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Research article

Finsler warped product metrics with isotropic E-curvature

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Abstract: The E-curvature is one of the most important non-Riemannian quantities in Finsler geometry. In this paper, we study Finsler warped product metrics with isotropic E-curvature and constant E-curvature. The equations that characterize Finsler warped product metrics of the above E-curvature are given. Moreover, some specific metrics with isotropic E-curvature are constructed.

Keywords: Finsler warped product metric; E-curvature; S-curvature; (α, β) -metric
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1. Introduction

Finsler geometry has plenty of colorful geometric properties, though it can be regarded as an extension of Riemannian geometry, freed from the constraints of quadratic forms. This fundamental difference endows Finsler geometry with a plethora of unique properties that are absent in Riemannian geometry. Riemann quantities usually refer to the quantities defined in Riemannian geometry, such as Riemann curvature, Ricci curvature, etc. However, some geometric quantities, which are called non-Riemannian quantities, vanish in the Riemannian case, such as Berwald curvature, S-curvature, E-curvature, and so on.

E-curvature is also called mean Berwald curvature, which was first introduced in [1]. Since S-curvature is related to the volume of the manifold, then by (2.3), E-curvature is also related to the volume. Thus, it is meaningful to explore further properties of these geometric quantities in order to better understand the global geometry of the manifold.

On an n-dimensional smooth Finsler manifold (M, F), for E-curvature defined in (2.3), if

$$E_{AB} = \frac{n+1}{2} cF_{y^A y^B},$$

then the corresponding Finsler metric $F = (x, y)$ is called the *isotropic E-curvature*, where $c = c(x)$ is a scalar function. Specially, F is of *constant E-curvature* if c is a constant.

In recent years, there has been some progress in the study of the relationship between E-curvature and other curvatures. In 2003, X. Cheng and Z. Shen showed that the S-curvature is isotropic, and so is the E-curvature [2]. Then, they discussed Douglas metrics with relatively isotropic E-curvature and introduced the Finsler metrics of isotropic E-curvature [3]. For some special metric types, some results on E-curvature were obtained in recent years. B. Tiwari, R. Gangopadhyay, and G.K. Prajapati showed that a special class of generalized (α,β)-metric F has isotropic S-curvature if and only if it has isotropic E-curvature [4]. M. Li and L. Zhang showed that Finsler manifolds with vanishing Berwald scalar curvature have zero E-curvature [5]. Z. Yang, Y. He, and X.L. Zhang studied the doubly-warped product of Finsler manifolds with isotropic E-curvature [6]. K. Wang and C. Zhong proved that the (α,β)-metric has vanishing S-curvature if and only if F has vanishing E-curvature [7]. M. Crampin showed that the S-curvature of a Finsler space vanishes if and only if the E-curvature vanishes if and only if the Berwald scalar curvature vanishes [8]. A. Tayebi studied the E-curvature of the class of homogeneous Finsler manifolds and proved three rigidity theorems [9].

Recently, Finsler warped product metrics have garnered a lot of attention in research since some metrics with nice properties were found in this metric class. In 2018, M. Gabrani and B. Rezaei obtained a differential equation which characterizes a general (α,β)-metric with isotropic E-curvature, and solved the equation in a particular case [10]. In 2019, H. Liu and X. Mo found and solved the differential equation characterizing Finsler warped product metrics with vanishing Douglas curvature [11]. H. Liu, X. Mo, and H. Zhang found equations for Finsler warped product metrics of constant flag curvature and explicitly constructed many new warped product Douglas metrics of constant Ricci curvature using related known metrics [12]. In 2022, H. Liu, X. Mo, and L. Zhu found the equation which serves to characterize Finsler warped product metrics with isotropic S-curvature [13]. D. Zheng obtained the differential equations that characterize Landsberg-Finsler warped product metrics [14].

Approaches based on E-curvature and Finsler warped product metrics have garnered significant interest in recent research. In this paper, we mainly study E-curvature for Finsler warped product metrics which are the natural extension of Riemannian warped product metrics. Let I be an interval of R, and $\check{M}$ be an $(n-1)$-dimensional Riemannian manifold equipped with a Riemannian metric $\check{\alpha}$. We stipulate that

$$1 \le A, B, C, D \ldots \le n, \quad 2 \le i, j \ldots \le n.$$

Let $\check{M}$ be an $(n-1)$-dimensional smooth manifold ($n \ge 3$) with a Riemannian metric $\check{\alpha}$. A *Finsler warped product metric* on $M := \mathbb{R} \times \check{M}$ is a Finsler metric of the form

$$F(x, y) := \check{\alpha}(\check{x}, \check{y})\phi(r, s),$$

where $r = x^1$, $s = y^1/\check{\alpha}$, $x = (x^1, \check{x}) = (x^1, x^2, ..., x^n)$, $y = (y^1, \check{y}) = (y^1, y^2, ..., y^n)$, and $\phi(r, s)$ is a suitable function defined on a domain of R^2. It is the Finsler version of the warped product Riemann metric. In fact, by letting $\phi(r, s) = \sqrt{s^2 + u(r)}$, a Riemann metric is obtained, i.e.,

$$F(x, y) = \sqrt{(y^1)^2 + u(r)\check{\alpha}^2(\check{x}, \check{y})},$$

where $\check{\alpha}(\check{x}, \check{y}) = \sqrt{g_{ij}(\check{x})y^i y^j}$, $u(r)$ is a positive function.

