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Research article

Investigation of the impact of delayed damping on the asymptotic behavior of solutions to Emden–Fowler neutral differential equations

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Abstract: This paper examines the oscillatory behavior of solutions to Emden–Fowler neutral differential equations that incorporate a delayed damping term. To our knowledge, there are no prior results addressing the impact of delayed damping on the oscillatory behavior of neutral equations. We establish a new sufficient criterion to confirm that non-oscillatory solutions exhibit asymptotic behavior. Then, we refine and extend this criterion to include the ordinary case. Applying our results to an Euler-type equation shows that they are an extension and complement to the results reported in the literature.

Keywords: differential equations; a delayed damping term; Emden–Fowler equations; neutral equation; asymptotic properties
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1. Introduction

The Emden–Fowler equation is attributed to the English astronomer Sir Ralph Howard Fowler (1889–1944) and the Swiss astrophysicist Jacob Robert Emden (1862–1940). Fowler analyzed this equation to elucidate several phenomena in fluid mechanics [1]. The generalization of this equation and its application in simulating various physical processes have garnered significant interest [2, 3]. Neutral differential equations (NDEs) emerge in several phenomena, including issues related to electric networks with lossless transmission lines (as seen in high-speed computers), the analysis of vibrating masses connected to an elastic bar, and the resolution of variational problems involving temporal delays; see [4].

This work aims to examine the asymptotic and oscillatory behavior of solutions to the nonlinear Emden–Fowler NDE

$$z''(t) + p(t) z'(h_1(t)) + q(t) \mathcal{F}(y(h_2(t))) = 0, \tag{1.1}$$

where $t \geq t_0$, $z(t) = y(t) + \eta(t) y(h_0(t))$, and the following hypotheses are satisfied:

(A1) $\eta, p, q \in \mathbf{C}(\mathbb{I}_{t_0}, [0, \infty))$, $\eta(t) \leq \ell < 1$, and $q(t) > 0$, where $\ell \in \mathbb{R}$ and $\mathbb{I}_l = [l, \infty)$;
(A2) $h_i \in \mathbf{C}(\mathbb{I}_{t_0}, \mathbb{R})$, $\lim_{t\to\infty} h_i(t) = \infty$ for $i = 0, 1, 2$, h_0 and h_1 are delay functions, h_2 is a delay or advanced function, and h_j is nondecreasing for $j = 1, 2$;
(A3) $\mathcal{F} \in \mathbf{C}(\mathbb{R}, \mathbb{R})$ and $\mathcal{F}(u)/u \geq k$ for all $u \neq 0$ and for some $k \in \mathbb{R}^+$.

A function $y \in \mathbf{C}\left(\mathbb{I}_{t_y}, \mathbb{R}\right)$, for all $t_y \geq t_0$, is said to be a solution of (1.1) if $z \in \mathbf{C}^2\left(\mathbb{I}_{t_y}, \mathbb{R}\right)$, y satisfies (1.1) on $\mathbb{I}_{t_y}$, and $\sup\{|y(t)| : t \geq t_*\} > 0$ for any $t_* \in \mathbb{I}_{t_0}$. This solution has oscillatory behavior if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

Oscillation theory is a vital subdiscipline of qualitative theory and a fundamental component of the qualitative analysis of differential equations (DEs). This theory analyzes the oscillatory features of solutions to NDs, together with their asymptotic and monotonic properties (see [5–9]).

The concept “damping” refers to the condition of the solution that arises from the inclusion of this term, wherein the amplitude of the solution progressively diminishes. Studying the oscillatory behavior of equations with a delayed damping involves examining the oscillation of solutions to a system in which the damping term is applied to a past time rather than the current state. These equations are used to model systems in which the damping effect lags behind the instantaneous state of the system (see [4, 10, 11]).

