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Research article

Second order neutral multi-delay differential equations and its oscillatory criteria derived by linearization

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Abstract: In this paper, we investigate the oscillatory behavior of second-order neutral multi-delay differential equations

$$\left(r(t)\left[\left(u(t)+\sum_{i=1}^{n} b_i(t)u(\psi_i(t))\right)'\right]^{\alpha}\right)' + Q(t, u(\tau(t))) = 0.$$

We introduce new monotonic properties of the non-oscillatory solutions of the equation, which are then used to linearize the equation and derive new oscillatory criteria. Our results are further supported by a numerical simulation example.

Keywords: second-order neutral multi-delay differential equations; asymptotic properties; linearization; oscillation theory
Mathematics Subject Classification: 34K40, 34K25, 34K06, 34K11

1. Introduction

The aim of our paper is to investigate the asymptotic and oscillatory behavior of solutions for second-order neutral multi-delay differential equations

$$\left(r(t)\left[\left(u(t)+\sum_{i=1}^{n} b_i(t)u(\psi_i(t))\right)'\right]^{\alpha}\right)' + Q(t, u(\tau(t))) = 0, t \geq t_0. \tag{1.1}$$

To abbreviate notation, we introduce $v(t) = u(t) + \sum_{i=1}^{n} b_i(t)u(\psi_i(t))$.

We make the following assumptions throughout the paper:

H1. $r(t) \in C^1([t_0, \infty))$, $r(t) > 0$, and α is a ratio of two positive odd integers.

H2. $b_i(t) \in C^1([t_0,\infty))$, $b_i \geq 0$, $\sup_{t\geq t_0}\sum_{i=1}^{n} b_i(t) < 1$ and there exists a positive and continuous function p such that $\frac{Q(t,u)}{u^\alpha} \geq p(t)$ for all u and $Q(t,u)$ is an odd function for $u \neq 0$.

H3. $\psi_i(t) \in C([t_0,\infty))$ and $\psi_i(t) \leq t$, $\lim_{t\to\infty} \psi_i(t) = \infty$.

H4. $\tau(t) \in C([t_0,\infty))$ and $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$.

Additionally, we require

$$R(t) = \int_{t_0}^{t} \frac{dy}{r^{1/\alpha}(y)} \to \infty \text{ as } t \to \infty, \tag{1.2}$$

which is known as the canonical condition.

A solution of (1.1) is defined to be a function $u(t) \in C^1([T_0,\infty))$ with $T_0 \geq t_0$, satisfying $r(t)(v'(t))^\alpha \in C^1([T_0,\infty))$ and Eq (1.1) on $[T_0,\infty)$. We only consider solutions $u(t)$ of (1.1) that satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for all $T \geq T_0$, and assume that such solutions exist. A solution of (1.1) is classified as oscillatory if it has arbitrarily large zeros on $[T_0,\infty)$, and non-oscillatory otherwise. The equation is said to be oscillatory if all of its solutions are oscillatory.

Second-order delay differential equations have been extensively investigated due to their theoretical significance and practical applications. Neutral delay differential equations constitute a class of delay differential equations in which the highest derivative of the solution appears both with and without delay. Beyond their theoretical importance, their qualitative analysis is of considerable practical relevance. Such equations arise in diverse contexts, including the vibration of masses attached to elastic bars, variational problems with time delays, and electrical networks containing lossless transmission lines—for instance, in high-speed computers where these lines interconnect switching circuits [1]. A substantial amount of research has been devoted to establishing oscillation criteria for various classes of such equations. Prominent monographs in the field include those by B. Baculikova, R. P. Agarwal, P. Temtek, et al., [2–9].

A well-known criterion for second-order delay differential equations, proposed by Koplatadze et al. [10], is based on the monotonic properties of positive solutions: specifically, the increasing behavior of u(t) and the decreasing behavior of $\frac{u(t)}{t}$ of positive solutions of $u''(t) + c(t)u(\tau(t)) = 0$. These properties have been effectively employed in the literature to investigate oscillatory phenomena in various delay differential equations. In addition, recent studies in oscillation theory [11–14] can be referenced.