The *Funk metric* on the unit open ball $\mathbb{B}^n$ is

$$F = \frac{\sqrt{|y|^2 - \left(|x|^2|y|^2 - \langle x, y\rangle^2\right)} + \langle x, y\rangle}{1 - |x|^2},$$

whose Finsler warped product form can be written as

$$F = \check{\alpha}_+ \frac{\sqrt{s^2 + r^2(1 - r^2)} + sr}{1 - r^2},$$

where $\check{\alpha}_+$ is the sphere metric [15]. As is known, the Funk metric has constant E-curvature.

In this paper, we find equivalent equations for warped product metrics with isotropic E-curvature and give several concrete examples. For convenience, the following notations are used, where the subscripts denote derivatives:

$$H := \frac{s\phi_r\phi_{ss} + \phi_s(\phi_r - s\phi_{rs})}{2\phi\phi_{ss}}, \quad X := sH - \frac{\phi_r - s\phi_{rs}}{2\phi_{ss}}, \tag{1.1}$$

$$\bar{H} := \int H \, \mathrm{ds}, \quad \Phi := \int \phi \, \mathrm{ds}. \tag{1.2}$$

Set

$$\omega = \frac{\phi_r - s\phi_{rs}}{2\phi_{ss}}. \tag{1.3}$$

Then, simplifying H and X yields

$$H = \frac{s\phi_r}{2\phi} + \frac{\phi_s}{\phi}\omega, \tag{1.4}$$

and

$$X = sH - \omega. \tag{1.5}$$

We first prove the following theorem.

Theorem 1.1. *Let $(\check{M}, \check{\alpha})$ be an $(n-1)$-dimensional Riemannian manifold with $n \geq 3$, and let $F(x, y) = \check{\alpha}(\check{x}, \check{y})\phi(r, s)$ be a Finsler metric on $M = I \times \check{M}$. Then, F is of isotropic E-curvature with $c = c(x)$, and $c = c(r)$ if and only if*

$$X = sH - (n+1)\bar{H} + (n+1)c(r)\Phi + \frac{1}{2}q(r)s^2 + l(r), \tag{1.6}$$

where $l(r)$ and $q(r)$ are some C^∞ function. In particular, when $c(r) = 0$, (1.6) *is simplified to*

$$X = sH - (n+1)\bar{H} + \frac{1}{2}q(r)s^2 + l(r), \tag{1.7}$$

which means F is of zero E-curvature.

As shown below, besides the Funk metric, some Finsler metrics of isotropic E-curvature are found by choosing special H.

Theorem 1.2. *(i) Let* $\frac{4f(r)(1+f(r))l_1(r)}{f_r(r)}$ *be a negative constant and*

$$\phi(r,s) = f(r)s + \sqrt{f(r)(1+f(r))s^2 - \frac{4f^2(r)\,(1+f(r))\,l_1(r)}{f_r(r)}},$$

where $f(r)$ *is a positive* C^∞ *function and* $l_1(r)$ *is a* C^∞ *function satisfying* $\frac{4f(r)(1+f(r))l_1(r)}{f_r(r)} < 0$. *Then, the following Randers warped product metric*

$$F(x,y) = \breve{\alpha}(\check{x},\check{y})\phi(r,s)$$

has isotropic E-curvature, i.e., its E-curvature is given by

$$E_{AB} = \frac{n+1}{8}\frac{f_r(r)}{f(r)(1+f(r))}F_{y^Ay^B}.$$

(ii) Let

$$\breve{\alpha} := \sqrt{(y^2)^2 + (y^3)^2}, \quad \phi(r,s) := \sqrt{s^2+e^{2r}}\tilde{\phi}(\frac{s}{\sqrt{s^2+e^{2r}}}),$$

and the positive C^∞ *function* $\tilde{\phi} = \tilde{\phi}(\tilde{s})$ *satisfies*

$$\tilde{\Phi} = -2(n+1)k_1\frac{\tilde{\phi}\Delta^2}{b^2-\tilde{s}^2},$$

for some constant k_1*, where*

$$\tilde{\Phi} := -(Q - \tilde{s}Q')\{n\Delta + 1 + \tilde{s}Q'\} - (b^2-\tilde{s}^2)(1+\tilde{s}Q)Q'',$$

and

$$\Delta := 1 + \tilde{s}Q + (b^2-\tilde{s}^2)Q', \quad Q := \tilde{\phi}'/(\tilde{\phi} - \tilde{s}\tilde{\phi}').$$

Then, F is of constant E-curvature.

Case(i) and Case(ii) will be proved in Section 4.

