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*Research article*

# On the study of multifractal analysis of functions in a metric space

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**Abstract:** Multi-fractal analysis plays a crucial role in understanding the complex behaviors of functions across different fields. In this study, we presented an innovative approach to examining the multi-fractal formalism. Specifically, we introduced new multi-fractal Hausdorff and packing measures, enabling the exploration of the multi-fractal spectrum within a metric space and offering a novel proof that extended the classical results in this setting. As an application, we focused on the Birkhoff averages when the multi-fractal formalism did not hold.

**Keywords:** multi-fractal formalism; Hausdorff and packing measures; Hausdorff and packing dimensions
**Mathematics Subject Classification:** 28A78, 28A80

## 1. Introduction

Multi-fractal analysis, inspired by B. Mandelbrot's work [1, 2] and developed around 1980, is commonly applied to characterize objects exhibiting scale invariance. Since then, it has produced significant theoretical and practical results. In some cases, a measure $\mu$ gives rise to sets of points where $\mu$ exhibits a local density with exponent $\alpha$. The dimensions of these sets represent the distribution of singularities in the measure. More precisely, for a finite measure $\mu$ defined on a separable metric space $(\mathbb{X}, d)$, the lower and upper local dimensions of $\mu$ at $x$ are defined as

$$\underline{\alpha}_\mu(x) = \liminf_{\varepsilon\to 0} \frac{\log \mu(\mathsf{B}(x,\varepsilon))}{\log \varepsilon} \qquad \text{and} \qquad \overline{\alpha}_\mu(x) = \limsup_{\varepsilon\to 0} \frac{\log \mu(\mathsf{B}(x,\varepsilon))}{\log \varepsilon},$$

where $\mathsf{B}(x,\varepsilon)$ denotes the open ball centered at $x \in \mathbb{X}$ with radius $\varepsilon > 0$. If $\underline{\alpha}_\mu(x) = \overline{\alpha}_\mu(x)$, we refer to the common value as the local dimension of $\mu$ at $x$ and denote it by $\alpha_\mu(x)$. For $\alpha \geq 0$, define

$$X_\mu(\alpha) = \Big\{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\Big\},$$

where $\operatorname{supp}\mu$ is the topological support of $\mu$. The level set $X_\mu(\alpha)$ contains crucial information about the geometrical properties of $\mu$. The goal of multi-fractal analysis is to estimate the size of $X_\mu(\alpha)$. This is achieved by relating the Hausdorff and packing dimensions of these level sets, denoted respectively by $\dim_H(X_\mu(\alpha))$ and $\dim_P(X_\mu(\alpha))$, to the Legendre transform of some convex function [3–5].

Olsen [6] developed a general formalism by incorporating the measure $\mu$ into the definitions of the standard Hausdorff and packing measures, denoted $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$, respectively, for parameters $q, t \in \mathbb{R}$. These measures are constructed using the dimension function $\mu(B(x,\varepsilon))^q\varepsilon^t$, which has since become widely adopted. Furthermore, numerous researchers have studied these measures, highlighting their significance in analyzing local fractal properties and fractal products [7–9]. Under appropriate conditions on the measure $\mu$ (such as doubling), the multi-fractal measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ are adequate for the study of the level set $X_\mu(\alpha)$. The doubling condition is only necessary in frameworks that require uniform scale-invariance (for instance constructing Gibbs measures with uniform constants), but it is not required for the upper and lower bounds of Hausdorff and packing dimensions as established in our theorems.

Readers may refer to [3, 10–13] for the multi-fractal formalism of various types of measures, including self-conformal, self-similar, self-affine, and Moran measures. Additionally, multi-fractal analysis of branching random walks on Galton–Watson trees and the thermodynamic formalism can be found in [14–16]. Recently, other results have been studied as [17–19]. However, consider the measure $\mu$ defined on $\operatorname{supp}\mu = [0, 1]$ with density (with respect to the Lebesgue measure) given by

$$f(x) = \frac{s|\log x|^{s-1}}{x}\, e^{-|\log x|^s}\, \mathbf{1}_{(0,1)}(x), \tag{1.1}$$

where $s$ is a positive real number. For $s = 2$ and $x = 0$, one obtains

$$\mu(B(x,\varepsilon)) = e^{-(\log\varepsilon)^2}.$$

This implies that $\alpha_\mu(0) = \infty$. In particular, $X_\mu(\alpha) = \emptyset$ for all $\alpha > 0$ [20, Example 5]. Hence, it may be more useful to consider a function $\alpha(\mu, h)$ for some parameter $h$, allowing us to define, for $\alpha \geq 0$, the set $\{x \mid \alpha(\mu, h) = \alpha\}$. This approach enables a valid multi-fractal formalism without relying on measures equivalent to power-law radii. Consequently, it is beneficial to develop a more general framework in which the function's restriction on balls can be any measure, not limited to those equivalent to $\varepsilon^\alpha$ [21–23]. This motivation underlies the exploration of multi-fractal measures based on a generalized dimension function. The purpose of this paper is to study the fractal dimension of the set

$$X_\varkappa(\alpha) = \Big\{x \in \mathbb{X} \;\Big|\; \lim_{\varepsilon\to 0} \frac{\tau(x,\varepsilon)}{-\zeta(\varepsilon)} = \alpha\Big\}, \tag{1.2}$$

where $\varkappa = (\tau, \zeta)$, $\tau : \mathbb{X} \times \mathbb{R}_+ \to \mathbb{R}$, and $\zeta : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\zeta \text{ is nondecreasing and } \zeta(\varepsilon) < 0 \text{ for } \varepsilon \text{ small enough.} \tag{1.3}$$

To address this problem, the mass distribution principle is commonly employed [15, 24]. This requires defining an auxiliary measure suitable for our sets, referred to as the Gibbs measure in the context of Birkhoff averages and the Mandelbrot measure in the study of branching random walks [25, 26]. In this work, a novel approach is presented for determining the fractal dimensions of a set $E \subseteq \mathbb{X}$. This method, originally introduced by Cutler and Tricot [27, 28] and later expanded by Ben Nasr et al. [29], provides a more general framework and allows the derivation of a more relevant formula for our problem, ensuring the validity of the multi-fractal formalism and enabling the study of cases where the formalism does not hold. To this end, we incorporate the functions $\tau$ and $\zeta$ into the definitions of the standard Hausdorff and packing measures, denoted by $\mathcal{H}^{q,t}_{\varkappa,\mu}$ and $\mathcal{P}^{q,t}_{\varkappa,\mu}$ (see Section 2), where $\mu \in \mathcal{M}(\mathbb{X})$. The generalized lower and upper local dimensions of $\mu$ relative to $\zeta$ are given, respectively, by

$$\underline{\alpha}_{\mu,\zeta}(x) := \underline{\lim}_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{-\zeta(\varepsilon)}, \quad \text{and} \quad \overline{\alpha}_{\mu,\zeta}(x) := \overline{\lim}_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{-\zeta(\varepsilon)}.$$

Let $\Lambda_{\varkappa,\mu}(E,q)$ be the separator function of the pre-packing measure $\overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E)$ (see Table 1). Moreover, if $\mu \equiv 1$ or $q = 0$, we denote them by $\Lambda_{\varkappa}(E,q)$ and $\Lambda_{\zeta,\mu}(E)$, respectively. We also consider the outer measure $\vartheta(E) := \mathcal{H}^{0}_{\zeta,\mu}(E)$, that is, the Hausdorff measure with dimension function $\mu$ (see (3.1)). The first result of this paper is the following, which we will prove in Section 3.

**Theorem 1.** *Let $E \subseteq \mathbb{X}$, $\mu \in \mathcal{M}(\mathbb{X})$, and let $\zeta$ satisfy condition* (1.3)*. Assume that $\vartheta(E) > 0$ and that $\Lambda_{\zeta,\mu}(E) \le 0$. Then, the Hausdorff and packing dimensions of E are given by*

$$\dim_H E = \sup_{\mu} \operatorname*{ess\,sup}_{x\in E,\vartheta} \underline{\alpha}_{\mu,\zeta}(x) \qquad \textit{and} \qquad \dim_P E = \sup_{\mu} \operatorname*{ess\,sup}_{x\in E,\vartheta} \overline{\alpha}_{\mu,\zeta}(x),$$

*where the essential supremum with respect to the outer measure $\vartheta$ is defined as*

$$\operatorname*{ess\,sup}_{x\in E,\vartheta} \varphi(x) := \inf\{t \in \mathbb{R} \mid \vartheta(E \cap \{\varphi > t\}) = 0\}.$$

**Remark 1.** *The condition $\Lambda_{\zeta,\mu}(E) \le 0$ will be used in Proposition 2 to estimate the upper bounds of the Hausdorff and packing dimensions. On the other hand, the condition $\vartheta(E) > 0$ will be used to obtain lower bounds for these dimensions.*

Theorem 1 generalizes classical results on Hausdorff and packing dimensions [28, 30] to a broader context. In particular, when $\zeta(\varepsilon) = \log(\varepsilon)$ in Euclidean spaces $\mathbb{R}^d$, it recovers the dimension exponents introduced by Cutler and Tricot [27, 28]. The condition $\Lambda_{\zeta,\mu}(E) \le 0$ is often satisfied in specific spaces or under particular choices of measures. For instance, it holds trivially when $q = 0$. In more general settings, however, this assumption must be stated explicitly.

**Corollary 1.** *[28, Theorem 1] Let $E \subseteq \mathbb{R}^d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then,*

$$\dim_H E = \sup_{\mu} \operatorname*{ess\,sup}_{x\in E,\vartheta} \underline{\alpha}_{\mu}(x), \qquad \textit{and} \qquad \dim_P E = \sup_{\mu} \operatorname*{ess\,sup}_{x\in E,\vartheta} \overline{\alpha}_{\mu}(x).$$

**Corollary 2.** *[31, Proposition 4.8] Let $E \subseteq \mathbb{X}$ and $\mu \in \mathcal{M}(\mathbb{X})$. Then,*

$$\dim_H E \le \sup_{x\in E} \underline{\alpha}_{\mu}(x), \quad \dim_P E \le \sup_{x\in E} \overline{\alpha}_{\mu}(x),$$

*and if $\vartheta(E) > 0$, one also has*

$$\dim_H E \ge \inf_{x\in E} \underline{\alpha}_{\mu}(x), \quad \dim_P E \ge \inf_{x\in E} \overline{\alpha}_{\mu}(x).$$

As an application of Theorem 1, we consider the multi-fractal formalism for the function $\tau$ relative to $\zeta$. To this end, we define the Legendre transform of a function $\psi : \mathbb{R} \to \overline{\mathbb{R}}$ by

$$\psi^*(\alpha) := \inf_{q \in \mathbb{R}} (q\alpha + \psi(q)), \qquad \alpha \in \mathbb{R}.$$

The function $q \mapsto \Lambda_\varkappa(q) := \Lambda_\varkappa(\mathbb{X}, q)$ is nonincreasing and convex. The following result deals with the case where this function is differentiable. When $\Lambda_\varkappa$ is not differentiable, the corresponding statements will be provided in Theorem 4. In that case, we consider the left derivative $\Lambda_{\varkappa,l}$ and the right derivative $\Lambda_{\varkappa,r}$ of $\Lambda_\varkappa$.