In 1992, Grace [12] examined the impact of incorporating a damping feature on the oscillatory behavior of solutions to ordinary DEs

$$(r\,\psi(y)\,y')' + p\,y' + q\,\mathcal{F}(y) = 0,$$

where $r \in \mathbf{C}(\mathbb{I}_{t_0}, (0, \infty))$, $p, q \in \mathbf{C}(\mathbb{I}_{t_0}, \mathbb{R})$, $\mathcal{F}, \psi \in \mathbf{C}(\mathbb{R}, \mathbb{R})$, $y\mathcal{F}(y) > 0$, and $\psi(y) > 0$ for $y \neq 0$. For the more general equation

$$(r\,K_1(y, y'))' + p\,K_2(y, y')\,y' + q\,\mathcal{F}(y) = L(t),$$

results in [13, 14] tested its oscillatory behavior when $p(t) \geq 0$, $q(t) \geq 0$, $vK_1(u, v) \geq c_1 |K_1(u, v)|^{1+1/\alpha}$, $vK_2(u, v) \mathcal{F}^{1/\alpha}(u) \geq c_2 |K_1(u, v)|^{1+1/\alpha}$, and

$$\frac{\mathcal{F}'(y)}{|\mathcal{F}(y)|^{(\alpha-1)/\alpha}} \geq c_3 > 0,$$

for all $\alpha > 0$ and for some $c_i > 0$, $i = 1, 2, 3$. Later, these results were improved by Li et al. [15], who considered a less general case on the form

$$\left(r\,(y')^{\beta}\right)' + p\,(y')^{\beta} + q\,\mathcal{F}(y) = 0,$$

where β is a ratio of odd positive integers, $p(t) \geq 0$, $q(t) \geq 0$, and $\mathcal{F}(y)/y \geq c > 0$. Studies [16–18] also contributed to the development of the oscillation criteria for damped ordinary DE.

On the other hand, Fu et al. [19], Bohner and Saker [20], and Qin and Ren [21] investigated the influence of the damping term on the delay DE

$$(r\,y')' + p\,y' + q\,\mathcal{F}(y(g)) = 0,$$

where $g(t) \leq t$, $p(t) \geq 0$, and $q(t) > 0$. Equations with delayed damping have received less attention than equations with non-delayed damping due to the analytical challenges that arise when investigating the oscillatory behavior of this type of equation. Grace [22] and Saker et al. [23] established sufficient conditions to verify that the non-oscillatory solutions of equation

$$(r\,y')' + p\,y'(h) + q\,\mathcal{F}(y(g)) = 0. \tag{1.2}$$

converge to zero. Recently, Moaaz and Ramos [24] offered a criterion that ensures that all solutions of (1.2) oscillate, and this criterion improved the results in [22, 23].

Very recently, Alomair and Muhib [25], Arab et al. [26], and Liu et al. [27] derived improved criteria for checking the oscillation of solutions to NDEs. Alomair and Muhib [25] extended the results of [28] to the neutral case and considered the equation

$$z'' + +q\,y(g) = 0.$$

In [26], Arab et al. studied the eqution with sublinear neutral terms

$$\left(r\,\left([y + \eta\,y^{\gamma}(h_0)]'\right)^{\beta}\right)' + q\,y^{\beta}(g) = 0,$$

where γ is a ratio of odd positive integers and $\gamma \in (0, 1]$. Meanwhile, the results in [27] tested the oscillation of the equation with non-delayed damping term

$$\left(r\,\left|z'\right|^{\beta-1} z'\right)' + p\,\left|z'\right|^{\beta-1} z' + q\,\left|y(g)\right|^{\gamma-1} y(g) = 0,$$

where β and γ are positive real numbers.

On the other hand, the examination of the oscillatory properties of solutions to third/fourth-order NDEs has shown notable advancement. Third-order neutral equations had their share of development and investigation (see [29–31]). Studies [32–34] explored the oscillatory characteristics of even-order equations.

To the best of our knowledge, the effect of delayed damping on the oscillatory behavior of solutions of neutral differential equations has not been investigated before. Previous studies in the literature have only addressed the effect of non-delayed damping or the effect of delayed damping for non-neutral equations. Therefore, investigating the oscillatory behavior of neutral differential equations after adding a delayed damping term is interesting and also involves many analytical difficulties. Moreover, for a non-neutral case, our results remove some of the restrictions imposed by previous results on delay functions, such as $h_1(t) \neq t$ and $h_2(t) \leq h_1(t)$.