This paper aims to develop new comparison theorems and oscillation conditions for second-order neutral multi-delay differential equations by utilizing the monotonicity of non-oscillatory solutions. Baculikova and Dzurina [2] studied half-linear second-order delay differential equations of the form

$$(r(t)(y'(t))^\alpha)' + p(t)y^\alpha(\tau(t)) = 0.$$

They gave monotonic properties of its nonoscillatory solutions and used them for linearization of the considered equation, which leads to oscillatory criteria. The equation we are considering is more general than theirs. Building on earlier results by Baculikova and Dzurina [2], we derive novel monotonic properties that facilitate the linearization of the original equation. This linearization enables the transfer of oscillatory properties from the linearized form to the original nonlinear equation.

Our first objective is to establish new comparison theorems for Eq (1.1). Subsequently, we propose new oscillation criteria based on the linear forms of the equation. Finally, we demonstrate the applicability and strength of the derived criteria through a concrete example.

The results obtained in this study represent a significant enhancement to existing oscillation theory and contribute to a novel analytical framework for the study of second-order neutral delay differential equations.

2. Preliminary results

We begin with some useful lemmas regarding the monotonic properties of non-oscillatory solutions for the equations under consideration.

Throughout the work, the positivity of u is assumed in the proofs, and analogous reasoning applies to the case of $-u$.

Lemma 1. *Let $u(t)$ be a positive solution of (1.1). Then*

i. $r(t)(v'(t))^{\alpha} > 0$,
ii. $\frac{v(t)}{R(t)}$ *is decreasing*

for $t \geq t_1 \geq t_0$. Moreover if

$$\int_{t_0}^{\infty} R^{\alpha}(\tau(y)p(y))\left(1-\sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha} dy = \infty \tag{2.1}$$

holds, then

iii. $\lim_{t\to\infty} \frac{v(t)}{R(t)} = 0$.

Proof. Suppose that $u(t)$ is a positive solution of (1.1). Then

$$\left(r(t)(v'(t))^{\alpha}\right)' < 0,$$

and there exists $t_1 \geq t_0$ such that $r(t)(v'(t))^{\alpha}$ has a constant sign for $t \geq t_1$.

(i) Now we assume the opposite, i.e., $r(t)(v'(t))^{\alpha} < 0$. This implies the existence of a constant $k > 0$ such that $r(t)(v'(t))^{\alpha} \leq -k < 0$. Integrating the previous inequality from t_1 to t and using (1.2), we obtain

$$v(t) \leq v(t_1) - k^{1/\alpha} R(t) \to -\infty \ as \ t \to \infty.$$

Hence, we have shown that assuming $r(t)(v'(t))^{\alpha} < 0$ leads to a contradiction, and therefore we can conclude that $r(t)(v'(t))^{\alpha} > 0$.

(ii) Using the monotonicity of $r^{1/\alpha}(t)v'(t)$, we get

$$v(t) \geq \int_{t_1}^{t} \frac{r^{1/\alpha}(y)v'(y)}{r^{1/\alpha}(y)} dy \geq r^{1/\alpha}(t)v'(t)R(t), \tag{2.2}$$

which implies $\left(\frac{v(t)}{R(t)}\right)' < 0$.

On the other hand, as $\frac{v(t)}{R(t)}$ is a positive and decreasing function, there exists a positive constant δ such that

$$\lim_{t\to\infty} \frac{v(t)}{R(t)} = \delta \geq 0.$$

(iii) Assume, on the contrary, that $\delta > 0$. Then $\frac{v(t)}{R(t)} \geq \delta$ for $t \geq t_1$. From the definition of $v(t)$ and H3 we obtain

$$\begin{aligned} u(t) &= v(t) - \sum_{i=1}^{n} b_i(t)\, u(\psi_i(t)) \\ &\geq v(t) - \sum_{i=1}^{n} b_i(t)\, v(\psi_i(t)) \\ &\geq v(t)\left(1 - \sum_{i=1}^{n} b_i(t)\right). \end{aligned} \tag{2.3}$$

Integrating (1.1) from t_1 to t, we obtain

$$r(t_1)\left(v'(t_1)\right)^{\alpha} \geq \delta^{\alpha} \int_{t_1}^{t} p(y)\, R^{\alpha}(\tau(y)) \left(1 - \sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha} dy,$$

which for $t \to \infty$ contradicts with (2.1). So, $\lim_{t\to\infty} \frac{v(t)}{R(t)} = 0$. The proof is complete. □

