}_2}, \mathscr{NR}^{-}_{\mathcal{D}_2} + i\mathscr{NI}^{-}_{\mathcal{D}_2} \right\rangle$$

be three BCIFNs, then

(1) $\mathcal{D}^c$ is a also a BCIFN.
(2) $\mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{D}_1 \cap \mathcal{D}_2$ are also BCIFN.

Proof. (1) Since

$$0 \le (\mathscr{MR}^{+}_{\mathcal{D}}) + (\mathscr{NR}^{+}_{\mathcal{D}}) \le 1, \;\; 0 \le (\mathscr{MI}^{+}_{\mathcal{D}}) + (\mathscr{NI}^{+}_{\mathcal{D}}) \le 1, \;\; 0 \le \left|\mathscr{MR}^{-}_{\mathcal{D}}\right| + \left|\mathscr{NR}^{-}_{\mathcal{D}}\right| \le 1,$$

and

$$0 \le |\mathscr{MI}^{-}_{\mathcal{D}}| + |\mathscr{NI}^{-}_{\mathcal{D}}| \le 1,$$

then

$$0 \le (\mathscr{NR}^{+}_{\mathcal{D}}) + (\mathscr{MR}^{+}_{\mathcal{D}}) = (\mathscr{MR}^{+}_{\mathcal{D}}) + (\mathscr{NR}^{+}_{\mathcal{D}}) \le 1,$$

$$0 \le (\mathscr{NI}^{+}_{\mathcal{D}}) + (\mathscr{MI}^{+}_{\mathcal{D}}) = (\mathscr{MI}^{+}_{\mathcal{D}}) + (\mathscr{NI}^{+}_{\mathcal{D}}) \le 1,$$

$$0 \leq \left|-\left|\mathscr{N}\mathscr{R}_{\mathcal{D}}^{-}\right|\right| + \left|-\left|\mathscr{M}\mathscr{R}_{\mathcal{D}}^{-}\right|\right| = \left|\mathscr{M}\mathscr{R}_{\mathcal{D}}^{-}\right| + \left|\mathscr{N}\mathscr{R}_{\mathcal{D}}^{-}\right| \leq 1,$$

and

$$0 \leq \left|-|\mathscr{N}\mathscr{I}_{\mathcal{D}}^{-}|\right| + \left|-|\mathscr{M}\mathscr{I}_{\mathcal{D}}^{-}|\right| = |\mathscr{M}\mathscr{I}_{\mathcal{D}}^{-}| + |\mathscr{N}\mathscr{I}_{\mathcal{D}}^{-}| \leq 1.$$

Hence, $\mathcal{D}^c$ is a BCIFN.

(2) Given that the following inequalities are satisfied:

$$0 \leq \mathscr{M}\mathscr{R}_{\mathcal{D}_1}^{+}, \mathscr{M}\mathscr{I}_{\mathcal{D}_1}^{+}, \mathscr{M}\mathscr{R}_{\mathcal{D}_2}^{+}, \mathscr{M}\mathscr{I}_{\mathcal{D}_2}^{+}, \mathscr{N}\mathscr{R}_{\mathcal{D}_1}^{+}, \mathscr{N}\mathscr{I}_{\mathcal{D}_1}^{+}, \mathscr{N}\mathscr{R}_{\mathcal{D}_2}^{+}, \mathscr{N}\mathscr{I}_{\mathcal{D}_2}^{+} \leq 1,$$

and

$$0 \leq |\mathscr{M}\mathscr{R}_{\mathcal{D}_1}^{-}|, |\mathscr{M}\mathscr{I}_{\mathcal{D}_1}^{-}|, |\mathscr{M}\mathscr{R}_{\mathcal{D}_2}^{-}|, |\mathscr{M}\mathscr{I}_{\mathcal{D}_2}^{-}|, |\mathscr{N}\mathscr{R}_{\mathcal{D}_1}^{-}|, |\mathscr{N}\mathscr{I}_{\mathcal{D}_1}^{-}|, |\mathscr{N}\mathscr{R}_{\mathcal{D}_2}^{-}|, |\mathscr{N}\mathscr{I}_{\mathcal{D}_2}^{-}| \leq 1,$$

it follows that the following conditions are satisfied:

$$0 \leq \left(\max\left\{\mathscr{M}\mathscr{R}_{\mathcal{D}_1}^{+}, \mathscr{M}\mathscr{R}_{\mathcal{D}_2}^{+}\right\}\right) + \left(\min\left\{\mathscr{N}\mathscr{R}_{\mathcal{D}_1}^{+}, \mathscr{N}\mathscr{R}_{\mathcal{D}_2}^{+}\right\}\right) \leq 1,$$

$$0 \leq \left(\max\left\{\mathscr{M}\mathscr{I}_{\mathcal{D}_1}^{+}, \mathscr{M}\mathscr{I}_{\mathcal{D}_2}^{+}\right\}\right) + \left(\min\left\{\mathscr{N}\mathscr{I}_{\mathcal{D}_1}^{+}, \mathscr{N}\mathscr{I}_{\mathcal{D}_2}^{+}\right\}\right) \leq 1,$$

$$0 \leq \left(\min\left\{|\mathscr{M}\mathscr{R}_{\mathcal{D}_1}^{-}|, |\mathscr{M}\mathscr{R}_{\mathcal{D}_2}^{-}|\right\}\right) + \left(\max\left\{|\mathscr{N}\mathscr{R}_{\mathcal{D}_1}^{-}|, |\mathscr{N}\mathscr{R}_{\mathcal{D}_2}^{-}|\right\}\right) \leq 1,$$

and

$$0 \leq \left(\max\left\{|\mathscr{N}\mathscr{I}_{\mathcal{D}_1}^{-}|, |\mathscr{N}\mathscr{I}_{\mathcal{D}_2}^{-}|\right\}\right) + \left(\min\left\{|\mathscr{M}\mathscr{I}_{\mathcal{D}_1}^{-}|, |\mathscr{M}\mathscr{I}_{\mathcal{D}_2}^{-}|\right\}\right) \leq 1.$$

Hence, $\mathcal{D}_1 \cup \mathcal{D}_2$ qualifies as a BCIFN, and by analogous reasoning, $\mathcal{D}_1 \cap \mathcal{D}_2$ also satisfies the conditions of a BCIFN.

□

Theorem 4. *Consider the following BCIFNs defined as*

$$\mathcal{D} = \left\langle \mathscr{M}\mathscr{R}_{\mathcal{D}}^{+} + i\mathscr{M}\mathscr{I}_{\mathcal{D}}^{+}, \mathscr{N}\mathscr{R}_{\mathcal{D}}^{+} + i\mathscr{N}\mathscr{I}_{\mathcal{D}}^{+}, \mathscr{M}\mathscr{R}_{\mathcal{D}}^{-} + i\mathscr{M}\mathscr{I}_{\mathcal{D}}^{-}, \mathscr{N}\mathscr{R}_{\mathcal{D}}^{-} + i\mathscr{N}\mathscr{I}_{\mathcal{D}}^{-} \right\rangle,$$

$$\mathcal{D}_1 = \left\langle \mathscr{M}\mathscr{R}_{\mathcal{D}_1}^{+} + i\mathscr{M}\mathscr{I}_{\mathcal{D}_1}^{+}, \mathscr{N}\mathscr{R}_{\mathcal{D}_1}^{+} + i\mathscr{N}\mathscr{I}_{\mathcal{D}_1}^{+}, \mathscr{M}\mathscr{R}_{\mathcal{D}_1}^{-} + i\mathscr{M}\mathscr{I}_{\mathcal{D}_1}^{-}, \mathscr{N}\mathscr{R}_{\mathcal{D}_1}^{-} + i\mathscr{N}\mathscr{I}_{\mathcal{D}_1}^{-} \right\rangle,$$

and

$$\mathcal{D}_2 = \left\langle \mathscr{M}\mathscr{R}_{\mathcal{D}_2}^{+} + i\mathscr{M}\mathscr{I}_{\mathcal{D}_2}^{+}, \mathscr{N}\mathscr{R}_{\mathcal{D}_2}^{+} + i\mathscr{N}\mathscr{I}_{\mathcal{D}_2}^{+}, \mathscr{M}\mathscr{R}_{\mathcal{D}_2}^{-} + i\mathscr{M}\mathscr{I}_{\mathcal{D}_2}^{-}, \mathscr{N}\mathscr{R}_{\mathcal{D}_2}^{-} + i\mathscr{N}\mathscr{I}_{\mathcal{D}_2}^{-} \right\rangle.$$

The subsequent key properties are satisfied

(1) $\mathcal{D}_2 \oplus \mathcal{D}_1 = \mathcal{D}_1 \oplus \mathcal{D}_2$.
(2) $\mathcal{D}_2 \otimes \mathcal{D}_1 = \mathcal{D}_1 \otimes \mathcal{D}_2$.
(3) $\mathcal{D} \oplus (\mathcal{D}_1 \oplus \mathcal{D}_2) = (\mathcal{D} \oplus \mathcal{D}_1) \oplus \mathcal{D}_2$.
(4) $\mathcal{D} \otimes (\mathcal{D}_1 \otimes \mathcal{D}_2) = (\mathcal{D} \otimes \mathcal{D}_1) \otimes \mathcal{D}_2$.
(5) $\mathcal{D} \oplus (\mathcal{D}_1 \cup \mathcal{D}_2) = (\mathcal{D} \oplus \mathcal{D}_1) \cup (\mathcal{D} \oplus \mathcal{D}_2)$.
(6) $\mathcal{D} \oplus (\mathcal{D}_1 \cap \mathcal{D}_2) = (\mathcal{D} \oplus \mathcal{D}_1) \cap (\mathcal{D} \oplus \mathcal{D}_2)$.
(7) $\mathcal{D} \otimes (\mathcal{D}_1 \cup \mathcal{D}_2) = (\mathcal{D} \otimes \mathcal{D}_1) \cup (\mathcal{D} \otimes \mathcal{D}_2)$.
(8) $\mathcal{D} \otimes (\mathcal{D}_1 \cap \mathcal{D}_2) = (\mathcal{D} \otimes \mathcal{D}_1) \cap (\mathcal{D} \otimes \mathcal{D}_2)$.

(9) $(\mathcal{D}_1 \cup \mathcal{D}_2) \oplus (\mathcal{D}_1 \cap \mathcal{D}_2) = \mathcal{D}_1 \oplus \mathcal{D}_2$.
(10) $(\mathcal{D}_1 \cup \mathcal{D}_2) \otimes (\mathcal{D}_1 \cap \mathcal{D}_2) = \mathcal{D}_1 \otimes \mathcal{D}_2$.

Proof. The proofs for parts (1), (3), and (5) are provided below, with the other parts following by analogous reasoning.
(1) The operation $\mathcal{D}_1 \oplus \mathcal{D}_2$ is defined as

$$\begin{aligned}\mathcal{D}_1\oplus\mathcal{D}_2 = \Big\langle &((\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})+(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2})-(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2}))+i((\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})+(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2})-(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2})),\\ &(\mathscr{N}\mathscr{R}^+_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^+_{\mathcal{D}_2}) + i(\mathscr{N}\mathscr{I}^+_{\mathcal{D}_1}\mathscr{N}\mathscr{I}^+_{\mathcal{D}_2}), -(\mathscr{M}\mathscr{R}^-_{\mathcal{D}_1}\mathscr{M}\mathscr{R}^-_{\mathcal{D}_2}) + i(-(\mathscr{M}\mathscr{I}^-_{\mathcal{D}_1}\mathscr{M}\mathscr{I}^-_{\mathcal{D}_2})),\\ &-(|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}|+|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}|-|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}|)+i(-(|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}|+|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}|-|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}|))\Big\rangle,\end{aligned}$$

alternatively, it can be represented as

$$\begin{aligned}&\Big\langle((\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2}) + (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1}) - (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2})(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})) + i((\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}) + (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1}) - (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2})(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})),\\ &(\mathscr{N}\mathscr{R}^+_{\mathcal{D}_2}\mathscr{N}\mathscr{R}^+_{\mathcal{D}_1}) + i(\mathscr{N}\mathscr{I}^+_{\mathcal{D}_2}\mathscr{N}\mathscr{I}^+_{\mathcal{D}_1}), -(\mathscr{M}\mathscr{R}^-_{\mathcal{D}_2}\mathscr{M}\mathscr{R}^-_{\mathcal{D}_1}) + i(-(\mathscr{M}\mathscr{I}^-_{\mathcal{D}_2}\mathscr{M}\mathscr{I}^-_{\mathcal{D}_1})),\\ &-(|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}| + |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}| - |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}||\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}|) + i(-(|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}| + |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}| - |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}||\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}|))\Big\rangle.\end{aligned}$$

Hence, it can be concluded that

$$\mathcal{D}_1 \oplus \mathcal{D}_2 = \mathcal{D}_2 \oplus \mathcal{D}_1.$$

(3)

$$\begin{aligned}&\mathcal{D} \oplus (\mathcal{D}_1 \oplus \mathcal{D}_2)\\ =&\langle \mathscr{M}\mathscr{R}^+_{\mathcal{D}} + i\mathscr{M}\mathscr{I}^+_{\mathcal{D}}, \mathscr{M}\mathscr{R}^-_{\mathcal{D}} + i\mathscr{M}\mathscr{I}^-_{\mathcal{D}}, \mathscr{N}\mathscr{R}^+_{\mathcal{D}} + i\mathscr{N}\mathscr{I}^+_{\mathcal{D}}, \mathscr{N}\mathscr{R}^-_{\mathcal{D}} + i\mathscr{N}\mathscr{I}^-_{\mathcal{D}}\rangle\\ \oplus&\Big\langle((\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2})) + i((\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2})),\\ &(\mathscr{N}\mathscr{R}^+_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^+_{\mathcal{D}_2}) + i(\mathscr{N}\mathscr{I}^+_{\mathcal{D}_1}\mathscr{N}\mathscr{I}^+_{\mathcal{D}_2}), -(\mathscr{M}\mathscr{R}^-_{\mathcal{D}_1}\mathscr{M}\mathscr{R}^-_{\mathcal{D}_2}) + i(-(\mathscr{M}\mathscr{I}^-_{\mathcal{D}_1}\mathscr{M}\mathscr{I}^-_{\mathcal{D}_2})),\\ &-(|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}|) + i(-(|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}|))\Big\rangle\\ =&\Big\langle((\mathscr{M}\mathscr{R}^+_{\mathcal{D}}) + ((\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2})) - (\mathscr{M}\mathscr{R}^+_{\mathcal{D}})((\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2})\\ &-(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_2}))) + i((\mathscr{M}\mathscr{I}^+_{\mathcal{D}}) + ((\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}))\\ &-(\mathscr{M}\mathscr{I}^+_{\mathcal{D}})((\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}))), (\mathscr{N}\mathscr{R}^+_{\mathcal{D}}\mathscr{N}\mathscr{R}^+_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^+_{\mathcal{D}_2})\\ &+i(\mathscr{N}\mathscr{I}^+_{\mathcal{D}}\mathscr{N}\mathscr{I}^+_{\mathcal{D}_1}\mathscr{N}\mathscr{I}^+_{\mathcal{D}_2}), -(|\mathscr{M}\mathscr{R}^-_{\mathcal{D}}||\mathscr{M}\mathscr{R}^-_{\mathcal{D}_1}||\mathscr{M}\mathscr{R}^-_{\mathcal{D}_2}|) + i(-(|\mathscr{M}\mathscr{I}^-_{\mathcal{D}}||\mathscr{M}\mathscr{I}^-_{\mathcal{D}_1}||\mathscr{M}\mathscr{I}^-_{\mathcal{D}_2}|)),\\ &-(|\mathscr{N}\mathscr{R}^-_{\mathcal{D}}| + (|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}|)\\ &-|\mathscr{N}\mathscr{R}^-_{\mathcal{D}}|(|\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{R}^-_{\mathcal{D}_2}|))\\ &+i(-(|\mathscr{N}\mathscr{I}^-_{\mathcal{D}}| + (|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}|)\\ &-|\mathscr{N}\mathscr{I}^-_{\mathcal{D}}|(|\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}||\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}|)))\Big\rangle\\ =&\Big\langle((\mathscr{M}\mathscr{R}^+_{\mathcal{D}}) + (\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1}) - (\mathscr{M}\mathscr{R}^+_{\mathcal{D}})(\mathscr{M}\mathscr{R}^+_{\mathcal{D}_1})) + i((\mathscr{M}\mathscr{I}^+_{\mathcal{D}}) + (\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1}) - (\mathscr{M}\mathscr{I}^+_{\mathcal{D}})(\mathscr{M}\mathscr{I}^+_{\mathcal{D}_1})),\\ &(\mathscr{N}\mathscr{R}^+_{\mathcal{D}}\mathscr{N}\mathscr{R}^+_{\mathcal{D}_1}) + i(\mathscr{N}\mathscr{I}^+_{\mathcal{D}}\mathscr{N}\mathscr{I}^+_{\mathcal{D}_1}), -(\mathscr{M}\mathscr{R}^-_{\mathcal{D}}\mathscr{M}\mathscr{R}^-_{\mathcal{D}_1}) + i(-(\mathscr{M}\mathscr{I}^-_{\mathcal{D}}\mathscr{M}\mathscr{I}^-_{\mathcal{D}_1})),\\ &-(|\mathscr{N}\mathscr{R}^-_{\mathcal{D}}| + |\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}| - |\mathscr{N}\mathscr{R}^-_{\mathcal{D}}||\mathscr{N}\mathscr{R}^-_{\mathcal{D}_1}|) + i(-(|\mathscr{N}\mathscr{I}^-_{\mathcal{D}}| + |\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}| - |\mathscr{N}\mathscr{I}^-_{\mathcal{D}}||\mathscr{N}\mathscr{I}^-_{\mathcal{D}_1}|))\Big\rangle\\ \oplus&\langle \mathscr{M}\mathscr{R}^+_{\mathcal{D}_2} + i\mathscr{M}\mathscr{I}^+_{\mathcal{D}_2}, \mathscr{M}\mathscr{R}^-_{\mathcal{D}_2} + i\mathscr{M}\mathscr{I}^-_{\mathcal{D}_2}, \mathscr{N}\mathscr{R}^+_{\mathcal{D}_2} + i\mathscr{N}\mathscr{I}^+_{\mathcal{D}_2}, \mathscr{N}\mathscr{R}^-_{\mathcal{D}_2} + i\mathscr{N}\mathscr{I}^-_{\mathcal{D}_2}\rangle\\ =&(\mathcal{D} \oplus \mathcal{D}_1) \oplus \mathcal{D}_2.\end{aligned}$$

(5)

$$
\begin{aligned}
&\mathcal{D}\oplus(\mathcal{D}_1\cup\mathcal{D}_2)\\
=&\langle \mathscr{MR}^+_{\mathcal{D}}+i\mathscr{MI}^+_{\mathcal{D}},\mathscr{MR}^-_{\mathcal{D}}+i\mathscr{MI}^-_{\mathcal{D}},\mathscr{NR}^+_{\mathcal{D}}+i\mathscr{NI}^+_{\mathcal{D}},\mathscr{NR}^-_{\mathcal{D}}+i\mathscr{NI}^-_{\mathcal{D}}\rangle\\
\oplus&\Big\langle \max\{\mathscr{MR}^+_{\mathcal{D}_1},\mathscr{MR}^+_{\mathcal{D}_2}\}+i\max\{\mathscr{MI}^+_{\mathcal{D}_1},\mathscr{MI}^+_{\mathcal{D}_2}\},\min\{\mathscr{NR}^+_{\mathcal{D}_1},\mathscr{NR}^+_{\mathcal{D}_2}\}+i\min\{\mathscr{NI}^+_{\mathcal{D}_1},\mathscr{NI}^+_{\mathcal{D}_2}\},\\
&\min\{\mathscr{MR}^-_{\mathcal{D}_1},\mathscr{MR}^-_{\mathcal{D}_2}\}+i\min\{\mathscr{MI}^-_{\mathcal{D}_1},\mathscr{MI}^-_{\mathcal{D}_2}\},\max\{\mathscr{NR}^-_{\mathcal{D}_1},\mathscr{NR}^-_{\mathcal{D}_2}\}+i\max\{\mathscr{NI}^-_{\mathcal{D}_1},\mathscr{NI}^-_{\mathcal{D}_2}\}\Big\rangle\\
=&\Big\langle ((\mathscr{MR}^+_{\mathcal{D}})+\max\{\mathscr{MR}^+_{\mathcal{D}_1},\mathscr{MR}^+_{\mathcal{D}_2}\}-(\mathscr{MR}^+_{\mathcal{D}})\max\{\mathscr{MR}^+_{\mathcal{D}_1},\mathscr{MR}^+_{\mathcal{D}_2}\})^{\frac{1}{q}}\\
&+i((\mathscr{MI}^+_{\mathcal{D}})+\max\{\mathscr{MI}^+_{\mathcal{D}_1},\mathscr{MI}^+_{\mathcal{D}_2}\}-(\mathscr{MI}^+_{\mathcal{D}})\max\{\mathscr{MI}^+_{\mathcal{D}_1},\mathscr{MI}^+_{\mathcal{D}_2}\})^{\frac{1}{q}},\\
&(\mathscr{NR}^+_{\mathcal{D}}\min\{\mathscr{NR}^+_{\mathcal{D}_1},\mathscr{NR}^+_{\mathcal{D}_2}\})+i(\mathscr{NI}_{\mathcal{D}}\min\{\mathscr{NI}^+_{\mathcal{D}_1},\mathscr{NI}^+_{\mathcal{D}_2}\}),\\
&-(\mathscr{MR}^-_{\mathcal{D}}\min\{\mathscr{MR}^-_{\mathcal{D}_1},\mathscr{MR}^-_{\mathcal{D}_2}\})+i(-(\mathscr{MI}^-_{\mathcal{D}}\min\{\mathscr{MI}^-_{\mathcal{D}_1},\mathscr{MI}^-_{\mathcal{D}_2}\})),\\
&-(|\mathscr{NR}^-_{\mathcal{D}}|+|\max\{\mathscr{NR}^-_{\mathcal{D}_1},\mathscr{NR}^-_{\mathcal{D}_2}\}|-|\mathscr{NR}^-_{\mathcal{D}}||\max\{\mathscr{NR}^-_{\mathcal{D}_1},\mathscr{NR}^-_{\mathcal{D}_2}\}|)^{\frac{1}{q}}\\
&+i(-(|\mathscr{NI}^-_{\mathcal{D}}|+|\max\{\mathscr{NI}^-_{\mathcal{D}_1},\mathscr{NI}^-_{\mathcal{D}_2}\}|-|\mathscr{NI}^-_{\mathcal{D}}||\max\{\mathscr{NI}^-_{\mathcal{D}_1},\mathscr{NI}^-_{\mathcal{D}_2}\}|)^{\frac{1}{q}})\Big\rangle\\
=&\Big\langle ((\mathscr{MR}^+_{\mathcal{D}})+(1-(\mathscr{MR}^+_{\mathcal{D}}))\max\{(\mathscr{MR}^+_{\mathcal{D}_1}),(\mathscr{MR}^+_{\mathcal{D}_2})\})+\\
&i((\mathscr{MI}^+_{\mathcal{D}})+(1-(\mathscr{MI}^+_{\mathcal{D}}))\max\{(\mathscr{MI}^+_{\mathcal{D}_1}),(\mathscr{MI}^+_{\mathcal{D}_2})\}),\\
&\min\{\mathscr{NR}^+_{\mathcal{D}}\mathscr{NR}^+_{\mathcal{D}_1},\mathscr{NR}^+_{\mathcal{D}}\mathscr{NR}^+_{\mathcal{D}_2}\}+i(\min\{\mathscr{NI}^+_{\mathcal{D}}\mathscr{NI}^+_{\mathcal{D}_1},\mathscr{NI}^+_{\mathcal{D}}\mathscr{NI}^+_{\mathcal{D}_2}\}),\\
&\min\{-(\mathscr{MR}^-_{\mathcal{D}}\mathscr{MR}^-_{\mathcal{D}_1}),-(\mathscr{MR}^-_{\mathcal{D}}\mathscr{MR}^-_{\mathcal{D}_2})\}+i(\min\{-(\mathscr{MI}^-_{\mathcal{D}}\mathscr{MI}^-_{\mathcal{D}_1}),-(\mathscr{MI}^-_{\mathcal{D}}\mathscr{MI}^-_{\mathcal{D}_2})\}),\\
&-(|\mathscr{NR}^-_{\mathcal{D}}|+(1-|\mathscr{NR}^-_{\mathcal{D}}|)\max\{|\mathscr{NR}^-_{\mathcal{D}_1}|,|\mathscr{NR}^-_{\mathcal{D}_2}|\})\\
&+i(-(|\mathscr{NI}^-_{\mathcal{D}}|+(1-|\mathscr{NI}^-_{\mathcal{D}}|)\max\{|\mathscr{NI}^-_{\mathcal{D}_1}|,|\mathscr{NI}^-_{\mathcal{D}_2}|\}))\Big\rangle.
\end{aligned}
$$