**Theorem 2.** *Let $q \in \mathbb{R}$ and $\zeta : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (1.3)*. Assume that $\mathcal{H}_\varkappa^{q,\Lambda_\varkappa(q)}(\mathbb{X}) > 0$ and that the function $\Lambda_\varkappa$ is differentiable at $q$. Then, for $\alpha = -\Lambda_\varkappa'(q)$, we have $b_\varkappa(q) = B_\varkappa(q) = \Lambda_\varkappa(q)$, and*

$$\dim_H \mathrm{X}_\varkappa(\alpha) = \dim_p \mathrm{X}_\varkappa(\alpha) = \Lambda_\varkappa^*(\alpha), \tag{1.4}$$

*where $b_\varkappa(q) := b_\varkappa(\mathbb{X}, q)$ and $B_\varkappa(q) := B_\varkappa(\mathbb{X}, q)$.*

Our approach allows the independent estimation of both Hausdorff and packing dimensions. Consequently, it provides results even in cases where the multi-fractal formalism fails to hold; that is, when the multi-fractal Hausdorff and packing dimensions differ. A detailed discussion of this approach is provided in Section 3. We also give examples related to the multi-fractal analysis of Birkhoff averages in symbolic spaces, where the multi-fractal formalism may not hold. The condition $\mathcal{H}_\varkappa^{q,\Lambda_\varkappa(q)}(\mathbb{X}) > 0$ is ensured whenever one can construct a suitable Frostman measure (for instance, a Gibbs measure) supported on $\mathbb{X}$. In this case, the measure provides the required lower bounds on the generalized Hausdorff content, thereby guaranteeing the positivity of the measure (see Remark 6).

The most common example in our study pertains to the case of examining the multi-fractal analysis of measures. Specifically, consider

$$\tau(x, \varepsilon) = -\log \nu(\mathsf{B}(x, \varepsilon)), \quad \text{and} \quad \zeta(\varepsilon) = \log \varepsilon, \quad \nu \in \mathcal{M}(\mathbb{X}).$$

In this framework, the measure $\nu$ is compared to powers of the diameter. This formulation revisits the classical multi-fractal formalism for measures supported on compact sets, as developed by Olsen [6], and later explored in [22, 32]. In particular, in this case Theorem 1 reduces to [29, Lemma 3.1]; see also [29, Theorem 3.4] for a comparison with Theorem 2.

In this paper, we examine the multi-fractal analysis of Birkhoff averages for a given function. More precisely, let $\mathbb{X} = \{0, 1\}^{\mathbb{N}}$ and let $g$ be a dyadic Hölder continuous function with period 1 and $g(0) = 1$. We consider the following specific case

$$\tau(x, \varepsilon) = -\log \prod_{s=0}^{n-1} g(\sigma^s x),$$

where $2^{-n} \leq \varepsilon < 2^{-n-1}$ and $\sigma$ denotes the left-shift operator defined as $\sigma^s(x_1, x_2, x_3, \dots) = (x_{s+1}, x_{s+2}, \dots)$. We define also

$$\zeta(\varepsilon) = \log(|I_n(x)|),$$

where $I_n(x)$ denotes the cylinder set with base $(x_1, \ldots, x_n)$ for all $x \in \{0,1\}^{\mathbb{N}}$. In this context, the level sets

$$\mathrm{X}_{\varkappa}(\alpha) := \Big\{ x \in [0,1) \;\Big|\; \lim_{n\to\infty} \frac{1}{-n} \sum_{s=0}^{n-1} \log_2 g(\sigma^s x) = \alpha \Big\}, \quad \alpha \in \mathbb{R}.$$

The main objective of the multi-fractal analysis is to study the asymptotic behavior of these Birkhoff averages. Initially, it was analyzed in the symbolic space for Hölder potentials by Pesin and Weiss [33], and for general continuous potentials by Fan et al. [26]. Later, Liao and Wu [34] studied continuous potentials for conformal expanding maps. Other relevant contributions include [35–37], each under slightly different assumptions. In the present example, we examine the asymptotic behavior of multi-periodic functions [26, 38]. Typically, the multi-fractal dimension of these sets is related to the topological pressure of the continuous function $\log g$, defined by

$$P(q) := \lim_{n\to\infty} \frac{1}{n} \log \Bigg( \sum_{I\in\mathcal{F}_n} \sup_{x\in I} \prod_{s=0}^{n-1} g(\sigma^s x)^q \Bigg),$$

where $\mathcal{F}_n$ denotes the collection of cylinder sets of generation $n$. It is known that this limit exists and that $P$ is convex and differentiable [38]. Moreover, in Section 4, we establish that for all $\alpha = P'(q)$,

$$\dim_H \mathrm{X}_{\varkappa}(\alpha) = \dim_P \mathrm{X}_{\varkappa}(\alpha) = \frac{P(q) - q\alpha}{\log 2}.$$

This paper is organized as follows. In Section 2, we recall the definitions of multi-fractal Hausdorff and packing measures, together with the associated dimensions, and discuss their fundamental properties. Section 3 is devoted to the presentation of our main results concerning Hausdorff and packing dimensions in the multi-fractal framework. In Section 3.1, we apply these results to the study of Birkhoff averages, a central tool in ergodic theory and dynamical systems. There, we provide a new proof that relies on the methods developed in this work and extend the classical multi-fractal formalism to this setting.

## 2. Construction of fractal measures and dimensions and preliminaries results

In this section, we introduce the mutual multi-fractal (MM) Hausdorff and packing measures, incorporating certain modifications to the standard definitions (see [16]) for technical reasons. We begin by presenting the notations and preliminary definitions that will be used throughout the paper. Let $(\mathbb{X}, d)$ be a separable metric space, and define the open ball with center $x \in \mathbb{X}$ and radius $\varepsilon > 0$ by

$$\mathsf{B}(x, \varepsilon) := \{ y \in \mathbb{X} \mid d(x, y) < \varepsilon \}.$$

Throughout this work, we assume that $\mathbb{X}$ satisfies the Besicovitch covering property (see appendix, Theorem 7). This is a theorem by Besicovitch where a Euclidean space fulfills this condition. It is obvious that an ultrametric space also has this property. In a general metric space, however, the center $x$ and the radius $\varepsilon$ of a ball $\mathsf{B}(x,\varepsilon)$ are not uniquely determined by the set itself. For instance, in the case where $d$ is the discrete metric, one has $\mathsf{B}(x,r) = \mathsf{B}(x,s)$ for all $r, s \in (0,1]$, and moreover $\mathsf{B}(x,r) = \mathsf{B}(y,s)$ for all $r, s > 1$. For this reason, we emphasize that a ball will always be considered together with its defining data $(x, \varepsilon)$, i.e., as an ordered pair consisting of a center and a radius. We denote by $\mathcal{B}(\mathbb{X})$ the collection of all open balls in $\mathbb{X}$, and by $\mathcal{M}(\mathbb{X})$ the set of maps from $\mathcal{B}(\mathbb{X})$ to $[0, +\infty)$, i.e., set functions defined on the collection of balls. Finally, for any measure on the space $\mathbb{X}$, we define the support of $\mu$, denoted $\mathcal{S}_\mu$, as the complement of the set $\bigcup\{\mathsf{B} \in \mathcal{B}(\mathbb{X}) \mid \mu(\mathsf{B}) = 0\}$. Let $\tau : \mathbb{X}\times\mathbb{R}_+ \to \mathbb{R}$ and $\zeta : \mathbb{R}_+ \to \mathbb{R}$ be an function satisfying (1.3). For $\varkappa = (\tau, \zeta)$ and $t, q \in \mathbb{R}$, we write

$$\Psi_\varkappa^{q,t}(x, \varepsilon) = \exp\Big(-q\tau(x, \varepsilon) + t\,\zeta(\varepsilon)\Big).$$

Let $E$ be a nonempty subset of $\mathbb{X}$, $\varepsilon > 0$ and $\mu \in \mathcal{M}(\mathbb{X})$. We define the function by

$$\overline{\mathcal{H}}^{q,t}_{\varkappa,\mu,\varepsilon}(E) = \inf_{(x_i,\varepsilon_i)\in\mathbb{X}\times\mathbb{R}_+} \sum_i \Psi_\varkappa^{q,t}(x_i, \varepsilon_i)\, \mu(\mathsf{B}(x_i, \varepsilon_i)),$$

where the infimum is taken over all constituents $\pi = \{(x_i, \varepsilon_i)\}_i$ such that $\pi$ a (centered) $\varepsilon$-cover of $E$; that is, $E \subseteq \bigcup_{(x,\varepsilon)\in\pi} \mathsf{B}(x, \varepsilon)$ and $\varepsilon_i \le \varepsilon$. We set $\overline{\mathcal{H}}^{q,t}_{\varkappa,\mu,\varepsilon}(\emptyset) = 0$ and we define the MM-Hausdorff measure $\mathcal{H}^{q,t}_{\varkappa,\mu}$ by

$$\overline{\mathcal{H}}^{q,t}_{\varkappa,\mu}(E) = \sup_{\varepsilon>0} \overline{\mathcal{H}}^{q,t}_{\varkappa,\mu,\varepsilon}(E) \quad \text{and} \quad \mathcal{H}^{q,t}_{\varkappa,\mu}(E) = \sup_{A\subseteq E} \overline{\mathcal{H}}^{q,t}_{\varkappa,\mu}(A).$$

When $\mu(\mathsf{B}) = 1$ for all $\mathsf{B} \in \mathcal{B}(\mathbb{X})$ or $q = 0$ then $\mathcal{H}^{q,t}_{\varkappa,\mu}$ will be denoted by $\mathcal{H}^{q,t}_{\varkappa}$ and $\mathcal{H}^{t}_{\zeta,\mu}$ respectively.