Since $R(t)$ is increasing, there exists $\lambda \geq 1$ such that

$$\frac{R(t)}{R(\tau(t))} \geq \lambda. \tag{2.4}$$

Theorem 2. *Let (2.1) hold, and let there exist* $\beta > 1$ *such that*

$$\frac{1}{\alpha} R^{\alpha}(\tau(t))\, r^{1/\alpha}(t)\, R(t)\, p(t) \left(1 - \sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} > \beta \text{ for } t \geq t_0. \tag{2.5}$$

If $u(t)$ *is a positive solution of (1.1), then*

(i) $\frac{v(t)}{R^{1-\beta}(t)}$ *is decreasing for* $t \geq t_1$,
(ii) $\frac{v(t)}{R^{\beta_0}(t)}$ *is increasing for* $t \geq t_1$, *where* $\beta_0 = \beta^{1/\alpha} \lambda^{\beta}$.

Proof. Assume that $u(t)$ is a positive solution of (1.1). It is important to note that condition (iii) of Lemma 1 implies

$$\lim_{t\to\infty} r^{1/\alpha}(t)\, v'(t) = 0. \tag{2.6}$$

Therefore, an integration of (1.1) yields

$$r^{1/\alpha}(t)\, v'(t) = \left(\int_{t}^{\infty} Q(y, u(\tau(y)))\, dy\right)^{1/\alpha}. \tag{2.7}$$

It is easy to see that

$$\left[\left(r^{1/\alpha}(t)\, v'(t)\right)^{\alpha}\right]' = \alpha \left(r^{1/\alpha}(t)\, v'(t)\right)^{\alpha-1} \left(r^{1/\alpha}(t)\, v'(t)\right)'. \tag{2.8}$$

Setting into (1.1), we have

$$\left(r^{1/\alpha}(t)\,v'(t)\right)' + \frac{1}{\alpha}\left(r^{1/\alpha}(t)\,v'(t)\right)^{1-\alpha} Q(t,u(\tau(t))) = 0. \tag{2.9}$$

Then $w(t) = r^{1/\alpha}(t)\,v'(t)$ is positive decreasing and satisfies

$$w'(t) + \frac{1}{\alpha}w^{1-\alpha}(t)\,Q(t,u(\tau(t))) = 0,$$

and from H2 we get

$$w'(t) + \frac{1}{\alpha}w^{1-\alpha}(t)\,p(t)\,u^{\alpha}(\tau(t)) \leq 0. \tag{2.10}$$

On the other hand (2.2) implies

$$v(t) \geq r^{1/\alpha}(t)\,v'(t)\,R(t) \geq w(t)\,R(t).$$

Then by (2.3),

$$u(\tau(t)) \geq w(t)\,R(\tau(t))\left(1 - \sum_{i=1}^{n} b_i(\tau(t))\right). \tag{2.11}$$

Substituting the last inequality into (2.10), we get

$$w'(t) + \frac{1}{\alpha}p(t)\,R^{\alpha}(\tau(t))\left(1 - \sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} w(t) \leq 0$$

and

$$w'(t) + \frac{\beta}{R(t)\,r^{1/\alpha}(t)}w(t) \leq 0,$$

which implies

$$-w'(t)\,R(t) \geq \frac{\beta}{r^{1/\alpha}(t)}w(t) = \beta v'(t).$$

The rest of the proof is the same as in Theorem 2.3 in [2], so we omit it. □

Remark 3. *We take $b_i(t) = 0$ in Theorem 2; our results reduce to the results of Theorem 2.3 in [2]. The previous results did not distinguish between the cases $\alpha < 1$ and $\alpha \geq 1$. However, in order to provide oscillatory criteria for (1.1), we need to consider these cases separately.*

3. Oscillation results for $\alpha \geq 1$

To simplify our notation, let us denote

$$\kappa = \frac{(1-\beta)^{1-\alpha}\,\lambda^{\beta(\alpha-1)}}{\alpha}.$$

Now we will provide new comparison principles that significantly simplify the examination of neutral delay differential equations.