The differences and difficulties in studying the oscillatory behavior of the neutral Eq (1.1) lie in several aspects: Finding the relationship between the solution y and its corresponding function z in all cases of the signs of the derivatives of z, deducing monotonic properties of z to overcome the restrictions imposed on delay functions, and using different approaches that allow us to apply the results to special cases that have not been studied before.

This article examines the asymptotic characteristics of the solutions of NDE (1.1). In the main results section, we start by inferring some characteristics of positive solutions. We then present a novel criterion that guarantees the convergence of all nonoscillatory solutions to zero. The subsequent theorem enhances and expands upon this criterion due to its deficiencies. Finally, we apply the results to an Euler equation.

2. Main results

The examination of oscillatory behavior often commences with the categorization of the positive solutions to the relevant equation. It is not difficult to verify that the corresponding function to every positive solution of (1.1) is monotonic. If not, the oscillation of z' implies that $z'' > 0$ for every zero of $z'(h_1(t))$, which necessarily means that z' cannot have an additional zero after the first one. Consequently, z belongs to one of the following two categories:

$$\text{(C1)}\ \ z > 0, z' > 0, \text{ and } z'' < 0,$$

and

$$\text{(C2)}\ \ z > 0,\ z' < 0.$$

The class of all positive solutions whose corresponding function belongs to the class (Ci) is denoted by (Si) for i $= 1, 2$. Moreover, we define $\widetilde{h}_2(t) = \min\{t, h_2(t)\}$ and $f^{[2]} = f \circ f$. The function $\widetilde{h}_2(t)$ aims to facilitate the display of results instead of classifying the study into the two cases $h_2(t) \leq t$ and $h_2(t) > t$. Moreover, we define

$$\mu(t) := \exp\left(\int_{t_1}^{t} p(l)\,\mathrm{d}l\right),$$

$$\widetilde{a}_1(t) := t + \int_{t_1}^{t} l\,p(l)\,\mathrm{d}l,$$

and

$$\widetilde{a}_2(t) := q(t)(1 - \eta(h_2(t))),$$

for $t \geq t_1$, where $t_1 \geq t_0$ is large enough.

Next, we give two lemmas that deduce some of the monotonic and asymptotic properties of positive solutions.

Lemma 2.1. *Let* $y \in$ (S1) *and*

$$\int_{t_0}^{\infty} \widetilde{h}_2(t)\,q(l)\,\mathrm{d}l = \infty. \tag{2.1}$$

Then, $(z/t)' \leq 0$ *and* $z/t \to 0$ *as* $t \to \infty$.

Proof. Let $y \in$ (S1). Hence, from (1.1), we deduce that

$$\left(t^2\left[\frac{z}{t}\right]'\right)' = (t\,z' - z)' = t\,z'' = -t\left[p\,z'(h_1) + q\,\mathcal{F}(y(h_2))\right],$$

and so

$$\left(t^2\left[\frac{z}{t}\right]'\right)' \leq -k\,t\,q\,y(h_2). \tag{2.2}$$

The integration of (2.2) produces

$$-t^2\left[\frac{z}{t}\right]' \geq M + k\int_{t_1}^{t} l\,q(l)\,y(h_2(l))\,\mathrm{d}l \geq M + k\,y(h_2(t_1))\int_{t_1}^{t} l\,q(l)\,\mathrm{d}l, \tag{2.3}$$

where $M = z(t_1) - t_1 z'(t_1)$. In view of (2.1), we obtain $(z/t)' \leq 0$.