**Remark 2.** *The function $\mathcal{H}^{q,t}_{\varkappa,\mu}$ can be interpreted as a Carathéodory-type structure, originally introduced by Carathéodory [39] and later generalized by Pesin [16] in order to define various dimension-related characteristics. Using the notation from Section 5, set*

$$\psi = \operatorname{diam}(\mathsf{B}(x_i, \varepsilon_i)), \qquad \eta = \exp(\zeta(\varepsilon_i)), \qquad \xi(x_i, \varepsilon_i) = \exp\Big(-q\,\tau(x_i, \varepsilon_i) + \log\mu(\mathsf{B}(x_i, \varepsilon_i))\Big).$$

*With these definitions, the family $\mathcal{A} = (\mathcal{B}(\mathbb{X}), \xi, \eta, \psi)$ forms a Carathéodory structure. In particular, the functional $\mathcal{H}^{q,t}_{\varkappa,\mu}$ is a Carathéodory measure.*

The MM-Hausdorff measure $\mathcal{H}^{q,t}_{\varkappa,\mu}$ assigns a dimension to each subset $E \subseteq \mathbb{X}$. More precisely, there exists a unique number $b_{\varkappa,\mu}(E, q)$ such that

$$\mathcal{H}^{q,t}_{\varkappa,\mu}(E) = \begin{cases} \infty & \text{if } t < b_{\varkappa,\mu}(E, q), \\ 0 & \text{if } t > b_{\varkappa,\mu}(E, q), \end{cases}$$

with the convention $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$. We call $b_{\varkappa,\mu}(\,\cdot\,, q)$ the MM-Hausdorff dimension of the set $E$. Similarly, we define the function

$$\overline{\mathcal{P}}^{q,t}_{\varkappa,\varepsilon}(E) = \sup_{(x_i,\varepsilon_i)\in\mathbb{X}\times\mathbb{R}_+} \sum_i \Psi_\varkappa^{q,t}(x_i, \varepsilon_i)\, \mu(\mathsf{B}(x_i, \varepsilon_i)),$$

where the supremum is taken over all constituents $\pi = \{(x_i, \varepsilon_i)\}_i$ such that $\pi$ is an $\varepsilon$-packing of $E$, i.e., $\varepsilon_i \le \varepsilon$ and $\mathsf{B}(x, \varepsilon) \cap \mathsf{B}(x', \varepsilon') = \emptyset$ for all $(x, \varepsilon) \ne (x', \varepsilon') \in \pi$. We set $\overline{\mathcal{P}}^{q,t}_{\varkappa,\mu,\varepsilon}(\emptyset) = 0$ and define the MM-packing measures $\mathcal{P}^{q,t}_{\varkappa,\mu}$ by

$$\overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E) = \inf_{\varepsilon>0} \overline{\mathcal{P}}^{q,t}_{\varkappa,\mu,\varepsilon}(E), \qquad \mathcal{P}^{q,t}_{\varkappa,\mu}(E) = \inf_{E\subseteq\bigcup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E_i).$$

If $\mu(\mathsf{B}) = 1$ for all $\mathsf{B} \in \mathcal{B}(\mathbb{X})$ or if $q = 0$, then $\mathcal{P}^{q,t}_{\varkappa,\mu}$ will be denoted by $\mathcal{P}^{q,t}_{\varkappa}$ and $\mathcal{P}^{t}_{\zeta,\mu}$, respectively. The function $\mathcal{P}^{q,t}_{\varkappa,\mu}$ satisfies the conditions of a $\sigma$-subadditive outer measure [10]. We call $\mathcal{P}^{q,t}_{\varkappa,\mu}(E)$ the MM-packing measure of $E$, and $\overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E)$ the MM-pre-packing measure. It follows from the definitions of the MM-packing and MM-pre-packing measures that there exist critical values $B_{\varkappa,\mu}(E, q)$ and $\Lambda_{\varkappa,\mu}(E, q)$ such that

$$\mathcal{P}^{q,t}_{\varkappa,\mu}(E) = \begin{cases} \infty & \text{if } t < B_{\varkappa,\mu}(E, q), \\ 0 & \text{if } t > B_{\varkappa,\mu}(E, q), \end{cases}$$

and

$$\overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E) = \begin{cases} \infty & \text{if } t < \Lambda_{\varkappa,\mu}(E, q), \\ 0 & \text{if } t > \Lambda_{\varkappa,\mu}(E, q). \end{cases}$$

From the definitions of the MM-packing and MM-pre-packing measures, it also follows that for any nonempty set $E$ it is true that $\mathcal{P}^{q,t}_{\varkappa,\mu}(E) \leq \overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E)$. Since $\mathbb{X}$ is a metric space satisfying the Besicovitch covering property, there exists an integer $\Theta \in \mathbb{N}$ such that $\mathcal{H}^{q,t}_{\varkappa,\mu}(E) \leq \Theta\, \mathcal{P}^{q,t}_{\varkappa,\mu}(E)$. Moreover, if $q \leq 0$, or if $q > 0$ and $\mu$ and $\Psi$ are blanketed [40], then, by Vitali's covering lemma [41], the following inequalities hold:

$$\mathcal{H}^{q,t}_{\varkappa,\mu}(E) \leq \mathcal{P}^{q,t}_{\varkappa,\mu}(E) \leq \overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}(E).$$

For any set $E \subseteq \mathbb{X}$, it follows from the definitions of MM-Hausdorff and MM-packing measures that

$$\mathcal{H}^{q,t}_{\varkappa,\mu}(E) \geq \mathcal{H}^{p,t}_{\varkappa,\mu}(E), \quad \mathcal{P}^{q,t}_{\varkappa,\mu}(E) \geq \mathcal{P}^{p,t}_{\varkappa,\mu}(E), \qquad \text{if } q \leq p,$$

and

$$\mathcal{H}^{q,t}_{\varkappa,\mu}(E) \geq \mathcal{H}^{p,s}_{\varkappa,\mu}(E), \quad \mathcal{P}^{q,t}_{\varkappa,\mu}(E) \geq \mathcal{P}^{p,s}_{\varkappa,\mu}(E), \qquad \text{if } t \leq s.$$

Therefore, the functions $B_{\varkappa,\mu}(E, q)$ and $\Lambda_{\varkappa,\mu}(E, q)$ are nonincreasing in both $q$ and $t$, and they are convex. The proofs of the following results rely on techniques developed in [6, 16].

If $\mu(\mathsf{B}) = 1$ for all $\mathsf{B} \in \mathcal{B}(\mathbb{X})$, or if $q = 0$, then the functions $b_{\varkappa,\mu}(E, q)$ will be denoted by $b_{\varkappa}(E, q)$ and $b_{\zeta,\mu}(E)$, respectively; similarly, $B_{\varkappa,\mu}(E, q)$ will be denoted by $B_{\varkappa}(E, q)$ and $B_{\zeta,\mu}(E)$, respectively. In particular, when $\mu$ is identically 1 and $q = 0$, one has

$$b_{\varkappa,\mu}(E, q) = \dim_{\zeta}(E) \qquad \text{and} \qquad B_{\varkappa,\mu}(E, q) = \mathrm{Dim}_{\zeta}(E).$$

**Remark 3.** *In the special case where $q = 0$ and $(\mu, \zeta) \equiv (1, \log r)$, we recover the classical Hausdorff and packing measures $\mathcal{H}^t$ and $\mathcal{P}^t$, as well as the corresponding dimensions $\dim_H$ and $\dim_P$ (see [3] for precise definitions). In particular, we have*

$$\mathcal{H}^{q,t}_{\varkappa,\mu} = \mathcal{H}^t, \qquad \mathcal{P}^{q,t}_{\varkappa,\mu} = \mathcal{P}^t,$$

*and consequently,*

$$b_{\varkappa,\mu}(E, q) = \dim_H(E), \qquad B_{\varkappa,\mu}(E, q) = \dim_P(E).$$

We now define the corresponding separator functions by

$$b_{\varkappa,\mu}(q) := b_{\varkappa,\mu}(\mathbb{X}, q), \quad B_{\varkappa,\mu}(q) := B_{\varkappa,\mu}(\mathbb{X}, q), \quad \Lambda_{\varkappa,\mu}(q) := \Lambda_{\varkappa,\mu}(\mathbb{X}, q).$$

It follows directly from the definitions that $b_{\varkappa,\mu}(q) \le B_{\varkappa,\mu}(q) \le \Lambda_{\varkappa,\mu}(q)$. We also obtain, from the definitions, that for $q = 0$

$$\mathsf{b}_{\xi,\varphi}(q) = \mathsf{b}_{\xi,\varphi}(q) = \Lambda_{\xi,\varphi}(q) = 0. \tag{2.1}$$

The separator function $\Lambda_{\varkappa,\mu}(\cdot, q)$ plays a central role in the study of the multi-fractal formalism. However, obtaining an explicit expression for this function is often challenging. To address this, for $\varepsilon > 0$, $\lambda > 1$, and $E \subset \mathbb{X}$ bounded, we define

$$L^{q,t}_{\varkappa,\mu,\varepsilon}(E) := \sup_{\pi} \sum_j \Psi^{q,t}_{\varkappa}(x_j, \varepsilon_j)\, \mu(\mathsf{B}(x_j, \varepsilon_j)),$$

where the supremum is taken over all $\varepsilon$-packings $\pi = \{(x_j, \varepsilon_j)\}_j$ of $E$ satisfying $\frac{\varepsilon}{\lambda} < \varepsilon_j \le \varepsilon$ with $\lambda \ge 2$.

We then define

$$P^{q,t}_{\varkappa,\mu}(E) := \overline{\lim_{\varepsilon \to 0}}\, L^{q,t}_{\varkappa,\mu,\varepsilon}(E),$$

and the associated separator function

$$\Delta_{\varkappa,\mu}(E, q) := \inf\left\{t \in \mathbb{R} \,\middle|\, P^{q,t}_{\varkappa,\mu}(E) = 0\right\}, \quad \Delta_{\varkappa,\mu}(q) := \Delta_{\varkappa,\mu}(\mathbb{X}, q).$$

**Remark 4.** *The function $P^{q,t}_{\varkappa,\mu}$ provides an alternative way to compute the separator function $\Lambda_{\varkappa,\mu}$. Indeed, following the same lines as [42, Lemma 1], one can prove that $\Lambda_{\varkappa,\mu} = \Delta_{\varkappa,\mu}$ if $\zeta$ is normal (see [23]); that is, for all $\varepsilon > 0$, there exists $\rho > 0$ such that*

$$\sum_{j \ge 0} e^{\varepsilon \tilde{\zeta}(\rho \lambda^{-j})} < \infty,$$

*where $\tilde{\zeta}(t) := \inf_{t/\lambda \le \varepsilon < t} \zeta(\varepsilon)$.*

The notation for the MM-functions and their separators is summarized in the Table 1.