Theorem 4. *Let $\alpha > 1$, and (2.1), (2.5) hold. Then (1.1) is oscillatory provided that*

$$\left(r^{1/\alpha}(t)\,v'(t)\right)' + \kappa R^{\alpha-1}(\tau(t))\,p(t)\left(1-\sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} v(\tau(t)) = 0 \tag{3.1}$$

is oscillatory.

Proof. Suppose the opposite of the desired result, that is, $u(t)$ is a positive solution of (1.1). By Theorem 2 (i), $\frac{v(t)}{R^{1-\beta}(t)}$ is decreasing, we have the inequality

$$v(t) \geq \frac{r^{1/\alpha}(t)\,v'(t)}{(1-\beta)} R(t), \tag{3.2}$$

which for $\alpha > 1$ yields

$$v^{1-\alpha}(t) \leq \frac{\left(r^{1/\alpha}(t)\,v'(t)\right)^{1-\alpha}}{(1-\beta)^{1-\alpha}} R^{1-\alpha}(t).$$

Hence

$$\left(r^{1/\alpha}(t)\,v'(t)\right)^{1-\alpha} \geq (1-\beta)^{1-\alpha}\,\frac{v^{1-\alpha}(t)}{R^{1-\alpha}(t)}. \tag{3.3}$$

By Theorem 2 (i), we obtain

$$v^{1-\alpha}(t) \geq \frac{v^{1-\alpha}(\tau(t))}{R^{(1-\alpha)(1-\beta)}(\tau(t))} R^{(1-\alpha)(1-\beta)}(t). \tag{3.4}$$

Substituting (3.4) from (3.3), we have in view of (2.4) that

$$\begin{aligned}\left(r^{1/\alpha}(t)\,v'(t)\right)^{1-\alpha} &\geq (1-\beta)^{1-\alpha}\,\frac{R^{(1-\alpha)(1-\beta)}(t)}{R^{(1-\alpha)(1-\beta)}(\tau(t))}\,\frac{v^{1-\alpha}(\tau(t))}{R^{1-\alpha}(t)} \\ &\geq (1-\beta)^{1-\alpha}\,\lambda^{\beta(\alpha-1)}\,\frac{v^{1-\alpha}(\tau(t))}{R^{1-\alpha}(\tau(t))}.\end{aligned} \tag{3.5}$$

By combining (2.10) and (3.5), we can derive that $v(t)$ satisfies the linear differential inequality

$$\left(r^{1/\alpha}(t)\,v'(t)\right)' + \kappa p(t)\,R^{\alpha-1}(\tau(t))\left(1-\sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} v(\tau(t)) \leq 0. \tag{3.6}$$

In contrast, by Corollary 1 in [15], it is ensured that the corresponding differential equation (3.1) has a positive solution, which contradicts the assumption made earlier, thus completing the proof. □

Remark 5. *We take $b_i(t) = 0$ in Theorem 4; our results reduce to the results of Theorem 3.1 in [2].*

We will utilize the outcomes of the previous theorem to establish novel oscillatory criteria.

Theorem 6. *Let* $\alpha > 1$*, and (2.1), (2.5) hold. If*

$$\limsup_{t\to\infty}\left\{R^{\beta-1}(\tau(t))\int_{t_1}^{\tau(t)} p(y)R(y)R^{\alpha-\beta}(\tau(y))\left(1-\sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha} dy\right.$$
$$+R^{\beta}(\tau(t))\int_{\tau(t)}^{t} p(y)R^{\alpha-\beta}(\tau(y))\left(1-\sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha} dy$$
$$\left.+R^{1-\beta_0}(\tau(t))\int_{t}^{\infty} p(y)R^{\alpha+\beta_0-1}(\tau(y))\left(1-\sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha} dy\right\} > \frac{1}{\kappa}$$

then (1.1) is oscillatory.

Proof. The proof of the theorem closely follows the approach taken in Theorem 4.1 in [2], making it unnecessary to provide a full proof here. □

For practical applications, we often need criteria that are easier to verify. The next theorem provides such conditions:

Theorem 7. *Let* $\alpha > 1$*, and (2.1), (2.5) hold. If*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(y)R^{\alpha}(\tau(y))\left(1-\sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha} dy > \frac{1}{\kappa e}$$

then (1.1) is oscillatory.