However,

$$
\begin{aligned}
&(\mathcal{D}\oplus\mathcal{D}_1)\cup(\mathcal{D}\oplus\mathcal{D}_2)\\
=&\Big\langle ((\mathscr{MR}^+_{\mathcal{D}})+(\mathscr{MR}^+_{\mathcal{D}_1})-(\mathscr{MR}^+_{\mathcal{D}})(\mathscr{MR}^+_{\mathcal{D}_1}))+i((\mathscr{MI}^+_{\mathcal{D}})+(\mathscr{MI}^+_{\mathcal{D}_1})-(\mathscr{MI}^+_{\mathcal{D}})(\mathscr{MI}^+_{\mathcal{D}_1})),\\
&(\mathscr{NR}^+_{\mathcal{D}}\mathscr{NR}^+_{\mathcal{D}_1})+i(\mathscr{NI}^+_{\mathcal{D}}\mathscr{NI}^+_{\mathcal{D}_1}),-(\mathscr{MR}^-_{\mathcal{D}}\mathscr{MR}^-_{\mathcal{D}_1})+i(-(\mathscr{MI}^-_{\mathcal{D}}\mathscr{MI}^-_{\mathcal{D}_1})),\\
&-(|\mathscr{NR}^-_{\mathcal{D}}|+|\mathscr{NR}^-_{\mathcal{D}_1}|-|\mathscr{NR}^-_{\mathcal{D}}||\mathscr{NR}^-_{\mathcal{D}_1}|)+i(-(|\mathscr{NI}^-_{\mathcal{D}}|+|\mathscr{NI}^-_{\mathcal{D}_1}|-|\mathscr{NI}^-_{\mathcal{D}}||\mathscr{NI}^-_{\mathcal{D}_1}|))\Big\rangle\\
\cup&\Big\langle ((\mathscr{MR}^+_{\mathcal{D}})+(\mathscr{MR}^+_{\mathcal{D}_2})-(\mathscr{MR}^+_{\mathcal{D}})(\mathscr{MR}^+_{\mathcal{D}_2}))+i((\mathscr{MI}^+_{\mathcal{D}})+(\mathscr{MI}^+_{\mathcal{D}})-(\mathscr{MI}^+_{\mathcal{D}})(\mathscr{MI}^+_{\mathcal{D}_2})),\\
&(\mathscr{NR}^+_{\mathcal{D}}\mathscr{NR}^+_{\mathcal{D}_2})+i(\mathscr{NI}^+_{\mathcal{D}}\mathscr{NI}^+_{\mathcal{D}_2}),-(\mathscr{MR}^-_{\mathcal{D}}\mathscr{MR}^-_{\mathcal{D}_2})+i(-(\mathscr{MI}^-_{\mathcal{D}}\mathscr{MI}^-_{\mathcal{D}_2})),\\
&-(|\mathscr{NR}^-_{\mathcal{D}}|+|\mathscr{NR}^-_{\mathcal{D}_2}|-|\mathscr{NR}^-_{\mathcal{D}}||\mathscr{NR}^-_{\mathcal{D}_2}|)+i(-(|\mathscr{NI}^-_{\mathcal{D}}|+|\mathscr{NI}^-_{\mathcal{D}_2}|-|\mathscr{NI}^-_{\mathcal{D}}||\mathscr{NI}^-_{\mathcal{D}_2}|))\Big\rangle\\
=&\Big\langle \max\{((\mathscr{MR}^+_{\mathcal{D}})+(\mathscr{MR}^+_{\mathcal{D}_1})-(\mathscr{MR}^+_{\mathcal{D}})(\mathscr{MR}^+_{\mathcal{D}_1})),((\mathscr{MR}^+_{\mathcal{D}})+(\mathscr{MR}^+_{\mathcal{D}_2})-(\mathscr{MR}^+_{\mathcal{D}})(\mathscr{MR}^+_{\mathcal{D}_2}))\}\\
&+i(\max\{((\mathscr{MI}^+_{\mathcal{D}})+(\mathscr{MI}^+_{\mathcal{D}_1})-(\mathscr{MI}^+_{\mathcal{D}})(\mathscr{MI}^+_{\mathcal{D}_1})),((\mathscr{MI}^+_{\mathcal{D}})+(\mathscr{MI}^+_{\mathcal{D}})-(\mathscr{MI}^+_{\mathcal{D}})(\mathscr{MI}^+_{\mathcal{D}_2}))\}),\\
&\min\{\mathscr{NR}^+_{\mathcal{D}}\mathscr{NR}^+_{\mathcal{D}_1},\mathscr{NR}^+_{\mathcal{D}}\mathscr{NR}^+_{\mathcal{D}_2}\}+i(\min\{\mathscr{NI}^+_{\mathcal{D}}\mathscr{NI}^+_{\mathcal{D}_1},\mathscr{NI}^+_{\mathcal{D}}\mathscr{NI}^+_{\mathcal{D}_2}\}),\\
&\min\{-(\mathscr{MR}^-_{\mathcal{D}}\mathscr{MR}^-_{\mathcal{D}_1}),-(\mathscr{MR}^-_{\mathcal{D}}\mathscr{MR}^-_{\mathcal{D}_2})\}+i(\min\{-(\mathscr{MI}^-_{\mathcal{D}}\mathscr{MI}^-_{\mathcal{D}_1}),-(\mathscr{MI}^-_{\mathcal{D}}\mathscr{MI}^-_{\mathcal{D}_2})\}),\\
&\max\{-(|\mathscr{NR}^-_{\mathcal{D}}|+|\mathscr{NR}^-_{\mathcal{D}_1}|-|\mathscr{NR}^-_{\mathcal{D}}||\mathscr{NR}^-_{\mathcal{D}_1}|),-(|\mathscr{NR}^-_{\mathcal{D}}|+|\mathscr{NR}^-_{\mathcal{D}_2}|-|\mathscr{NR}^-_{\mathcal{D}}||\mathscr{NR}^-_{\mathcal{D}_2}|)\}\\
&+i(\max\{-(|\mathscr{NI}^-_{\mathcal{D}}|+|\mathscr{NI}^-_{\mathcal{D}_1}|-|\mathscr{NI}^-_{\mathcal{D}}||\mathscr{NI}^-_{\mathcal{D}_1}|),-(|\mathscr{NI}^-_{\mathcal{D}}|+|\mathscr{NI}^-_{\mathcal{D}_2}|-|\mathscr{NI}^-_{\mathcal{D}}||\mathscr{NI}^-_{\mathcal{D}_2}|)\})\Big\rangle\\
=&\Big\langle ((\mathscr{MR}^+_{\mathcal{D}})+(1-(\mathscr{MR}^+_{\mathcal{D}}))\max\{(\mathscr{MR}^+_{\mathcal{D}_1}),(\mathscr{MR}^+_{\mathcal{D}_2})\})+i((\mathscr{MI}^+_{\mathcal{D}})+(1-(\mathscr{MI}^+_{\mathcal{D}}))\max\{(\mathscr{MI}^+_{\mathcal{D}_1}),
\end{aligned}
$$

$$(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2})\}), \min\{\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}, \mathscr{N}\mathscr{R}^{+}_{\mathcal{D}}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}\} + i(\min\{\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}}\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_1}, \mathscr{N}\mathscr{I}^{+}_{\mathcal{D}}\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_2}\}),$$
$$\min\{-(\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}}\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_1}), -(\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}}\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_2})\} + i(\min\{-(\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}}\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_1}), -(\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}}\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_2})\}),$$
$$-(|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}}| + (1 - |\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}}|)\max\{|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1}|, |\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2}|\}) + i(-(|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}}| + (1 - |\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}}|)\max\{|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1}|, |\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2}|\})) \Big\rangle.$$

□

Theorem 5. *Let*

$$\mathcal{D} = \langle \mathscr{M}\mathscr{R}^{+}_{\mathcal{D}} + i\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}}, \mathscr{N}\mathscr{R}^{+}_{\mathcal{D}} + i\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}}, \mathscr{M}\mathscr{R}^{-}_{\mathcal{D}} + i\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}}, \mathscr{N}\mathscr{R}^{-}_{\mathcal{D}} + i\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}} \rangle,$$

$$\mathcal{D}_1 = \Big\langle \mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1} + i\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_1}, \mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1} + i\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_1}, \mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_1} + i\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_1}, \mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1} + i\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1} \Big\rangle,$$

and

$$\mathcal{D}_2 = \Big\langle \mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2} + i\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2}, \mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2} + i\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_2}, \mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_2} + i\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_2}, \mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2} + i\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2} \Big\rangle$$

be BCIFNs. For any $\mathscr{S} > 0$, the subsequent properties hold true

(1) $(\mathcal{D}_1 \oplus \mathcal{D}_2)^c = \mathcal{D}_1^c \otimes \mathcal{D}_2^c$.
(2) $(\mathcal{D}_1 \otimes \mathcal{D}_2)^c = \mathcal{D}_1^c \oplus \mathcal{D}_2^c$.
(3) $(\mathcal{D}^c)^{\mathscr{S}} = (\mathscr{S}\mathcal{D})^c$.
(4) $\mathscr{S}(\mathcal{D})^c = (\mathcal{D}^{\mathscr{S}})^c$.

Proof. Items (1) and (3) are demonstrated here, with the remaining parts following by a similar approach.

(1)

$$(\mathcal{D}_1 \oplus \mathcal{D}_2)^c$$
$$= \Big\langle ((\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1})(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2})) + i((\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_1})(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2})),$$
$$(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}) + i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_2}), -(\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_1}\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_2}) + i(-(\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_1}\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_2})),$$
$$-(|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1}||\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2}|) + i(-(|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1}||\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2}|)) \Big\rangle^c$$
$$= \Big\langle (\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}) + i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_1}\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_2}), ((\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1})(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}))$$
$$+ i((\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_1}) + (\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2}) - (\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_1})(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2})), -|-(|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1}||\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2}|)|$$
$$+ i(-|-(|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1}| + |\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2}| - |\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1}||\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2}|)|), -|-(\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_1}\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_2})| + i(-|-(\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_1}\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_2})|) \Big\rangle$$
$$= \langle (\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_1}) + i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_1}), (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_1}) + i(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_1}), -|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_1}| + i(-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_1}|), -|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_1}| + i(-|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_1}|) \rangle$$
$$\otimes \langle (\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_2}) + i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_2}), (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_2}) + i(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_2}), -|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_2}| + i(-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_2}|), -|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_2}| + i(-|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_2}|) \rangle$$
$$= \mathcal{D}_1^c \otimes \mathcal{D}_2^c.$$

(3)

$$(\mathcal{D}^c)^{\mathscr{S}} = \langle (\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}}) + i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}}), (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}}) + i(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}}), -|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}}| + i(-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}}|), -|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}}| + i(-|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}}|) \rangle^{\mathscr{S}}$$
$$= \Big\langle (\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}})^{\mathscr{S}} + i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}})^{\mathscr{S}}, (1 - (1 - (\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}}))^{\mathscr{S}}) + i(1 - (1 - (\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}}))^{\mathscr{S}}),$$
$$-(1 - (1 - |-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}}||)^{\mathscr{S}})^{\frac{1}{q}} + i(-(1 - (1 - |-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}}||)^{\mathscr{S}})), -|-|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}}||^{\mathscr{S}} + i(-|-|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}}||^{\mathscr{S}}) \Big\rangle$$

$$
\begin{aligned}
&=\left\langle ((({\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}})^{\mathscr{S}}) + i(({\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}})^{\mathscr{S}}), ((1-(1-({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}}))^{\mathscr{S}})^{\frac{1}{q}}) + i((1-(1-({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}}))^{\mathscr{S}})),\right.\\
&\quad \left. -|-(1-(1-|-|{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}}||^{\mathscr{S}})|+i(-|-(1-(1-|{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}}|)^{\mathscr{S}})|), -|-|{\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}}|^{\mathscr{S}}|+i(-|-|{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}}|^{\mathscr{S}}|)\right\rangle\\
&=\left\langle (1-(1-({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}}))^{\mathscr{S}})^{\frac{1}{q}} + i(1-(1-({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}}))^{\mathscr{S}}), ({\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}})^{\mathscr{S}} + i({\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}})^{\mathscr{S}},\right.\\
&\quad \left. -|{\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}}|^{\mathscr{S}} + i(-|{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}}|^{\mathscr{S}}), -(1-(1-|{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}}||)^{\mathscr{S}}) + i(-(1-(1-|{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}}||)^{\mathscr{S}}))\right\rangle^{c}\\
&=(\mathscr{S}\mathcal{D})^{c}.
\end{aligned}
$$

□

Theorem 6. *Let*

$$
\mathcal{D} = \langle {\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}} + i{\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}}, {\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}} + i{\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}}, {\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}} + i{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}}, {\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}} + i{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}} \rangle,
$$

$$
\mathcal{D}_1 = \left\langle {\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_1} + i{\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_1}, {\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_1} + i{\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_1}, {\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}_1} + i{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}_1}, {\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_1} + i{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_1} \right\rangle,
$$

and

$$
\mathcal{D}_2 = \left\langle {\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_2} + i{\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_2}, {\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_2} + i{\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_2}, {\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}_2} + i{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}_2}, {\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_2} + i{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_2} \right\rangle
$$

be BCIFNs. For any $\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2 > 0$*, the subsequent properties hold true*

(1) $\mathscr{S}(\mathcal{D}_1 \oplus \mathcal{D}_2) = \mathscr{S}\mathcal{D}_1 \oplus \mathscr{S}\mathcal{D}_2$.
(2) $(\mathscr{S}_1 + \mathscr{S}_2)\mathcal{D} = \mathscr{S}_1\mathcal{D} \oplus \mathscr{S}_2\mathcal{D}$.
(3) $(\mathcal{D}_1 \otimes \mathcal{D}_2)^{\mathscr{S}} = \mathcal{D}_1^{\mathscr{S}} \otimes \mathcal{D}_2^{\mathscr{S}}$.
(4) $\mathcal{D}^{(\mathscr{S}_1+\mathscr{S}_2)} = \mathcal{D}^{\mathscr{S}_1} \otimes \mathcal{D}^{\mathscr{S}_2}$.

Proof. Expressions (1) and (2) are explicitly demonstrated here, while the remaining parts can be derived similarly.

(1)

$$
\begin{aligned}
&\mathscr{S}(\mathcal{D}_1 \oplus \mathcal{D}_2)\\
&=\mathscr{S}\left\langle ((({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_1}) + ({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_2}) - ({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_1})({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_2})) + i(({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_1}) + ({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_2}) - ({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_1})({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_2})),\right.\\
&\quad ({\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_1}{\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_2}) + i({\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_1}{\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_2}), -({\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}_1}{\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}_2}) + i(-({\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}_1}{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}_2})),\\
&\quad \left. -(|{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_1}| + |{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_2}| - |{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_1}||{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_2}|) + i(-(|{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_1}| + |{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_2}| - |{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_1}||{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_2}|)))\right\rangle\\
&=\left\langle (1-(1-(({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_1}) + ({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_2}) - ({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_1})({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_2})))^{\mathscr{S}})\right.\\
&\quad +i(1-(1-(({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_1})+({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_2})-({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_1})({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_2})))^{\mathscr{S}}), ({\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_1}{\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_2})^{\mathscr{S}} + i({\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_1}{\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_2})^{\mathscr{S}},\\
&\quad -(|{\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}_1}||{\mathscr{M}\mathscr{R}}^{-}_{\mathcal{D}_2}|)^{\mathscr{S}}+i(-(|{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}_1}||{\mathscr{M}\mathscr{I}}^{-}_{\mathcal{D}_2}|)^{\mathscr{S}}), -(1-(1-(-(|{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_1}|+|{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_2}| - |{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_1}||{\mathscr{N}\mathscr{R}}^{-}_{\mathcal{D}_2}|)))^{\mathscr{S}})\\
&\quad \left. + i(-(1-(1-(-(|{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_1}| + |{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_2}| - |{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_1}||{\mathscr{N}\mathscr{I}}^{-}_{\mathcal{D}_2}|)))^{\mathscr{S}})))\rangle\right\rangle.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\mathscr{S}\mathcal{D}_1 \oplus \mathscr{S}\mathcal{D}_2\\
&=\left\langle (1-(1-({\mathscr{M}\mathscr{R}}^{+}_{\mathcal{D}_1}))^{\mathscr{S}}) + i(1-(1-({\mathscr{M}\mathscr{I}}^{+}_{\mathcal{D}_1}))^{\mathscr{S}}), ({\mathscr{N}\mathscr{R}}^{+}_{\mathcal{D}_1})^{\mathscr{S}} + i({\mathscr{N}\mathscr{I}}^{+}_{\mathcal{D}_1})^{\mathscr{S}},\right.
\end{aligned}
$$

$$
\begin{aligned}
& -|\mathscr{MR}^-_{\mathcal{D}_1}|^{\mathscr{S}} + i(-|\mathscr{MI}^-_{\mathcal{D}_1}|^{\mathscr{S}}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{S}}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{S}}))\Big\rangle \\
\oplus & \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{S}}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{S}}), (\mathscr{NR}^+_{\mathcal{D}_2})^{\mathscr{S}} + i(\mathscr{NI}^+_{\mathcal{D}_2})^{\mathscr{S}}, \\
& -|\mathscr{MR}^-_{\mathcal{D}_2}|^{\mathscr{S}} + i(-|\mathscr{MI}^-_{\mathcal{D}_2}|^{\mathscr{S}}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{S}}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{S}}))\Big\rangle \\
= & \Big\langle ((1-(1-(\mathscr{MR}^+_{\mathcal{D}_1}))^{\mathscr{S}}) + (1-(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{S}}) - (1-(1-(\mathscr{MR}^+_{\mathcal{D}_1}))^{\mathscr{S}})(1-(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{S}})) \\
& + i((1-(1-(\mathscr{MI}^+_{\mathcal{D}_1}))^{\mathscr{S}}) + (1-(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{S}}) - (1-(1-(\mathscr{MI}^+_{\mathcal{D}_1}))^{\mathscr{S}})(1-(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{S}})), \\
& (\mathscr{NR}^+_{\mathcal{D}_1}\mathscr{NR}^+_{\mathcal{D}_2})^{\mathscr{S}} + i(\mathscr{NI}^+_{\mathcal{D}_1}\mathscr{NI}^+_{\mathcal{D}_2})^{\mathscr{S}}, -(|\mathscr{MR}^-_{\mathcal{D}_1}||\mathscr{MR}^-_{\mathcal{D}_2}|)^{\mathscr{S}} + i(-(|\mathscr{MI}^-_{\mathcal{D}_1}||\mathscr{MI}^-_{\mathcal{D}_2}|)^{\mathscr{S}}), \\
& -((1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{S}}) + (1-(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{S}}) - (1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{S}})(1-(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{S}})) \\
& + i(-((1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{S}}) + (1-(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{S}}) - (1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{S}})(1-(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{S}})))\Big\rangle \\
= & \mathscr{S}(\mathcal{D}_1 \oplus \mathcal{D}_2).
\end{aligned}
$$

(2)

$$
\begin{aligned}
& (\mathscr{S}_1 + \mathscr{S}_2)\mathcal{D} \\
= & \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_1+\mathscr{S}_2}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_1+\mathscr{S}_2}), (\mathscr{NR}^+_{\mathcal{D}})^{\mathscr{S}_1+\mathscr{S}_2} + i(\mathscr{NI}^+_{\mathcal{D}})^{\mathscr{S}_1+\mathscr{S}_2}, \\
& -|\mathscr{MR}^-_{\mathcal{D}}|^{\mathscr{S}_1+\mathscr{S}_2} + i(-|\mathscr{MI}^-_{\mathcal{D}}|^{\mathscr{S}_1+\mathscr{S}_2}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_1+\mathscr{S}_2}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_1+\mathscr{S}_2}))\Big\rangle \\
= & \Big\langle ((1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_1}) + (1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_2}) - (1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_1})(1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_2})) \\
& + i((1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_1}) + (1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_2}) - (1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_1})(1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_2})), \\
& (\mathscr{NR}^+_{\mathcal{D}})^{\mathscr{S}_1}(\mathscr{NR}^+_{\mathcal{D}})^{\mathscr{S}_2} + i(\mathscr{NI}^+_{\mathcal{D}})^{\mathscr{S}_1}(\mathscr{NI}^+_{\mathcal{D}})^{\mathscr{S}_2}, -((-|\mathscr{MR}^-_{\mathcal{D}}|^{\mathscr{S}_1})(-|\mathscr{MR}^-_{\mathcal{D}}|^{\mathscr{S}_2})) \\
& + i(-((-|\mathscr{MI}^-_{\mathcal{D}}|^{\mathscr{S}_1})(-|\mathscr{MI}^-_{\mathcal{D}}|^{\mathscr{S}_2}))), -(|-(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_1})| + |-(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_2})| \\
& -|-(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_1})||-(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_2})|) + i((-(|-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_1})| \\
& + |-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_2})| - |-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_1})||-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_2})|)))\Big\rangle \\
= & \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_1}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_1}), (\mathscr{NR}^+_{\mathcal{D}})^{\mathscr{S}_1} + i(\mathscr{NI}^+_{\mathcal{D}})^{\mathscr{S}_1}, \\
& -|\mathscr{MR}^-_{\mathcal{D}}|^{\mathscr{S}_1} + i(-|\mathscr{MI}^-_{\mathcal{D}}|^{\mathscr{S}_1}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_1}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_1}))\Big\rangle \\
\oplus & \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}}))^{\mathscr{S}_2}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}}))^{\mathscr{S}_2}), (\mathscr{NR}^+_{\mathcal{D}})^{\mathscr{S}_2} + i(\mathscr{NI}^+_{\mathcal{D}})^{\mathscr{S}_2}, \\
& -|\mathscr{MR}^-_{\mathcal{D}}|^{\mathscr{S}_2} + i(-|\mathscr{MI}^-_{\mathcal{D}}|^{\mathscr{S}_2}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\mathscr{S}_2}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\mathscr{S}_2}))\Big\rangle \\
= & \mathscr{S}_1\mathcal{D} \oplus \mathscr{S}_2\mathcal{D}.
\end{aligned}
$$

□

4. Aggregation operators for bipolar complex intuitionistic fuzzy sets

This section focuses on the development and application of weighted average and weighted geometric aggregation operators tailored for BCIFS. It presents a detailed examination of the mathematical foundations of these operators and emphasizes their critical role in effectively aggregating complex, uncertain, and bipolar information within multi-attribute decision-making frameworks.

