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Research article

Existence and stability of limit cycles in a perturbed seventh-order differential equation

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Abstract: This paper investigates the existence and stability of limit cycles in a class of perturbed seventh-order non-autonomous differential equations that model multi-frequency oscillatory behavior with damping, commonly encountered in mechanical and control systems. The high order of the equation presents significant challenges in detecting periodic solutions and analyzing their stability. By applying first-order averaging theory, we reduced the original equation to a lower-dimensional averaged equation. We then derived explicit conditions under which isolated periodic solutions bifurcate from the equilibrium point. Furthermore, the stability of these limit cycles was examined by analyzing the Jacobian matrix of the averaged equation. These results extend the applicability of averaging methods to high-order nonlinear differential equations and provide valuable insights into the dynamics and control of oscillatory phenomena in science and engineering.

Keywords: seventh-order differential equation; periodic solution; averaging theory; stability
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1. Introduction

High-order differential equations play a critical role in modeling complex dynamical behaviors in science and engineering, particularly in systems involving elasticity, fluid mechanics, and control processes. Among these, seventh-order differential equations have drawn specific interest due to their ability to represent intricate physical phenomena, such as beam vibrations, wave propagation, and multiscale oscillations. Their inherent analytical and numerical challenges including issues of existence, uniqueness, stability, and solution convergence have motivated sustained mathematical inquiry. Over the years, researchers have made substantial contributions to both the theoretical and computational aspects of these equations [5, 9, 15, 17, 19]. These efforts have strengthened the general analytical framework for high-order systems and advanced the understanding of boundary value problems and nonlinear dynamics specific to seventh-order models. Additionally, foundational results in this area have facilitated more accurate approximation and simulation techniques for handling the high dimensionality and sensitivity of such systems.

The study of periodic solutions, particularly limit cycles, is central to the qualitative analysis of differential equations due to their role in describing long-term system behavior [3]. In high-order non-autonomous equations, detecting and analyzing such cycles is especially challenging, as classical results like the Poincaré–Bendixson theorem do not extend beyond two dimensions [14]. While several works have addressed seventh-order boundary value problems using functional analytic and spline-based techniques [7, 8], and others have improved numerical stability through periodized Shannon wavelets [16], these approaches largely focus on computational performance rather than the qualitative behavior of periodic orbits [4, 14].

Averaging theory has emerged as a rigorous analytical framework for studying complex, high-dimensional nonlinear systems. It reduces a complicated system to a simpler averaged system whose solutions closely approximate the long-term behavior of the original, facilitating the detection of bifurcations, limit cycles, and invariant sets that may be hidden by the system's complexity. Recent work demonstrates averaging's effectiveness in analyzing high-order oscillators, time-periodic control systems, and discontinuous switched systems [1, 6, 10]. These advances highlight averaging theory's advantages over classical methods such as Lyapunov functions and Melnikov analysis, which often fail in high-order or non-smooth settings. Building on these developments, this article applies averaging theory to seventh-order differential equations to rigorously study the existence, bifurcation, and stability of periodic solutions where traditional methods are insufficient.

In the present work, we consider a structurally rich class of perturbed seventh-order non-autonomous differential equations incorporating both dissipative and oscillatory parameters.

Recently, Tabet and Makhlouf [18] investigated periodic solutions for a class of seventh-order differential equations using the first-order averaging method. Their analysis focused on equations involving only odd-order derivatives and established the existence of limit cycles, without addressing their stability. In contrast, the present study is devoted to a more general class of perturbed seventh-order differential equations that incorporate both even- and odd-order derivatives. More precisely, we consider the following differential equation:

$$x^{(7)} - \lambda x^{(6)} + Ax^{(5)} - \lambda Ax^{(4)} + Cx^{(3)} - \lambda Cx^{(2)} + (C - A + 1)x' - \lambda(C - A + 1)x = \varepsilon f\big(t, x, x', x'', x^{(3)}, x^{(4)}, x^{(5)}, x^{(6)}\big), \quad (1.1)$$

where we assume

(H1) $\lambda \in \mathbb{R}$, $A = p^2 + q^2 + 1$, $C = p^2q^2 + p^2 + q^2$, $p, q \in \mathbb{Q} \setminus \{0, \pm 1\}$, $p \neq \pm q$, and $\varepsilon > 0$ is a small parameter.