Proof. Suppose, for the sake of contradiction, that (1.1) has a positive solution $u(t)$. By Theorem 6, Eq (3.1) is non-oscillatory, and we can assume that it has an eventually positive solution $v(t)$. Then, we can define $w(t) = r^{1/\alpha}(t)v'(t)$, which is positive and decreasing. Therefore, we have

$$v(t) = v(t_1) + \int_{t_1}^{t} v'(y)\,dy \geq \int_{t_1}^{t} \frac{r^{1/\alpha}(y)\,v'(y)}{r^{1/\alpha}(y)}dy$$
$$\geq r^{1/\alpha}(t)\,v'(t)\int_{t_1}^{t}\frac{dy}{r^{1/\alpha}(y)} = w(t)R(t).$$

By substituting the expression for $v(t)$ into (3.1), we obtain a first-order differential inequality of the form

$$w'(t) + \kappa R^{\alpha}(\tau(t))\,p(t)\left(1-\sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} w(\tau(t)) \leq 0 \tag{3.7}$$

with positive solution $w(t)$. According to Theorem 2.1.1 in [16], (3.7) has no positive solution if

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} \kappa p(y)R^{\alpha}(\tau(y))\left(1-\sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} dy > \frac{1}{e}.$$

This contradicts our assumption that (1.1) has a positive solution. Hence, the proof is complete. □

4. Oscillation results for $\alpha \in (0,1)$

We now turn our attention to the case where $0 < \alpha < 1$. This case requires different techniques due to the different nature of the non-linearity. The following results parallel those of the previous section, but with important modifications to account for the changed parameter range.

To simplify the notation, we define

$$\omega = \frac{\beta^{\frac{1-\alpha}{\alpha}} \lambda^{1-\alpha}}{\alpha (1-\beta_0)^{\frac{1-\alpha}{\alpha}}}.$$

Theorem 8. *Let* $0 < \alpha < 1$, *and (2.1), (2.5) hold. Then (1.1) is oscillatory provided that*

$$\left(r^{1/\alpha}(t) v'(t)\right)' + \omega R^{\alpha-1}(t) p(t) \left(1 - \sum_{i=1}^{n} b_i(\tau(t))\right)^{\alpha} v(\tau(t)) = 0 \tag{4.1}$$

is oscillatory.

Proof. Assume the contrary, that $u(t)$ is a positive solution of (1.1). Differentiating (2.7) with respect to t leads to the equation

$$\left(r^{1/\alpha}(t) v'(t)\right)' + \frac{1}{\alpha} \left(\int_t^{\infty} Q(y, u(\tau(y)))\, dy\right)^{\frac{1-\alpha}{\alpha}} Q(t, u(\tau(t))) = 0.$$

Employing the fact that $\frac{v(t)}{R^{\beta_0}(t)}$ is an increasing function, we can rewrite the previous inequality as follows:

$$\left(r^{1/\alpha}(t) v'(t)\right)' + \frac{1}{\alpha} \frac{v^{1-\alpha}(\tau(t))}{R^{(1-\alpha)\beta_0}(\tau(t))} \left(\int_t^{\infty} \frac{p(y) b_0^{\alpha}(t) dy}{R^{-\alpha\beta_0}(\tau(y))}\right)^{\frac{1-\alpha}{\alpha}} \frac{p(t) v^{\alpha}(\tau(t))}{b_0^{-\alpha}(t)} \le 0,$$

where $b_0(t) = 1 - \sum_{i=1}^{n} b_i(\tau(t))$. Consequently, we can conclude that $u(t)$ must satisfy the linear differential inequality

$$\left(r^{1/\alpha}(t) v'(t)\right)' + \frac{1}{\alpha} \left(\int_t^{\infty} \frac{p(y) b_0^{\alpha}(y) dy}{R^{-\alpha\beta_0}(\tau(y))}\right)^{\frac{1-\alpha}{\alpha}} \frac{b_0^{\alpha}(t) p(t) v(\tau(t))}{R^{(1-\alpha)\beta_0}(\tau(t))} \le 0. \tag{4.2}$$