Now we have that $w = z/t$ is positive and decreasing. So, $w \to m \geq 0$ as $t \to \infty$. From (2.3), we arrive at

$$-w' \geq \frac{M}{t^2} + \frac{k}{t^2} \int_{t_1}^{t} l\, q(l)\, y(h_2(l))\, \mathrm{d}l,$$

which, by integrating, produces

$$\begin{aligned} w(t_1) &\geq \frac{M}{t} + k\, m \int_{t_1}^{\infty} \frac{1}{v^2} \int_{t_1}^{v} l\, h_2(l)\, q(l)\, \mathrm{d}l \mathrm{d}v \\ &\geq \frac{M}{t} + k\, m \int_{t_1}^{\infty} h_2(l)\, q(l)\, \mathrm{d}l, \end{aligned}$$

which contradicts (2.1). Hence, $m = 0$.

The proof is therefore complete. □

Lemma 2.2. *Let* $y \in$ (S2)*, (2.1), and there is a* $\delta \in \mathbf{C}(\mathbb{I}_{t_0}, (0, \infty))$ *such that*

$$\delta \geq \widetilde{h}_2,\ \delta \text{ is nondecreasing, } \left(\frac{p(t)\,\delta(t)}{h_1'(t)} \right)' \leq 0, \tag{2.4}$$

and

$$\int_{t_0}^{\infty} \frac{1}{\delta(v)} \int_{t_0}^{v} \delta(l)\; q(l)\, \mathrm{d}l \mathrm{d}v = \infty. \tag{2.5}$$

Then, $y \to 0$ *as* $t \to \infty$.

Proof. Let $y(t) \in$ (S2) for $t \geq t_1 \in \mathbb{I}_{t_0}$. Thus, $z \to z_0 \geq 0$ as $t \to \infty$.

Suppose on the contrary that $z_0 > 0$. Hence, $z_0 < z(t) < z_0 + \epsilon$ for every $\epsilon > 0$ and $t \geq t_2$. By choosing $\epsilon < (z_0 - z_0 \ell)/\ell$, we obtain

$$y = z - \eta\, z(h_2) > z_0 - \ell(z_0 + \epsilon) = M,$$

where $M = z_0 - \ell(z_0 + \epsilon) > 0$. This, with (1.1), implies that

$$z'' + p\, z'(h_1) + k\, M\, q \leq 0. \tag{2.6}$$

Setting $\psi := \delta\, z'$, it follows from (2.6) that

$$\begin{aligned} \psi' &= \delta\, z'' + \delta'\, z' \\ &\leq -p\, \delta\, z'(h_1) - k\, M\, \delta\, q + \delta'\, z' \\ &\leq -p\, \delta\, z'(h_1) - k\, M\, \delta\, q. \end{aligned} \tag{2.7}$$

The integration of (2.7) gives

$$\psi(t) \leq \psi(t_1) - \int_{t_2}^{t} p(l)\, \delta(l)\, z'(h_1(l))\, \mathrm{d}l - k\, M \int_{t_2}^{t} \delta(l)\; q(l)\, \mathrm{d}l. \tag{2.8}$$

Using (2.4), we find

$$\begin{aligned}\int_{t_2}^{t} p(l)\,\delta(l)\,z'(h_1(l))\,\mathrm{d}l &\geq \frac{p(t_2)\,\delta(t_2)}{h_1'(t_2)}\int_{t_2}^{t} h_1'(l)\,z'(h_1(l))\,\mathrm{d}l\\ &= \frac{p(t_2)\,\delta(t_2)}{h_1'(t_2)}\left[z(h_1(t)) - z(h_1(t_2))\right]\mathrm{d}l\\ &\geq -\frac{p(t_2)\,\delta(t_2)}{h_1'(t_2)}z(h_1(t_2)),\end{aligned}$$

which with (2.8) yields

$$\psi(t) \leq K - k\,M\int_{t_2}^{t}\delta(l)\;q(l)\,\mathrm{d}l,$$

where

$$K = \psi(t_1) + \frac{p(t_2)\,\delta(t_2)}{h_1'(t_2)}z(h_1(t_2)).$$

In view of (2.1), we get

$$\int_{t_2}^{t}\delta(l)\;q(l)\,\mathrm{d}l \to \infty \text{ as } t \to \infty,$$

and hence

$$\psi(t) \leq -\frac{k\,M}{2}\int_{t_2}^{t}\delta(l)\;q(l)\,\mathrm{d}l,$$

or

$$z'(t) \leq -\frac{k\,M}{2}\frac{1}{\delta(t)}\int_{t_2}^{t}\delta(l)\;q(l)\,\mathrm{d}l. \tag{2.9}$$