**Table 1.** Mutual multi-fractal functions and their separators.

| MM-Functions | $\mu(\mathsf{B}) = 1$ | $q = 0$ | Separator on $E$ | Separator on $E$ with $q = 0$ |
|---|---|---|---|---|
| $\mathcal{H}^{q,t}_{\varkappa,\mu}$ | $\mathcal{H}^{q,t}_{\varkappa}$ | $\mathcal{H}^{t}_{\zeta,\mu}$ | $b_{\varkappa,\mu}(E, q)$ | $b_{\zeta,\mu}(E)$ |
| $\mathcal{P}^{q,t}_{\varkappa,\mu}$ | $\mathcal{P}^{q,t}_{\varkappa}$ | $\mathcal{P}^{t}_{\zeta,\mu}$ | $B_{\varkappa,\mu}(E, q)$ | $B_{\zeta,\mu}(E)$ |
| $\overline{\mathcal{P}}^{q,t}_{\varkappa,\mu}$ | $\overline{\mathcal{P}}^{q,t}_{\varkappa}$ | $\overline{\mathcal{P}}^{t}_{\zeta,\mu}$ | $\Lambda_{\varkappa,\mu}(E, q)$ | $\Lambda_{\zeta,\mu}(E)$ |
| $P^{q,t}_{\varkappa,\mu}$ | $P^{q,t}_{\varkappa}$ | $P^{t}_{\zeta,\mu}$ | $\Delta_{\varkappa,\mu}(E, q)$ | $\Delta_{\zeta,\mu}(E)$ |

**Example 1.** *Let $\mathbb{X}$ be the symbolic space $\partial\mathcal{A} = \{0, 1\}^{\mathbb{N}}$ endowed with the usual metric*

$$d(x, y) = 2^{-\min\{n \,:\, x_n \neq y_n\}}, \quad x = (x_n)_n,\ y = (y_n)_n.$$

*We define the shift map* $\sigma : \partial\mathcal{A} \to \partial\mathcal{A}$ *by* $\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$*. For* $x \in \partial\mathcal{A}$ *and* $n \geq 1$*, let*

$$I_n(x) = \{y \in \partial\mathcal{A} : y_i = x_i \quad i = 1, \dots, n\}$$

*denote the cylinder set with base* $(x_1, \dots, x_n)$*, and let* $\mathcal{F}_n$ *denote the set of cylinders at generation* $n$*. We say that a function* $h : \partial\mathcal{A} \times \mathbb{R}_+ \to \mathbb{R}$ *is a cylinder function if, for any cylinder* $[u]$*, one has* $h(x, r) = h(y, r)$ *for all* $x, y \in [u]$*. Let* $g$ *be a dyadic Hölder continuous function with period* 1 *and* $g(0) = 1$*. In this example, we define*

$$\tau(x, \varepsilon) = -\log \prod_{s=0}^{n-1} g(\sigma^s x), \qquad \zeta(\varepsilon) = \log |I_n(x)|, \tag{2.2}$$

*where* $2^{-n} \leq \varepsilon < 2^{-n-1}$*. Observe that the function* $\tau$ *is a cylinder function. The following proposition will be used in Section 4.*

**Proposition 1.** *Let* $\tau$ *and* $\zeta$ *be defined by* (2.2)*. Then:*

1. *For all* $\mu \in \mathcal{M}(\partial\mathcal{A})$*,* $q, t \in \mathbb{R}$*, and* $n \geq 1$*, one has*

$$L^{q,t}_{\varkappa,\mu,2^{-n}}(\partial\mathcal{A}) = 2^{-tn} \sum_{I \in \mathcal{F}_n} \left( \prod_{s=1}^{n-1} g(\sigma^s x) \right)^q \mu(I), \quad x \in I. \tag{2.3}$$

2. *For all* $\mu \in \mathcal{M}(\partial\mathcal{A})$ *and* $q \in \mathbb{R}$*, one has*

$$\Lambda_{\varkappa,\mu}(q) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log_2 \left( \sum_{I \in \mathcal{F}_n} \Big( \prod_{s=1}^{n-1} g(\sigma^s x) \Big)^q \mu(I) \right).$$

*Proof.* 1. Let $\{\mathsf{B}(x_j, \varepsilon_j)\}_j$ be a packing of $\partial\mathcal{A}$ such that $2^{-n-1} \leq \varepsilon_j < 2^{-n}$. Then,

$$\begin{aligned} \sum_j \varepsilon_j^t \, e^{-q\tau(x_j,\varepsilon_j)} \, \mu(\mathsf{B}(x_j, \varepsilon_j)) &\leq 2^{-tn} \sum_{I \in \mathcal{F}_n} e^{-q\tau(x_j,\varepsilon_j)} \, \mu(I) \\ &= 2^{-tn} \sum_{I \in \mathcal{F}_n} \left( \prod_{s=1}^{n-1} g(\sigma^s x) \right)^q \mu(I), \end{aligned}$$

where $x \in I$ is chosen arbitrarily. It follows that $L^{q,t}_{\varkappa,2^{-n}}(\partial\mathcal{A}) \leq 2^{-tn} \sum_{I \in \mathcal{F}_n} \left( \prod_{s=1}^{n-1} g(\sigma^s x) \right)^q \mu(I)$. On the other hand, since $\Big\{I, \ \ I \in \mathcal{F}_n\Big\}$ is a $2^{-n}$-packing of $\partial\mathcal{A}$, we get the other side as required.

2. By Remark 4, we have $\Lambda_{\varkappa,\mu}(q) = \Delta_{\varkappa,\mu}(q)$. Define

$$f(q) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log_2 \left( \sum_{I \in \mathcal{F}_n} \Big( \prod_{s=1}^{n-1} g(\sigma^s x) \Big)^q \mu(I) \right).$$

If $t > f(q)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{1}{n} \log_2 \sum_{I \in \mathcal{F}_n} \Big( \prod_{s=1}^{n-1} g(\sigma^s x) \Big)^q \mu(I) \leq t.$$

Hence,

$$\sum_{I\in\mathcal{F}_n}\Big(\prod_{s=1}^{n-1} g(\sigma^s x)\Big)^q \mu(I)2^{-tn} \leq 1,$$

which implies $L^{q,t}_{\varkappa,\mu,2^{-n}}(\partial\mathcal{A}) \leq 1$, and thus $P^{q,t}_{\varkappa,\mu}(\partial\mathcal{A}) < \infty$, yielding $\Lambda_{\varkappa,\mu}(q) \leq f(q)$. Conversely, if $t < f(q)$, there exists a subsequence $(n_k)_k$ such that

$$\sum_{I\in\mathcal{F}_{n_k}}\Big(\prod_{s=1}^{n_k-1} g(\sigma^s x)\Big)^q \mu(I)2^{-n_k t} \geq 1,$$

which implies $L^{q,t}_{\varkappa,\mu,2^{-n_k}}(\partial\mathcal{A}) > 0$ and therefore $P^{q,t}_{\varkappa,\mu}(\partial\mathcal{A}) > 0$, giving $\Lambda_{\varkappa,\mu}(q) \geq f(q)$. The result follows.

□

## 3. Multi-fractal formalism: A new approach

In this section, we study the multi-fractal analysis of the function $\tau$ with respect to $\zeta$. This analysis aims to characterize the local variations in regularity by examining the distribution of Hölder-type singularities at small scales. Specifically, this heterogeneity can be described through the lower and upper $\zeta$-local dimensions of $\tau$ at a point $x \in \mathbb{X}$, defined, respectively, as

$$\underline{\alpha}_\varkappa(x) = \varliminf_{\varepsilon\to 0}\frac{\tau(x,\varepsilon)}{-\zeta(\varepsilon)}, \qquad \overline{\alpha}_\varkappa(x) = \varlimsup_{\varepsilon\to 0}\frac{\tau(x,\varepsilon)}{-\zeta(\varepsilon)}.$$

When $\underline{\alpha}_\varkappa(x) = \overline{\alpha}_\varkappa(x)$, the common value is denoted by $\alpha_\varkappa(x)$ and is called the $\zeta$-local dimension of $\tau$ at the point $x$. For $\alpha,\beta \geq 0$, we define the level sets

$$\underline{\mathrm{X}}_\varkappa(\alpha) = \{x \in \mathbb{X} \mid \underline{\alpha}_\varkappa(x) \geq \alpha\}, \qquad \overline{\mathrm{X}}_\varkappa(\beta) = \{x \in \mathbb{X} \mid \overline{\alpha}_\varkappa(x) \leq \beta\}.$$

Finally, we define

$$\mathrm{X}_\varkappa(\alpha,\beta) = \underline{\mathrm{X}}_\varkappa(\alpha) \cap \overline{\mathrm{X}}_\varkappa(\beta), \qquad \mathrm{X}_\varkappa(\alpha) = \mathrm{X}_\varkappa(\alpha,\alpha).$$

### *3.1. Proof of Theorem 1*

Here, we establish upper and lower bounds for the Hausdorff and packing dimensions of a given set $E \subseteq \mathbb{X}$. Recall that $\mathbb{X}$ fulfills the Besicovitch covering property and then the following proposition provides an upper bound, which will be applied later in Theorem 3 to determine the exact upper bound for the dimension of the multi-fractal set $\mathrm{X}_\varkappa(\alpha)$. Notice that $\mu$ is not required to be a measure, allowing this approach to extend classical methods in multi-fractal analysis.