Additionally, by utilizing (2.4) and (2.5), we can derive

$$\begin{aligned}
\int_t^{\infty} \frac{p(y) \left(1 - \sum_{i=1}^{n} b_i(\tau(y))\right)^{\alpha}}{R^{-\alpha\beta_0}(\tau(y))} dy &\ge \int_t^{\infty} \alpha\beta \frac{R^{\alpha\beta_0}(\tau(y))}{R^{\alpha}(\tau(y)) r^{1/\alpha}(y) R(y)} dy \\
&\ge \alpha\beta\lambda^{\alpha(\beta_0-1)} \int_t^{\infty} \frac{R^{\alpha(\beta_0-1)-1}(y)}{r^{1/\alpha}(y)} dy \\
&\ge \frac{\beta\lambda^{\alpha(\beta_0-1)}}{1-\beta_0} R^{\alpha(\beta_0-1)}(t).
\end{aligned}$$

Substituting (4.2), we get

$$\left(r^{1/\alpha}(t)\,v'(t)\right)' + \frac{\beta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}\frac{R^{(1-\alpha)(\beta_0-1)}(t)}{R^{(1-\alpha)\beta_0}(\tau(t))}\left(1-\sum_{i=1}^{n}b_i(\tau(t))\right)^{\alpha} p(t)\,v(\tau(t)) \le 0.$$

Based on Corollary 1 in [15], we may infer that the corresponding differential equation (4.1) also has a positive solution. This contradicts the assumption that (4.1) has no positive solutions, and therefore the proof is complete. □

Remark 9. *Assume that $b_i(t) = 0$ in Theorem 8; the results of Theorem 8 reduce to the results of Theorem 3.2 in [2].*

Theorem 10. *Let $0 < \alpha < 1$, and (2.1), (2.5) hold. If*

$$\limsup_{t\to\infty}\Bigg\{R^{\beta-1}(\tau(t))\int_{t_1}^{\tau(t)} R^{1-\beta}(\tau(y))\,R^{\alpha}(y)\,p(y)\left(1-\sum_{i=1}^{n}b_i(\tau(y))\right)^{\alpha}dy$$
$$+R^{\beta}(\tau(t))\int_{\tau(t)}^{t} R^{1-\beta}(\tau(y))\,R^{\alpha-1}(y)\,p(y)\left(1-\sum_{i=1}^{n}b_i(\tau(y))\right)^{\alpha}dy$$
$$+R^{1-\beta_0}(\tau(t))\int_{t}^{\infty} R^{\beta_0}((y))\,R^{\alpha-1}(y)\,p(y)\left(1-\sum_{i=1}^{n}b_i(\tau(y))\right)^{\alpha}dy\Bigg\} > \frac{1}{\omega},$$

then (1.1) is oscillatory.

Proof. The proof of the theorem closely follows the approach used in Theorem 4.2 of [2]. □

Theorem 11. *Let $0 < \alpha < 1$, and (2.1), (2.5) hold. If*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} R(\tau(y))\,R^{\alpha-1}(y)\,p(y)\left(1-\sum_{i=1}^{n}b_i(\tau(y))\right)^{\alpha}dy > \frac{1}{\omega e},$$

then (1.1) is oscillatory.

Proof. Since the proof of the theorem closely resembles that of Theorem 7, it may be omitted. □

5. Example

Example 12. *Consider a second-order neutral delay differential equation*

$$\left(t^{k}(v'(t))^{\alpha}\right)' + \left(\frac{c_0}{t^{(\alpha-k+1)}} + u^2(\tau_0 t)\right)u^{\alpha}(\tau_0 t) = 0, \tag{5.1}$$

where $v(t) = u(t) + b_0\sum_{i=1}^{n} u(\psi_i(t))$ with $c_0 > 0$, ψ and $\tau_0 \in (0,1)$, and $0 \le b_0 < 1$.