Remark 1.3. *In **Case(i)**,* $c(r) = \frac{f_r(r)}{4f(r)(1+f(r))}$*, taking* $f(r) = 1/(C_1e^{-4\lambda r} - 1)$*, where* C_1 *is an arbitrary positive constant, leads to* $c(r) = \lambda =$ *constant. In this condition, F is of constant E-curvature.*

In recent years, some applications of the Berwald metric (with vanishing Berwald curvature) to spacetimes in physics have been discussed [16–18]. The isotropic E-curvature condition in Finsler geometry, being a less restrictive requirement than vanishing Berwald curvature, may lead to promising physical applications in forthcoming studies. This suggests that our conclusions could yield meaningful physical applications in the future.

2. Preliminaries

In this section, the basic knowledge used in this paper is introduced.

Let M be an n-dimensional smooth manifold and $F = F(x,y)$ be a non-negative function on its tangent bundle TM. F is called a *Finsler metric* if it satisfies the following three conditions:

(i) Homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$;
(ii) Smoothness: F is a C^∞ function on $TM \backslash \{0\}$;
(iii) Strong convexity: For $\forall y \neq 0$, $(g_{AB}(x, y)) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^A \partial y^B}\right)$ is a positive definite matrix.

In this case, (M, F) is called a *Finsler manifold*. In particular, F is called a *Riemann metric* when $g_{ij}(x, y) = g_{ij}(x)$ is independent of y, and F is called a *Minkowski norm* (or*Minkowski metric*) when $F(x, y) = F(y)$ is independent of x.

In Theorem 1.2, an (α,β)-metric is used to construct the warped product metric. Here, we introduce the definition of the (α,β)-metric:

$$F(x, y) := \alpha(x, y)\tilde{\phi}(\tilde{s}), \quad \tilde{s} = \frac{\beta(x, y)}{\alpha(x, y)},$$

F is called an (α,β)-*metric*, where $\tilde{\phi}$ is an arbitrary C^∞ function that satisfies the following positive definiteness condition.

Lemma 2.1. *[19] $F(x, y) = \alpha(x, y)\tilde{\phi}(\tilde{s})$ is positive definite if and only if it satisfies*

$$\tilde{\phi}(\tilde{s}) > 0, \quad \tilde{\phi}(\tilde{s}) - \tilde{s}\tilde{\phi}'(\tilde{s}) + (\rho^2 - \tilde{s}^2)\tilde{\phi}'' > 0, \tag{2.1}$$

where $\tilde{s}$ and ρ are arbitrary real numbers satisfying $|\tilde{s}| \leq \rho < b_0$.

There are also positive definite judgment conditions of the Finsler warped product metric.

Lemma 2.2. *[15] Let $F(x, y) = \check{\alpha}(\check{x}, \check{y})\phi(r, s)$ be a Finsler warped product metric on $M = I \times \check{M}$. Then, the function ϕ satisfies*

$$\phi > 0, \quad \phi - s\phi_s > 0, \quad \phi_{ss} > 0, \tag{2.2}$$

where ϕ_s represents the partial derivative of ϕ with respect to s, and ϕ_{ss} is the second derivative of ϕ with respect to s.

The geodesics of F are characterized by the following ODEs:

$$\frac{d^2 x^A}{dt^2} + 2G^A\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^A = G^A(x, y)$ are called *geodesic coefficients* given by

$$G^A := \frac{1}{4} g^{AB}\left\{[F^2]_{y^B x^C} y^C - [F^2]_{x^B}\right\}.$$

A coordinate volume form dV for M is an n-form on a coordinate neighborhood

$$dV_F = \sigma_F(x) dx^1 \ldots dx^n,$$

where $\sigma_F(x) > 0$ is called the *volume measure function*. *S-curvature* is the rate of change of the Finsler metric distortion along a geodesic line. It can be calculated from the geodesic coefficient and volume measure function that

$$S(x, y) := \sum_A \frac{\partial G^A}{\partial y^A} - y^A \frac{\partial}{\partial x^A}(\ln \sigma_F(x)).$$

If $S = (n+1)c(x)F$, then F is called of *isotropic S-curvature*, where $c(x)$ is a scalar function. In particular, when $c(x) = c$ is a constant, then F is said to be of *constant S-curvature*.

A second-order symmetric tensor $E := E_{AB}(x,y)dx^A \otimes dx^B$ is called an *E-tensor* on a Finsler manifold. $\mathbf{E} := \{\mathbf{E}_y | y \in TM\backslash\{0\}\}$ is called the *E-curvature* or *mean Berwald curvature*, where $\mathbf{E}_y(u,v) := E_{AB}(x,y)u^i v^j$. It can be calculated from S-curvature that

$$E_{AB} = \frac{1}{2} S_{y^A y^B} = \frac{1}{2}\left[\frac{\partial G^C}{\partial y^C}\right]_{y^A y^B}. \tag{2.3}$$

By the above definition and (2.3), it is easy to see that F is of isotropic E-curvature $E_{AB} = \frac{(n+1)}{2} c F_{y^A y^B}$ if F is of isotropic S-curvature $S = (n+1)cF$.