Definition 7. *Consider a collection of BCIFNs, denoted as* $\mathcal{D}_i = \left\{\left\langle \mathscr{MR}^{+}_{\mathcal{D}_i} + i\mathscr{MI}^{+}_{\mathcal{D}_i}, \mathscr{NR}^{+}_{\mathcal{D}_i} + i\mathscr{NI}^{+}_{\mathcal{D}_i}, \mathscr{MR}^{-}_{\mathcal{D}_i} + i\mathscr{MI}^{-}_{\mathcal{D}_i}, \mathscr{NR}^{-}_{\mathcal{D}_i} + i\mathscr{NI}^{-}_{\mathcal{D}_i}\right\rangle\right\}$, $\forall i \in \{1, 2, \ldots, k\}$ *where each* $\mathcal{D}_i$ *denotes a BCIFN. Correspondingly, a weight vector* $\mathscr{H}$ *linked to these elements is defined as*

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_k)^T,$$

where each weight $\mathscr{H}_i$ *is positive* $\mathscr{H}_i > 0$ *and the weights sum to one, that is,* $\sum_{i=1}^{k} \mathscr{H}_i = 1$*. Using this setup, two key aggregation operators are defined:*

(1) The BCnonmembership operator is a mapping BCnonmembership: $\mathcal{D}^k \to \mathcal{D}$ *that aggregates a set of BCIFNs using a weighted average scheme. It is defined as*

$$BCnonmembership\,(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) = \bigoplus_{i=1}^{k} \mathscr{H}_i\mathcal{D}_i = \mathscr{H}_1\mathcal{D}_1 \oplus \mathscr{H}_2\mathcal{D}_2 \oplus \cdots \oplus \mathscr{H}_k\mathcal{D}_k. \quad (4.1)$$

(2) The BCIFWG operator is a mapping BCIFWG: $\mathcal{D}^k \to \mathcal{D}$*, and is expressed as*

$$BCIFWG\,(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) = \bigotimes_{i=1}^{k} \mathcal{D}_i^{\mathscr{H}_i} = \mathcal{D}_1^{\mathscr{H}_1} \otimes \mathcal{D}_2^{\mathscr{H}_2} \otimes \cdots \otimes \mathcal{D}_k^{\mathscr{H}_k}. \quad (4.2)$$

This operator aggregates BCIFNs through a weighted geometric combination, capturing a distinct aggregation behavior compared to the averaging operator.

Theorem 7. *Let* $\mathcal{D}_i$ *denote a collection of BCIFNs, with each element expressed as* $\mathcal{D}_i = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_i} + i\mathscr{MI}^{+}_{\mathcal{D}_i}, \mathscr{NR}^{+}_{\mathcal{D}_i} + i\mathscr{NI}^{+}_{\mathcal{D}_i}, \mathscr{MR}^{-}_{\mathcal{D}_i} + i\mathscr{MI}^{-}_{\mathcal{D}_i}, \mathscr{NR}^{-}_{\mathcal{D}_i} + i\mathscr{NI}^{-}_{\mathcal{D}_i}\right\rangle$, $i = 1, 2, \ldots, k$*. Furthermore, let* $\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_k)^T$ *denote the associated weight vector, satisfying* $\mathscr{H}_i > 0$ *and* $\sum_{i=1}^{k} \mathscr{H}_i = 1$*. Under this formulation, the BCIF weighted averaging and weighted geometric aggregation operators are expressed as*

(1) The BCIFWA operator performs aggregation through a weighted average of the BCIFNs, which can be represented in the following form:

$$\begin{aligned} &BCIFWA\,(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \\ &= \left\langle \left(1 - \prod_{i=1}^{k}\left(1 - \mathscr{MR}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i}\right) + i\left(1 - \prod_{i=1}^{k}\left(1 - \mathscr{MI}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i}\right), \prod_{i=1}^{k}\left(\mathscr{NR}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i} + i\prod_{i=1}^{k}\left(\mathscr{NI}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i}, \right. \\ &\left. -\prod_{i=1}^{k}\left|\mathscr{MR}^{-}_{\mathcal{D}_i}\right|^{\mathscr{H}_i} + i\left(-\prod_{i=1}^{k}\left|\mathscr{MI}^{-}_{\mathcal{D}_i}\right|^{\mathscr{H}_i}\right), -\left(1 - \prod_{i=1}^{k}\left(1 - \left|\mathscr{NR}^{-}_{\mathcal{D}_i}\right|\right)^{\mathscr{H}_i}\right) + i\left(-\left(1 - \prod_{i=1}^{k}\left(1 - \left|\mathscr{NI}^{-}_{\mathcal{D}_i}\right|\right)^{\mathscr{H}_i}\right)\right)\right\rangle. \end{aligned} \quad (4.3)$$

(2) The BCIFWG operator performs aggregation using a weighted geometric approach, and its formulation is expressed as

$$\begin{aligned} &BCIFWG\,(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \\ &= \left\langle \prod_{i=1}^{k}\left(\mathscr{MR}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i} + i\prod_{i=1}^{k}\left(\mathscr{MI}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i}, 1 - \prod_{i=1}^{k}\left(1 - \mathscr{NR}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i} + i\left(1 - \prod_{i=1}^{k}\left(1 - \mathscr{NI}^{+}_{\mathcal{D}_i}\right)^{\mathscr{H}_i}\right), \right. \\ &\left. -\left(1 - \prod_{i=1}^{k}\left(1 - \left|\mathscr{MR}^{-}_{\mathcal{D}_i}\right|\right)^{\mathscr{H}_i}\right) + i\left(-\left(1 - \prod_{i=1}^{k}\left(1 - \left|\mathscr{MI}^{-}_{\mathcal{D}_i}\right|\right)^{\mathscr{H}_i}\right)\right), -\prod_{i=1}^{k}\left|\mathscr{NR}^{-}_{\mathcal{D}_i}\right|^{\mathscr{H}_i} + i\left(-\prod_{i=1}^{k}\left|\mathscr{NI}^{-}_{\mathcal{D}_i}\right|^{\mathscr{H}_i}\right)\right\rangle. \end{aligned} \quad (4.4)$$

Proof. (1) To verify the result through mathematical induction, the process starts with the base case $k = 2$. In this initial step, the expression simplifies, allowing a straightforward validation of the

$$\begin{aligned}
&\text{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2) = \mathscr{H}_1\mathcal{D}_1 \oplus \mathscr{H}_2\mathcal{D}_2\\
&= \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}_1}))^{\mathscr{H}_1}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}_1}))^{\mathscr{H}_1}), (\mathscr{NR}^+_{\mathcal{D}_1})^{\mathscr{H}_1} + i(\mathscr{NI}^+_{\mathcal{D}_1})^{\mathscr{H}_1}, -|\mathscr{MR}^-_{\mathcal{D}_1}|^{\mathscr{H}_1}\\
&\quad + i(-|\mathscr{MI}^-_{\mathcal{D}_1}|^{\mathscr{H}_1}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1}))\Big\rangle\\
&\oplus \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}), (\mathscr{NR}^+_{\mathcal{D}_2})^{\mathscr{H}_2} + i(\mathscr{NI}^+_{\mathcal{D}_2})^{\mathscr{H}_2},\\
&\quad -|\mathscr{MR}^-_{\mathcal{D}_2}|^{\mathscr{H}_2} + i(-|\mathscr{MI}^-_{\mathcal{D}_2}|^{\mathscr{H}_2}), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2}))\Big\rangle\\
&= \Big\langle ((1-(1-(\mathscr{MR}^+_{\mathcal{D}_1}))^{\mathscr{H}_1}) + (1-(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}) - (1-(1-(\mathscr{MR}^+_{\mathcal{D}_1}))^{\mathscr{H}_1})(1-(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}))\\
&\quad + i((1-(1-(\mathscr{MI}^+_{\mathcal{D}_1}))^{\mathscr{H}_1}) + (1-(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}) - (1-(1-(\mathscr{MI}^+_{\mathcal{D}_1}))^{\mathscr{H}_1})(1-(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{H}_2})),\\
&\quad (\mathscr{NR}^+_{\mathcal{D}_1})^{\mathscr{H}_1}(\mathscr{NR}^+_{\mathcal{D}_2})^{\mathscr{H}_2} + i(\mathscr{NI}^+_{\mathcal{D}_1})^{\mathscr{H}_1}(\mathscr{NI}^+_{\mathcal{D}_2})^{\mathscr{H}_2}, -(|\mathscr{MR}^-_{\mathcal{D}_1}|^{\mathscr{H}_1}|\mathscr{MR}^-_{\mathcal{D}_2}|^{\mathscr{H}_2})\\
&\quad + i(-(|\mathscr{MI}^-_{\mathcal{D}_1}|^{\mathscr{H}_1}|\mathscr{MI}^-_{\mathcal{D}_2}|^{\mathscr{H}_2})), -((1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1}) + (1-(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2})\\
&\quad -(1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1})(1-(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2})) + i(-((1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1}) + (1-(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2})\\
&\quad -(1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1})(1-(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2})))\Big\rangle\\
&= \Big\langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}_1}))^{\mathscr{H}_1}(1-(\mathscr{MR}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}_1}))^{\mathscr{H}_1}(1-(\mathscr{MI}^+_{\mathcal{D}_2}))^{\mathscr{H}_2}),\\
&\quad (\mathscr{NR}^+_{\mathcal{D}_1})^{\mathscr{H}_1}(\mathscr{NR}^+_{\mathcal{D}_2})^{\mathscr{H}_2} + i((\mathscr{NI}^+_{\mathcal{D}_1})^{\mathscr{H}_1}(\mathscr{NI}^+_{\mathcal{D}_2})^{\mathscr{H}_2}), -((|\mathscr{MR}^-_{\mathcal{D}_1}|^{\mathscr{H}_1})(|\mathscr{MR}^-_{\mathcal{D}_2}|^{\mathscr{H}_2}))\\
&\quad + i(-((|\mathscr{MI}^-_{\mathcal{D}_1}|^{\mathscr{H}_1})(|\mathscr{MI}^-_{\mathcal{D}_2}|^{\mathscr{H}_2}))), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1}(1-|\mathscr{NR}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2})\\
&\quad + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}_1}|)^{\mathscr{H}_1}(1-|\mathscr{NI}^-_{\mathcal{D}_2}|)^{\mathscr{H}_2}))\Big\rangle\\
&= \Big\langle (1-\prod\nolimits_{i=1}^{2}(1-(\mathscr{MR}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) + i(1-\prod\nolimits_{i=1}^{2}(1-(\mathscr{MI}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}), \prod\nolimits_{i=1}^{2}(\mathscr{NR}^+_{\mathcal{D}_i})^{\mathscr{H}_i} + i\prod\nolimits_{i=1}^{2}(\mathscr{NI}^+_{\mathcal{D}_i})^{\mathscr{H}_i},\\
&\quad -(\prod\nolimits_{i=1}^{2}|\mathscr{MR}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) + i(-(\prod\nolimits_{i=1}^{2}|\mathscr{MI}^-_{\mathcal{D}_i}|^{\mathscr{H}_i})), -(1-\prod\nolimits_{i=1}^{2}(1-|\mathscr{NR}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i})\\
&\quad + i(-(1-\prod\nolimits_{i=1}^{2}(1-|\mathscr{NI}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}))\Big\rangle.
\end{aligned}$$

Assume that the theorem holds for $k = l$, meaning the formula is valid for a collection of l elements. Specifically, we consider

$$\begin{aligned}
&\text{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_l) = \mathscr{H}_1\mathcal{D}_1 \oplus \mathscr{H}_2\mathcal{D}_2 \oplus \cdots \oplus \mathscr{H}_l\mathcal{D}_l\\
&= \Big\langle (1-\prod\nolimits_{i=1}^{l}(1-(\mathscr{MR}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) + i(1-\prod\nolimits_{i=1}^{l}(1-(\mathscr{MI}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}), \prod\nolimits_{i=1}^{l}(\mathscr{NR}^+_{\mathcal{D}_i})^{\mathscr{H}_i} + i\prod\nolimits_{i=1}^{l}(\mathscr{NI}^+_{\mathcal{D}_i})^{\mathscr{H}_i},\\
&\quad -(\prod\nolimits_{i=1}^{l}|\mathscr{MR}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) + i(-(\prod\nolimits_{i=1}^{l}|\mathscr{MI}^-_{\mathcal{D}_i}|^{\mathscr{H}_i})), -(1-\prod\nolimits_{i=1}^{l}(1-|\mathscr{NR}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}) + i(-(1-\prod\nolimits_{i=1}^{l}(1-|\mathscr{NI}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}))\Big\rangle.
\end{aligned}$$

To extend the proof to $k = l + 1$, the inductive hypothesis is applied. Under this condition, the expression for $k = l + 1$ takes the form:

$$\begin{aligned}
&\text{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{l+1}) = \mathscr{H}_1\mathcal{D}_1 \oplus \mathscr{H}_2\mathcal{D}_2 \oplus \cdots \oplus \mathscr{H}_{l+1}\mathcal{D}_{l+1}\\
&= \Big\langle (1-\prod\nolimits_{i=1}^{l}(1-(\mathscr{MR}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) + i(1-\prod\nolimits_{i=1}^{l}(1-(\mathscr{MI}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}),\\
&\quad \prod\nolimits_{i=1}^{l}(\mathscr{NR}^+_{\mathcal{D}_i})^{\mathscr{H}_i} + i\prod\nolimits_{i=1}^{l}(\mathscr{NI}^+_{\mathcal{D}_i})^{\mathscr{H}_i}, -(\prod\nolimits_{i=1}^{l}|\mathscr{MR}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) + i(-(\prod\nolimits_{i=1}^{l}|\mathscr{MI}^-_{\mathcal{D}_i}|^{\mathscr{H}_i})),
\end{aligned}$$

$$
\begin{aligned}
&-(1-\prod_{i=1}^{l}(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})+i(-(1-\prod_{i=1}^{l}(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i}))\Big\rangle\\
\oplus&\Big\langle(1-(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_{l+1}}))^{\mathscr{H}_{l+1}})+i(1-(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_{l+1}}))^{\mathscr{H}_{l+1}}),(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_{l+1}})^{\mathscr{H}_{l+1}}+i(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_{l+1}})^{\mathscr{H}_{l+1}},\\
&-|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_{l+1}}|^{\mathscr{H}_{l+1}}+i(-|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_{l+1}}|^{\mathscr{H}_{l+1}}),-(1-(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_{l+1}}|)^{\mathscr{H}_{l+1}})+i(-(1-(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_{l+1}}|)^{\mathscr{H}_{l+1}}))\Big\rangle\\
=&\Big\langle((1-\prod_{i=1}^{l}(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})+(1-(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_{l+1}}))^{\mathscr{H}_{l+1}})\\
&-(1-\prod_{i=1}^{l}(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})(1-(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_{l+1}}))^{\mathscr{H}_{l+1}}))^{\frac{1}{q}}+i((1-\prod_{i=1}^{l}(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})\\
&+(1-(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_{l+1}}))^{\mathscr{H}_{l+1}})-(1-\prod_{i=1}^{l}(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})(1-(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_{l+1}}))^{\mathscr{H}_{l+1}}))^{\frac{1}{q}},\\
&(\prod_{i=1}^{l}(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i}(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_{l+1}})^{\mathscr{H}_{l+1}})+i(\prod_{i=1}^{l}(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i}(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_{l+1}})^{\mathscr{H}_{l+1}}),\\
&-((-(\prod_{i=1}^{l}|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i}))(-|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_{l+1}}|^{\mathscr{H}_{l+1}}))+i(-((-(\prod_{i=1}^{l}|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i}))(-|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_{l+1}}|^{\mathscr{H}_{l+1}}))),\\
&-(|-(1-\prod_{i=1}^{l}(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})|+(|-(1-(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_{l+1}}|)^{\mathscr{H}_{l+1}})|)\\
&-(|-(1-\prod_{i=1}^{l}(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})|)(|-(1-(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_{l+1}}|)^{\mathscr{H}_{l+1}})|))\\
&+i(-(|-(1-\prod_{i=1}^{l}(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})|+(|-(1-(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_{l+1}}|)^{\mathscr{H}_{l+1}})|)\\
&-(|-(1-\prod_{i=1}^{l}(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})|)(|-(1-(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_{l+1}}|)^{\mathscr{H}_{l+1}})|)))\Big\rangle\\
=&\Big\langle(1-\prod_{i=1}^{l+1}(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})+i(1-\prod_{i=1}^{l+1}(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i}),\\
&\prod_{i=1}^{l+1}(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i}+i\prod_{i=1}^{l+1}(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i},-(\prod_{i=1}^{l+1}|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i})+i(-(\prod_{i=1}^{l+1}|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i})),\\
&-(1-\prod_{i=1}^{l+1}(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})+i(-(1-\prod_{i=1}^{l+1}(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i}))\Big\rangle.
\end{aligned}
$$

This aligns with the expression for $k = l + 1$, thereby completing the proof through mathematical induction.

(2) The proof proceeds using the same approach as that employed in demonstrating (1).