By integrating (2.9), we arrive at

$$z(t_2) \geq \frac{k\,M}{2}\int_{t_2}^{\infty}\frac{1}{\delta(v)}\int_{t_2}^{v}\delta(l)\;q(l)\,\mathrm{d}l\mathrm{d}v,$$

which contradicts (2.5). Consequently, $z_0 = 0$ and $y \to 0$ as $t \to \infty$.

The proof is therefore complete. □

The following theorem establishes a novel criterion that guarantees that all nonoscillatory solutions to (1.1) converge to zero.

Theorem 2.1. *All solutions of NDE (1.1) oscillate or converge to zero if there is a* $\delta \in \mathbf{C}(\mathbb{I}_{t_0}, (0,\infty))$ *such that (2.1), (2.4), and (2.5) hold, and one of the subsequent conditions is satisfied:*

(a) $h_2(t) < t$ *and*

$$\limsup_{t\to\infty}\left(\frac{h_2^{[2]}(t)}{\mu(h_2(t))}\int_{h_2(t)}^{t}\mu(l)\,q(l)\,(1-\eta(h_2(l)))\,\mathrm{d}l\right) > \frac{1}{k}; \tag{2.10}$$

(b) $h_2(t) > t$ *and*

$$\limsup_{t\to\infty}\left(\frac{t}{\mu(t)}\int_{t}^{h_2(t)}\mu(l)\,q(l)\,(1-\eta(h_2(l)))\,\mathrm{d}l\right) > \frac{1}{k}. \tag{2.11}$$

Proof. Suppose the opposite, that $y > 0$ is a solution of (1.1). Hence, $y \in (S1) \cup (S2)$.

Let $y \in (S2)$. From Lemma 2.2, $y \to 0$ as $t \to \infty$.

Let $y \in (S1)$. Thus, $y > (1 - \eta)\, z$. So, equation (1.1) implies that

$$z'' + p\, z' + k\, q\ (1 - \eta\,(h_2))\, z\,(h_2) \leq 0, \tag{2.12}$$

or

$$(\mu\, z')' + k\, \mu\, q\ (1 - \eta\,(h_2))\, z\,(h_2) \leq 0. \tag{2.13}$$

The integration of (2.13) gives

$$\mu\,(u)\, z'\,(u) \geq k\, z\,(h_2\,(u)) \int_u^v \mu\,(l)\, q\,(l)\,(1 - \eta\,(h_2\,(l)))\, \mathrm{d}l. \tag{2.14}$$

Now we consider the following two cases:

(a) Assume that $h_2 < t$. Using (2.14) with $u = h_2$ and $v = t$, we obtain

$$\mu\,(h_2\,(t))\, z'\,(h_2\,(t)) \geq kz\left(h_2^{[2]}\,(t)\right) \int_{h_2(t)}^t \mu\,(l)\, q\,(l)\,(1 - \eta\,(h_2\,(l)))\, \mathrm{d}l. \tag{2.15}$$

Since $(z/t)' \leq 0$, $z'' \leq 0$, and $h_2 < t$, we have that

$$z\left(h_2^{[2]}\right) > h_2^{[2]}\, z'\left(h_2^{[2]}\right) \geq h_2^{[2]}\, z'\,(h_2)\,,$$

which with (2.15) yields

$$1 \geq k \frac{h_2^{[2]}\,(t)}{\mu\,(h_2\,(t))} \int_{h_2(t)}^t \mu\,(l)\, q\,(l)\,(1 - \eta\,(h_2\,(l)))\, \mathrm{d}l.$$

This contradicts (2.10).