3. The E-curvature of Finsler warped product metric

In this section, the expression for the E-curvature of Finsler warped product metric is given, and then Theorem 1.1 can be proved.

Before we prove Theorem 1.1, let us recall the geodesics coefficients of warped product metrics, which were given by B. Chen, Z. Shen and L. Zhao in [15].

Lemma 3.1. *[15] The expression of geodesic coefficients of the warped product metrics are as follows:*

$$G^1 = X\breve{\alpha}^2, \quad G^i = G^i_{\breve{\alpha}} + H\breve{\alpha}y^i,$$

where $G^i_{\breve{\alpha}}$ is the geodesic coefficient of the Riemannian metric $\breve{\alpha}$.

Taking the third-order derivative of G^A yields the following results, in particular where $(G^m)_{y^A y^B y^m}$ represents the sum of the m indices:

$$\begin{aligned}
(G^1)_{y^1y^1y^1} &= \frac{X_{sss}}{\alpha}, \quad (G^1)_{y^1y^1y^i} = -\frac{X_{sss}y^1 y_i}{\alpha^3},\\
(G^1)_{y^1y^iy^j} &= \frac{X_{sss}(y^1)^2 y_i y_j}{\alpha^5} + \frac{X_{ss}y^1 y_i y_j}{\alpha^4} - \frac{X_{ss}y^1 a_{ij}}{\alpha^2} + \frac{X_s a_{ij}}{\alpha} - \frac{X_s y_i y_j}{\alpha^3},\\
(G^m)_{y^my^1y^1} &= -\frac{H_{sss}y^1}{\alpha^2} + \frac{(n-2)H_{ss}}{\alpha}, \quad (G^m)_{y^my^iy^1} = \frac{H_{sss}(y^1)^2 y_i}{\alpha^4} + \frac{(2-n)H_{ss}y^1 y_i}{\alpha^3},\\
(G^m)_{y^my^jy^i} &= \frac{\left(nH\alpha^2 - nH_s\alpha y^1 + H_{ss}(y^1)^2\right)a_{ij}}{\alpha^3}\\
&\quad - \frac{\left(nH\alpha^3 - nH_s\alpha^2 y^1 - (n-3)H_{ss}\alpha(y^1)^2) + H_{sss}(y^1)^3\right)y_i y_j}{\alpha^6}.
\end{aligned}$$

By (2.3), the E-curvature of F can be directly obtained by

$$\begin{aligned}
E_{11} &= \frac{1}{2}\left(\frac{\partial^3 G^1}{\partial y^1\partial y^1\partial y^1} + \frac{\partial^3 G^m}{\partial y^m\partial y^1\partial y^1}\right), \quad E_{1i} = \frac{1}{2}\left(\frac{\partial^3 G^1}{\partial y^1\partial y^1\partial y^i} + \frac{\partial^3 G^m}{\partial y^m\partial y^1\partial y^i}\right),\\
E_{i1} &= \frac{1}{2}\left(\frac{\partial^3 G^1}{\partial y^1\partial y^i\partial y^1} + \frac{\partial^3 G^m}{\partial y^m\partial y^i\partial y^1}\right), \quad E_{ij} = \frac{1}{2}\left(\frac{\partial^3 G^1}{\partial y^1\partial y^i\partial y^j} + \frac{\partial^3 G^m}{\partial y^m\partial y^i\partial y^j}\right).
\end{aligned} \tag{3.1}$$

For simplicity, let $A_{ij} = \alpha^2 a_{ij} - y_i y_j$ and $s = \frac{y^1}{\alpha}$. Substituting the previous expression of $(G^m)_{y^m y^A y^B}$ into (3.1) yields

$$E_{11} = \frac{1}{2\alpha}\left(X_{sss} + (n-2)H_{ss} - sH_{sss}\right), \tag{3.2}$$

$$E_{1i} = E_{i1} = -\frac{sy_i}{2\alpha^2}\left(X_{sss} + (n-2)H_{ss} - sH_{sss}\right) = -\frac{sy_i}{\alpha}E_{11}, \tag{3.3}$$

$$\begin{aligned} E_{ij} =& \frac{1}{2\alpha^3}\Big(nH + X_s - s(nH_s + X_{ss}) + s^2 H_{ss}\Big)A_{ij} \\ &+ \frac{s^2 y_i y_j}{2\alpha^3}\left[X_{sss} + (n-2)H_{ss} - sH_{sss}\right]. \end{aligned}$$

It is easy to see that E_{ij} can be simply written into

$$E_{ij} = \frac{1}{2\alpha^3}\Big[nH + X_s - s(nH_s + X_{ss}) + s^2 H_{ss}\Big]A_{ij} + \frac{s^2 y_i y_j}{\alpha^2}E_{11}. \tag{3.4}$$