**Proposition 2.** *Let $E \subseteq \mathbb{X}$, $\mu \in \mathcal{M}(\mathbb{X})$, and let $\zeta : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (1.3)*. Assume that $\Lambda_{\zeta,\mu}(E) \leq 0$. Then,*

$$\dim_H E \leq \sup_{x\in E}\underline{\alpha}_{\mu,\zeta}(x), \qquad \textit{and} \qquad \dim_P E \leq \sup_{x\in E}\overline{\alpha}_{\mu,\zeta}(x).$$

*Proof.* Let $\delta > \sup_{x\in E} \underline{\alpha}_{\mu,\zeta}(x)$ and $\gamma > 0$ be arbitrary. Since $B_{\zeta,\mu}(E) \le \Lambda_{\zeta,\mu}(E) \le 0$, it follows that $\mathcal{P}^{\gamma/2}_{\zeta,\mu}(E) = 0$. Consequently, there exists a sequence of subsets $(E_j)_j$ such that $E = \bigcup_j E_j$, and for each $j$,

$$\sum_j \overline{\mathcal{P}}^{\gamma/2}_{\zeta,\mu}(E_j) < 1, \quad \text{and} \quad \sum_j \overline{\mathcal{P}}^{\gamma}_{\zeta,\mu}(E_j) = 0.$$

Let $\eta$ be a positive number and $A$ be a subset of $E_j$. For $x \in A$, there exists $0 < \varepsilon \le \eta$ such that

$$\mu(\mathsf{B}(x,\varepsilon)) \ge e^{\delta\zeta(\varepsilon)}.$$

By the Besicovitch covering theorem (Theorem 7), one can select a collection of balls $\{\mathsf{B}(x_i,\varepsilon_i)\}_i$ that is a centered $\eta$-cover of $A$, which can be decomposed into $\Theta$ disjoint packings satisfying $\mu(\mathsf{B}(x_i,\varepsilon_i)) \ge e^{\delta\zeta(\varepsilon_i)}$. Therefore, from (1.3), we obtain

$$\overline{\mathcal{H}}^{\delta+\eta}(A) \le C \sum_i \mu(\mathsf{B}(x_i,\varepsilon_i)) e^{\eta\zeta(\varepsilon_i)} \le C\,\Theta\, \overline{\mathcal{P}}^{\eta}_{\zeta,\mu}(E_j),$$

which implies $\overline{\mathcal{H}}^{\delta+\eta}(A) = 0$, and, thus, $\mathcal{H}^{\delta+\eta}(E_j) = 0$. Consequently, $\mathcal{H}^{\delta+\eta}(E) = 0$, yielding $\dim_H E \le \delta + \eta$. Since $\eta > 0$ is arbitrary, we deduce $\dim_H E \le \delta$. Taking the infimum over all $\delta > \sup_{x\in E} \underline{\alpha}_{\mu,\zeta}(x)$ gives the desired upper bound.

Now, we will prove the second assertion. Let $\delta > \sup_{x\in E} \overline{\alpha}_{\mu,\zeta}(x)$. Since $B_{\zeta,\mu}(E) \le \Lambda_{\zeta,\mu}(E) \le 0$, there exists a partition $(E_j)_j$ of $E$ such that

$$\sum_j \overline{\mathcal{P}}^{\gamma}_{\zeta,\mu}(E_j) = 0.$$

For each $x \in E$, there exists $\eta > 0$ such that for all $0 < \varepsilon \le \eta$, $\mu(\mathsf{B}(x,\varepsilon)) \ge e^{\delta\zeta(\varepsilon)}$. Define, for each $m \ge 1$, the set

$$E(m) = \left\{ x \in E \;\middle|\; \forall\, 0 < \varepsilon \le 1/m,\ \ \mu(\mathsf{B}(x,\varepsilon)) \ge e^{(\delta-\eta)\zeta(\varepsilon)} \right\}.$$

Let $\{\mathsf{B}(x_i,\varepsilon_i)\}_i$ be a centered $\eta$-packing of $E_j \cap E(m)$. Then, from (1.3), we obtain

$$\sum_i \varepsilon_i^{\delta+\eta} \le C \sum_i \mu(\mathsf{B}(x_i,\varepsilon_i))\, e^{\eta\zeta(\varepsilon_i)} \le C\, \overline{\mathcal{P}}^{\gamma}_{\zeta,\mu}(E_j).$$

It follows that

$$\mathcal{P}^{\delta+\eta}(E_j \cap E(m)) = 0, \quad \text{and hence} \quad \mathcal{P}^{\delta+\eta}(E(m)) = 0.$$

Since $E = \bigcup_{m\ge1} E(m)$, we conclude that $\dim_p E \le \delta$. Finally, taking the infimum over all $\delta > \sup\limits_{x\in E} \overline{\alpha}_{\mu,\zeta}(x)$ yields the desired upper bound for the packing dimension. □

In the following proposition, we establish a lower bound for the dimension of a given set $E \subseteq \mathbb{X}$. This result will be applied in Theorem 3 to determine the exact lower bound for the dimension of the set $\mathrm{X}_\varkappa(\alpha)$. Let $\mu \in \mathcal{M}(\mathbb{X})$, and consider the outer measure $\vartheta$ on $\mathbb{X}$ associated with the dimension function $\mu$, defined as follows:

$$\overline{\vartheta}_\varepsilon(E) := \overline{\mathcal{H}}^{0}_{\zeta,\mu,\varepsilon}(E), \qquad \overline{\vartheta}(E) := \overline{\mathcal{H}}^{0}_{\zeta,\mu}(E), \qquad \text{and} \qquad \vartheta(E) := \mathcal{H}^{0}_{\zeta,\mu}(E). \tag{3.1}$$

In particular, when the metric $d$ is ultrametric, the outer measures $\overline{\vartheta}$ and $\vartheta$ coincide, that is,

$$\overline{\vartheta}(E) = \vartheta(E).$$

**Proposition 3.** *Let $E \subseteq \mathbb{X}$, $\mu \in \mathcal{M}(\mathbb{X})$, and let $\zeta : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (1.3)*. Suppose that $\vartheta(E) > 0$. Then,*

$$\dim_H E \geq \operatorname*{ess\,sup}_{x\in E,\vartheta} \underline{\alpha}_{\mu,\zeta}(x), \qquad \dim_P E \geq \operatorname*{ess\,sup}_{x\in E,\vartheta} \overline{\alpha}_{\mu,\zeta}(x),$$

*where* $\operatorname*{ess\,sup}_{x\in E,\vartheta} \varphi(x) = \inf\{t \in \mathbb{R};\ \ \vartheta(E \cap \{\varphi > t\}) = 0\}$.

*Proof.* Let $\delta < \operatorname*{ess\,sup}_{x\in E,\vartheta} \underline{\alpha}_{\mu,\zeta}(x)$. Define the set

$$A := \Big\{x \in E \;\Big|\; \underline{\alpha}_{\mu,\zeta}(x) > \delta\Big\}.$$

Then $\vartheta(A) > 0$. For each $x \in A$, there exists $\eta > 0$ such that for all $0 < \varepsilon \leq \eta$, $\mu(\mathsf{B}(x,\varepsilon)) \leq e^{\delta\zeta(\varepsilon)}$. We write $A = \bigcup_{m\geq 1} A(m)$, where

$$A(m) := \Big\{x \in A \;\Big|\; \forall 0 < \varepsilon \leq 1/m,\ \mu(\mathsf{B}(x,\varepsilon)) \leq e^{\delta\zeta(\varepsilon)}\Big\}.$$

Since $\vartheta(A) > 0$, there exists $m \in \mathbb{N}$ such that $\vartheta(A(m)) > 0$. Therefore, there exists a subset $F \subseteq A(m)$ with $\overline{\vartheta}(F) > 0$. Consider a centered $\eta$-covering $(\mathsf{B}(x_i,\varepsilon_i))_i$ of $F$ with $0 < \eta \leq 1/m$. Then

$$\overline{\vartheta}_\varepsilon(F) \leq \sum_i \mu(\mathsf{B}(x_i,\varepsilon_i)) \leq \sum_i e^{\delta\zeta(\varepsilon_i)}.$$

It follows that

$$0 < \overline{\vartheta}_\varepsilon(F) \leq \overline{\mathcal{H}}^{\delta}_{\zeta,\varepsilon}(F) \leq \overline{\mathcal{H}}^{\delta}_{\zeta,\varepsilon}(E),$$

and hence $\dim_H(E) \geq \delta$.

For the packing dimension, let $\delta < \operatorname*{ess\,sup}_{x\in E,\vartheta} \overline{\alpha}_{\mu,\zeta}(x)$, and define

$$F = \Big\{x \in E \;\Big|\; \overline{\alpha}_{\mu,\zeta}(x) > \delta\Big\}.$$

Then $\vartheta(F) > 0$, so there exists a subset $F' \subseteq F$ with $\vartheta(F') > 0$. For subset $E \subseteq F'$, for each $x \in E$ and each $\eta > 0$, there exists $0 < \varepsilon \leq \eta$ such that

$$\mu(\mathsf{B}(x,\varepsilon)) \leq e^{\delta\,\zeta(\varepsilon)}.$$

Using the Besicovitch covering property, there exists a finite collection of $\eta$-packings $(\mathsf{B}(x_{ij},\varepsilon_{ij}))_i$, $0 \leq j \leq \Theta$, which together cover $E$ and satisfy $\mu(\mathsf{B}(x_{ij},\varepsilon_{ij})) \leq e^{\delta\zeta(\varepsilon_{ij})}$. Hence,

$$\overline{\vartheta}_\varepsilon(E) \leq \sum_{i,j} \mu(\mathsf{B}(x_{ij},\varepsilon_{ij})) \leq \Theta\,\overline{\mathcal{P}}^{\delta}_{\zeta,\varepsilon}(E),$$

and taking limits as $\varepsilon \to 0$ gives $\overline{\vartheta}(E) \leq \Theta\,\overline{\mathcal{P}}^{\delta}_{\zeta}(E)$. If $F' = \bigcup_i E_i$, then

$$0 < \overline{\vartheta}(F') \leq \sum_i \overline{\mathcal{P}}^{\delta}_{\zeta}(E_i),$$

which implies that $\mathcal{P}^{\delta}_{\zeta}(E) \geq \mathcal{P}^{\delta}_{\zeta}(F') > 0$, and, therefore, $\dim_P(E) \geq \delta$. □

**Remark 5.** *If $\mu \in \mathcal{M}(\mathbb{X})$ is a Borel measure, the key observation is that in our proofs of Propositions 2 and 3 we do not require uniform control of $\mu$ across different scales. Instead, we rely only on the local estimates of $\mu(B(x,\varepsilon))$ at each point $x \in E$, the definitions of the local scaling exponents $\underline{\alpha}_{\mu,\zeta}(x)$ and $\overline{\alpha}_{\mu,\zeta}(x)$, and the Besicovitch covering theorem to construct suitable packings or coverings. Consequently, our multi-fractal formalism holds for $\tau = -\log\nu$ with any measure $\nu$, even if it is not doubling.*