(b) Assume that $h_2 > t$. Using (2.14) with $u = t$ and $v = h_2$, we obtain

$$\mu\,(t)\, z'\,(t) \geq k\, z\,(h_2\,(t)) \int_t^{h_2(t)} \mu\,(l)\, q\,(l)\,(1 - \eta\,(h_2\,(l)))\, \mathrm{d}l. \tag{2.16}$$

Based on the facts that $(z/t)' \leq 0$ and $z' > 0$, we obtain $t\, z' < z < z\,(h_2)$, and then

$$1 \geq k \frac{t}{\mu\,(t)} \int_t^{h_2(t)} \mu\,(l)\, q\,(l)\,(1 - \eta\,(h_2\,(l)))\, \mathrm{d}l,$$

which contradicts (2.11).

The proof is therefore complete. □

Example 2.1. *Consider the Emden–Fowler NDE*

$$z''\,(t) + \frac{\alpha}{t} z'\,(\kappa t) + \frac{\beta}{t^2} y\,(\lambda t) = 0, \tag{2.17}$$

where $z\,(t) = y\,(t) + \ell y\,(h_0\,(t))\,$, $\alpha \in [0, 1)\,$, $\beta > 0$, $\kappa \in (0, 1]$, *and* $\lambda > 0$. *By selecting* $\delta\,(t) = t$, *it is straightforward to confirm that criteria (2.1), (2.4), and (2.5) are fulfilled. In view of Theorem 2.1, every solution of (2.17) oscillates or converges to zero if one of the subsequent conditions is satisfied:*

(a) $\lambda \in (0,1)$ *and*

$$\begin{cases} \beta\lambda\,(1-\ell)\ln(1/\lambda) > 1, & \text{for } \alpha = 1, \\ \beta\lambda^{2-\alpha}\,(1-\ell)\left(\frac{1-\lambda^{\alpha-1}}{\alpha-1}\right) > 1, & \text{for } \alpha \neq 1; \end{cases} \tag{2.18}$$

(b) $\lambda > 1$ *and*

$$\begin{aligned} &\beta\,(1-l)\ln(\lambda) > 1, && \text{for } \alpha = 1, \\ &\beta\,(1-l)\left(\tfrac{\lambda^{\alpha-1}-1}{\alpha-1}\right) > 1, && \text{for } \alpha \neq 1. \end{aligned}$$

Remark 2.1. *Consider Eq (2.17) when* $\eta(t) = \ell = 0$. *It is not difficult to verify the following:*

(i) The results in [22] cannot be applied if $\kappa = 1$.
(ii) The results in [23] can only be applied if $\lambda \leq \kappa \leq 1$.

Therefore, our results complement and extend the results in [22, 23] and cover many cases that have not been studied before.

In the case where $p(t) = 0$, Eq (1.1) becomes the canonical case, and so there are no positive solutions with a decreasing corresponding function, i.e., (S2) = $\emptyset$. Hence, the criteria in Theorem 2.1 ensure that all solutions oscillate. However, the results of this theorem fail to apply to the delay equation

$$y''(t) + \frac{1}{t^2} y\left(\frac{t}{\mathrm{e}}\right) = 0, \tag{2.19}$$

since (2.18) requires that $\beta > \mathrm{e}$. It is known that all solutions to (2.19) exhibit oscillatory behavior (see, e.g., [37]). Consequently, in the subsequent theorem, we shall enhance Theorem 2.1 to derive more effective criteria.

Theorem 2.2. *All solutions of NDE (1.1) oscillate or converge to zero if there is a* $\delta \in \mathbf{C}(\mathbb{I}_{t_0}, (0,\infty))$ *such that (2.1), (2.4), and (2.5) hold, and one of the subsequent conditions is satisfied:*

(a) $h_2(t) \leq t$ *and*

$$\limsup_{t\to\infty}\left[\frac{1}{h_2(t)}\int_{t_1}^{h_2(t)} l\,h_2(l)\widetilde{a}_2(l)\,\mathrm{d}l + \frac{\widetilde{a}_1(h_2(t))}{h_2(t)}\int_{h_2(t)}^{t} h_2(l)\widetilde{a}_2(l)\,\mathrm{d}l + \widetilde{a}_1(h_2(t))\int_t^\infty \widetilde{a}_2(l)\,\mathrm{d}l.\right] > \frac{1}{k}; \tag{2.20}$$