Then

$$R(t) = \frac{\alpha t^{\left(1-\frac{k}{\alpha}\right)}}{\alpha-k},\quad R(\tau_0 t) = \frac{\alpha(\tau_0 t)^{\left(1-\frac{k}{\alpha}\right)}}{\alpha-k},\quad \lambda = \frac{R(t)}{R(\tau_0 t)} = \tau_0^{\left(\frac{k}{\alpha}-1\right)},$$

$$\beta = \frac{c_0 \tau_0^{(\alpha-k)}}{\alpha} \left(\frac{\alpha}{\alpha - k} \right)^{(\alpha+1)} (1 - b_0)^{\alpha}, \ \ \beta_0 = \frac{c_0^{\left(\frac{1}{\alpha}\right)} \alpha \tau_0^{\left(1-\frac{k}{\alpha}\right)(1-\beta)}}{(\alpha - k)^{\left(1+\frac{1}{\alpha}\right)}} (1 - b_0)$$

$$\kappa = \frac{(1-\beta)^{(1-\alpha)} \, \tau_0^{\beta(1-\alpha)\left(1-\frac{k}{\alpha}\right)}}{\alpha}$$

and

$$\omega = \frac{\beta^{\frac{1}{\alpha}-1} \tau_0^{(1-\alpha)\left(\frac{k}{\alpha}-1\right)}}{\alpha \, (1 - \beta_0)^{\frac{1}{\alpha}-1}}.$$

Equation (5.1) with $\alpha > 1$ is oscillatory, provided that

$$\alpha\beta \left\{ \frac{\tau_0^{\beta\left(\frac{k}{\alpha}-1\right)}}{1-\beta} + \frac{\tau_0^{\beta\left(\frac{k}{\alpha}-1\right)} - 1}{\beta} + \frac{1}{1-\beta_0} \right\} > \frac{1}{\kappa} \tag{5.2}$$

by Theorem 6, or

$$\beta \, (\alpha - k) \ln \frac{1}{\tau_0} > \frac{1}{e\kappa} \tag{5.3}$$

by Theorem 7.

Equation (5.1) with $0 < \alpha < 1$ is oscillatory, provided that

$$\alpha\beta\tau_0^{(1-\alpha)\left(1-\frac{k}{\alpha}\right)} \left\{ \frac{\tau_0^{\beta\left(\frac{k}{\alpha}-1\right)}}{1-\beta} + \frac{\tau_0^{\beta\left(\frac{k}{\alpha}-1\right)} - 1}{\beta} + \frac{1}{1-\beta_0} \right\} > \frac{1}{\omega} \tag{5.4}$$

by Theorem 10 or

$$\beta \, (\alpha - k) \, \tau_0^{(1-\alpha)\left(1-\frac{k}{\alpha}\right)} \ln \frac{1}{\tau_0} > \frac{1}{e\omega} \tag{5.5}$$

by Theorem 11.

Let us assume the special case for Eq (5.1),

$$(t \, (v'(t))^3)' + \left(\frac{12}{t^3} + u^2(0.5t) \right) u^3 \, (0.5t) = 0, \tag{5.6}$$

where $v(t) = u(t) + 0.2u(0.5t) + 0.2u(0.8t)$ for $n = 2$ and $k = 1$, $\alpha = 3$, $c_0 = 12 > 0$, $\tau_0 = 0.5 \in (0, 1)$ *and* $b_0 = 0.2 \in (0, 1)$.

Then $R(t) \to \infty$ as $t \to \infty$, conditions (2.1) and (2.5) also hold. Since $\beta \approx 1.093$ and $\kappa \approx 105.35$, inequality (5.3) holds, i.e.,

$$\beta \, (\alpha - k) \ln \frac{1}{\tau_0} > \frac{1}{e\kappa}, \quad 1.515 > 0.00349.$$

Therefore differential equation (5.6) is oscillatory by Theorem 7.

As a further illustration of Eq (5.1) for the case $0 < \alpha < 1$, we now consider the following differential equation,

$$(t^{0.2}\,(v'\,(t))^{0.6})' + \left(\frac{1}{t^{1.4}} + u^2(0.3t)\right) u^{0.6}\,(0.3t) = 0, \tag{5.7}$$

where $v(t) = u(t) + 0.2u(0.5t) + 0.2u(0.8t)$ for $n = 2$ and $k = 0.2$, $\alpha = 0.6$, $c_0 = 1 > 0$, $\tau_0 = 0.5 \in (0, 1)$ *and* $b_0 = 0.2 \in (0, 1)$.