□

Example 2. Take into account the following BCIFNs:

$$\mathcal{D}_1 = \langle .20 + i(.84), .71 + i(.10), -.89 + i(-.52), -.01 + i(-.30)\rangle,$$

$$\mathcal{D}_2 = \langle .79 + i(.71), .12 + i(.11), -.44 + i(-.60), -.49 + i(-.31)\rangle,$$

and

$$\mathcal{D}_3 = \langle .52 + i(.21), .27 + i(.33), -.79 + i(-.24), -.12 + i(-.71)\rangle,$$

along with the corresponding weight vector $\mathscr{H} = (.295, .408, .297)^T$ respectively. Next, the process continues as follows:

(1).

$$\mathrm{BCIFWA}(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3) = \Big\langle(1-\prod_{i=1}^{3}(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})+i(1-\prod_{i=1}^{3}(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i}),$$

$$\prod_{i=1}^{3}(\mathcal{NR}^{+}_{\mathcal{D}_i})^{\mathcal{H}_i}+i\prod_{i=1}^{3}(\mathcal{NI}^{+}_{\mathcal{D}_i})^{\mathcal{H}_i},-(\prod_{i=1}^{3}|\mathcal{MR}^{-}_{\mathcal{D}_i}|^{\mathcal{H}_i})+i(-(\prod_{i=1}^{3}|\mathcal{MI}^{-}_{\mathcal{D}_i}|^{\mathcal{H}_i})),$$
$$-(1-\prod_{i=1}^{3}(1-|\mathcal{NR}^{-}_{\mathcal{D}_i}|)^{\mathcal{H}_i})+i(-(1-\prod_{i=1}^{3}(1-|\mathcal{NI}^{-}_{\mathcal{D}_i}|)^{\mathcal{H}_i}))\Big\rangle$$
$$\approx\langle .6017+.6723i,.2580+.1482i,-.6445+(-.4382)i,-.2707+(-.4643)i\rangle.$$

(2).

$$\mathrm{BCIFWG}\,(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)=\Big\langle\prod_{i=1}^{3}(\mathcal{MR}^{+}_{\mathcal{D}_i})^{\mathcal{H}_i}+i(\prod_{i=1}^{3}\mathcal{MI}^{+}_{\mathcal{D}_i})^{\mathcal{H}_i},(1-\prod_{i=1}^{3}(1-(\mathcal{NR}^{+}_{\mathcal{D}_i}))^{\mathcal{H}_i})$$
$$+i(1-\prod_{i=1}^{3}(1-(\mathcal{NI}^{+}_{\mathcal{D}_i}))^{\mathcal{H}_i}),-(1-\prod_{i=1}^{3}(1-|\mathcal{MR}^{-}_{\mathcal{D}_i}|)^{\mathcal{H}_i})+i(-(1-\prod_{i=1}^{3}(1-|\mathcal{MI}^{-}_{\mathcal{D}_i}|)^{\mathcal{H}_i})),$$
$$-(\prod_{i=1}^{3}|\mathcal{NR}^{-}_{\mathcal{D}_i}|_i^{\mathcal{H}_i})+i(-(\prod_{i=1}^{3}|\mathcal{NI}^{-}_{\mathcal{D}_i}|_i^{\mathcal{H}_i}))\Big\rangle$$
$$\approx\langle .4653+.5196i,.4000+.1793i,-.7411+(-.4893)i,-.1024+(-.3927)i\rangle.$$

Theorem 8. *The outcomes resulting from the application of the operators* $\mathrm{BCIFWA}\,(\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_k)$ *and* $\mathrm{BCIFWG}\,(\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_k)$ *are both characterized as BCIFNs.*

Proof. The assertion that the application of the BCIFWA and BCIFWG operators results in BCIFNs is directly established by Theorems 1 and 2. □

Theorem 9. *(Idempotency) A collection of BCIFNs is defined as* $\mathcal{D}_i=\Big\{\Big\langle\mathcal{MR}^{+}_{\mathcal{D}_i}+i\mathcal{MI}^{+}_{\mathcal{D}_i},\mathcal{NR}^{+}_{\mathcal{D}_i}+i\mathcal{NI}^{+}_{\mathcal{D}_i},\mathcal{MR}^{-}_{\mathcal{D}_i}+i\mathcal{MI}^{-}_{\mathcal{D}_i},\mathcal{NR}^{-}_{\mathcal{D}_i}+i\mathcal{NI}^{-}_{\mathcal{D}_i}\Big\rangle\Big\}$. *Let* $\mathcal{H}=(\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_k)^T$ *denote the weight vector corresponding to* $\mathcal{D}_i$*, where each weight satisfies* $\mathcal{H}_i>0$ *and the weights sum to unity, that is,* $\sum_{i=1}^{k}\mathcal{H}_i=1$. *If every element* $\mathcal{D}_i$ *equals the same BCIFN* $\mathcal{D}$*, represented by* $\mathcal{D}=\langle\mathcal{MR}^{+}_{\mathcal{D}}+i\mathcal{MI}^{+}_{\mathcal{D}},\mathcal{NR}^{+}_{\mathcal{D}}+i\mathcal{NI}^{+}_{\mathcal{D}},\mathcal{MR}^{-}_{\mathcal{D}}+i\mathcal{MI}^{-}_{\mathcal{D}},\mathcal{NR}^{-}_{\mathcal{D}}+i\mathcal{NI}^{-}_{\mathcal{D}}\rangle$*, then*

(1) $BCIFWA(\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_k)=\mathcal{D}$.
(2) $BCIFWG(\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_k)=\mathcal{D}$.

Proof. Establishing the first case suffices since the other cases can be demonstrated through analogous reasoning. Sine $\mathcal{D}_i=\mathcal{D}$, $\forall i=1,2,\ldots,k$, then the following expression is obtained:

$$\mathrm{BCIFWA}\,(\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_k)$$
$$=\Big\langle(1-\prod_{i=1}^{k}(1-(\mathcal{MR}^{+}_{\mathcal{D}_i}))^{\mathcal{H}_i})+i(1-\prod_{i=1}^{k}(1-(\mathcal{MI}^{+}_{\mathcal{D}_i}))^{\mathcal{H}_i}),$$
$$\prod_{i=1}^{k}(\mathcal{NR}^{+}_{\mathcal{D}_i})^{\mathcal{H}_i}+i\prod_{i=1}^{k}(\mathcal{NI}^{+}_{\mathcal{D}_i})^{\mathcal{H}_i},-(\prod_{i=1}^{k}|\mathcal{MR}^{-}_{\mathcal{D}_i}|^{\mathcal{H}_i})+i(-(\prod_{i=1}^{k}|\mathcal{MI}^{-}_{\mathcal{D}_i}|^{\mathcal{H}_i})),$$
$$-(1-\prod_{i=1}^{k}(1-|\mathcal{NR}^{-}_{\mathcal{D}_i}|)^{\mathcal{H}_i})+i(-(1-\prod_{i=1}^{k}(1-|\mathcal{NI}^{-}_{\mathcal{D}_i}|)^{\mathcal{H}_i}))\Big\rangle$$
$$=\Big\langle(1-\prod_{i=1}^{k}(1-(\mathcal{MR}^{+}_{\mathcal{D}}))^{\mathcal{H}_i})+i(1-\prod_{i=1}^{k}(1-(\mathcal{MI}^{+}_{\mathcal{D}}))^{\mathcal{H}_i}),$$
$$\prod_{i=1}^{k}(\mathcal{NR}^{+}_{\mathcal{D}})^{\mathcal{H}_i}+i\prod_{i=1}^{k}(\mathcal{NI}^{+}_{\mathcal{D}})^{\mathcal{H}_i},-(\prod_{i=1}^{k}|\mathcal{MR}^{-}_{\mathcal{D}}|^{\mathcal{H}_i})+i(-(\prod_{i=1}^{k}|\mathcal{MI}^{-}_{\mathcal{D}}|^{\mathcal{H}_i})),$$
$$-(1-\prod_{i=1}^{k}(1-|\mathcal{NR}^{-}_{\mathcal{D}}|)^{\mathcal{H}_i})+i(-(1-\prod_{i=1}^{k}(1-|\mathcal{NI}^{-}_{\mathcal{D}}|)^{\mathcal{H}_i}))\Big\rangle$$
$$=\Big\langle(1-(1-(\mathcal{MR}^{+}_{\mathcal{D}}))^{\sum_{i=1}^{k}\mathcal{H}_i})+i(1-(1-(\mathcal{MI}^{+}_{\mathcal{D}}))^{\sum_{i=1}^{k}\mathcal{H}_i}),$$

$$(\mathscr{NR}^+_{\mathcal{D}})^{\sum_{i=1}^k \mathscr{H}_i} + i(\mathscr{NI}^+_{\mathcal{D}})^{\sum_{i=1}^k \mathscr{H}_i}, -(|\mathscr{MR}^-_{\mathcal{D}}|^{\sum_{i=1}^k \mathscr{H}_i}) + i(-(|\mathscr{MI}^-_{\mathcal{D}}|^{\sum_{i=1}^k \mathscr{H}_i})),$$
$$-(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)^{\sum_{i=1}^k \mathscr{H}_i}) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)^{\sum_{i=1}^k \mathscr{H}_i}))\Big\rangle$$
$$= \langle (1-(1-(\mathscr{MR}^+_{\mathcal{D}}))) + i(1-(1-(\mathscr{MI}^+_{\mathcal{D}}))), (\mathscr{NR}^+_{\mathcal{D}}) + i(\mathscr{NI}^+_{\mathcal{D}}),$$
$$-(|\mathscr{MR}^-_{\mathcal{D}}|) + i(-(|\mathscr{MI}^-_{\mathcal{D}}|)), -(1-(1-|\mathscr{NR}^-_{\mathcal{D}}|)) + i(-(1-(1-|\mathscr{NI}^-_{\mathcal{D}}|)))\rangle$$
$$= \langle \mathscr{MR}^+_{\mathcal{D}} + i\mathscr{MI}^+_{\mathcal{D}}, \mathscr{NR}^+_{\mathcal{D}} + i\mathscr{NI}^+_{\mathcal{D}}, \mathscr{MR}^-_{\mathcal{D}} + i\mathscr{MI}^-_{\mathcal{D}}, \mathscr{NR}^-_{\mathcal{D}} + i\mathscr{NI}^-_{\mathcal{D}}\rangle,$$

where $-|\mathscr{MR}^-_{\mathcal{D}}| = \mathscr{MR}^-_{\mathcal{D}}$, $-|\mathscr{NR}^-_{\mathcal{D}}| = \mathscr{NR}^-_{\mathcal{D}}$, $-|\mathscr{MI}^-_{\mathcal{D}}| = \mathscr{MI}^-_{\mathcal{D}}$, and $-|\mathscr{NI}^-_{\mathcal{D}}| = \mathscr{NI}^-_{\mathcal{D}}$. □

Theorem 10. *(Boundedness) A collection of BCIFNs can be expressed as* $\mathcal{D}_i = \left\{\left\langle \mathscr{MR}^+_{\mathcal{D}_i} + i\mathscr{MI}^+_{\mathcal{D}_i}, \mathscr{NR}^+_{\mathcal{D}_i} + i\mathscr{NI}^+_{\mathcal{D}_i}, \mathscr{MR}^-_{\mathcal{D}_i} + i\mathscr{MI}^-_{\mathcal{D}_i}, \mathscr{NR}^-_{\mathcal{D}_i} + i\mathscr{NI}^-_{\mathcal{D}_i}\right\rangle\right\}$, $\forall i = 1, 2, \ldots, k$. *Define the weight vector as* $\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_k)^T$ *where each component satisfies* $\mathscr{H}_i > 0$ *and the weights are normalized such that* $\sum_{i=1}^k \mathscr{H}_i = 1$. *Now, consider two BCIFNs,* $\overline{\mathcal{D}}$ *and* $\underline{\mathcal{D}}$, *defined as follows:* $\overline{\mathcal{D}} = \left\langle \mathscr{MR}^{+*}_{\overline{\mathcal{D}}} + i\mathscr{MI}^{+*}_{\overline{\mathcal{D}}}, \mathscr{NR}^{+\bullet}_{\overline{\mathcal{D}}} + i\mathscr{NI}^{+\bullet}_{\overline{\mathcal{D}}}, \mathscr{MR}^{-\bullet}_{\overline{\mathcal{D}}} + i\mathscr{MI}^{-\bullet}_{\overline{\mathcal{D}}}, \mathscr{NR}^{-*}_{\overline{\mathcal{D}}} + i\mathscr{NI}^{-*}_{\overline{\mathcal{D}}}\right\rangle$, *which can alternatively be expressed using the maximum and minimum values of the given BCIFNs as* $\overline{\mathcal{D}} = \Big\langle \max(\mathscr{MR}^+_{\overline{\mathcal{D}_i}}) + i\max(\mathscr{MI}^+_{\overline{\mathcal{D}_i}}), \min(\mathscr{NR}^+_{\overline{\mathcal{D}_i}}) + i\min(\mathscr{NI}^+_{\overline{\mathcal{D}_i}}), \min(\mathscr{MR}^-_{\overline{\mathcal{D}_i}}) + i\min(\mathscr{MI}^-_{\overline{\mathcal{D}_i}}), \max(\mathscr{NR}^-_{\overline{\mathcal{D}_i}}) + i\max(\mathscr{NI}^-_{\overline{\mathcal{D}_i}})\Big\rangle$, *where* $1 \le i \le k$. *Likewise, the BCIFN* $\underline{\mathcal{D}}$ *is specified as:* $\underline{\mathcal{D}} = \left\langle \mathscr{MR}^{+\bullet}_{\underline{\mathcal{D}}} + i\mathscr{MI}^{+\bullet}_{\underline{\mathcal{D}}}, \mathscr{NR}^{+*}_{\underline{\mathcal{D}}} + i\mathscr{NI}^{+*}_{\underline{\mathcal{D}}}, \mathscr{MR}^{-*}_{\underline{\mathcal{D}}} + i\mathscr{MI}^{-*}_{\underline{\mathcal{D}}}, \mathscr{NR}^{-\bullet}_{\underline{\mathcal{D}}} + i\mathscr{NI}^{-\bullet}_{\underline{\mathcal{D}}}\right\rangle$, *which can alternatively be expressed using the corresponding minimum and maximum values as:* $\underline{\mathcal{D}} = \Big\langle \min(\mathscr{MR}^+_{\underline{\mathcal{D}}_i}) + i\min(\mathscr{MI}^+_{\underline{\mathcal{D}}_i}), \max(\mathscr{NR}^+_{\underline{\mathcal{D}}_i}) + i\max(\mathscr{NI}^+_{\underline{\mathcal{D}}_i}), \max(\mathscr{MR}^-_{\underline{\mathcal{D}}_i}) + i\max(\mathscr{MI}^-_{\underline{\mathcal{D}}_i}), \min(\mathscr{NR}^-_{\underline{\mathcal{D}}_i}) + i\min(\mathscr{NI}^-_{\underline{\mathcal{D}}_i})\Big\rangle$, *where* $1 \le i \le k$. *Then,*

(1) $\underline{\mathcal{D}} \le \mathrm{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \le \overline{\mathcal{D}}$.
(2) $\underline{\mathcal{D}} \le \mathrm{BCIFWG}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \le \overline{\mathcal{D}}$.

Proof. The demonstration begins with the first result, as the remaining ones can be derived using similar reasoning. To verify this initial part, it is necessary to ensure that:

$$\mathscr{MR}^{+\bullet}_{\underline{\mathcal{D}}} \le (1 - \textstyle\prod_{i=1}^k (1 - (\mathscr{MR}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) \le \mathscr{MR}^{+*}_{\underline{\mathcal{D}}},$$

$$\mathscr{MI}^{+\bullet}_{\underline{\mathcal{D}}} \le (1 - \textstyle\prod_{i=1}^k (1 - (\mathscr{MI}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) \le \mathscr{MI}^{+*}_{\underline{\mathcal{D}}},$$

$$\mathscr{NR}^{+*}_{\underline{\mathcal{D}}} \ge \textstyle\prod_{i=1}^k (\mathscr{NR}^+_{\mathcal{D}_i})^{\mathscr{H}_i} \ge \mathscr{NR}^{+\bullet}_{\underline{\mathcal{D}}},$$

$$\mathscr{NI}^{+*}_{\underline{\mathcal{D}}} \ge \textstyle\prod_{i=1}^k (\mathscr{NI}^+_{\mathcal{D}_i})^{\mathscr{H}_i} \ge \mathscr{NI}^{+\bullet}_{\underline{\mathcal{D}}},$$

$$\mathscr{MR}^{-*}_{\underline{\mathcal{D}}} \ge -(\textstyle\prod_{i=1}^k |\mathscr{MR}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) \ge \mathscr{MR}^{-\bullet}_{\underline{\mathcal{D}}},$$

$$\mathscr{MI}^{-*}_{\underline{\mathcal{D}}} \ge -(\textstyle\prod_{i=1}^k |\mathscr{MI}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) \ge \mathscr{MI}^{-\bullet}_{\underline{\mathcal{D}}},$$

$$\mathscr{NR}^{-\bullet}_{\underline{\mathcal{D}}} \le -(1 - \textstyle\prod_{i=1}^k (1 - |\mathscr{NR}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}) \le \mathscr{NR}^{-*}_{\underline{\mathcal{D}}},$$

and

$$\mathscr{NI}^{-\bullet}_{\underline{\mathcal{D}}} \le -(1 - \textstyle\prod_{i=1}^k (1 - |\mathscr{NI}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}) \le \mathscr{NI}^{-*}_{\underline{\mathcal{D}}}.$$

Starting from the following inequalities:

$$\mathscr{M}\mathscr{R}^{+\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i} \leq \mathscr{M}\mathscr{R}^{+*}_{\overline{\mathcal{D}}}, \quad \mathscr{M}\mathscr{R}^{-\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_i} \leq \mathscr{M}\mathscr{R}^{-*}_{\overline{\mathcal{D}}},$$

$$\mathscr{N}\mathscr{R}^{+\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_i} \leq \mathscr{N}\mathscr{R}^{+*}_{\overline{\mathcal{D}}}, \quad \mathscr{N}\mathscr{R}^{-\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i} \leq \mathscr{N}\mathscr{R}^{-*}_{\overline{\mathcal{D}}}.$$

In a similar manner, the following relations hold for the remaining terms:

$$\mathscr{M}\mathscr{I}^{+\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i} \leq \mathscr{M}\mathscr{I}^{+*}_{\overline{\mathcal{D}}}, \quad \mathscr{M}\mathscr{I}^{-\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_i} \leq \mathscr{M}\mathscr{I}^{-*}_{\overline{\mathcal{D}}},$$

$$\mathscr{N}\mathscr{I}^{+\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_i} \leq \mathscr{N}\mathscr{I}^{+*}_{\overline{\mathcal{D}}}, \quad \mathscr{N}\mathscr{I}^{-\bullet}_{\underline{\mathcal{D}}} \leq \mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i} \leq \mathscr{N}\mathscr{I}^{-*}_{\overline{\mathcal{D}}}.$$

From these relations, it follows that:

$$\begin{aligned}\mathscr{M}\mathscr{R}^{+^\bullet}_{\mathcal{D}} =&(1-(1-(\mathscr{M}\mathscr{R}^{+^\bullet}_{\mathcal{D}}))^{\sum_{i=1}^{k}\mathscr{H}_i}) = (1-\prod_{i=1}^{k}(1-(\mathscr{M}\mathscr{R}^{+^\bullet}_{\mathcal{D}}))^{\mathscr{H}_i}) \leq (1-\prod_{i=1}^{k}(1-(\mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i})\\ \leq&(1-\prod_{i=1}^{k}(1-(\mathscr{M}\mathscr{R}^{+^*}_{\mathcal{D}}))^{\mathscr{H}_i}) = (1-(1-(\mathscr{M}\mathscr{R}^{+^*}_{\mathcal{D}}))^{\sum_{i=1}^{k}\mathscr{H}_i}) = \mathscr{M}\mathscr{R}^{+^*}_{\mathcal{D}},\end{aligned}$$

$$\begin{aligned}\mathscr{N}\mathscr{R}^{+^\bullet}_{\mathcal{D}} =&(\mathscr{N}\mathscr{R}^{+^\bullet}_{\mathcal{D}})^{\sum_{i=1}^{k}\mathscr{H}_i} = \prod_{i=1}^{k}(\mathscr{N}\mathscr{R}^{+^\bullet}_{\mathcal{D}})^{\mathscr{H}_i} \leq \prod_{i=1}^{k}(\mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i}\\ \leq&\prod_{i=1}^{k}(\mathscr{N}\mathscr{R}^{+^*}_{\mathcal{D}})^{\mathscr{H}_i} = (\mathscr{N}\mathscr{R}^{+^*}_{\mathcal{D}})^{\sum_{i=1}^{k}\mathscr{H}_i} = \mathscr{N}\mathscr{R}^{+^*}_{\mathcal{D}},\end{aligned}$$

$$\begin{aligned}\mathscr{M}\mathscr{R}^{-^\bullet}_{\mathcal{D}} =& -(|\mathscr{M}\mathscr{R}^{-^\bullet}_{\mathcal{D}}|^{\sum_{i=1}^{k}\mathscr{H}_i}) = -(\prod_{i=1}^{k}(|\mathscr{M}\mathscr{R}^{-^\bullet}_{\mathcal{D}}|)^{\mathscr{H}_i}) \leq -(\prod_{i=1}^{k}(|\mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})\\ \leq& -(\prod_{i=1}^{k}(|\mathscr{M}\mathscr{R}^{-^*}_{\mathcal{D}}|)^{\mathscr{H}_i}) = -(|\mathscr{M}\mathscr{R}^{-^*}_{\mathcal{D}}|^{\sum_{i=1}^{k}\mathscr{H}_i}) = -|\mathscr{M}\mathscr{R}^{-^*}_{\mathcal{D}}| = \mathscr{M}\mathscr{R}^{-^*}_{\mathcal{D}},\end{aligned}$$

and

$$\begin{aligned}\mathscr{N}\mathscr{R}^{-^\bullet}_{\mathcal{D}} =& -(1-(1-|\mathscr{N}\mathscr{R}^{-^\bullet}_{\mathcal{D}}|)^{\sum_{i=1}^{k}\mathscr{H}_i}) = -(1-\prod_{i=1}^{k}(1-|\mathscr{N}\mathscr{R}^{-^\bullet}_{\mathcal{D}}|)^{\mathscr{H}_i}) \leq -(1-\prod_{i=1}^{k}(1-|\mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})\\ \leq& -(1-\prod_{i=1}^{k}(1-|\mathscr{N}\mathscr{R}^{-^*}_{\mathcal{D}}|)^{\mathscr{H}_i}) = -(1-(1-(\mathscr{N}\mathscr{R}^{-^*}_{\mathcal{D}}))^{\sum_{i=1}^{k}\mathscr{H}_i}) = -|\mathscr{N}\mathscr{R}^{-^*}_{\mathcal{D}}| = \mathscr{N}\mathscr{R}^{-^*}_{\mathcal{D}}.\end{aligned}$$

By applying the same reasoning, it can also be demonstrated that:

$$\mathscr{M}\mathscr{I}^{+\bullet}_{\underline{\mathcal{D}}} \leq (1-\prod_{i=1}^{k}(1-(\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i}) \leq \mathscr{M}\mathscr{I}^{+*}_{\underline{\mathcal{D}}},$$

$$\mathscr{N}\mathscr{I}^{+*}_{\underline{\mathcal{D}}} \geq \prod_{i=1}^{k}(\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i} \geq \mathscr{N}\mathscr{I}^{+\bullet}_{\underline{\mathcal{D}}},$$

$$\mathscr{M}\mathscr{I}^{-*}_{\underline{\mathcal{D}}} \geq -(\prod_{i=1}^{k}|\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i}) \geq \mathscr{M}\mathscr{I}^{-\bullet}_{\underline{\mathcal{D}}},$$

and

$$\mathscr{N}\mathscr{I}^{-\bullet}_{\underline{\mathcal{D}}} \leq -(1-\prod_{i=1}^{k}(1-|\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i}) \leq \mathscr{N}\mathscr{I}^{-*}_{\underline{\mathcal{D}}}.$$

□

Theorem 11. *(Monotonicity) Let* $\mathcal{D}_i = \left\{\langle \mathscr{M}\mathscr{R}^{+}_{\mathcal{D}_i} + i\mathscr{M}\mathscr{I}^{+}_{\mathcal{D}_i}, \mathscr{N}\mathscr{R}^{+}_{\mathcal{D}_i} + i\mathscr{N}\mathscr{I}^{+}_{\mathcal{D}_i}, \mathscr{M}\mathscr{R}^{-}_{\mathcal{D}_i} + i\mathscr{M}\mathscr{I}^{-}_{\mathcal{D}_i}, \mathscr{N}\mathscr{R}^{-}_{\mathcal{D}_i} + i\mathscr{N}\mathscr{I}^{-}_{\mathcal{D}_i}\rangle\right\}$ *and* $\tilde{\mathcal{D}}_i = \left\{\langle \mathscr{M}\mathscr{R}^{+}_{\tilde{\mathcal{D}}_i} + i\mathscr{M}\mathscr{I}^{+}_{\tilde{\mathcal{D}}_i}, \mathscr{N}\mathscr{R}^{+}_{\tilde{\mathcal{D}}_i} + i\mathscr{N}\mathscr{I}^{+}_{\tilde{\mathcal{D}}_i}, \mathscr{M}\mathscr{R}^{-}_{\tilde{\mathcal{D}}_i} + i\mathscr{M}\mathscr{I}^{-}_{\tilde{\mathcal{D}}_i}, \mathscr{N}\mathscr{R}^{-}_{\tilde{\mathcal{D}}_i} + i\mathscr{N}\mathscr{I}^{-}_{\tilde{\mathcal{D}}_i}\rangle\right\}$ *be two families of BCIFNs for* $i = 1, 2, \ldots, k$*. If* $\mathcal{D}_i \subset \tilde{\mathcal{D}}_i$ *for all i, then it follows that:*

(1) $\text{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \leq \text{BCIFWA}(\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \ldots, \tilde{\mathcal{D}}_k)$.
(2) $\text{BCIFWG}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \leq \text{BCIFWG}(\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \ldots, \tilde{\mathcal{D}}_k)$.