On the other hand, let $y_i = a_{ij}y^j$. Then, $F_{y^A y^B}$ can be represented by

$$F_{y^1 y^1} = \frac{\phi_{ss}}{\alpha}, \tag{3.5}$$

$$F_{y^1 y^i} = F_{y^i y^1} = -\frac{sy_i\phi_{ss}}{\alpha^2} = -\frac{sy_i}{\alpha}F_{y^1 y^1}, \tag{3.6}$$

$$F_{y^i y^j} = \frac{1}{\alpha^3}\left(\phi - s\phi_s\right)A_{ij} + \frac{y_i y_j s^2 \phi_{ss}}{\alpha^3}. \tag{3.7}$$

Proof of Theorem 1.1. Before the proof of "necessity" and "sufficiency", we need to perform some computations. By the assumption $c = c(x)$,

$$E_{AB} = \frac{1}{2}(n+1)c(x)F_{y^A y^B}, \tag{3.8}$$

which together with (3.3) and (3.6) implies that $E_{11} = \frac{1}{2}(n+1)c(x)F_{11}$ is equivalent to $E_{1i} = \frac{1}{2}(n+1)c(x)F_{1i}$. Then, (3.8) is equivalent to

$$E_{11} = \frac{1}{2}(n+1)c(x)F_{11}, \quad E_{ij} = \frac{1}{2}(n+1)c(x)F_{ij}.$$

Substituting (3.4), (3.5), and (3.7) into the above formulas leads to

$$E_{11} = \frac{n+1}{2}c(x)\frac{\phi_{ss}}{\alpha}, \tag{3.9}$$

$$\begin{aligned} E_{ij} =& \frac{1}{2\alpha^3}\Big[nH + X_s - s(nH_s + X_{ss}) + s^2 H_{ss}\Big]A_{ij} + \frac{s^2 y_i y_j}{\alpha^2}E_{11} \\ =& \frac{n+1}{2\alpha^3}c(x)\Big[\left(\phi - s\phi_s\right)A_{ij} + y_i y_j s^2 \phi_{ss}\Big]. \end{aligned} \tag{3.10}$$

Now, we first prove the "necessity". Substituting (3.9) into (3.10) yields

$$\Big[nH + X_s - s(nH_s + X_{ss}) + s^2 H_{ss}\Big]A_{ij} = (n+1)c(x)\left(\phi - s\phi_s\right)A_{ij}.$$

Contracting both sides of the above equation by a^{ij}, and noting that $A_{ij}a^{ij} = (n-2)\alpha^2 \neq 0$ gives

$$(nH + X_s) - s(nH_s + X_{ss}) + s^2 H_{ss} = (n+1)c(x)(\phi - s\phi_s). \tag{3.11}$$

Since H, X, and ϕ all are functions of r and s, $c(x) = c(r, \check{x})$ must be a function of r and s, and moreover it is independent of y. Hence,

$$c(x) = c(r).$$

Equation (3.11) can be rewritten into

$$(-s^2)\left[\frac{nH + X_s}{s}\right]_s + s^2 H_{ss} = (n+1)c(r)\left(\frac{\phi}{s}\right)_s (-s^2),$$

which is equivalent to

$$s^2\left(H_s + (n+1)\,c(r)\left(\frac{\phi}{s}\right) - \left[\frac{nH + X_s}{s}\right]\right)_s = 0,$$

which implies

$$\left(H_s + (n+1)\,c(r)\left(\frac{\phi}{s}\right) - \left[\frac{nH + X_s}{s}\right]\right)_s = 0.$$

Integrating both sides of the above equation by s gives

$$X_s = (sH)_s - (n+1)H + (n+1)c(r)\phi + q(r)s. \tag{3.12}$$

which leads to (1.6) by an integration with respect to s.

Next, we prove the "sufficiency". Obviously, (1.6) implies (3.12). Then, differentiating X_s in (3.12) with respect to s yields

$$\begin{aligned} X_{ss} &= (sH)_{ss} - (n+1)H_s + (n+1)c(r)\phi_s + q(r), \\ X_{sss} &= -(n-2)H_{ss} + sH_{sss} + (n+1)c(r)\phi_{ss}. \end{aligned}$$

A direct computation shows that the above X_s, X_{ss}, and X_{sss} together with (3.2) lead to that (3.9) and (3.10) hold true. Thus, the E-curvature is isotropic if X satisfies (1.6).

In particular, when $c(r) = 0$, it is a special case, and it can be similarly proved that (1.7) holds true.

4. Some explicit metrics of isotropic E-curvature

In this section, some special solutions of equivalent equations of isotropic E-curvature and constant E-curvature are found. Then, by checking the positive definiteness, whether the required ϕ can induce a Finsler metric can be determined.