Then $R(t) \to \infty$ as $t \to \infty$, conditions (2.1) and (2.5) also hold. Since $\beta \approx 1.722$ and $\omega \approx 0.791$, inequality (5.5) holds, i.e.,

$$\beta\,(\alpha - k)\,\tau_0^{(1-\alpha)\left(1-\frac{k}{\alpha}\right)} \ln\frac{1}{\tau_0} > \frac{1}{e\omega}, \quad 0.558 > 0.465.$$

Hence differential equation (5.7) is oscillatory by Theorem 11.

6. Conclusions

In this paper, we established new oscillation criteria for second-order neutral multi-delay differential equations by utilizing the monotonic properties of non-oscillatory solutions. Our approach generalized previous results by considering a neutral term and a more general nonlinearity. The presented theorems provided effective tools for determining the oscillatory behavior of these equations. It is worth noting that when we set $b_i(t) = 0$ (eliminating the neutral term) and $Q(t, u(t)) = c(t)u^{\alpha}(t)$, our results reduced to those obtained by Baculikova and Dzurina [2]. This confirmed the consistency of our approach while demonstrating its broader applicability to more complex equations.

Use of Generative-AI tools declaration

During the preparation of this work, the authors used ChatGPT to improve the readability and language of the article. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. J. K. Hale, *Functional differential equations*, Berlin: Springer, 1971.
2. B. Baculikova, J. Dzurina, Oscillatory criteria via linearization of half linear second order delay differential equations, *Opuscula Math.*, **40** (2020), 523–536. https://doi.org/10.7494/OpMath.2020.40.5.523
3. B. Baculikova, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.*, **91** (2019), 68–75. https://doi.org/10.1016/j.aml.2018.11.021

4. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Springer Dordrecht, 2002. https://doi.org/10.1007/978-94-017-2515-6
5. P. Temtek, A. Tiryaki, Oscillation criteria for a certain second-order nonlinear perturbed differential equations, *J. Inequal. Appl.*, **2013** (2013), 524. https://doi.org/10.1186/1029-242X-2013-524
6. P. Temtek, Oscillation criteria for second-order nonlinear perturbed differential equations, *Electron. J. Differ. Eq.*, **2014** (2014), 1–10.
7. J. Dzurina, E. Thandapani, B. Baculikova, C. Dharuman, N. Prabaharan, Oscillation of second order nonlinear differential equations with several sub-linear neutral terms, *Nonlinear Dyn. Syst. Theory*, **19** (2019), 124–132.
8. L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation theory for functional differential equations*, Routledge, 2017.
9. M. T. Şenel, T. Candan, B. Çına, Existence of nonoscillatory solutions of second-order nonlinear neutral differential equations with distributed deviating arguments, *J. Taibah Univ. Sci.*, **13** (2019), 998–1005. https://doi.org/10.1080/16583655.2019.1668101
10. R. Koplatadze, G. Kvinikadze, I. P. Stavroulakis, Properties *A* and *B* of *n*th order linear differential equations with deviating argument, *Georgian Math. J.*, **6** (1999), 553–566. https://doi.org/10.1515/GMJ.1999.553
11. O. Moaaz, A. Al-Jaser, Functional differential equations of the neutral type: oscillatory features of solutions, *AIMS Math.*, **9** (2024), 16544–16563. https://doi.org/10.3934/math.2024802
12. L. Hua, J. Sun, W. Li, Y. Pang, Oscillation of solutions for second-order neutral multi-delay differential equations, *AIMS Math.*, **10** (2025), 18586–18602. https://doi.org/10.3934/math.2025830
13. A. A. El-Gaber, M. M. A. El-Sheikh, H. M. Rezk, M. Zakarya, G. AlNemer, E. I. El-Saedy, On the oscillatory behavior of solutions of second-order damped differential equations with several sub-linear neutral terms, *Mathematics*, **12** (2024), 3217. https://doi.org/10.3390/math12203217
14. O. Bazighifan, N. Alshammari, F. Aldosari, L. F. Iambor, Oscillation solutions for nonlinear second-order neutral differential equations, *Mathematics*, **13** (2025), 247. https://doi.org/10.3390/math13020247
15. T. Kusano, M. Naito, Comparison theorems for functional differential equations with deviating arguments, *J. Math. Soc. Japan*, **33** (1981), 509–532. https://doi.org/10.2969/jmsj/03330509
16. G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcl Dekker, 1987.