*3.2. The multi-fractal formalism is valid: Proof of Theorem 2*

The purpose of multi-fractal analysis of functions is to study the size of $\mathrm{X}_\varkappa(\alpha)$. This is done by computing the multi-fractal Hausdorff spectrum $f_\varkappa : \alpha \longmapsto \dim_H \mathrm{X}_\varkappa(\alpha)$ and the multi-fractal packing spectrum $F_\varkappa : \alpha \longmapsto \dim_P \mathrm{X}_\varkappa(\alpha)$. In this paragraph, we will prove Theorem 2, which establishes the validity of the extended joint multi-fractal formalism. We define

$$\underline{\alpha} = \sup_{q>0} -\frac{b_\varkappa(q)}{q} \qquad \text{and} \qquad \overline{\alpha} = \inf_{q<0} -\frac{b_\varkappa(q)}{q}.$$

**Theorem 3.** *For any $\alpha \in (\underline{\alpha}, \overline{\alpha})$, we have*

$$\operatorname{Dim} \mathrm{X}_\varkappa(\alpha) \leq B^*_\varkappa(\alpha) \quad \textit{and} \quad \dim_H \mathrm{X}_\varkappa(\alpha) \leq b^*_\varkappa(\alpha).$$

*For $\alpha \notin (\underline{\alpha}, \overline{\alpha})$, the spectra are trivial since $\mathrm{X}_\varkappa(\alpha)$ is empty.*

*Proof.* We prove only the first inequality. Let $q \geq 0$ and define the function

$$\mu(\mathsf{B}(x,\varepsilon)) := \Psi_\varkappa^{q,B_\varkappa(q)}(x,\varepsilon) = \exp\Big(-q\,\tau(x,\varepsilon) + B_\varkappa(q)\,\zeta(\varepsilon)\Big).$$

By a simple calculation, we get $B_{\varkappa,\mu}(t) = B_\varkappa(q+t) - B_\varkappa(q)$. For any $x \in \overline{\mathrm{X}}_\varkappa(\alpha)$, we have

$$\overline{\alpha}_{\mu,\zeta}(x) = \overline{\lim_{\varepsilon\to 0}} -q\frac{\tau(x,\varepsilon)}{\zeta(\varepsilon)} + B_\varkappa(q) \leq q\alpha + B_\varkappa(q).$$

Applying Proposition 2, it follows that

$$\dim_P \mathrm{X}_\varkappa(\alpha) \leq \dim_P \overline{\mathrm{X}}_\varkappa(\alpha) \leq q\alpha + B_\varkappa(q),$$

which proves the desired inequality. □

To establish the lower bound in multi-fractal analysis, we begin by proving a key lemma that will guide the process. This lemma provides fundamental insights into the support of the measure $\vartheta$.

**Lemma 1.** *Let $\varphi(t) := \Lambda_{\varkappa,\mu}(E,t)$, and assume that $\varphi(0) = 0$ and $\vartheta(\mathsf{S}_\mu) > 0$. Then, we have*

$$\vartheta\Big(\big(\mathrm{X}_\varkappa(-\varphi'_l(0), -\varphi'_r(0))\big)^c\Big) = 0,$$

*where $\varphi'_l$ and $\varphi'_r$ denote the left and right derivatives of $\varphi$, respectively.*

*Proof.* We begin by proving that

$$\vartheta\Big(\{x \in \mathbb{X} \mid \overline{\alpha}_\varkappa(x) > -\varphi'_l(0)\}\Big) = 0.$$

Given $\delta > -\varphi'_l(0)$, there exist $\delta'$ and $t$ such that $\delta > \delta' > -\varphi'_l(0)$ and $\varphi(-t) < \delta' t$. Clearly, we have $\mathcal{P}_{\varkappa,\mu}^{-t,\delta' t}(\mathbb{X}) = 0$. We can choose a countable partition $\mathbb{X} = \bigcup_j E_j$ such that

$$\sum_j \overline{\mathcal{P}}_{\varkappa,\mu}^{-t,\delta' t}(E_j) \leq 1 \qquad \text{and} \qquad \overline{\mathcal{P}}_{\varkappa,\mu}^{-t,\delta t}(E_j) = 0.$$

Define the set

$$K_\delta = \Big\{x \in \mathbb{X} \;\Big|\; \overline{\lim_{\varepsilon\to 0}} \frac{\tau(x,\varepsilon)}{-\zeta(\varepsilon)} > \delta\Big\}.$$

If $x \in K_\delta$, then for all $\eta > 0$, there exists $\varepsilon \le \eta$ such that $e^{-\tau(x,\varepsilon)} \le e^{\delta\zeta(\varepsilon)}$. Let $F$ be a subset of $K_\delta$ and set $F_j = F \cap E_j$. For $\eta > 0$ and for each $j$, one can find a Besicovitch $\eta$-cover $\{\mathsf{B}(x_{jk}, \varepsilon_{jk})\}_k$ of $F_j$ such that

$$e^{-\tau(x_{jk},\varepsilon_{jk})} \le e^{\delta\zeta(\varepsilon_{jk})}.$$

It follows that

$$\begin{aligned} \overline{\vartheta}_\eta(F_j) &\le \sum_k \mu(\mathsf{B}(x_{jk}, \varepsilon_{jk})) \\ &= \sum_k e^{-t\tau(x_{jk},\varepsilon_{jk})}\, e^{t\tau(x_{jk},\varepsilon_{jk})}\, \mu(\mathsf{B}(x_{jk}, \varepsilon_{jk})) \\ &\le \sum_k \Psi_\varkappa^{t,\delta}(x_{jk}, \varepsilon_{jk})\, \mu(\mathsf{B}(x_{jk}, \varepsilon_{jk})), \end{aligned}$$

which, together with the Besicovitch property, implies

$$\overline{\vartheta}_\eta(F_j) \le \overline{\mathcal{P}}^{t,\delta t}_{\varkappa,\mu,\eta}(E_j).$$

Hence, $\overline{\vartheta}(F_j) \le \overline{\mathcal{P}}^{t,\delta t}_{\varkappa,\mu}(E_j)$. Consequently, $\vartheta(F) = 0$, and therefore $\vartheta(K_\delta) = 0$. By a similar argument, one proves that

$$\vartheta\Big(\{x \in \mathbb{X} \;\Big|\; \underline{\alpha}_\varkappa(x) < \varphi_l'(0)\}\Big) = 0.$$

□

**Proposition 4.** *Let $\varphi(t) = \Lambda_{\varkappa,\mu}(E, t)$ and assume that $\varphi(0) = 0$ and $\vartheta(\mathcal{S}_\mu) > 0$. Then, for all $x \in$ $\mathrm{X}_\varkappa(\varphi_l'(0), \varphi_r'(0))$, we have*

$$\dim_H \mathrm{X}_\varkappa(\varphi_l'(0), \varphi_r'(0)) \;\ge\; \inf_x \underline{\alpha}_{\mu,\zeta}(x),$$

*and*

$$\dim_P \mathrm{X}_\varkappa(\varphi_l'(0), \varphi_r'(0)) \;\ge\; \inf_x \overline{\alpha}_{\mu,\zeta}(x).$$

*Proof.* The result follows directly from Proposition 3 combined with Lemma 1. □

**Theorem 4.** *Let $q \in \mathbb{R}$ and let $\zeta$ satisfy condition* (1.3). *Assume that $\mathcal{H}_\varkappa^{q,\Lambda_\varkappa(q)}(\mathbb{X}) > 0$. Then:*

*i) If $q \in \mathbb{R}_+$, then*

$$\dim_H \mathrm{X}_\varkappa(\Lambda'_{\varkappa,l}(q), \Lambda'_{\varkappa,r}(q)) \;\ge\; -q\,\Lambda'_{\varkappa,l}(q) + \Lambda_{\varkappa,l}(q).$$

*ii) If $q \in \mathbb{R}_-$, then*

$$\dim_H \mathrm{X}_\varkappa(\Lambda'_{\varkappa,l}(q), \Lambda'_{\varkappa,r}(q)) \;\ge\; -q\,\Lambda'_{\varkappa,r}(q) + \Lambda_{\varkappa,r}(q).$$

*iii) If $\Lambda_\varkappa$ is differentiable at q, then*

$$\dim_H \mathrm{X}_\varkappa(\Lambda'_\varkappa(q)) = \dim_p \mathrm{X}_\varkappa(\Lambda'_\varkappa(q)) = b^*_\varkappa(\alpha) = B^*_\varkappa(\alpha) = \Lambda^*_\varkappa(\alpha).$$

*Proof.* The result follows directly from Proposition 4. Indeed, for $q \geq 0$, one may consider the measure

$$\mu(\mathsf{B}(x,\varepsilon)) = \Psi_\varkappa^{q,\Lambda_\varkappa(q)}(x,\varepsilon) = \exp\Big(-q\,\tau(x,\varepsilon) + \Lambda_\varkappa(q)\,\zeta(\varepsilon)\Big),$$

which satisfies the assumptions of Proposition 4. The same reasoning applies for $q < 0$, leading to the stated inequalities. Finally, when $\Lambda_\varkappa$ is differentiable at $q$, the multi-fractal formalism holds in full strength, which establishes the equalities in (iii). □

### *3.3. The multi-fractal formalism is not valid*

In general, we have $\dim_H \mathrm{X}_\varkappa(\alpha) \neq \dim_P \mathrm{X}_\varkappa(\alpha)$. We now address the situation in which the Hausdorff and packing multi-fractal functions, $b_\varkappa(q)$ and $B_\varkappa(q)$, do not necessarily coincide. To this end, we introduce the following sets, defined for all $\alpha,\beta \geq 0$:

$$\widetilde{\mathrm{X}}_\varkappa(\alpha,\beta) = \Big\{x \in \mathbb{X} \;\Big|\; \underline{\alpha}_\varkappa(x) \leq \alpha \;\text{ and }\; \beta \leq \overline{\alpha}_\varkappa(x)\Big\}, \qquad \widetilde{\mathrm{X}}_\varkappa(\alpha) = \mathrm{X}_\varkappa(\alpha,\alpha).$$

**Theorem 5.** *Let $q \in \mathbb{R}$ and let $\zeta : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (1.3). *Assume that $\mathcal{H}_\varkappa^{q,b_\varkappa(q)}(\mathbb{X}) > 0$, and that the function $b_\varkappa$ is differentiable at q. Then, for $\alpha = -b'_\varkappa(q)$, one has*

$$\dim_H \widetilde{\mathrm{X}}_\varkappa(\alpha) = b_\varkappa(q) - q\, b'_\varkappa(q).$$

For $q, t \in \mathbb{R}$ and a function $\chi : \mathbb{R} \to \mathbb{R}$, we define the left and right derivatives of $\chi$ in the following sense:

$$\chi^l(q) = \overline{\lim_{t\to 0^-}} \frac{\chi(q+t) - \chi(q)}{t} \qquad \text{and} \qquad \chi^r(q) = \overline{\lim_{t\to 0^+}} \frac{\chi(q+t) - \chi(q)}{t}.$$

**Proposition 5.** *Let $\chi(t) = b_{\varkappa,\mu}(E,t)$ and assume that $\chi(0) = 0$ and $\vartheta(\mathbb{X}) > 0$. Then, one has*

$$\vartheta\Big((\mathrm{X}_\varkappa(-\chi^l(0), -\chi^r(0)))^c\Big) = 0,$$

*where $\vartheta$ denotes the outer measure defined in* (3.1).