(b) $h_2(t) \geq t$ *and*

$$\limsup_{t\to\infty}\left[\frac{1}{h_2(t)}\int_{t_1}^{t} l\,h_2(l)\widetilde{a}_2(l)\,\mathrm{d}l + \int_t^{h_2(t)} l\widetilde{a}_2(l)\,\mathrm{d}l + \widetilde{a}_1(h_2(t))\int_{h_2(t)}^\infty \widetilde{a}_2(l)\,\mathrm{d}l\right] > \frac{1}{k}. \tag{2.21}$$

Proof. Suppose the opposite, that $y > 0$ is a solution of (1.1). Hence, $y \in$ (S1) $\cup$ (S2).

Let $y \in$ (S2). From Lemma 2.2, $y \to 0$ as $t \to \infty$.

Let $y \in$ (S1). It follows from (2.12) that

$$z'' + k\,q\,(1-\eta(h_2))\,z(h_2) \leq -p\,z' \leq 0.$$

By integrating this inequality, we arrive at

$$z'(t) \geq k \int_t^\infty q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l. \tag{2.22}$$

Setting $w = z - t z' > 0$, we get

$$w' = -t z'' \geq t\, p\, z' + k\, t\, q\, (1-\eta(h_2))\, z(h_2).$$

Integrating this inequality and using the facts $z'' < 0$ and $(z/t)' < 0$, we find

$$w(t) \geq z'(t) \int_{t_1}^t l\, p(l)\,\mathrm{d}l + k \int_{t_1}^t l\, q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l,$$

which with (2.22) yields

$$\begin{aligned} z(t) \ \geq\ & k\widetilde{a}_1(t) \int_t^\infty q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l \\ & + k \int_{t_1}^t l\, q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l. \end{aligned} \tag{2.23}$$

Now we consider the following two cases:

(a) Assume that $h_2 \leq t$. Then, (2.23) reduces to

$$\begin{aligned} z(h_2(t)) \ \geq\ & k \int_{t_1}^{h_2(t)} l\, q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l \\ & + k\widetilde{a}_1(h_2(t)) \int_{h_2(t)}^t q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l \\ & + k\widetilde{a}_1(h_2(t)) \int_t^\infty q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l. \end{aligned}$$

In view of the facts $z' > 0$ and $(z/t)' < 0$, we arrive at

$$\begin{aligned} \frac{1}{k} \ \geq\ & \frac{1}{h_2(t)} \int_{t_1}^{h_2(t)} l\, h_2(l)\, q(l)(1-\eta(h_2(l)))\,\mathrm{d}l \\ & + \frac{\widetilde{a}_1(h_2(t))}{h_2(t)} \int_{h_2(t)}^t q(l)\, h_2(l)(1-\eta(h_2(l)))\,\mathrm{d}l \\ & + \widetilde{a}_1(h_2(t)) \int_t^\infty q(l)(1-\eta(h_2(l)))\,\mathrm{d}l. \end{aligned}$$

This contradicts (2.20).

(b) Assume that $h_2 \geq t$. Then, (2.23) reduces to

$$\begin{aligned} z(h_2(t)) \ \geq\ & k \int_{t_1}^t l\, q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l \\ & + k \int_t^{h_2(t)} l\, q(l)(1-\eta(h_2(l)))\, z(h_2(l))\,\mathrm{d}l \end{aligned}$$

$$+k\widetilde{a}_1\left(h_2\left(t\right)\right)\int_{h_2(t)}^{\infty}q\left(l\right)\left(1-\eta\left(h_2\left(l\right)\right)\right)z\left(h_2\left(l\right)\right)\mathrm{d}l.$$

In view of the facts $z' > 0$ and $(z/t)' < 0$, we arrive at

$$\begin{aligned}\frac{1}{k} \geq\ & \frac{1}{h_2\left(t\right)}\int_{t_1}^{t} l\,h_2\left(l\right)q\left(l\right)\left(1-\eta\left(h_2\left(l\right)\right)\right)\mathrm{d}l \\ & +\int_{t}^{h_2(t)} l\,q\left(l\right)\left(1-\eta\left(h_2\left(l\right)\right)\right)\mathrm{d}l \\ & +\widetilde{a}_1\left(h_2\left(t\right)\right)\int_{h_2(t)}^{\infty}q\left(l\right)\left(1-\eta\left(h_2\left(l\right)\right)\right)\mathrm{d}l,\end{aligned}$$

which contradicts (2.21).