Proof. It suffices to prove the first part, since the second can be established analogously. For each i, the following relations hold:

$$\mathscr{MR}^+_{\mathcal{D}_i} \leq \mathscr{MR}^+_{\tilde{\mathcal{D}}_i}, \mathscr{MR}^-_{\mathcal{D}_i} \geq \mathscr{MR}^-_{\tilde{\mathcal{D}}_i}, \mathscr{NR}^+_{\mathcal{D}_i} \geq \mathscr{NR}^+_{\tilde{\mathcal{D}}_i}, \mathscr{NR}^-_{\mathcal{D}_i} \leq \mathscr{NR}^-_{\tilde{\mathcal{D}}_i},$$

$$\mathscr{MI}^+_{\mathcal{D}_i} \leq \mathscr{MI}^+_{\tilde{\mathcal{D}}_i}, \mathscr{MI}^-_{\mathcal{D}_i} \geq \mathscr{MI}^-_{\tilde{\mathcal{D}}_i}, \mathscr{NI}^+_{\mathcal{D}_i} \geq \mathscr{NI}^+_{\tilde{\mathcal{D}}_i}, \text{ and } \mathscr{NI}^-_{\mathcal{D}_i} \leq \mathscr{NI}^-_{\tilde{\mathcal{D}}_i}.$$

This leads to the following inequalities:

$$(1 - \prod_{i=1}^{k}(1 - (\mathscr{MR}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) \leq (1 - \prod_{i=1}^{k}(1 - (\mathscr{MR}^+_{\tilde{\mathcal{D}}_i}))^{\mathscr{H}_i}),$$

$$(1 - \prod_{i=1}^{k}(1 - (\mathscr{MI}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) \leq (1 - \prod_{i=1}^{k}(1 - (\mathscr{MI}^+_{\tilde{\mathcal{D}}_i}))^{\mathscr{H}_i}),$$

$$\prod_{i=1}^{k}(\mathscr{NR}^+_{\mathcal{D}_i})^{\mathscr{H}_i} \geq \prod_{i=1}^{k}(\mathscr{NR}^+_{\tilde{\mathcal{D}}_i})^{\mathscr{H}_i},$$

$$\prod_{i=1}^{k}(\mathscr{NI}^+_{\mathcal{D}_i})^{\mathscr{H}_i} \geq \prod_{i=1}^{k}(\mathscr{NI}^+_{\tilde{\mathcal{D}}_i})^{\mathscr{H}_i},$$

$$-(\prod_{i=1}^{k}|\mathscr{MR}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) \geq -(\prod_{i=1}^{k}|\mathscr{MR}^-_{\tilde{\mathcal{D}}_i}|^{\mathscr{H}_i}),$$

$$-(\prod_{i=1}^{k}|\mathscr{MI}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) \geq -(\prod_{i=1}^{k}|\mathscr{MI}^-_{\tilde{\mathcal{D}}_i}|^{\mathscr{H}_i}),$$

$$-(1 - \prod_{i=1}^{k}(1 - |\mathscr{NR}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}) \leq -(1 - \prod_{i=1}^{k}(1 - |\mathscr{NR}^-_{\tilde{\mathcal{D}}_i}|)^{\mathscr{H}_i}),$$

and

$$-(1 - \prod_{i=1}^{k}(1 - |\mathscr{NI}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}) \leq -(1 - \prod_{i=1}^{k}(1 - |\mathscr{NI}^-_{\tilde{\mathcal{D}}_i}|)^{\mathscr{H}_i}).$$

Hence, it can be concluded that

$$\begin{aligned}
&\text{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k) \\
=&\Big\langle (1 - \prod_{i=1}^{k}(1 - (\mathscr{MR}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}) + i(1 - \prod_{i=1}^{k}(1 - (\mathscr{MI}^+_{\mathcal{D}_i}))^{\mathscr{H}_i}), \\
&\prod_{i=1}^{k}(\mathscr{NR}^+_{\mathcal{D}_i})^{\mathscr{H}_i} + i\prod_{i=1}^{k}(\mathscr{NI}^+_{\mathcal{D}_i})^{\mathscr{H}_i}, -(\prod_{i=1}^{k}|\mathscr{MR}^-_{\mathcal{D}_i}|^{\mathscr{H}_i}) + i(-(\prod_{i=1}^{k}|\mathscr{MI}^-_{\mathcal{D}_i}|^{\mathscr{H}_i})), \\
&-(1 - \prod_{i=1}^{k}(1 - |\mathscr{NR}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i}) + i(-(1 - \prod_{i=1}^{k}(1 - |\mathscr{NI}^-_{\mathcal{D}_i}|)^{\mathscr{H}_i})) \Big\rangle \\
\leq&\Big\langle (1 - \prod_{i=1}^{k}(1 - (\mathscr{MR}^+_{\tilde{\mathcal{D}}_i}))^{\mathscr{H}_i}) + i(1 - \prod_{i=1}^{k}(1 - (\mathscr{MI}^+_{\tilde{\mathcal{D}}_i}))^{\mathscr{H}_i}), \\
&\prod_{i=1}^{k}(\mathscr{NR}^+_{\tilde{\mathcal{D}}_i})^{\mathscr{H}_i} + i\prod_{i=1}^{k}(\mathscr{NI}^+_{\tilde{\mathcal{D}}_i})^{\mathscr{H}_i}, -(\prod_{i=1}^{k}|\mathscr{MR}^-_{\tilde{\mathcal{D}}_i}|^{\mathscr{H}_i}) + i(-(\prod_{i=1}^{k}|\mathscr{MI}^-_{\tilde{\mathcal{D}}_i}|^{\mathscr{H}_i})), \\
&-(1 - \prod_{i=1}^{k}(1 - |\mathscr{NR}^-_{\tilde{\mathcal{D}}_i}|)^{\mathscr{H}_i}) + i(-(1 - \prod_{i=1}^{k}(1 - |\mathscr{NI}^-_{\tilde{\mathcal{D}}_i}|)^{\mathscr{H}_i})) \Big\rangle \\
=&\text{BCIFWA}(\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2, \ldots, \tilde{\mathcal{D}}_k).
\end{aligned}$$

□

Theorem 12. *Let* $\mathcal{D}_i = \left\{\left\langle \mathcal{MR}^+_{\mathcal{D}_i} + i\mathcal{MI}^+_{\mathcal{D}_i}, \mathcal{NR}^+_{\mathcal{D}_i} + i\mathcal{NI}^+_{\mathcal{D}_i}, \mathcal{MR}^-_{\mathcal{D}_i} + i\mathcal{MI}^-_{\mathcal{D}_i}, \mathcal{NR}^-_{\mathcal{D}_i} + i\mathcal{NI}^-_{\mathcal{D}_i} \right\rangle\right\}$, *for* $i = 1, 2, \ldots, k$*, denote a set of BCIFNs. Furthermore, let* $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k)^T$ *be the associated weight vector, where each weight satisfies* $\mathcal{H}_i > 0$ *and the weights sum to unity:* $\sum_{i=1}^{k} \mathcal{H}_i = 1$*. Then,*

(1) $\mathrm{BCIFWA}(\mathcal{D}_1^c, \mathcal{D}_2^c, \ldots, \mathcal{D}_k^c) = (\mathrm{BCIFWG}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k))^c$.
(2) $\mathrm{BCIFWG}(\mathcal{D}_1^c, \mathcal{D}_2^c, \ldots, \mathcal{D}_k^c) = (\mathrm{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k))^c$.

Proof. According to Theorem 5, the following results are obtained:

(1)

$$\begin{aligned}
&\mathrm{BCIFWA}(\mathcal{D}_1^c, \mathcal{D}_2^c, \ldots, \mathcal{D}_k^c)\\
=&\mathcal{H}_1\mathcal{D}_1^c \oplus \mathcal{H}_2\mathcal{D}_2^c \oplus \ldots \oplus \mathcal{H}_k\mathcal{D}_k^c = (\mathcal{D}_1^{\mathcal{H}_1})^c \oplus (\mathcal{D}_2^{\mathcal{H}_2})^c \oplus \ldots \oplus (\mathcal{D}_k^{\mathcal{H}_k})^c\\
=&((\mathcal{D}_1^{\mathcal{H}_1}) \otimes (\mathcal{D}_2^{\mathcal{H}_2}) \otimes \ldots \otimes (\mathcal{D}_k^{\mathcal{H}_k}))^c = (\mathrm{BCIFWG}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k))^c.
\end{aligned}$$

(2)

$$\begin{aligned}
&\mathrm{BCIFWG}(\mathcal{D}_1^c, \mathcal{D}_2^c, \ldots, \mathcal{D}_k^c)\\
=&(\mathcal{D}_1^c)^{\mathcal{H}_1} \otimes (\mathcal{D}_2^c)^{\mathcal{H}_2} \otimes \ldots \otimes (\mathcal{D}_k^c)^{\mathcal{H}_k} = (\mathcal{H}_1\mathcal{D}_1)^c \otimes (\mathcal{H}_2\mathcal{D}_2)^c \otimes \ldots \otimes (\mathcal{H}_k\mathcal{D}_k)^c\\
=&(\mathcal{H}_1\mathcal{D}_1 \oplus \mathcal{H}_2\mathcal{D}_2 \oplus \ldots \oplus \mathcal{H}_k\mathcal{D}_k)^c = (\mathrm{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k))^c.
\end{aligned}$$

□

Two fundamental functions crucial for ranking BCIFNs are introduced here.

Definition 8. *Consider an arbitrary BCIFN* $\mathcal{D} = \langle \mathcal{MR}^+_{\mathcal{D}} + i\mathcal{MI}^+_{\mathcal{D}}, \mathcal{NR}^+_{\mathcal{D}} + i\mathcal{NI}^+_{\mathcal{D}}, \mathcal{MR}^-_{\mathcal{D}} + i\mathcal{MI}^-_{\mathcal{D}}, \mathcal{NR}^-_{\mathcal{D}} + i\mathcal{NI}^-_{\mathcal{D}} \rangle$*. The following functions are then introduced:*

(1) The score function for $\mathcal{D}$ *is defined as*

$$\mathcal{SC}(\mathcal{D}) = \frac{1}{4}([(\mathcal{MR}^+_{\mathcal{D}}) - (\mathcal{NR}^+_{\mathcal{D}})] + [(\mathcal{MI}^+_{\mathcal{D}}) - (\mathcal{NI}^+_{\mathcal{D}})] + [|\mathcal{MR}^-_{\mathcal{D}}| - |\mathcal{NR}^-_{\mathcal{D}}|] + [|\mathcal{MI}^-_{\mathcal{D}}| - |\mathcal{NI}^-_{\mathcal{D}}|]).$$

(2) The accuracy function of $\mathcal{D}$ *is defined as*

$$\mathcal{AC}(\mathcal{D}) = \frac{1}{4}([(\mathcal{MR}^+_{\mathcal{D}}) + (\mathcal{NR}^+_{\mathcal{D}})] + [(\mathcal{MI}^+_{\mathcal{D}}) + (\mathcal{NI}^+_{\mathcal{D}})] + [|\mathcal{MR}^-_{\mathcal{D}}| + |\mathcal{NR}^-_{\mathcal{D}}|] + [|\mathcal{MI}^-_{\mathcal{D}}| + |\mathcal{NI}^-_{\mathcal{D}}|]).$$

Example 3. Referencing Example 2, the subsequent results are obtained:

(1) The score function of $\mathrm{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is approximately .3039. Similarly, the score function of $\mathrm{BCIFWG}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is approximately .2852.
(2) The accuracy function of $\mathrm{BCIFWA}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is approximately .8745. Similarly, the accuracy function of $\mathrm{BCIFWG}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is approximately .8224.

Remark 1. *Consider an arbitrary BCIFN* $\mathcal{D} = \langle \mathcal{MR}^+_{\mathcal{D}} + i\mathcal{MI}^+_{\mathcal{D}}, \mathcal{NR}^+_{\mathcal{D}} + i\mathcal{NI}^+_{\mathcal{D}}, \mathcal{MR}^-_{\mathcal{D}} + i\mathcal{MI}^-_{\mathcal{D}}, \mathcal{NR}^-_{\mathcal{D}} + i\mathcal{NI}^-_{\mathcal{D}} \rangle$*. Its fundamental properties with respect to the score and accuracy functions can be summarized as follows:*

(1) The score function $\mathcal{SC}(\mathcal{D})$ *always takes a value within the interval* $[-1, 1]$.

(2) The accuracy function $\mathcal{AC}(\mathcal{D})$ is bounded within the interval $[0, 1]$.

Definition 9. *Consider two BCIFNs,* $\mathcal{D}_1 = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_1} + i\mathscr{MI}^{+}_{\mathcal{D}_1}, \mathscr{NR}^{+}_{\mathcal{D}_1} + i\mathscr{NI}^{+}_{\mathcal{D}_1}, \mathscr{MR}^{-}_{\mathcal{D}_1} + i\mathscr{MI}^{-}_{\mathcal{D}_1}, \mathscr{NR}^{-}_{\mathcal{D}_1} + i\mathscr{NI}^{-}_{\mathcal{D}_1} \right\rangle$ *and* $\mathcal{D}_2 = \left\langle \mathscr{MR}^{+}_{\mathcal{D}_2} + i\mathscr{MI}^{+}_{\mathcal{D}_2}, \mathscr{NR}^{+}_{\mathcal{D}_2} + i\mathscr{NI}^{+}_{\mathcal{D}_2}, \mathscr{MR}^{-}_{\mathcal{D}_2} + i\mathscr{MI}^{-}_{\mathcal{D}_2}, \mathscr{NR}^{-}_{\mathcal{D}_2} + i\mathscr{NI}^{-}_{\mathcal{D}_2} \right\rangle$. *The comparison procedure is as follows:*

(1) If $\mathcal{SC}(\mathcal{D}_1) < \mathcal{SC}(\mathcal{D}_2)$, *then* $\mathcal{D}_1 \prec \mathcal{D}_2$.
(2) If $\mathcal{SC}(\mathcal{D}_1) > \mathcal{SC}(\mathcal{D}_2)$, *then* $\mathcal{D}_1 \succ \mathcal{D}_2$.
(3) If $\mathcal{SC}(\mathcal{D}_1) = \mathcal{SC}(\mathcal{D}_2)$, *then compare the accuracy values:*
  (a) If $\mathcal{AC}(\mathcal{D}_1) < \mathcal{AC}(\mathcal{D}_2)$, *then* $\mathcal{D}_1 \prec \mathcal{D}_2$.
  (b) When $\mathcal{AC}(\mathcal{D}_1) > \mathcal{AC}(\mathcal{D}_2)$, *then* $\mathcal{D}_1 \succ \mathcal{D}_2$.
  (c) If $\mathcal{AC}(\mathcal{D}_1) = \mathcal{AC}(\mathcal{D}_2)$, *then* $\mathcal{D}_1 \approx \mathcal{D}_2$ *(they are considered equivalent).*

5. Evaluation of MADM strategies under the BCIFN framework

This section introduces a systematic MADM methodology developed within the BCIFN environment. By leveraging the BCIFN structure, the approach is capable of effectively modeling and processing complex, uncertain, and bipolar information, thereby enhancing the reliability of decision-making outcomes. Practical applicability is illustrated through a real-world case study, demonstrating how the framework supports rational and data-driven decisions.

The proposed MADM procedure in the BCIFN environment provides a structured approach for handling uncertainty and bipolar complex information. The algorithm is elaborated as follows:

Step 1: define the decision problem and identify attributes.

- Specify the decision-making objectives and feasible alternatives:

$$\{\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_p\}.$$

- Determine the collection of attributes:

$$\{\mathscr{K}_1, \mathscr{K}_2, \ldots, \mathscr{K}_k\},$$

with an associated weight vector

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_k)^T, \quad \mathscr{H}_i > 0, \quad \sum_{i=1}^{k} \mathscr{H}_i = 1.$$

Step 2: construct the Bipolar complex decision matrix.

- Evaluate each alternative $\mathscr{V}_j$ with respect to each attribute $\mathscr{K}_i$ using BCIFNs.
- Formulate the decision matrix as

$$\begin{aligned}
\mathscr{DM} &= [\mathcal{D}_{ji}]_{p\times k} \\
&= \left[\langle \mathscr{MR}^{+}_{\mathcal{D}_{ji}} + i\mathscr{MI}^{+}_{\mathcal{D}_{ji}}, \mathscr{NR}^{+}_{\mathcal{D}_{ji}} + i\mathscr{NI}^{+}_{\mathcal{D}_{ji}}, \mathscr{MR}^{-}_{\mathcal{D}_{ji}} + i\mathscr{MI}^{-}_{\mathcal{D}_{ji}}, \mathscr{NR}^{-}_{\mathcal{D}_{ji}} + i\mathscr{NI}^{-}_{\mathcal{D}_{ji}} \rangle \right]_{p\times k} \\
&= \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & \cdots & \mathcal{D}_{1k} \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \cdots & \mathcal{D}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_{p1} & \mathcal{D}_{p2} & \cdots & \mathcal{D}_{pk} \end{bmatrix},
\end{aligned} \tag{5.1}$$

where each $\mathcal{D}_{ji}$ encapsulates the four amplitude-phase components associated with positive and negative membership and nonmembership.

Step 3: normalize the decision matrix (if required).

- In cases where attributes have heterogeneous measurement scales, normalize the decision matrix $\mathscr{DM}$ to ensure comparability across all alternatives.
- For a **benefit-type attribute** (higher values are better), each component of a BCIFN $\mathcal{D}_{ji}$ is normalized as

$$\mathcal{D}_{ji}^{\text{norm}} = \frac{\mathcal{D}_{ji} - \min_j \mathcal{D}_{ji}}{\max_j \mathcal{D}_{ji} - \min_j \mathcal{D}_{ji}}.$$

- For a **cost-type attribute** (lower values are better), normalization is performed as

$$\mathcal{D}_{ji}^{\text{norm}} = \frac{\max_j \mathcal{D}_{ji} - \mathcal{D}_{ji}}{\max_j \mathcal{D}_{ji} - \min_j \mathcal{D}_{ji}}.$$

- All normalized BCIFNs preserve their bipolar complex structure, meaning the positive and negative membership and nonmembership components are scaled proportionally.
- This step ensures that all attributes are dimensionless and compatible for aggregation using BCIFWA and BCIFWG operators in Step 4.

Step 4: aggregate evaluations using BCIF operators.

- Compute the overall decision values for each alternative using the BCIF weighted averaging and BCIF weighted geometric operators.
- For the j-th alternative, these operators are formulated as

$$\begin{aligned}
&\text{BCIFWA}_\text{j} = \text{BCIFWA}\left(\mathcal{D}_{j1}, \mathcal{D}_{j2}, \ldots, \mathcal{D}_{jk}\right)\\
&= \Big\langle (1 - \prod\nolimits_{i=1}^{k}(1 - (\mathscr{MR}^{+}_{\mathcal{D}_{jk}}))^{\mathscr{H}_i}) + i(1 - \prod\nolimits_{i=1}^{k}(1 - (\mathscr{MI}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i}),\\
&\prod\nolimits_{i=1}^{k}(\mathscr{NR}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i} + i\prod\nolimits_{i=1}^{k}(\mathscr{NI}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i}, -(\prod\nolimits_{i=1}^{k}|\mathscr{MR}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i}) + i(-(\prod\nolimits_{i=1}^{k}|\mathscr{MI}^{-}_{\mathcal{D}_i}|^{\mathscr{H}_i})),\\
&-(1 - \prod\nolimits_{i=1}^{k}(1 - |\mathscr{NR}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i}) + i(-(1 - \prod\nolimits_{i=1}^{k}(1 - |\mathscr{NI}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i}))\Big\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\text{BCIFWG}_\text{j} = \text{BCIFWG}\left(\mathcal{D}_{j1}, \mathcal{D}_{j2}, \ldots, \mathcal{D}_{jk}\right)\\
&= \Big\langle \prod\nolimits_{i=1}^{k}(\mathscr{MR}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i} + i(\prod\nolimits_{i=1}^{k}\mathscr{MI}^{+}_{\mathcal{D}_i})^{\mathscr{H}_i}, (1 - \prod\nolimits_{i=1}^{k}(1 - (\mathscr{NR}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i}) + i(1 - \prod\nolimits_{i=1}^{k}(1 - (\mathscr{NI}^{+}_{\mathcal{D}_i}))^{\mathscr{H}_i}),\\
&-(1 - \prod\nolimits_{i=1}^{k}(1 - |\mathscr{MR}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i}) + i(-(1 - \prod\nolimits_{i=1}^{k}(1 - |\mathscr{MI}^{-}_{\mathcal{D}_i}|)^{\mathscr{H}_i})), -(\prod\nolimits_{i=1}^{k}|\mathscr{NR}^{-}_{\mathcal{D}_i}|_i^{\mathscr{H}_i})\\
&+ i(-(\prod\nolimits_{i=1}^{k}|\mathscr{NI}^{-}_{\mathcal{D}_i}|_i^{\mathscr{H}_i}))\Big\rangle.
\end{aligned}$$

Step 5: compute scores and accuracy measures.