*Proof.* Let $\delta > -\chi^l(0) = \underline{\lim}_{t\to 0} \dfrac{\chi(-t)}{t}$, and choose $t$ such that $\delta t > -\chi(-t)$. It follows that $\mathcal{H}_{\varkappa,\mu}^{-t,\delta t}(\mathbb{X}) = 0$. Define the set

$$K_\delta = \left\{x \in \mathbb{X} \;\middle|\; \underline{\lim}_{\varepsilon\to 0} \frac{\tau(x,\varepsilon)}{-\zeta(\varepsilon)} > \delta\right\}.$$

If $x \in K_\delta$, then for every $\eta > 0$, there exists $0 < \varepsilon \leq \eta$ such that $e^{-\tau(x,\varepsilon)} \leq e^{\delta\zeta(\varepsilon)}$. Now, for $m \in \mathbb{N}$, set

$$K(m) = \Big\{x \in \mathbb{X} \;\Big|\; \forall\, \varepsilon \leq 1/m,\;\; \tau(x,\varepsilon) \leq \delta\,\zeta(\varepsilon)\Big\}.$$

Let $E \subseteq K(m)$ and consider a centered cover $\{\mathsf{B}(x_i,\varepsilon_i)\}_i$ of $E$, with $\delta < 1/n$. For sufficiently small $\varepsilon_i$, we have

$$\overline{\vartheta}^0_\delta(E) \;\leq\; \sum_i \mu(\mathsf{B}(x_i,\varepsilon_i))$$

$$\leq \sum_i e^{-t\tau(x_i,\varepsilon_i)} e^{t\tau(x_i,\varepsilon_i)} \mu(\mathsf{B}(x_i,\varepsilon_i))$$

$$\leq \sum_i e^{-t\tau(x_i,\varepsilon_i)} e^{t\delta\zeta(\varepsilon_i)} \mu(\mathsf{B}(x_i,\varepsilon_i)).$$

Hence, $\overline{\vartheta}(K(m)) \leq \overline{\mathcal{H}}^{t,t\delta}_{\varkappa,\mu}(\mathbb{X}) = 0$. Consequently, $\vartheta(K(m)) = 0$, and since $E \subseteq K(m)$ was arbitrary, we also have $\vartheta(E) = 0$. □

## 4. Application

In this section, we investigate the multi-fractal analysis of Birkhoff averages in a specific setting involving two functions, $\tau$ and $\zeta$. This framework allows us to establish the validity of the relative multi-fractal formalism stated in Theorem 2. More precisely, let $\mathbb{X} = \{0,1\}^{\mathbb{N}}$ denote the space of infinite binary sequences, and let $g$ be a positive, dyadic Hölder continuous function with period 1, normalized such that $g(0) = 1$. We focus on the following particular case:

$$\tau(x,\varepsilon) = -\log \prod_{s=0}^{n-1} g(\sigma^s x), \qquad \zeta(\varepsilon) = \log(|I_n(x)|),$$

where $2^{-n} \leq \varepsilon < 2^{-n-1}$, and $I_n(x)$ denotes the cylinder set with base $(x_1,\ldots,x_n)$. We consider the level sets

$$\mathrm{X}_\varkappa(\alpha) := \left\{ x \in [0,1) \;\middle|\; \lim_{n\to\infty} \frac{1}{-n} \sum_{s=0}^{n-1} \log_2 g(\sigma^s x) = \alpha \right\},$$

for each $\alpha \in \mathbb{R}$. From [38, Proposition 2], the topological pressure associated with $\log g$ is defined by

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \left( \prod_{s=0}^{n-1} g(\sigma^s x) \right)^q dx \; + \log 2.$$

It is well-known that this limit exists, and that the function $P(q)$ is differentiable and convex (see [38]). For almost all $x \in [0,1)$, the sequence $\sigma^s x(\mathrm{mod}1)$ is uniformly distributed on $[0,1)$ (see [43]). Moreover, if $g(x) > 0$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \log g(\sigma^s x) = \int_0^1 \log g(t)\, dt \qquad \text{for almost all } x \in [0,1). \tag{4.1}$$

It immediately follows that $P(q) = q \lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \log g(\sigma^s x) + \log 2$, for almost all $x \in [0,1)$. Furthermore, there exists a unique Borel probability measure $\mu_q$ such that, for some constant $C > 0$ and for all $x \in \partial\mathcal{A}$ and $n \geq 1$,

$$C^{-1} \exp\Big(nP(q) + q \sum_{s=0}^{n-1} \log g(\sigma^s x)\Big) \leq \mu_q(I_n(x)) \leq C \exp\Big(nP(q) + q \sum_{s=0}^{n-1} \log g(\sigma^s x)\Big), \tag{4.2}$$

where $I_n(x)$ denotes the cylinder set of length $n$. We call $\mu_q$ the *Gibbs measure* associated with $g$. In this setting, for $t, q \in \mathbb{R}$, it follows from Proposition 1 that

$$\begin{aligned}
\Lambda_{\varkappa,\mu}(t) &= \lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{I\in\mathcal{F}_n} \left( \prod_{s=1}^{n-1} g(\sigma^s x) \right)^t \mu(I) \\
&= \lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{I\in\mathcal{F}_n} \left( \prod_{s=1}^{n-1} g(\sigma^s x) \right)^t \exp\left( -nP(q) + \sum_{s=1}^{n-1} \log g(\sigma^s x)^q \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{I} \left( e^{-nP(q)} \Big( \prod_{s=1}^{n-1} g(\sigma^s x) \Big)^{t+q} \right) \\
&= \frac{-P(q)}{\log 2} + \lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{I} \left( \prod_{s=1}^{n-1} g(\sigma^s x)^{t+q} \right) \\
&= \frac{-P(q)}{\log 2} + \frac{P(q+t)}{\log 2}.
\end{aligned}$$

From the definition of pressure $P(q)$ and Proposition 1, we thus conclude $\Lambda_{\varkappa,\mu}(t) = -\Lambda_{\varkappa}(q) + \Lambda_{\varkappa}(q+t)$. It is clear that $\Lambda_{\varkappa,\mu}(0) = 0$, and $\Lambda_{\varkappa,\mu}$ is differentiable with $\Lambda'_{\varkappa,\mu}(0) = \Lambda'_{\varkappa}(q) = \frac{P'(q)}{\log 2}$.

**Theorem 6.** *Let $g$ be a positive Hölder continuous function with period* 1*. Then, for every $\alpha = P'(q)$, one has*

$$\dim_H \mathrm{X}_{\varkappa}(\alpha) = \dim_P \mathrm{X}_{\varkappa}(\alpha) = \frac{P(q) - q\alpha}{\log 2}.$$

*Proof.* Let $\alpha = P'(q)$ and consider the Gibbs measure $\mu_q$ defined in (4.2). By the Gibbs property, for $\mu_q$-almost every $x \in [0, 1]$, we have

$$\lim_{n\to\infty} \frac{\log \mu_q(I_n(x))}{\log |I_n(x)|} = -\frac{1}{\log 2} \left( \lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} q \log g(\sigma^s x) - P(q) \right).$$

Since $\mu_q$ is ergodic (see [30]), the Birkhoff ergodic theorem implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \log g(\sigma^s x) = \int_0^1 \log g(x)\, d\mu_q(x), \qquad \text{for } \mu_q\text{-a.e. } x.$$

By the definition of pressure $P(q)$ and properties of the convex conjugate, $P(q) = P^*(\alpha) + q\alpha$, we deduce $\alpha = \int_0^1 \log g(x)\, d\mu_q(x)$. Hence, $\mu_q(\mathrm{X}_{\varkappa}(\alpha)) = 1$, and for all $x \in \mathrm{X}_{\varkappa}(\alpha)$, we have

$$\lim_{n\to\infty} \frac{\log \mu_q(I_n(x))}{\log |I_n(x)|} = \frac{P(q) - q\alpha}{\log 2}.$$

It follows from (2.1) that $\Lambda_{\zeta,\mu}(\mathrm{X}_{\varkappa}(\alpha)) \le \Lambda_{\zeta,\mu}(\mathbb{X}) \le 0$, and by Proposition 2, we obtain

$$\dim_P \mathrm{X}_{\varkappa}(\alpha) \le \frac{P(q) - q\alpha}{\log 2}.$$

Finally, since $\mu_q(X_\varkappa(\alpha)) > 0$, we have $\vartheta(X_\varkappa(\alpha)) > 0$, and Proposition 3 yields

$$\dim_H X_\varkappa(\alpha) \geq \frac{P(q) - q\alpha}{\log 2}.$$

Combining the upper and lower bounds, we conclude

$$\dim_H X_\varkappa(\alpha) = \dim_P X_\varkappa(\alpha) = \frac{P(q) - q\alpha}{\log 2}.$$

□

**Remark 6.** *The measure $\mu_q$ is a Gibbs measure, as it satisfies inequality* (4.2)*. Consequently, the generalized Hausdorff measure is positive, that is,*

$$\mathcal{H}_\varkappa^{q,\Lambda_\varkappa(q)}(\mathbb{X}) > 0.$$

## 5. Conclusions

In this paper, we introduced a new approach to the multi-fractal analysis of functions in a metric space by defining generalized Hausdorff and packing measures based on functions $\tau$ and $\zeta$. This framework extends classical results and provides a unified setting for studying the local regularity of functions and measures. As an application, we examined Birkhoff averages and established the validity of the multi-fractal formalism in this context.