By (1.3) and (1.4), the relation between ϕ and H can be found in the following steps. By (1.4), ϕ_r can be expressed by

$$\phi_r = 2\frac{\phi H - \phi_s \omega}{s}. \tag{4.1}$$

Differentiating (4.1) with respect to s yields

$$\phi_{rs} = 2\left[\frac{\phi_s H + \phi H_s - \phi_{ss}\omega - \phi_s \omega_s}{s} - \frac{\phi H - \phi_s \omega}{s^2}\right]. \tag{4.2}$$

Since $(2.2)_3$ ($\phi_{ss} > 0$), then rewriting (1.3) gives

$$\phi_r - s\phi_{rs} - 2\omega\phi_{ss} = 0.$$

Substituting (4.1) and (4.2) into the above equation yields

$$\frac{2\phi H - s\phi H_s + \phi_s[s\omega_s - sH - 2\omega]}{s} = 0, \tag{4.3}$$

whose solution can be obtained as

$$\phi(r, s) = e^{\int \frac{sH_s - 2H}{s\omega_s - sH - 2\omega}\,\mathrm{d}s + m(r)},$$

where $m(r)$ is an arbitrary C^∞ function of r.

For general H, because of the high nonlinearity, it is hard to express ϕ explicitly. Here, the explicit expression for ϕ is important to check whether it satisfies (2.2), which ensures the corresponding $F = \breve{\alpha}\phi$ is a Finsler metric. Here, some special examples are given. By assuming that H in Theorem 1.1 is given in a special form as

$$H = c(r)\phi(r, s) + 2c(r)s, \quad \text{where} \quad c(r) = \frac{f_r(r)}{4f(r)(1 + f(r))}, \tag{4.4}$$

then the derivative of H and the integral of H which is $\bar{H}$ in (1.2) are given by

$$H_s = c(r)\phi_s(r, s) + 2c(r), \quad \bar{H} = c(r)\Phi + c(r)s^2 + \eta(r), \tag{4.5}$$

where $\eta(r)$ is an arbitrary C^∞ function. Substituting X in (1.5) into (1.6) in Theorem 1.1 yields

$$\omega = (n + 1)\bar{H} - (n + 1)c(r)\Phi - \frac{1}{2}q(r)s^2 - l(r),$$

which implies

$$\omega_s = (n + 1)H - (n + 1)c(r)\Phi_s - q(r)s - l(r).$$

Then, substituting $\bar{H}$ in (4.5) into the above equation, yields

$$\omega = \left[(n + 1)c(r) - \frac{1}{2}q(r)\right] s^2 - l_1(r),$$

where $l_1(r) = l(r) - (n + 1)\eta(r)$. Since $l(r)$ and $\eta(r)$ are arbitrary C^∞ functions, then so is $l_1(r)$. Then, substituting H in (4.4), H_s in (4.5), and ω and ω_s in the above equations into (4.3) gives

$$c(r)\phi^2 + c(r)s\phi - c(r)s\phi\phi_s - c(r)s^2\phi_s + l_1(r)\phi_s = 0.$$

The solution to the above equation is

$$\phi(r, s) = \left(c(r)s + \sqrt{m(r)c(r)s^2 + (c(r) - m(r))l_1(r)}\right) / \left(m(r) - c(r)\right).$$

$m(r)$ is an arbitrary C^∞ function, and by letting $m(r) = (1 + \frac{1}{f(r)})c(r)$, where $f(r) > 0$, we get

$$\phi(r, s) = f(r)s + \sqrt{f(r)(1 + f(r))s^2 - \frac{f(r)l_1(r)}{c(r)}}. \tag{4.6}$$

Obviously, as ϕ is given by (4.6), $F = \check{\alpha}\phi(r, s)$ is a Randers metric, which will be showed being with isotropic E-curvature in the following proof.

Proof of case (1.2) in Theorem 1.2. Let $c(r) = \frac{f_r(r)}{4f(r)(1+f(r))}$ as in (4.4). Then, by the above discussions and (4.6), we only need to verify the positive definiteness of $F = \check{\alpha}(\check{x}, \check{y})\phi(r, s)$. A direct computation yields

$$\phi - s\phi_s = -\frac{f(r)l_1(r)}{c(r)\sqrt{f(r)(1+f(r))s^2 - \frac{f(r)l_1(r)}{c(r)}}}, \quad \phi_{ss} = -\frac{f^2(r)(1+f(r)l_1(r))}{c(r)(f(r)(1+f(r))s^2 - \frac{f(r)l_1(r)}{c(r)})^{\frac{3}{2}}}.$$

These expressions and Lemma 2.2 tell that $\frac{l_1(r)}{c(r)} < 0$.

Now the E-curvature of $F = \check{\alpha}(\check{x}, \check{y})\phi(r, s)$ can be given. Substituting (4.6) into H and X in (1.1), and then calculating ϕ_{ss}, X_{sss}, H_{ss}, and H_{sss}, gives

$$X_{sss} + (n-2)H_{ss} - sH_{sss} = \frac{(n+1)f_r(r)}{4f(r)(1+f(r))}\phi_{ss},$$

which, compared with (3.2) and (3.5), implies

$$E_{11} = \frac{n+1}{8}\frac{f_r(r)}{f(r)(1+f(r))}F_{y^1y^1}. \tag{4.7}$$

According to (3.3) and (3.6), it is obvious that

$$E_{1i} = \frac{n+1}{8}\frac{f_r(r)}{f(r)(1+f(r))}F_{y^1y^i}.$$

Similarly, a direct computation shows that

$$nH + X_s - s(nH_s + X_{ss}) + s^2H_{ss} = \frac{(n+1)f_r(r)}{4f(r)(1+f(r))}(\phi - s\phi_s),$$