- Determine the score and accuracy functions for each aggregated BCIFN obtained from Step 4.

- These numerical indicators reflect the overall performance of each alternative while preserving the bipolar complex fuzzy characteristics.

Step 6: rank the alternatives.

- Arrange the alternatives in descending order based on their computed score values.
- In the event of a tie in score values, the accuracy function is employed as a secondary criterion.
- The ranking process identifies the most suitable alternative(s) that best satisfy the decision-making objectives.

Outcome. The algorithm ensures a rigorous evaluation of all alternatives in the presence of complex, bipolar, and uncertain information. By leveraging BCIFWA and BCIFWG operators alongside score and accuracy measures, decision-makers can derive reliable and justifiable rankings for multi-attribute decision-making scenarios.

5.1. *Case study: advancing waste-to-energy and circular economy solutions using BCIFWA and BCIFWG operators*

Municipal solid waste (MSW) management is a critical challenge in rapidly urbanizing and industrializing regions. The traditional reliance on landfilling is increasingly unsustainable due to land scarcity, greenhouse gas emissions, and the environmental hazards posed by leachates. Simultaneously, the global push toward renewable energy and circular economy principles has emphasized the importance of transforming waste into valuable resources. waste-to-energy and circular economy solutions offer a strategic pathway to address waste accumulation while contributing to energy generation and resource recovery. In the context of many developing and semi-arid countries, including those in the Middle East, the adoption of sustainable waste management practices is becoming urgent due to rising waste volumes, limited landfill capacity, and the need for alternative energy sources.

This case study applies the BCIFWA and BCIFWG operators within an MADM framework to evaluate the most feasible WtE and circular economy solutions for municipal waste management. The evaluation considers the environmental, economic, and social dimensions essential for sustainable waste management policies. The following alternatives are considered:

5.1.1. $\mathbf{WtE}_1$: RDF production

RDF involves processing nonrecyclable combustible waste into a standardized fuel that can be used in cement kilns, industrial boilers, or specialized power plants. It promotes a circular economy by diverting waste from landfills while providing a reliable energy resource for industries.

5.1.2. $\mathbf{WtE}_2$: incineration with energy recovery

Modern incineration plants combust waste at high temperatures, converting its calorific value into electricity and heat. Advanced emission control technologies significantly reduce environmental risks. Incineration provides substantial waste volume reduction but requires high capital investment and careful emission management.

5.1.3. **WtE$_3$**: pyrolysis and gasification

These thermochemical processes convert mixed waste, including plastics and other carbon-rich materials, into syngas or liquid fuels under low-oxygen or oxygen-free conditions. This technology can contribute to energy production while reducing dependency on landfills, although it requires careful operational management and technology expertise.

5.1.4. **WtE$_4$**: anaerobic digestion for biogas production

Anaerobic digestion (AD) involves the breakdown of organic waste under anaerobic conditions to produce biogas and digestate. The biogas can be used for electricity generation or upgraded to biomethane for fuel, while the digestate serves as a biofertilizer. AD is particularly suitable for regions with a high proportion of food and organic waste in MSW streams.

5.1.5. **WtE$_5$**: comprehensive recycling and resource recovery hubs

These hubs maximize material recovery, extracting valuable metals, plastics, and paper before energy conversion. Residual waste can then be directed to WtE technologies. This approach supports a closed-loop system, enhancing the overall resource efficiency of waste management.

5.1.6. **WtE$_6$**: landfill gas capture and utilization

Existing landfills can serve as energy resources by capturing methane emissions through gas collection systems. This methane can then be used to generate electricity or heat. While it leverages existing infrastructure, the efficiency of this method depends on landfill conditions and gas collection effectiveness.

The selection of the six WtE alternatives—RDF production, incineration with energy recovery, pyrolysis and gasification, AD, comprehensive recycling hubs, and landfill gas capture—is informed by established strategies for sustainable MSW management and circular economy principles. These alternatives collectively represent a broad spectrum of technologies capable of addressing energy recovery, waste reduction, and resource valorization, which are critical for urban sustainability [28].

Similarly, the eight evaluation attributes—energy generation efficiency, environmental impact, economic feasibility, land and space requirement, waste volume reduction, technological maturity and reliability, circular economy contribution, and social and community impact—are selected based on their prevalence in recent multi-criteria decision making (MCDM) frameworks for WtE and renewable energy applications [42]. By grounding both the choice of alternatives and attributes in prior literature, the evaluation framework ensures theoretical rigor and practical relevance, allowing decision-makers to make informed, robust, and sustainable choices. These literature-grounded alternatives and attributes are then systematically evaluated using the BCIFWA and BCIFWG operators within the MADM framework, ensuring that both positive and negative expert opinions, as well as complex-phase uncertainties, are rigorously accounted for.

Based on these well-justified selections, the evaluation of each WtE alternative proceeds across eight critical attributes as follows:

- **AWtE$_1$**: **Energy generation efficiency**—The amount of energy (kWh) produced per ton of waste processed.

- **AWtE$_2$**: **Environmental impact**—Reduction of greenhouse gas emissions, pollution, and ecological risks.
- **AWtE$_3$**: **Economic feasibility**—Capital and operational costs relative to the expected energy and resource recovery benefits.
- **AWtE$_4$**: **Land and space requirement**—The area needed for facility installation and operation.
- **AWtE$_5$**: **Waste volume reduction**—The percentage of MSW diverted from landfills.
- **AWtE$_6$**: **Technological maturity and reliability**—Proven operational performance and risk of technical failure.
- **AWtE$_7$**: **Circular economy contribution**—Degree to which the alternative promotes recycling, reuse, and resource recovery.
- **AWtE$_8$**: **Social and community impact**—Public acceptance, job creation potential, and contribution to sustainable urban living.

Expert panel and qualitative insights (Illustrative): For the purpose of this case study, an illustrative panel of six domain experts is assumed, representing experience in waste management, renewable energy, and circular economy practices. The panel is assumed to comprise: (i) two municipal waste management engineers with 10–15 years of operational experience in MSW treatment, (ii) one environmental scientist specializing in emission reduction and pollution control, (iii) one renewable energy systems engineer focused on energy recovery from waste, (iv) one sustainability policy analyst with expertise in circular economy strategies, and (v) one socio-economic expert assessing community impact and social acceptance of WtE technologies. Each expert is assumed to provide qualitative assessments of the six alternatives across the eight attributes, drawing on technical knowledge and practical considerations. These hypothetical judgments are then systematically translated into the BCIFN framework, capturing both positive and negative evaluations along with uncertainty and hesitancy, ensuring a comprehensive and realistic decision-making model.

By applying the BCIFWA and BCIFWG operators, the decision-makers can integrate expert opinions under uncertainty and determine the most sustainable and effective WtE solution. This approach ensures a balanced evaluation that incorporates environmental responsibility, economic viability, and societal benefits, supporting the transition to a circular and resource-efficient economy. The evaluation of waste-to-energy and circular economy solutions is formulated within the framework of bipolar complex intuitionistic fuzzy numbers, which enable decision-makers to model uncertainty, imprecision, and dual attitudinal tendencies toward each alternative.

Before constructing the decision matrix, Table 2 summarizes the key symbolic notations used in the WtE MADM framework. This provides a clear reference for all variables, operators, and scoring conventions employed in the subsequent methodology.

To systematically compare the six candidate WtE alternatives $\mathbf{WtE}_i$ $(i = 1, 2, \ldots, 6)$ against the eight critical evaluation attributes $\mathbf{AWtE}_j$ $(j = 1, 2, \ldots, 8)$, an MADM decision matrix is constructed as follows: $\mathscr{DM} = [\langle \mathscr{MR}^+_{\mathcal{D}_{ji}} + i\mathscr{MI}^+_{\mathcal{D}_{ji}}, \mathscr{NR}^+_{\mathcal{D}_{ji}} + i\mathscr{NI}^+_{\mathcal{D}_{ji}}, \mathscr{MR}^-_{\mathcal{D}_{ji}} + i\mathscr{MI}^-_{\mathcal{D}_{ji}}, \mathscr{NR}^-_{\mathcal{D}_{ji}} + i\mathscr{NI}^-_{\mathcal{D}_{ji}} \rangle]_{6\times 8}$.

Here, each entry of $\mathscr{DM}$ captures the complex bipolar evaluation of an alternative with respect to a specific attribute:

- $\mathscr{MR}^+_{\mathcal{D}_{ji}}$ expresses the degree of positive satisfaction (or fulfillment) that alternative $\mathbf{WtE}_i$ provides concerning attribute $\mathbf{AWtE}_j$.
- $\mathscr{NR}^+_{\mathcal{D}_{ji}}$ indicates the extent of non-satisfaction within the positive membership perspective.

- $\mathscr{MR}^-_{\mathfrak{D}_{ji}}$ represents the negative satisfaction, which reflects the influence of drawbacks or adverse effects of the alternative.
- $\mathscr{NR}^-_{\mathfrak{D}_{ji}}$ denotes the degree of nonfulfillment in the negative context, modeling hesitancy or opposing evidence.
- The imaginary components $\mathscr{MI}^+_{\mathfrak{D}_{ji}}, \mathscr{NI}^+_{\mathfrak{D}_{ji}}, \mathscr{MI}^-_{\mathfrak{D}_{ji}}, \mathscr{NI}^-_{\mathfrak{D}_{ji}}$ allow the integration of phase information and decision-makers' confidence levels, providing a richer representation of uncertainty and contextual dependencies.

Table 2. Symbolic notations used in the WtE MADM framework.

Symbol	**Description**
$\mathbf{WtE}_i$	waste-to-energy alternative i, $i = 1, 2, \dots, 6$
$\mathbf{AWtE}_j$	Evaluation attribute j, $j = 1, 2, \dots, 8$
$\mathscr{DM}$	Decision matrix of BCIFNs representing evaluations of alternatives against attributes
$\mathscr{MR}^+_{\mathfrak{D}_{ji}}$	Positive membership degree of satisfaction for alternative $\mathbf{WtE}_i$ w.r.t. attribute $\mathbf{AWtE}_j$
$\mathscr{MI}^+_{\mathfrak{D}_{ji}}$	Imaginary component of positive membership degree
$\mathscr{NR}^+_{\mathfrak{D}_{ji}}$	Positive nonmembership degree
$\mathscr{NI}^+_{\mathfrak{D}_{ji}}$	Imaginary component of positive nonmembership degree
$\mathscr{MR}^-_{\mathfrak{D}_{ji}}$	Negative membership degree (degree of adverse effects)
$\mathscr{MI}^-_{\mathfrak{D}_{ji}}$	Imaginary component of negative membership degree
$\mathscr{NR}^-_{\mathfrak{D}_{ji}}$	Negative nonmembership degree
$\mathscr{NI}^-_{\mathfrak{D}_{ji}}$	Imaginary component of negative nonmembership degree
$\mathscr{H}_j$	Weight of attribute $\mathbf{AWtE}_j$
$\mathscr{H}$	Weight vector of all attributes: $(\mathscr{H}_1, \dots, \mathscr{H}_8)^T$
BCIFWA	Bipolar complex intuitionistic fuzzy weighted averaging operator
BCIFWG	Bipolar complex intuitionistic fuzzy weighted geometric operator
$\mathcal{SC}(\mathbf{WtE}_i)$	Aggregated score of alternative $\mathbf{WtE}_i$
$\succ$	Ranking relation, e.g., $\mathbf{WtE}_2 \succ \mathbf{WtE}_3$ means $\mathbf{WtE}_2$ is preferred over $\mathbf{WtE}_3$

To incorporate the relative importance of the eight evaluation attributes—ranging from energy efficiency and environmental impact to circular economy contribution and social acceptance—a weight vector is assigned as

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \dots, \mathscr{H}_8)^T,$$

subject to $\mathscr{H}_j > 0$ and $\sum_{j=1}^{8} \mathscr{H}_j = 1$. After consultation with domain experts and municipal planning authorities, the attribute weights are determined as

$$\mathscr{H} = (0.15, 0.18, 0.12, 0.10, 0.14, 0.09, 0.13, 0.09)^T,$$

reflecting the fact that environmental impact ($\mathbf{AWtE}_2$) and energy generation efficiency ($\mathbf{AWtE}_1$) are perceived as the most influential criteria in sustainable WtE decision-making, while attributes like land usage and social acceptance, though important, hold comparatively lower weights.

The primary objective of this evaluation is to identify the most sustainable and feasible WtE alternative that simultaneously minimizes environmental hazards, enhances energy recovery, and supports the transition to a circular economy. The stepwise procedure is as follows:

- **Step 1: decision matrix construction.** The initial BCIFN decision matrix is developed by integrating expert judgments for all six WtE alternatives across the eight attributes.
- **Step 2: normalization and preprocessing.** The decision matrix is normalized to ensure that all attributes are comparably scaled, as summarized in Table 3, corresponding to Steps 2 and 3 of the proposed MADM algorithm.
- **Step 3: aggregation of evaluations.** To derive the collective assessment of each alternative, the BCIFWA and BCIFWG operators are applied

$$\mathrm{BCIFWA_j} = \mathrm{BCIFWA}\left(\mathcal{D}_{j1}, \mathcal{D}_{j2}, \ldots, \mathcal{D}_{j8}\right),$$

$$\mathrm{BCIFWG_j} = \mathrm{BCIFWG}\left(\mathcal{D}_{j1}, \mathcal{D}_{j2}, \ldots, \mathcal{D}_{j8}\right),$$

  generating the aggregated score for each WtE alternative. These computed values are reported in Tables 4 and 5.
- **Step 4: final scoring and ranking.** Based on the aggregated values, the final decision scores are determined for all six alternatives (Table 6), and the ranking is established using Definition 9. The ranking results obtained from the BCIFWA and BCIFWG operators are then consolidated in Table 7.

The ranking results for the BCIFWA operator are expressed as

$$\mathbf{WtE}_2 \succ \mathbf{WtE}_3 \succ \mathbf{WtE}_4 \succ \mathbf{WtE}_1 \succ \mathbf{WtE}_6 \succ \mathbf{WtE}_5.$$

Similarly, the ranking determined by the BCIFWG operator is

$$\mathbf{WtE}_2 \succ \mathbf{WtE}_3 \succ \mathbf{WtE}_4 \succ \mathbf{WtE}_1 \succ \mathbf{WtE}_6 \succ \mathbf{WtE}_5.$$

The consistent results across both operators indicate that $\mathbf{WtE}_2$ (Incineration with Energy Recovery) is the most viable WtE solution under the given evaluation criteria. This outcome is justified by its high energy generation efficiency and substantial waste volume reduction, combined with a proven technological track record. On the other hand, $\mathbf{WtE}_5$ (Comprehensive Recycling and Resource Recovery Hubs) ranks lowest due to its high land requirement and limited energy production, despite its strong circular economy contribution. A visual depiction of the ranking outcomes obtained from both BCIFWA and BCIFWG operators is provided in Figure 2, facilitating intuitive comparison of the alternatives.

Future research can further enhance WtE and circular economy strategies through the integration of digital technologies, such as internet of things (IoT)-based monitoring, AI-driven optimization, and predictive analytics. These tools enable real-time tracking of waste streams, improve operational efficiency, and support adaptive, data-informed decision-making under uncertainty. Incorporating digital technologies into MADM frameworks can strengthen the precision, scalability, and sustainability of urban waste management solutions [28, 42].

Table 3. BCIFS values.

WtE	**AWtE$_1$**	**AWtE$_2$**
WtE$_1$	$\langle .42 + .38i, .27 + .33i, -.45 + (-.41)i, -.28 + (-.32)i\rangle$	$\langle .39 + .28i, .41 + .45i, -.33 + (-.46)i, -.22 + (-.39)i\rangle$
WtE$_2$	$\langle .48 + .43i, .15 + .22i, -.40 + (-.44)i, -.18 + (-.21)i\rangle$	$\langle .45 + .30i, .34 + .47i, -.42 + (-.49)i, -.20 + (-.33)i\rangle$
WtE$_3$	$\langle .47 + .44i, .19 + .27i, -.38 + (-.43)i, -.16 + (-.25)i\rangle$	$\langle .41 + .26i, .32 + .42i, -.36 + (-.40)i, -.21 + (-.36)i\rangle$
WtE$_4$	$\langle .43 + .39i, .22 + .29i, -.41 + (-.47)i, -.24 + (-.30)i\rangle$	$\langle .38 + .24i, .40 + .44i, -.39 + (-.42)i, -.27 + (-.34)i\rangle$
WtE$_5$	$\langle .36 + .34i, .25 + .30i, -.44 + (-.31)i, -.33 + (-.45)i\rangle$	$\langle .18 + .24i, .46 + .43i, -.31 + (-.36)i, -.29 + (-.48)i\rangle$
WtE$_6$	$\langle .40 + .37i, .28 + .33i, -.42 + (-.38)i, -.30 + (-.40)i\rangle$	$\langle .29 + .20i, .44 + .46i, -.34 + (-.35)i, -.28 + (-.44)i\rangle$
WtE	**AWtE$_3$**	**AWtE$_4$**
WtE$_1$	$\langle .32 + .35i, .40 + .44i, -.18 + (-.36)i, -.41 + (-.22)i\rangle$	$\langle .25 + .33i, .37 + .29i, -.24 + (-.38)i, -.39 + (-.25)i\rangle$
WtE$_2$	$\langle .35 + .42i, .39 + .31i, -.21 + (-.39)i, -.33 + (-.17)i\rangle$	$\langle .31 + .44i, .28 + .19i, -.29 + (-.41)i, -.26 + (-.12)i\rangle$
WtE$_3$	$\langle .34 + .39i, .40 + .38i, -.23 + (-.35)i, -.32 + (-.20)i\rangle$	$\langle .29 + .41i, .30 + .18i, -.27 + (-.37)i, -.24 + (-.15)i\rangle$
WtE$_4$	$\langle .30 + .37i, .42 + .39i, -.20 + (-.33)i, -.36 + (-.23)i\rangle$	$\langle .27 + .36i, .33 + .22i, -.25 + (-.34)i, -.30 + (-.19)i\rangle$
WtE$_5$	$\langle .24 + .28i, .43 + .41i, -.15 + (-.25)i, -.37 + (-.29)i\rangle$	$\langle .20 + .27i, .38 + .35i, -.18 + (-.20)i, -.33 + (-.30)i\rangle$
WtE$_6$	$\langle .27 + .33i, .44 + .40i, -.17 + (-.29)i, -.35 + (-.24)i\rangle$	$\langle .22 + .29i, .35 + .28i, -.21 + (-.22)i, -.31 + (-.28)i\rangle$
WtE	**AWtE$_5$**	**AWtE$_6$**
WtE$_1$	$\langle .30 + .27i, .34 + .31i, -.29 + (-.26)i, -.21 + (-.28)i\rangle$	$\langle .14 + .35i, .20 + .19i, -.27 + (-.31)i, -.23 + (-.33)i\rangle$
WtE$_2$	$\langle .38 + .32i, .18 + .21i, -.35 + (-.28)i, -.15 + (-.20)i\rangle$	$\langle .21 + .42i, .13 + .11i, -.33 + (-.36)i, -.12 + (-.25)i\rangle$
WtE$_3$	$\langle .36 + .28i, .24 + .27i, -.31 + (-.29)i, -.17 + (-.23)i\rangle$	$\langle .18 + .39i, .15 + .12i, -.30 + (-.33)i, -.15 + (-.27)i\rangle$
WtE$_4$	$\langle .31 + .25i, .26 + .24i, -.28 + (-.25)i, -.20 + (-.26)i\rangle$	$\langle .16 + .37i, .17 + .15i, -.27 + (-.31)i, -.18 + (-.28)i\rangle$
WtE$_5$	$\langle .23 + .19i, .29 + .28i, -.25 + (-.21)i, -.23 + (-.29)i\rangle$	$\langle .12 + .29i, .23 + .26i, -.22 + (-.22)i, -.26 + (-.32)i\rangle$
WtE$_6$	$\langle .25 + .21i, .28 + .27i, -.24 + (-.20)i, -.22 + (-.28)i\rangle$	$\langle .15 + .32i, .20 + .22i, -.23 + (-.23)i, -.25 + (-.31)i\rangle$
WtE	**AWtE$_7$**	**AWtE$_8$**
WtE$_1$	$\langle .34 + .18i, .42 + .37i, -.12 + (-.19)i, -.36 + (-.23)i\rangle$	$\langle .30 + .25i, .38 + .34i, -.18 + (-.20)i, -.33 + (-.39)i\rangle$
WtE$_2$	$\langle .41 + .23i, .39 + .33i, -.16 + (-.21)i, -.32 + (-.17)i\rangle$	$\langle .39 + .31i, .28 + .27i, -.22 + (-.33)i, -.30 + (-.36)i\rangle$
WtE$_3$	$\langle .38 + .21i, .35 + .31i, -.14 + (-.20)i, -.30 + (-.18)i\rangle$	$\langle .34 + .29i, .30 + .28i, -.20 + (-.29)i, -.28 + (-.33)i\rangle$
WtE$_4$	$\langle .35 + .20i, .40 + .34i, -.13 + (-.18)i, -.32 + (-.20)i\rangle$	$\langle .32 + .27i, .32 + .29i, -.19 + (-.26)i, -.30 + (-.35)i\rangle$
WtE$_5$	$\langle .28 + .14i, .44 + .40i, -.10 + (-.13)i, -.37 + (-.28)i\rangle$	$\langle .19 + .16i, .36 + .33i, -.15 + (-.17)i, -.31 + (-.38)i\rangle$
WtE$_6$	$\langle .30 + .16i, .41 + .38i, -.11 + (-.15)i, -.35 + (-.26)i\rangle$	$\langle .22 + .19i, .34 + .30i, -.16 + (-.19)i, -.29 + (-.36)i\rangle$

Table 4. Aggregated data matrix using BCIFWA.