## Appendix

*Carathéodory's construction of fractal measures*

Let $\mathcal{F}$ be a family of subsets of the metric space $(\mathbb{X}, d)$ such that $\emptyset \in \mathcal{F}$. We define two functions $\eta, \psi : \mathcal{F} \longrightarrow \mathbb{R}_+$ satisfying the following properties:

**H1.** $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\eta(U), \psi(U) > 0$ for all nonempty $U \in \mathcal{F}$.
**H2.** For any $\delta > 0$, there exists $\gamma > 0$ such that if $\eta(U) \leq \delta$ for all $U \in \mathcal{F}$, then $\psi(U) \leq \gamma$.
**H3.** For any $\gamma > 0$, there exists a finite or countable subfamily $\mathcal{G} \subseteq \mathcal{F}$ covering $\mathbb{X}$ such that

$$\psi(\mathcal{G}) = \sup\{\psi(U) : U \in \mathcal{G}\}.$$

Let $\xi : \mathcal{F} \to \mathbb{R}_+$ be a function. Then the quadruple $\mathcal{A} = (\mathcal{F}, \xi, \eta, \psi)$ defines a Carathéodory structure on $\mathbb{X}$ (or C-structure) [16]. The outer measure associated with this structure is called Carathéodory's outer measure. Under suitable conditions on these functions, one can introduce a free-energy analogue for fractal measures by defining

$$\mathcal{H}(Z, \alpha) = \inf\left\{ \sum_{U \in \mathcal{G}} \xi(U)\, \eta(U)^\alpha \right\},$$

where $\mathcal{G}$ is any finite or countable covering of $Z \subseteq \mathbb{X}$ satisfying the conditions of the Carathéodory structure. This "measure" induces in the usual way a notion of dimension, known as the capacity of the set $Z$, which is a classical version of multi-fractal dimensions.

**Theorem 7.** *(Besicovitch Covering Theorem) [41] There exists an integer* $\xi \in \mathbb{N}$ *such that, for any subset* $A \subset \mathbb{R}^n$ *and any sequence* $(r_x)_{x\in A}$ *satisfying*

1. $r_x > 0$ *for all* $x \in A$,
2. $\sup_{x\in A} r_x < \infty$,

*there exist* $\gamma$ *finite or countable families* $B_1, \ldots, B_\gamma$ *of the collection* $\{B_x(r_x) : x \in A\}$ *such that*

1. $A \subset \bigcup_{i=1}^{\gamma} \bigcup_{B\in B_i} B$,
2. *each* $B_i$ *is a family of disjoint sets.*

## Author contributions

All authors contributed equally to the preparation of this manuscript. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

## References

1. B. Mandelbrot, *Les objets fractales: Forme, hasard et dimension*, Flammarion, 1975.
2. B. Mandelbrot, J. A. Wheeler, The fractal geometry of nature, *Am. J. Phys.*, **51** (1983), 286–287. https://doi.org/10.1119/1.13295
3. S. Jurina, N. MacGregor, A. Mitchell, L. Olsen, A. Stylianou, On the Hausdorff and packing measures of typical compact metric spaces, *Aequationes Math.*, **92** (2018), 709–735. https://doi.org/10.1007/s00010-018-0548-5

4. J. Peyrière, *Multifractal measures*, In: Proceedings of the NATO ASI on Probabilistic and Stochastic Methods in Analysis with Applications, Il Ciocco, NATO ASI Series, Series C, Math. Phys. Sci., **372** (1992), 175–186. https://doi.org/10.1007/978-94-011-2791-2-7

5. Z. Li, B. Selmi, H. Zyoudi, A comprehensive approach to multifractal analysis, *Expositiones Math.*, **43** (2025), 125690, https://doi.org/10.1016/j.exmath.2025.125690

6. L. Olsen, A multifractal formalism, *Adv. Math.*, **116** (1995), 82–196. https://doi.org/10.1006/aima.1995.1066

7. R. Guedri, N. Attia, A note on the generalized Hausdorff and packing measures of product sets in metric space, *Math. Inequal. Appl.*, **25** (2022), 335–358. https://dx.doi.org/10.7153/mia-2022-25-20

8. H. Haase, The dimension of analytic sets, *Acta Univ. Carolin. Math. Phys.*, **29** (1988), 15–18.

9. A. Mahjoub, N. Attia, A relative vectorial multifractal formalism, *Chaos Soliton. Fract.*, **160** (2022), 112221. https://doi.org/10.1016/j.chaos.2022.112221

10. K. J. Falconer, The multifractal spectrum of statistically self-similar measures, *J. Theor. Probab.*, **7** (1994), 681–702. https://doi.org/10.1007/BF02213576

11. D. J. Feng, K. S. Lau, Multifractal formalism for self-similar measures with weak separation condition, *J. Math. Pure. Appl.*, **92** (2009), 407–428. https://doi.org/10.1016/j.matpur.2009.05.009

12. L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$, *Pac. J. Math.*, **183** (1998), 143–199. https://doi.org/10.2140/pjm.1998.183.143

13. M. Wu, The multifractal spectrum of some Moran measures, *Sci. China Ser. A*, **48** (2005), 97–112. https://doi.org/10.1360/022004-10

14. N. Attia, On the multifractal analysis of covering number on the Galton-Watson tree, *J. Appl. Probab.*, **56** (2019), 265–281. https://doi.org/10.1017/jpr.2019.17

15. N. Attia, On the multifractal analysis of the branching random walk in $\mathbb{R}^d$, *J. Theor. Probab.*, **27** (2014), 1329–1349. https://doi.org/10.1007/s10959-013-0488-x

16. Y. Pesin, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, 1997. https://doi.org/10.1017/S0143385798128298

17. B. Selmi, S. Shen, Z. Yuan, Multifractal analysis of inhomogeneous multinomial measures with non-doubling projections, *Fractals*, **33** (2025), 2550027. https://doi.org/10.1142/S0218348X25500276

18. L. Guo, B. Selmi, Z. Li, H. Zyoudi, Probabilistic spaces and generalized dimensions: A multifractal approach, *Chaos Soliton. Fract.*, **192** (2025), 115953. https://doi.org/10.1016 j.chaos.2024.115953

19. B. Selmi, General multifractal dimensions of measures, *Fuzzy Set. Syst.*, **499** (2025), 109177. https://doi.org/10.1016/j.fss.2024.109177 .

20. M. Menceur, A. Ben Mabrouk, A joint multifractal analysis of vector valued non-Gibbs measures, *Chaos Soliton. Fract.*, **126** (2019). https://doi.org/10.1016/j.chaos.2019.05.010

21. A. Mahjoub, The relative multifractal analysis of a vector function in a metric space, *Dynam. Syst.*, **39** (2024), 1–23. https://doi.org/10.1080/14689367.2024.2360208

22. N. Attia, R. Guedri, O. Guizani, Note on the multifractal measures of Cartesian product sets, *Commun. Korean Math. S.*, **37** (2022), 1073–1097. https://doi.org/10.4134/CKMS.c210350

23. J. Peyrière, *A vectorial multifractal formalism*, Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot, Part 2, 2004, 217–230. https://doi.org/10.1090/pspum/072.2/2112124

24. N. Attia, On the multifractal analysis of a non-standard branching random walk, *Acta Sci. Math.*, **88** (2022), 697–722. https://doi.org/10.1007/s44146-022-00046-7

25. N. Attia, Hausdorff and packing dimensions of Mandelbrot measure, *Int. J. Math.*, **31** (2020), 2050068. https://doi.org/10.1142/S0129167X20500688

26. D. Feng, K. Liao, J. Wu, Ergodic limits on the conformal repellers, *Adv. Math.*, **169** (2002), 58–91. https://doi.org/10.1006/aima.2001.2054

27. D. Cutler, Strong and weak duality principles for fractal dimension in Euclidean space, *Math. Proc. Cambridge*, **118** (1995), 393–410. https://doi.org/10.1017/S0305004100073758

28. C. Tricot, Two definitions of fractional dimension, *Math. Proc. Cambridge*, **91** (1982), 54–74. https://doi.org/10.1017/S0305004100059119

29. F. B. Nasr, J. Peyrière, Revisiting the multifractal analysis of measures, *Revista Math. Iberoam.*, **25** (2013), 315–328. https://doi.org/10.4171/RMI/721

30. P. Billingsley, *Ergodic theory and information*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1965. https://doi.org/10.1002/bimj.19680100113

31. O. Zindulka, Hentschel-Procaccia spectra in separable metric spaces, *Real Anal. Exch.*, **26** (2002), 115–120.

32. O. Guizani, A. Mahjoub, N. Attia, Some relations between Hewitt-Stromberg premeasure and Hewitt-Stromberg measure, *Filomat*, **37** (2023), 13–20.https://doi.org/10.2298/FIL2301013A

33. Y. Pesin, H. Weiss, *The multifractal analysis of Birkhoff averages and large deviations*, Global Analysis of Dynamical Systems, CRC Press, 2001, 419–431.

34. A. H. Fan, L. M. Liao, B. W. Wang, J. Wu, On Khintchine exponents and Lyapunov exponents of continued fractions, *Ergodic Theor. Dyn. Syst.*, **29** (2009), 73–109. https://doi.org/10.1017/S0143385708000138

35. L. Barreira, *Dimension and recurrence in hyperbolic dynamics*, Progress in Mathematics, Basel: Birkhäuser, **272** (2008). https://doi.org/10.1007/978-3-7643-8882-9

36. A. H. Fan, D. J. Feng, J. Wu, Recurrence, entropy and dimension, *J. Lond. Math. Soc.*, **64** (2001), 229–244. https://doi.org/10.1017/S0024610701002137

37. A. H. Fan, L. M. Liao, J. Peyrière, Generic points in systems of specification and Banach-valued Birkhoff ergodic averages, *Discrete Cont. Dyn. Syst.*, **21** (2008), 1103–1128. https://doi.org/10.3934/dcds.2008.21.1103

38. A. H. Fan, K. S. Lau, Asymptotic behavior of multiperiodic functions $G(x) = \prod_{n=1}^{\infty} g(x/2^n)$, *J. Fourier Anal. Appl.*, **4** (1998), 129–150. https://doi.org/10.1007/BF02475985

39. C. Carathéodory, Über das Lineare Maß, *Göttingen Nachr.*, **29** (1914), 406–426.

40. G. A. Edgar, Centered densities and fractal measures, *New York J. Math.*, **13** (2007), 33–87.

41. P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*, Cambridge University Press, 1995. https://doi.org/10.1017/CBO9780511623813

42. N. Attia, A. Mahjoub, On the vectorial multifractal analysis in a metric space, *AIMS Math.*, **8** (2023), 23548–23565. https://doi.org/10.3934/math.20231197

43. P. Janardhan, D. Rosenblum, R. S. Strichartz, Numerical experiments in Fourier asymptotics of Cantor measures and wavelets, *Exp. Math.*, **1** (1992), 249–273.