which together with (3.4) implies

$$E_{ij} = \frac{(n+1)f_r(r)A_{ij}}{8\alpha^3 f(r)(1+f(r))}(\phi - s\phi_s) + \frac{s^2 y_i y_j}{\alpha^2}E_{11},$$

which comparing with (3.5), (3.7), and (4.7) implies

$$E_{ij} = \frac{n+1}{8}\frac{f_r(r)}{f(r)(1+f(r))}F_{y^iy^j}.$$

In summary, when $\phi(r, s) = f(r)s + \sqrt{f(r)(1+f(r))s^2 - \frac{4f^2(r)(1+f(r))l_1(r)}{f_r(r)}}$, the corresponding warped product metric $F = \check{\alpha}\phi(r, s)$ is of isotropic E-curvature.

The idea of case(ii) in Theorem 1.2 is from [20]. In [20], Cheng and Shen introduce a special (α, β)-metric with constant S-curvature as follows.

Lemma 4.1. *[20] Let $F = \alpha\tilde{\phi}(\frac{\beta}{\alpha})$ be an (α,β)-metric defined on an open subset in $\mathbb{R}^3$, where α and β are given by*

$$\alpha := \sqrt{(y^1)^2 + e^{2x^1}((y^2)^2 + (y^3)^2)}, \quad \beta := y^1.$$

Then, β satisfies

$$r_{ij} = \varepsilon\left\{b^2 a_{ij} - b_i b_j\right\}, \quad s_j = 0,$$

with $\varepsilon = 1$ and $b = 1$. Thus, if $\tilde{\phi} = \tilde{\phi}(\tilde{s})$ satisfies

$$\tilde{\Phi} = -2(n+1)k_1\frac{\tilde{\phi}\Delta^2}{b^2 - \tilde{s}^2}, \tag{4.8}$$

for some constant k_1, then F is of constant S-curvature, and $S = (n+1)c_1 F$ with $c_1 = k_1\varepsilon = k_1$. Here,

$$\tilde{\Phi} := -(Q - \tilde{s}Q')\{n\Delta + 1 + \tilde{s}Q'\} - (b^2 - \tilde{s}^2)(1 + \tilde{s}Q)Q'',$$

and

$$\Delta := 1 + \tilde{s}Q + (b^2 - \tilde{s}^2)Q', \quad Q := \tilde{\phi}'/(\tilde{\phi} - \tilde{s}\tilde{\phi}'), \quad \tilde{s} = \frac{\beta}{\alpha}.$$

In case (1.2) of Theorem 1.2, the key step is to verify (2.1) for $\tilde{\phi}$.

Proof of case (1.2) in Theorem 1.2. Since F in Lemma 4.1 is of constant S-curvature with $c(r) = k_1$, then F is of constant E-curvature with $c(r) = k_1$.

We claim that the metric in Lemma 4.1 can be written into a warped product metric. In fact,

$$\breve{\alpha} = \sqrt{(y^2)^2 + (y^3)^2}, \quad \phi(r,s) := \sqrt{s^2 + e^{2r}}\tilde{\phi}(\frac{s}{\sqrt{s^2 + e^{2r}}}).$$

Then,

$$\breve{\alpha}\phi(r,s) = \sqrt{(y^1)^2 + e^{2x^1}((y^2)^2 + (y^3)^2)}\tilde{\phi}(\frac{y^1}{\sqrt{(y^1)^2 + e^{2x^1}((y^2)^2 + (y^3)^2)}}),$$

which satisfies $\breve{\alpha}\phi(r,s) = \alpha\tilde{\phi}(\tilde{s})$ in Lemma 4.1, where

$$\tilde{s} = \frac{s}{\sqrt{s^2 + e^{2r}}}.$$

To ensure the positively definiteness of the metric, it is necessary to very $(2.2)_2$ in Lemma 2.2. A direct computation yields

$$\tilde{\phi} - \frac{s}{\sqrt{s + e^{2r}}}\tilde{\phi}' = \tilde{\phi} - \tilde{s}\tilde{\phi}' > 0,$$

where the last inequality is from (2.1) by setting $\rho = s$. Similarly, $(2.2)_3$ in Lemma 2.2 yields

$$\tilde{\phi} - \frac{s}{\sqrt{s + e^{2r}}}\tilde{\phi}' + \left(1 - \frac{s^2}{s^2 + e^{2r}}\right)\tilde{\phi}'' = \tilde{\phi} - \tilde{s}\tilde{\phi}' + \left(1 - \tilde{s}^2\right)\tilde{\phi}'' > 0.$$

From (2.1), the above equation obviously holds true. These satisfy (2.2) in Lemma 2.2. Thus, the metric is a warped product metric of constant E-curvature, and $E_{AB} = \frac{n+1}{2}k_1 F_{y^A y^B}$ with k_1 is the constant in (4.8).