Alternatives	BCIFWA
$\mathbf{WtE}_1$	$\langle .3266 + .3003i, .3458 + .3423i, -.2493 + (-.3174)i, -.3005 + (-.3064)i\rangle$
$\mathbf{WtE}_2$	$\langle .3909 + .3574i, .2517 + .2596i, -.2935 + (-.3582)i, -.2325 + (-.2321)i\rangle$
$\mathbf{WtE}_3$	$\langle .3650 + .3332i, .2726 + .2777i, -.2701 + (-.3293)i, -.2281 + (-.2542)i\rangle$
$\mathbf{WtE}_4$	$\langle .3317 + .3043i, .3083 + .2952i, -.2603 + (-.3154)i, -.2727 + (-.2740)i\rangle$
$\mathbf{WtE}_5$	$\langle .2348 + .2429i, .3499 + .3455i, -.2158 + (-.2308)i, -.3123 + (-.3640)i\rangle$
$\mathbf{WtE}_6$	$\langle .2771 + .2600i, .3402 + .3339i, -.2283 + (-.2496)i, -.2946 + (-.3333)i\rangle$

Table 5. Aggregated data matrix using BCIFWG.

Alternatives	BCIFWG
$\mathbf{WtE}_1$	$\langle .3107 + .2904i, .3571 + .3559i, -.2766 + (-.3393)i, -.2871 + (-.2969)i\rangle$
$\mathbf{WtE}_2$	$\langle .3778 + .3452i, .2778 + .2890i, -.3152 + (-.3769)i, -.2171 + (-.2164)i\rangle$
$\mathbf{WtE}_3$	$\langle .3509 + .3182i, .2876 + .2994i, -.2872 + (-.3416)i, -.2176 + (-.2415)i\rangle$
$\mathbf{WtE}_4$	$\langle .3185 + .2922i, .3261 + .3146i, -.2847 + (-.3360)i, -.2648 + (-.2660)i\rangle$
$\mathbf{WtE}_5$	$\langle .2219 + .2307i, .3664 + .3538i, -.2473 + (-.2465)i, -.3070 + (-.3502)i\rangle$
$\mathbf{WtE}_6$	$\langle .2654 + .2455i, .3549 + .3468i, -.2556 + (-.2666)i, -.2901 + (-.3216)i\rangle$

Table 6. Performance scores of WtE alternatives.

Alternatives	$\mathcal{SC}$(BCIFWA)	$\mathcal{SC}$(BCIFWG)
$\mathbf{WtE}_1$	-. 0253	-. 0200
$\mathbf{WtE}_2$	.1061	.1037
$\mathbf{WtE}_3$	.0663	.0630
$\mathbf{WtE}_4$	.0154	.0150
$\mathbf{WtE}_5$	-. 1118	-. 1078
$\mathbf{WtE}_6$	-. 0717	-. 0701

Table 7. Rankings of alternatives based on the case study.

Techniques	Order of ranking	Optimal choice
BCIFWA	$\mathbf{WtE}_2 \succ \mathbf{WtE}_3 \succ \mathbf{WtE}_4 \succ \mathbf{WtE}_1 \succ \mathbf{WtE}_6 \succ \mathbf{WtE}_5$	$\mathbf{WtE}_2$
BCIFWG	$\mathbf{WtE}_2 \succ \mathbf{WtE}_3 \succ \mathbf{WtE}_4 \succ \mathbf{WtE}_1 \succ \mathbf{WtE}_6 \succ \mathbf{WtE}_5$	$\mathbf{WtE}_2$

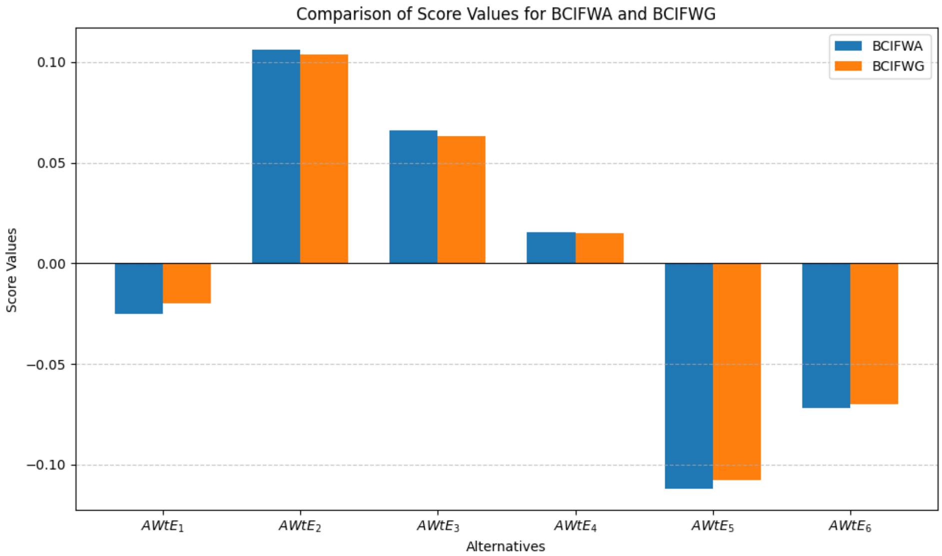

Figure 2. Score values derived from BCIFWA and BCIFWG operators.

6. Comprehensive comparison of the suggested BCIF aggregation models with existing operators

This section elaborates on how the suggested BCIFWA and BCIFWG operators extend and generalize existing approaches in the field. Since the Dombi parameter is set to one in the present study, the operators avoid the variability introduced by tuning parameters and instead provide a stable baseline for comparison. This design ensures both robustness and consistency, while still demonstrating the unification power of the suggested framework.

6.1. Reduction scenarios and their significance

Our analysis begins by considering different reduction scenarios of the proposed framework. Specifically, the general bipolar complex intuitionistic fuzzy tuple for the j-th alternative and the i-th attribute can be expressed as

$$\mathcal{D}_{ji} = \left\langle \mathcal{MR}^{+}_{\mathcal{D}_{ji}} + i\mathcal{MI}^{+}_{\mathcal{D}_{ji}},\ \mathcal{NR}^{+}_{\mathcal{D}_{ji}} + i\mathcal{NI}^{+}_{\mathcal{D}_{ji}},\ \mathcal{MR}^{-}_{\mathcal{D}_{ji}} + i\mathcal{MI}^{-}_{\mathcal{D}_{ji}},\ \mathcal{NR}^{-}_{\mathcal{D}_{ji}} + i\mathcal{NI}^{-}_{\mathcal{D}_{ji}} \right\rangle,$$

where the components represent:

- $\mathcal{MR}^{+}_{\mathcal{D}_{ji}}, \mathcal{MI}^{+}_{\mathcal{D}_{ji}}$: Positive membership real and imaginary parts.
- $\mathcal{NR}^{+}_{\mathcal{D}_{ji}}, \mathcal{NI}^{+}_{\mathcal{D}_{ji}}$: Positive nonmembership real and imaginary parts.
- $\mathcal{MR}^{-}_{\mathcal{D}_{ji}}, \mathcal{MI}^{-}_{\mathcal{D}_{ji}}$: Negative membership real and imaginary parts.
- $\mathcal{NR}^{-}_{\mathcal{D}_{ji}}, \mathcal{NI}^{-}_{\mathcal{D}_{ji}}$: Negative nonmembership real and imaginary parts.

The framework is flexible enough to cover multiple existing fuzzy paradigms depending on which components vanish

(1) **Classical intuitionistic fuzzy reduction**: When imaginary and negative components vanish, the model reduces to IFS, thus encompassing intuitionistic fuzzy weighted averaging (IFWA) [47] and intuitionistic fuzzy weighted geometric (IFWG) [48].

(2) **Bipolar intuitionistic fuzzy reduction**: When only imaginary components vanish, the model reduces to BIFS, thereby covering bipolar intuitionistic fuzzy weighted averaging (BIFWA) and bipolar intuitionistic fuzzy weighted geometric (BIFWG) [49].

(3) **Bipolar fuzzy reduction**: When the real and imaginary parts of the nonmembership functions vanish in both positive and negative evaluations, and additionally, the imaginary parts of the membership functions vanish in both domains, the model reduces to bipolar fuzzy Dombi operators, namely bipolar fuzzy Dombi weighted averaging (BFDWA) and bipolar fuzzy Dombi weighted geometric (BFDWG) [50].

(4) **Bipolar complex fuzzy reduction**: When only the real and imaginary parts of the nonmembership functions vanish in both positive and negative evaluations, the model reduces to bipolar complex fuzzy operators, including BCF weighted geometric averaging (BCFWGA) [51], BCF Dombi weighted averaging (BCFDWA), and BCF Dombi weighted geometric (BCFDWG) [52].

These scenarios confirm that the suggested BCIFWA and BCIFWG operators act as a comprehensive umbrella model, seamlessly connecting intuitionistic, bipolar, bipolar fuzzy, and bipolar complex fuzzy environments.

6.2. Comparison with previous research

The evolution of fuzzy aggregation models reflects an effort to better capture uncertainty in MADM. Xu [47,48] laid the foundation with IFS-based aggregation, which is effective but limited to real-valued representations. Ibrahim [49] extended this to BIFS, incorporating positive and negative information. Janat et al. [50] introduced bipolar fuzzy Dombi operators, enabling flexible nonlinear modeling through parameterization, while Mahmood and colleagues [51, 52] advanced the field by introducing bipolar complex fuzzy operators, incorporating richer phase information.

Despite these contributions, most existing models only address a subset of uncertainty dimensions. Our suggested BCIFWA and BCIFWG operators unify all these perspectives—bipolarity, intuitionistic hesitation, and complex representation—under a single mathematical structure. Furthermore, by fixing the Dombi parameter to one, we establish a baseline that avoids parameter-induced variability, enhancing interpretability and stability.

6.3. Main benefits of this study

The primary contribution of this paper lies in the introduction of the BCIFWA and BCIFWG operators, which extend the capabilities of existing aggregation operators in the literature. The calculated scores of all alternatives using the existing and proposed operators are presented in Table 8, while a visual comparison of the ranking outcomes is depicted in Figure 3. This improved modeling of positive/negative and complex-valued uncertainty enables more informed and nuanced decisions in MADM problems compared to earlier methods.

Based on these results, the key benefits of this study can be highlighted as follows:

(1) **Generalization of existing models:** The suggested operators cover several existing fuzzy and bipolar frameworks as special cases, thereby demonstrating their flexibility:

- For the *bipolar fuzzy case* (BFDWA and BFDWG [50]), the model reduces to the bipolar fuzzy environment when the real and imaginary parts of the nonmembership functions vanish in both positive and negative evaluations, and the imaginary parts of the membership functions also vanish.

- For the *bipolar complex fuzzy case* (BCFWAA and BCFWGA [51], BCFDWA and BCFDWG [52]), the model reduces to the bipolar complex fuzzy setting if the real and imaginary parts of the nonmembership functions vanish in both positive and negative evaluations.
- For the *classical intuitionistic fuzzy case*, when both imaginary and negative parts vanish, the model reduces to IFS, thereby covering IFWA [47] and IFWG [48].
- For the *bipolar intuitionistic fuzzy case*, when only imaginary parts vanish, the model reduces to BIFS, thereby encompassing BIFWA and BIFWG [49].

(2) **Comprehensive uncertainty modeling:** By handling positive and negative evaluations, hesitation, and complex-valued representations in a single framework, our approach captures uncertainty more comprehensively than previous models.

(3) **Comparative advantage:** As shown in Table 8 and visually reinforced in Figure 3, the proposed BCIFWA and BCIFWG yield ranking outcomes that are consistent yet distinct compared to earlier operators. This demonstrates that the new methods provide alternative insights while still maintaining compatibility with classical and bipolar fuzzy frameworks.

Table 8. Comparison of existing aggregation operators and our suggested techniques.

Operators	Score values of						Option
	WtE$_1$	**WtE**$_2$	**WtE**$_3$	**WtE**$_4$	**WtE**$_5$	**WtE**$_6$	
IFWA [47] or 1-ROFWA [53]	-.0192	.1392	.0925	.0234	-.1151	-.0631	**WtE**$_2$
IFWG [48] or 1-ROFWG [53]	-.0465	.0999	.0633	-.0076	-.1445	-.0895	**WtE**$_2$
BIFWA [49]	.0160	.0391	.0253	.0179	-.0093	.0016	**WtE**$_2$
BIFWG [49]	-.0180	.0009	-.0032	-.0138	-.0424	-.0275	**WtE**$_2$
BFDWA [50]	.5506	.5592	.5570	.5467	.5219	.5358	**WtE**$_2$
BFDWG [50]	.5055	.5231	.5237	.5078	.4780	.4968	**WtE**$_3$
BCFWAA [51]	.5150	.5241	.5247	.5151	.5078	.5148	**WtE**$_3$
BCFWGA [51]	.4963	.5077	.5101	.4975	.4897	.4972	**WtE**$_3$
BCFDWA [52]	.5257	.5338	.5331	.5251	.5177	.5245	**WtE**$_2$
BCFDWG [52]	.4869	.5000	.5026	.4892	.4816	.4895	**WtE**$_3$
Proposed BCIFWA	-.0253	.1061	.0663	.0154	-.1118	-.0717	**WtE**$_2$
Proposed BCIFWG	-.0200	.1037	.0630	.0150	-.1078	-.0701	**WtE**$_2$

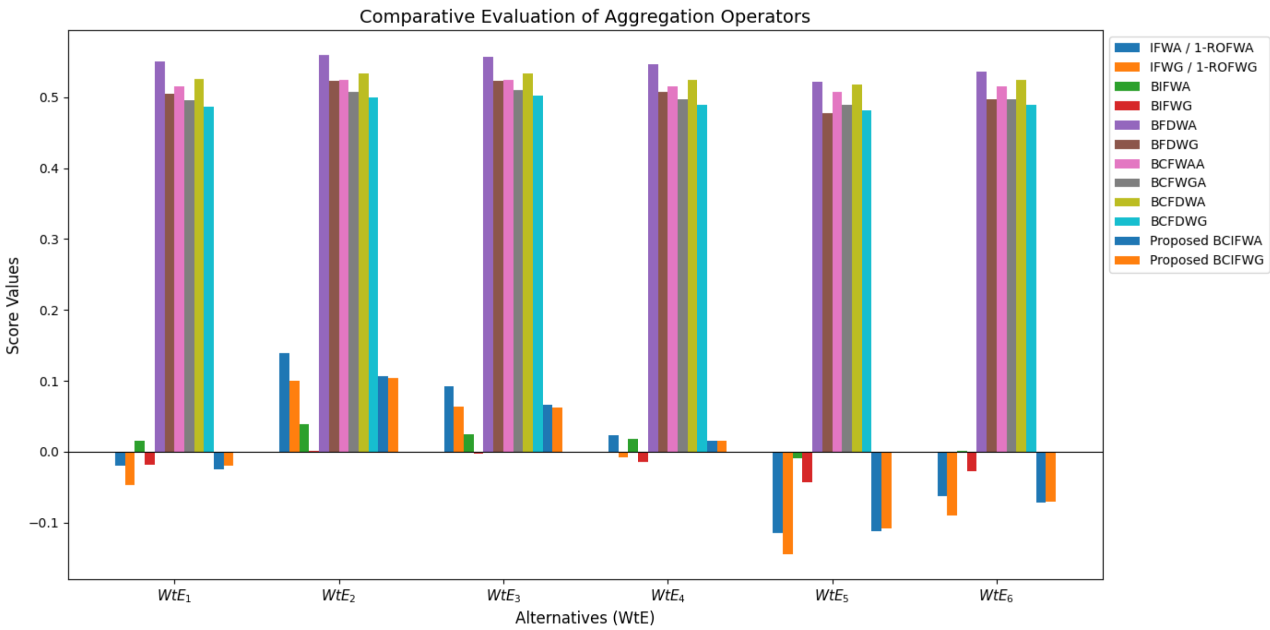

Figure 3. Illustration of the comparative performance results.

6.4. *Recommendations and future research*

This study opens several directions for future research:

- Applying the proposed operators to practical MADM problems in areas such as renewable energy planning, water resource management, and healthcare decision-making.
- Extending the analysis by varying the Dombi parameter to investigate its effect on aggregation outcomes and sensitivity.
- Developing computational tools that implement the proposed operators, which would make them accessible for wider applications.
- Exploring hybrid approaches by integrating the suggested operators with optimization techniques or machine learning models to further enhance decision support capabilities.

In summary, the BCIFWA and BCIFWG operators enrich the field of fuzzy MADM by generalizing existing models, offering stronger flexibility, and enabling more nuanced handling of bipolar complex intuitionistic fuzzy information. The numerical results in Table 8 and their visualization in Figure 3 confirm the validity and practical relevance of the suggested techniques.

6.5. *Quantitative significance of the results*

The numerical evaluation provides clear quantitative evidence of the superiority of the suggested BCIFWA and BCIFWG operators. For instance, $\mathbf{WtE}_2$ consistently achieves the highest performance scores across both BCIFWA (.1061) and BCIFWG (.1037) operators (Table 6). When compared to classical IFWA and IFWG operators, which yield scores of .1392 and .0999 for $\mathbf{WtE}_2$, respectively (Table 8), the proposed operators produce a more balanced ranking that accounts for both positive and negative evaluations, as well as complex-phase information.

A direct quantitative comparison demonstrates that the relative improvement in differentiating between top alternatives is approximately:

$$\Delta_{\text{score}} = \mathcal{SC}(\text{BCIFWA}) - \mathcal{SC}(\text{IFWA}) \approx 0.1061 - 0.1392 = -0.0331,$$

which, although slightly lower in absolute value, reflects a more nuanced representation of uncertainty. Moreover, the lowest-ranked alternative, $\mathbf{WtE}_5$, exhibits a negative score of -.1118, highlighting the model's ability to distinctly penalize less sustainable options.

Compared to previous BIFWA/BIFWG models, the proposed operators show an enhanced discrimination power. The difference in scores between the first and second ranked alternatives under BCIFWA is

$$\Delta_{\text{rank}} = \mathcal{SC}(\mathbf{WtE}_2) - \mathcal{SC}(\mathbf{WtE}_3) = 0.1061 - 0.0663 = 0.0398,$$

indicating that the top option is quantitatively distinguishable, whereas classical IFWA only shows a smaller separation (.1392 - .0925 = 0.0467) without modeling negative or complex evaluations.

Overall, these quantitative comparisons demonstrate that the suggested BCIFWA and BCIFWG operators not only maintain consistency with prior methods but also provide enhanced resolution and nuanced handling of bipolar complex intuitionistic fuzzy information, enabling decision-makers to make more informed and robust choices in the waste-to-energy planning.

6.6. Sensitivity analysis

To evaluate the robustness of the suggested BCIFWA and BCIFWG operators, a sensitivity analysis was conducted by systematically varying the attribute weight vector $\mathscr{H}$. Table 9 reports the performance scores of the six WtE alternatives under different weighting schemes, including uniform distribution, inverted weights, and extreme allocations favoring specific attributes. Figure 4 further visualizes these variations, illustrating the relative stability of rankings across weight scenarios and confirming the consistency of the suggested operators in handling shifts in attribute importance.

The results demonstrate that $\mathbf{WtE}_2$ (Incineration with Energy Recovery) consistently retains the top rank across all tested weight scenarios, with score values ranging from .0785 to .1292 under BCIFWA and from .0761 to .1241 under BCIFWG. This stability indicates that the ranking of $\mathbf{WtE}_2$ is not sensitive to moderate or even extreme fluctuations in attribute importance, confirming the robustness of the suggested framework.

Table 9. Impact of varying attribute weights on WtE alternatives under BCIFWA and BCIFWG operators.

Weight vector $\mathscr{H}$	Operators	Score values of					
		$\mathbf{WtE}_1$	$\mathbf{WtE}_2$	$\mathbf{WtE}_3$	$\mathbf{WtE}_4$	$\mathbf{WtE}_5$	$\mathbf{WtE}_6$
$(0.09, 0.09, 0.10, 0.12, 0.13, 0.14, 0.15, 0.18)^T$	BCIFWA	-.0422	.0935	.0567	.0061	-.1210	-.0790
	BCIFWG	-.0396	.0919	.0526	.0041	-.1199	-.0800
$(0.18, 0.15, 0.14, 0.13, 0.12, 0.10, 0.09, 0.09)^T$	BCIFWA	-.0171	.1181	.0783	.0259	-.1040	-.0624
	BCIFWG	-.0130	.1146	.0732	.0242	-.1006	-.0617
$(0.15, 0.18, 0.12, 0.10, 0.14, 0.09, 0.13, 0.09)^T$	BCIFWA	-.0253	.1061	.0663	.0154	-.1118	-.0717
	BCIFWG	-.0200	.1037	.0630	.0150	-.1078	-.0701
$(0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125, 0.125)^T$	BCIFWA	-.0304	.1046	.0662	.0146	-.1139	-.0720
	BCIFWG	-.0272	.1018	.0614	.0126	-.1115	-.0721
$(0.02, 0.05, 0.06, 0.07, 0.17, 0.20, 0.21, 0.22)^T$	BCIFWA	-.0558	.0785	.0422	-.0057	-.1282	-.0876
	BCIFWG	-.0568	.0761	.0375	-.0099	-.1301	-.0913
$(0.22, 0.21, 0.20, 0.17, 0.07, 0.06, 0.05, 0.02)^T$	BCIFWA	-.0035	.1292	.0885	.0345	-.0996	-.0563
	BCIFWG	-.0008	.1241	.0826	.0320	-.0951	-.0549
$(0.17, 0.05, 0.06, 0.02, 0.22, 0.20, 0.21, 0.07)^T$	BCIFWA	-.0203	.1140	.0744	.0240	-.0967	-.0597
	BCIFWG	-.0204	.1089	.0684	.0215	-.0952	-.0602
$(0.07, 0.21, 0.20, 0.22, 0.02, 0.06, 0.05, 0.17)^T$	BCIFWA	-.0416	.0915	.0548	.0022	-.1341	-.0869
	BCIFWG	-.0356	.0911	.0513	.0010	-.1298	-.0860

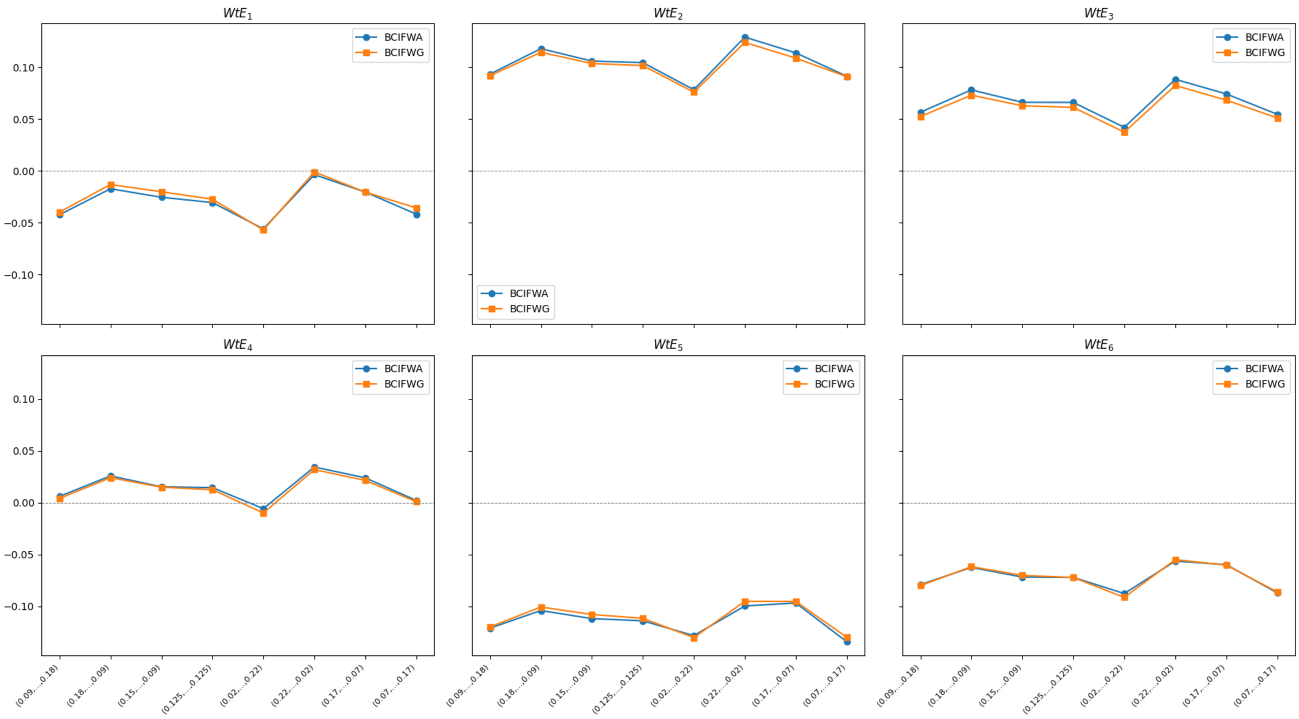

Figure 4. Sensitivity of WtE decision outcomes to changes in attribute weight vectors.