The proof is therefore complete. □

Condition (2.20) deals with the case where the function $h_2(t) \leq t$ is a delay function, meaning that past effects are less extended in time. Condition (2.21) deals with the opposite case, where the function $h_2(t) > t$ is an advanced function, meaning that past effects are longer. The two conditions describe a different effect on the oscillation condition for each region in $[t_1, \infty)$.

Remark 2.2. *Theorem 2.2 considers the ordinary case of $h_2(t)$, which is absent in Theorem 2.1. Moreover, Theorem 2.2 confirms that all solutions of (2.19) oscillate, since* (S2) $= \emptyset$. *This distinguishes our results from the previous related results [22, 23] that did not provide any criteria that guarantee oscillation for all solutions.*

Example 2.2. *Consider the Emden-Fowler NDE (2.17). Using Theorem 2.2, every solution of (2.17) oscillates or converges to zero if one of the subsequent conditions is satisfied:*

(a) $\lambda \in (0, 1]$ *and*

$$\beta\left(1-\ell\right)\left[\left(2+\alpha\right)\lambda+\left(1+\alpha\right)\lambda\ln\left(\frac{1}{\lambda}\right)\right] > 1; \tag{Cr1.1}$$

(b) $\lambda \geq 1$ *and*

$$\beta\left(1-\ell\right)\left[2+\alpha+\ln\left(\lambda\right)\right] > 1. \tag{Cr1.2}$$

Remark 2.3. *When $p(t) = 0$ and $h_2(t) \leq t$, Theorem 2.2 reduces to Theorem 2.1, with $n = 2$, in [37].*

Figure 1 shows some numerical solutions to Eq (1.1) when $p(t) = \alpha/t, q(t) = \beta/t^2, \eta(t) = 0.5$, and $h_i(t) = t - 1$ for $i = 0, 1, 2$.

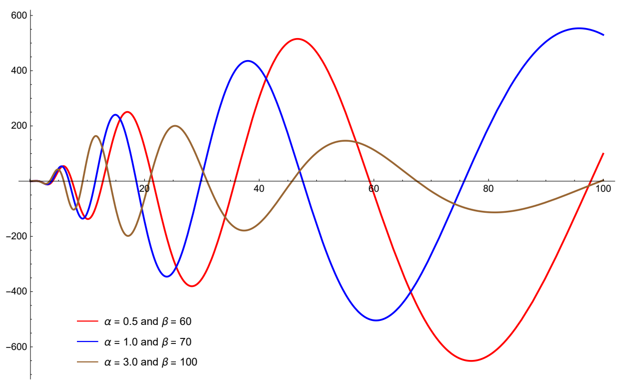

Figure 1. Numerical simulation of some solutions to Eq (1.1).

3. Conclusions

The analytical difficulties that arise from adding a delayed damping term to differential equations are the main reason why there are not enough studies that have discussed this type of problem. This study analyzed the asymptotic characteristics of nonoscillatory solutions to the Emden–Fowler NDE (1.1), which includes a delayed damping term. We proposed a novel criterion to verify that nonoscillatory solutions converge to zero. However, the new criterion does not take into account the ordinary case of h_2, and it does not provide efficient results when reducing the equation to the undamped case. Therefore, we enhanced and expanded this criterion to address those issues. Our findings extended and complemented the results in [37–39]. Finding the criteria that ensure the oscillation of every solution to (1.1) is an interesting research point.

Author contributions

Osama Moaaz: Conceptualization, formal analysis, methodology, writing-review and editing; Wedad Albalawi: Conceptualization, investigation, methodology, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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