5. Conclusions

We have investigated the E-curvature of Finsler warped product metrics. Our main findings are summarized as follows: (1) We derive the equivalent equation for warped product metrics with isotropic or zero E-curvature. (2) We present two concrete examples that it is of isotropic or constant E-curvature. Moreover, the proposed approach-prescribing the form of H and then invoking the equivalent equation provides a practical route to determining the explicit expression of ϕ. Nevertheless, several limitations remain. Owing to the high nonlinearity of the equivalent equation, we cannot solve for ϕ directly from it. Resolving these issues demands a more advanced theory for solving nonlinear PDEs.

Author contributions

Benling Li: Conceptualization, funding acquisition, methodology, resources, supervision, validation, writing-review & editing; Ke Xu: Data curation, investigation, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used generative artificial intelligence tools in the creation of this article.

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Conflict of interest

There is no conflict of interest.

References

1. Z. Shen, *Differential geometry of spray and Finsler spaces*, Dordrecht: Springer, 2001. https://doi.org/10.1007/978-94-015-9727-2
2. X. Chen, Z. Shen, Randers metrics with special curvature properties, *Osaka J. Math.*, **40** (2003), 87–101.
3. X. Chen, Z. Shen, On Douglas metrics, *Publ. Math. Debrecen*, **66** (2005), 503–512. https://doi.org/10.5486/PMD.2005.3192
4. B. Tiwari, R. Gangopadhyay, G. K. Prajapati, A class of Finsler spaces with general (α,β)-metrics, *Int. J. Geom. Methods Mod. Phys.*, **16** (2019), 1950102. https://doi.org/10.1142/S0219887819501020

5. M. Li, L. Zhang, Properties of Berwald scalar curvature, *Front. Math. China*, **15** (2020), 1143–1153. https://doi.org/10.1007/s11464-020-0872-7
6. Z. Yang, Y. He, X. L. Zhang, S-curvature of doubly warped product of Finsler manifolds, *Acta Math. Sin. English Ser.*, **36** (2020), 1292–1298. https://doi.org/10.1007/s10114-020-9427-9
7. K. Wang, C. Zhong, A characterization of weakly Berwald spaces with (α,β)-metrics, *Differ. Geom. Appl.*, **82** (2022), 101870. https://doi.org/10.1016/j.difgeo.2022.101870
8. M. Crampin, S-curvature, E-curvature, and Berwald scalar curvature of Finsler spaces, *Differ. Geom. Appl.*, **92** (2024), 102080. https://doi.org/10.1016/j.difgeo.2023.102080
9. A. Tayebi, On E-curvature of homogeneous Finsler manifolds, *Period. Math. Hung.*, **90** (2025), 140–155. https://doi.org/10.1007/s10998-024-00610-4
10. M. Gabrani, B. Rezaei, On general (α,β)-metrics with isotropic E-curvature, *J. Korean Math. Soc.*, **55** (2018), 415–424. https://doi.org/10.4134/JKMS.j170277
11. H. Liu, X. Mo, Finsler warped product metrics of Douglas type, *Can. Math. Bull.*, **62** (2019), 119–130. https://doi.org/10.4153/CMB-2017-077-0
12. H. Liu, X. Mo, H. Zhang, Finsler warped product metrics with special Riemannian curvature properties, *Sci. China Math.*, **63** (2020), 1391–1408. https://doi.org/10.1007/s11425-018-9422-4
13. H. Liu, X. Mo, L. Zhu, Finsler warped product metrics with isotropic S-curvature, *Differ. Geom. Appl.*, **81** (2022), 101865. https://doi.org/10.1016/j.difgeo.2022.101865
14. D. Zheng, Landsberg Finsler warped product metrics with zero flag curvature, *Differ. Geom. Appl.*, **93** (2024), 102082. https://doi.org/10.1016/j.difgeo.2023.102082
15. B. Chen, Z. Shen, L. Zhao, Constructions of Einstein Finsler metrics by warped product, *Int. J. Math.*, **29** (2018), 1850081. https://doi.org/10.1142/S0129167X18500817
16. S. Dhasmana, Z. K. Silagadze, Finsler spacetime in light of Segal's principle, *Mod. Phys. Lett. A*, **35** (2020), 2050019. https://doi.org/10.1142/S0217732320500194
17. R. G. Torromé, On singular generalized Berwald spacetimes and the equivalence principle, *Int. J. Geom. Methods Mod. Phys.*, **14** (2017), 1750091. https://doi.org/10.1142/S0219887817500918
18. M. Zhou, S. D. Liang, Finslerian geometrodynamics, *Int. J. Theor. Phys.*, **63** (2024), 158. https://doi.org/10.1007/s10773-024-05681-0
19. Z. Shen, Landsberg curvature, S-curvature and Riemann curvature, In: *A sampler of Riemann-Finsler geometry*, **50** (2004), 303–355.
20. X. Cheng, Z. Shen, A class of Finsler metrics with isotropic S-curvature, *Israel J. Math.*, **169** (2009), 317–340. https://doi.org/10.1007/s11856-009-0013-1