Meanwhile, $\mathbf{WtE}_5$ consistently ranks lowest across all scenarios, with strongly negative scores (from -.0996 to -.1341). Alternatives such as $\mathbf{WtE}_3$ and $\mathbf{WtE}_4$ exhibit minor score variations depending on attribute weighting, but they do not surpass $\mathbf{WtE}_2$. These findings suggest that while intermediate rankings are somewhat weight-dependent, the top- and bottom-ranked options remain unchanged, further underscoring the reliability of the proposed BCIFWA and BCIFWG operators in supporting decision-making under uncertainty.

Overall, this sensitivity analysis validates the robustness of the suggested model by showing that alternative rankings remain consistent under diverse weighting strategies.

6.7. Limitations and future directions

Despite the promising outcomes, several limitations of the proposed framework should be acknowledged. First, the study is restricted to a single case study of municipal solid waste-to-energy options, which may limit the generalizability of the findings to other sectors or geographic contexts. Second, while the BCIFWA and BCIFWG operators effectively capture bipolar, intuitionistic, and complex information, they rely on expert-provided weights and evaluations, which may introduce subjectivity or bias. Third, although sensitivity analysis confirmed ranking stability, the analysis did not incorporate dynamic changes in technological, economic, or regulatory environments that may affect long-term decision outcomes.

Future research could address these limitations in several ways. Expanding the framework to multiple case studies across different regions or application domains (e.g., renewable energy portfolios, smart city planning) would enhance its applicability. Integrating data-driven methods—such as machine learning or IoT-enabled monitoring systems—can reduce reliance on subjective expert input and support real-time updates of evaluation parameters. Moreover, coupling the proposed operators with scenario-based or stochastic simulations would allow researchers to model uncertainty stemming from policy shifts, market fluctuations, and technological advancements more realistically.

By addressing these limitations, future studies can further strengthen the scalability, objectivity, and practical impact of BCIFWA and BCIFWG operators for complex multi-attribute decision-making problems.

7. Conclusions and future work

This study introduced a comprehensive MADM framework based on BCIFNs, specifically designed to handle complex, uncertain, and dual (positive-negative) evaluations. The work began with the formulation of rigorous operational laws for BCIFNs, providing a solid mathematical foundation for two novel aggregation operators: the BCIFWA and BCIFWG operators. These operators were systematically derived to ensure logical consistency, robustness, and suitability for capturing both positive and negative attitudes, as well as complex-phase uncertainty, in decision-making processes. The proposed framework was applied to a real-world WtE and circular economy case study, evaluating six alternative strategies across environmental, economic, and social dimensions. The analysis consistently identified $\mathbf{WtE}_2$ as the most sustainable option, achieving a balanced trade-off between energy recovery, waste volume reduction, and technological feasibility. Comparative evaluations demonstrated that the BCIFWA and BCIFWG operators provide enhanced ranking stability and discrimination power compared to classical IFWA/IFWG operators, bipolar fuzzy (BIFWA/BIFWG) models, bipolar fuzzy Dombi operators, and bipolar complex fuzzy approaches (BCFWAA/BCFWGA and BCFDWA/BCFDWG), confirming the framework's superior ability to capture nuanced, bipolar, and complex-valued uncertainties in practical MADM scenarios.

The key contributions of this framework include its unified and generalized modeling capability, which allows the proposed operators to reduce to classical intuitionistic fuzzy, bipolar intuitionistic fuzzy, bipolar fuzzy, or bipolar complex fuzzy operators under specific conditions, demonstrating flexibility across diverse decision-making paradigms. By incorporating bipolarity, intuitionistic hesitation, and complex-valued membership and nonmembership functions, the framework enhances uncertainty representation, modeling complex information more comprehensively than existing approaches. Sensitivity analyses confirmed that rankings remain robust under varying attribute weights, with $\mathbf{WtE}_2$ consistently achieving top performance, while quantitative comparisons show improved discrimination between alternatives by simultaneously accounting for positive, negative, and oscillatory information. This makes the framework practically relevant for policymakers and urban planners seeking actionable guidance in selecting sustainable waste management strategies aligned with circular economy and renewable energy principles.

Compared to existing models, the BCIFN-based approach offers clear advantages. Unlike hybrid MCDM frameworks for renewable energy prioritization [54], which may not explicitly model bipolar evaluations, the proposed operators capture richer information on conflicting criteria and decision-maker hesitation. Relative to bipolar fuzzy rough set-based VIKOR extensions [55], which rely on δ-covering approximations, the BCIFWA and BCIFWG operators allow smooth aggregation of complex-valued membership information, improving adaptability and interpretability. Similarly, compared to Hamacher aggregation operators for Pythagorean fuzzy sets [56], the BCIF framework captures oscillatory and dual-phase uncertainties more comprehensively, providing nuanced insights into alternative rankings.

Despite these strengths, the study has limitations, including its focus on a single municipal WtE case study, reliance on expert-provided weights and evaluations that may introduce subjectivity, and sensitivity analyses that do not fully account for dynamic changes in technology, policy, or market conditions. Future research can extend this framework to additional domains such as renewable energy site selection, water resource management, healthcare, and climate-resilient infrastructure planning. Hybrid approaches integrating BCIFWA/BCIFWG with optimization algorithms, machine learning, or scenario-based simulations could enhance dynamic decision-making and reduce computational overhead. Development of computational tools for real-time implementation would improve accessibility, while exploration of variable Dombi parameters and alternative fuzzy paradigms (e.g., spherical or q-rung orthopair fuzzy sets) could further enhance modeling flexibility and aggregation precision. Integration with digital technologies, such as IoT-enabled monitoring and AI-driven predictive analytics, may also support data-driven updates in dynamic environments.

Author contributions

H. Z. Ibrahim: Writing-original draft, writing-review and editing, methodology, software, supervision, visualization, conceptualization, investigation, formal analysis; M. H. Alqahtani: Writing review and editing, project administration, formal analysis, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors disclose no conflict of interests in publishing this paper.

References

1. L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3

3. D. Ramot, R. Milo, M. Fiedman, A. Kandel, Complex fuzzy sets, *IEEE Trans. Fuzzy Syst.*, **10** (2002), 171–186. https://doi.org/10.1109/91.995119

4. A. Alkouri, A. Salleh, Complex intuitionistic fuzzy sets, *AIP Conf. Proc.*, **1482** (2012), 464–470. https://doi.org/10.1063/1.4757515

5. W. R. Zhang, Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis, *Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference*, 1994, 305–309. https://doi.org/10.1109/IJCF.1994.375115

6. W. R. Zhang, Yin (Yang) bipolar fuzzy sets, *Proceedings of IEEE International Conference on Fuzzy Systems Proceedings*, 1998, 835–840. https://doi.org/10.1109/FUZZY.1998.687599

7. D. Ezhilmaran, K. Sankar, Morphism of bipolar intuitionistic fuzzy graphs, *J. Discret. Math. Sci. C.*, **18** (2015), 605–621. https://doi.org/10.1080/09720529.2015.1013673

8. A. Alkouri, M. Massa'deh, M. Ali, On bipolar complex fuzzy sets and its application, *J. Intell. Fuzzy Syst.*, **39** (2020), 383–397. https://doi.org/10.3233/JIFS-191350

9. A. Al-Husban, A. Amourah, J. Jaber, Bipolar complex fuzzy sets and their properties, *Ital. J. Pure Appl. Math.*, **43** (2020), 75–761.

10. M. Gulistan, N. Yaqoob, A. Elmoasry, J. Alebraheem, Complex bipolar fuzzy sets: an application in a transport's company, *J. Intell. Fuzzy Syst.*, **40** (2021), 3981–3997. https://doi.org/10.3233/JIFS-200234

11. T. Mahmood, U. ur Rehman, A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures, *Int. J. Intell. Syst.*, **37** (2022), 535–567. https://doi.org/10.1002/int.22639

12. A. Al-Husban, Bipolar complex intuitionistic fuzzy sets, *Earthline Journal of Mathematical Sciences*, **8** (2022), 273–280. https://doi.org/10.34198/ejms.8222.273280

13. A. Alkouri, Z. Alshboul, Environmental impact assessment using bipolar complex intuitionistic fuzzy sets, *Soft Comput.*, **29** (2025), 3795–3809. https://doi.org/10.1007/s00500-025-10608-w

14. H. Liu, A novel decision algorithm for the innovation and optimization in university labor education courses using fuzzy information involving multiple experts, *Sci. Rep.*, **15** (2025), 24934. https://doi.org/10.1038/s41598-025-10160-5

15. P. Liu, P. Wang, Some improved linguistic intuitionistic fuzzy aggregation operators and their applications to multiple-attribute decision making, *Int. J. Inf. Tech. Decis.*, **16** (2017), 817–850. https://doi.org/10.1142/S0219622017500110

16. N. Altaweel, M. Rashid, O. Albalawi, M. Alshehri, N. Eljaneid, R. Albalawi, On the ideal convergent sequences in fuzzy normed space, *Symmetry*, **15** (2023), 936. https://doi.org/10.3390/sym15040936

17. A. Mahboob, M. Al-Tahan, G. Muhiuddin, Fuzzy (m, n)-filters based on fuzzy points in ordered semigroups, *Comput. Appl. Math.*, **42** (2023), 245. https://doi.org/10.1007/s40314-023-02381-z

18. G. Muhiuddin, N. Abughazalah, A. Mahboob, D. Al-Kadi, A novel study of fuzzy bi-ideals in ordered semirings, *Axioms*, **12** (2023), 626. https://doi.org/10.3390/axioms12070626

19. G. Muhiuddin, T. Mahapatra, M. Pal, O. Alshahrani, A. Mahboob, Integrity on m-polar fuzzy graphs and its application, *Mathematics*, **11** (2023), 1398. https://doi.org/10.3390/math11061398

20. M. Areshi, S. El-Tantawy, B. Alotaibi, S. Zaland, Study of fuzzy fractional third-order dispersive KdV equation in a plasma under Atangana-Baleanu derivative, *J. Funct. Space.*, **2022** (2022), 7922001. https://doi.org/10.1155/2022/7922001

21. R. Gul, S. Ayub, M. Shabir, A. Hummdi, A. Aljaedi, Z. Bassfar, A novel approach for fuzzification of rough sets based on fuzzy preference relation: Properties and application to medicine selection problem, *IEEE Access*, **12** (2024), 67681–67703. https://doi.org/10.1109/ACCESS.2024.3396526

22. S. Ayub, R. Gul, M. Shabir, A. Hummdi, A. Aljaedi, Z. Bassfar, A novel approach to roughness of linear Diophantine fuzzy sets by fuzzy relations and its application toward multi-criteria group decision-making, *IEEE Access*, **12** (2024), 53549–53564. https://doi.org/10.1109/ACCESS.2024.3386581

23. M. Cortez, A. Althobaiti, A. Aljohani, S. Althobaiti, Generalized fuzzy-valued convexity with Ostrowski's and Hermite-Hadamard type inequalities over inclusion relations and their applications, *Axioms*, **13** (2024), 471. https://doi.org/10.3390/axioms13070471

24. A. Mahboob, M. Al-Tahan, G. Muhiuddin, Characterizations of ordered semigroups in terms of fuzzy (m, n)-substructures, *Soft Comput.*, **28** (2024), 10827–10834. https://doi.org/10.1007/s00500-024-09880-z

25. G. Muhiuddin, M. Takallo, R. Borzooei, Y. Jun, m-polar fuzzy q-ideals in BCI-algebras, *J. King Saud Univ. Sci.*, **32** (2020), 2803–2809. https://doi.org/10.1016/j.jksus.2020.07.001

26. A. Alanazi, G. Muhiuddin, B. Alenazi, T. Mahapatra, M. Pal, Utilizing m-polar fuzzy saturation graphs for optimized allocation problem solutions, *Mathematics*, **11** (2023), 4136. https://doi.org/10.3390/math11194136

27. A. Alanazi, G. Muhiuddin, T. Mahapatra, Z. Bassfar, M. Pal, Inverse graphs in m-polar fuzzy environments and their application in robotics manufacturing allocation problems with new techniques of resolvability, *Symmetry*, **15** (2023), 1387. https://doi.org/10.3390/sym15071387

28. F. Mizrak, D. Şahin, Investment strategies for renewable energy technologies and harvesting systems in airport operations using spherical fuzzy MCDM models, *Sci. Rep.*, **15** (2025), 24075. https://doi.org/10.1038/s41598-025-08480-7

29. M. Wasim, A. Yousaf, H. Alolaiyan, M. Akbar, A. Alburaikan, H. Abd El-Wahed Khalifa, Optimizing decision-making with aggregation operators for generalized intuitionistic fuzzy sets and their applications in the tech industry, *Sci. Rep.*, **14** (2024), 16538. https://doi.org/10.1038/s41598-024-57461-9

30. S. Dai, Linguistic complex fuzzy sets, *Axioms*, **12** (2023), 328. https://doi.org/10.3390/axioms12040328

31. M. Khan, K. Ullah, A. Raza, T. Senapati, S. Moslem, Multi-attribute decision-making method based on complex T-spherical fuzzy frank prioritized aggregation operators, *Heliyon*, **10** (2024), e25368. https://doi.org/10.1016/j.heliyon.2024.e25368

32. Y. Yang, B. Ning, F. Tian, G. Lu, Some complex probabilistic hesitant fuzzy aggregation operators and their applications to multi-attribute decision making, *Sci. Rep.*, **15** (2025), 4342. https://doi.org/10.1038/s41598-024-84582-y

33. H. Alolaiyan, S. Abbas, A. Razaq, U. Shuaib, Q. Xin, A. Baidar, An innovative decision-making approach for unmanned aerial vehicle software selection using bipolar fuzzy aggregation operators, *Sci. Rep.*, **15** (2025), 13836. https://doi.org/10.1038/s41598-025-97832-4

34. J. Ahmmad, M. Khan, I. Aldayel, T. Mahmood, Significance and classification of AI-driven techniques in telecommunication sectors based on interval-valued bipolar fuzzy soft aggregation operators, *Sci. Rep.*, **15** (2025), 14126. https://doi.org/10.1038/s41598-025-97866-8

35. M. Akram, Shumaiza, A. N. Al-Kenani, Multi-criteria group decision-making for selection of green suppliers under bipolar fuzzy PROMETHEE process, *Symmetry*, **12** (2020), 77. https://doi.org/10.3390/sym12010077

36. E. Natarajan, F. Augustin, R. Saraswathy, S. Narayanamoorthy, S. Salahshour, A. Ahmadian, et al., A bipolar intuitionistic fuzzy decision-making model for selection of effective diagnosis method of tuberculosis, *Acta Trop.*, **252** (2024), 107132. https://doi.org/10.1016/j.actatropica.2024.107132

37. T. Mahmood, Z. Ali, Multi-attribute decision-making methods based on Aczel-Alsina power aggregation operators for managing complex intuitionistic fuzzy sets, *Comp. Appl. Math.*, **42** (2023), 87. https://doi.org/10.1007/s40314-023-02204-1

38. Z. Khan, I. Ullah, F. Hussain, T. Rahim, R. Jan, M. Khan, Multi-attribute decision-making problem using complex intuitionistic fuzzy interaction aggregation operators, *J. Appl. Math. Comput.*, **70** (2024), 5153–5189. https://doi.org/10.1007/s12190-024-02170-9

39. H. Garg, T. Mahmood, U. ur Rehman, G. Nguyen, Multi-attribute decision-making approach based on Aczel-Alsina power aggregation operators under bipolar fuzzy information & its application to quantum computing, *Alex. Eng. J.*, **82** (2023), 248–259. https://doi.org/10.1016/j.aej.2023.09.073

40. T. Mahmood, U. ur Rehman, Multi-attribute decision-making method based on bipolar complex fuzzy Maclaurin symmetric mean operators, *Comp. Appl. Math.*, **41** (2022), 331. https://doi.org/10.1007/s40314-022-02016-9

41. Z. Zhao, A. Hussain, N. Zhang, K. Ullah, S. Yin, A. Awsar, et al., Decision support system based on bipolar complex fuzzy Hamy mean operators, *Heliyon*, **10** (2024), e36461. https://doi.org/10.1016/j.heliyon.2024.e36461

42. W. Emam, H. Waqas, T. Mahmood, U. Rehman, D. Pamucar, AI-driven energy management system based on hesitant bipolar complex fuzzy Hamacher power aggregation operators and their applications in MADM, *Sci. Rep.*, **15** (2025), 13083. https://doi.org/10.1038/s41598-025-94340-3

43. H. Alolaiyan, M. Liaqat, A. Razaq, U. Shuaib, A. Baidar, Q. Xin, Optimal selection of diagnostic method for diabetes mellitus using complex bipolar fuzzy dynamic data, *Sci. Rep.*, **15** (2025), 3921. https://doi.org/10.1038/s41598-024-84460-7

44. T. Mahmood, U. ur Rehman, S. Shahab, Z. Ali, M. Anjum, Decision-making by using TOPSIS techniques in the framework of bipolar complex intuitionistic fuzzy N-soft sets, *IEEE Access*, **11** (2023), 105677–105697. https://doi.org/10.1109/ACCESS.2023.3316879

45. R. Nandhinii, D. Amsaveni, On bipolar complex intuitionistic fuzzy graphs, *TWMS J. Appl. Eng. Math.*, **12** (2022), 92–106.

46. S. Yin, Y. Wang, Q. Zhang, Mechanisms and implementation pathways for distributed photovoltaic grid integration in rural power systems: a study based on multi-agent game theory approach, *Energy Strateg. Rev.*, **60** (2025), 101801. https://doi.org/10.1016/j.esr.2025.101801

47. Z. Xu, Intuitionistic fuzzy aggregation operators, *IEEE Trans. Fuzzy Syst.*, **15** (2007), 1179–1187. https://doi.org/10.1109/TFUZZ.2006.890678

48. Z. Xu, R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, *Int. J. Gen. Syst.*, **35** (2006), 417–433. https://doi.org/10.1080/03081070600574353

49. H. Ibrahim, Multi-attribute group decision-making based on bipolar n, m-rung orthopair fuzzy sets, *Granul. Comput.*, **8** (2023), 1819–1836. https://doi.org/10.1007/s41066-023-00405-x

50. C. Jana, M. Pal, J. Wang, Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process, *J. Ambient Intell. Human Comput.*, **10** (2019), 3533–3549. https://doi.org/10.1007/s12652-018-1076-9

51. T. Mahmood, U. Rehman, Z. Ali, M. Aslam, R. Chinram, Identification and classification of aggregation operators using bipolar complex fuzzy settings and their application in decision support systems, *Mathematics*, **10** (2022), 1726. https://doi.org/10.3390/math10101726

52. T. Mahmood, U. Rehman, A method to multi-attribute decision making technique based on Dombi aggregation operators under bipolar complex fuzzy information, *Comp. Appl. Math.*, **41** (2022), 47. https://doi.org/10.1007/s40314-021-01735-9

53. P. Liu, P. Wang, Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making, *Int. J. Intell. Syst.*, **33** (2018), 259–280. https://doi.org/10.1002/int.21927

54. T. Jameel, Y. Yasin, M. Riaz, An integrated hybrid MCDM framework for renewable energy prioritization in sustainable development, *Spectrum of Decision Making and Applications*, **3** (2025), 124–150. https://doi.org/10.31181/sdmap31202640

55. R. Gul, An extension of VIKOR approach for MCDM using bipolar fuzzy preference δ-covering based bipolar fuzzy rough set model, *Spectrum of Operational Research*, **2** (2025), 72–91. https://doi.org/10.31181/sor21202511

56. M. Asif, U. Ishtiaq, I. K. Argyros, Hamacher aggregation operators for pythagorean fuzzy set and its application in multi-attribute decision-making problem, *Spectrum of Operational Research*, **2** (2025), 27–40. https://doi.org/10.31181/sor2120258