(H2) $f : \mathbb{R}^8 \to \mathbb{R}$ is a C^2 nonlinear, non-autonomous, and periodic excitation depending on time t and on the state variables up to the sixth derivative.

Our goal is to investigate the bifurcation and stability of limit cycles of the differential Eq (1.1), which emanate from the trivial equilibrium under the influence of a small, time-periodic perturbation. By applying the first-order averaging method, we reduce the original equation to an averaged equation of lower complexity and establish explicit criteria for the existence and local stability of periodic solutions. For further details on averaging theory, see Section 3 and the references therein, as well as the books [13, 20].

This contribution not only advances the theoretical understanding of high-order oscillatory dynamics but also provides a constructive analytical framework applicable to a wide range of systems in mathematical physics and engineering. To the best of our knowledge, this is the first study addressing limit cycle bifurcations in such seventh-order systems using averaging theory, thus offering a novel and valuable perspective in nonlinear dynamical analysis.

The remainder of this paper is organized as follows. Section 2 presents the statement of results. Section 3 recalls some fundamental results on the first-order averaging theory. Section 4 is devoted to the proofs of the main results. Finally, the paper ends with a conclusion summarizing the main findings. Some symbolic and algebraic computations were performed using Maple.

2. Statement of results

In this section, we establish rigorous analytical conditions for the existence of limit cycles in the perturbed seventh-order differential Eq (1.1).

Consider the function $F : \mathbb{R}^6 \to \mathbb{R}^6$, $F = (F_1, F_2, F_3, F_4, F_5, F_6)^T$ defined by

$$\begin{aligned}
F_1(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} \cos(qt)\, f(t,c)\, dt, & F_2(X_0) &= \frac{-1}{2m\pi}\int_0^{2m\pi} \sin(qt)\, f(t,c)\, dt,\\
F_3(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} \cos(pt)\, f(t,c)\, dt, & F_4(X_0) &= \frac{-1}{2m\pi}\int_0^{2m\pi} \sin(pt)\, f(t,c)\, dt,\\
F_5(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} \cos(t)\, f(t,c)\, dt, & F_6(X_0) &= \frac{-1}{2m\pi}\int_0^{2m\pi} \sin(t)\, f(t,c)\, dt,
\end{aligned} \tag{2.1}$$

where $X_0 = (X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0})^T$ and $c = (C_1, C_2, C_3, C_4, C_5, C_6, C_7)^T$. C_i, $i = \overline{1,7}$, are given in Appendix A. Then we have the following result.

Theorem 2.1. *Let $p = \frac{p_1}{p_2}$ and $q = \frac{q_1}{q_2}$ be nonzero rational numbers satisfying the conditions (H1) with p_1, p_2, q_1, q_2 as nonzero integers, and $\lambda \neq 0$ in the differential Eq (1.1). Assume that f is a $2m\pi$-periodic function satisfying the conditions (H2), where m is the least common multiple of p_2 and q_2. Then, for each simple zero solution $X_0^* = (X_{1,0}^*, X_{2,0}^*, X_{3,0}^*, X_{4,0}^*, X_{5,0}^*, X_{6,0}^*)^T$ of the system,*

$$F_i(X_0) = 0, \quad i = \overline{1,6} \tag{2.2}$$

satisfying the condition

$$\det\left(\left.\frac{\partial F}{\partial X_0}\right|_{X_0 = X_0^*}\right) \neq 0. \tag{2.3}$$

Equation (1.1) admits a $2m\pi$-periodic solution $x(t,\varepsilon)$ approaching the following periodic solution:

$$\begin{aligned}
\phi(t) = &-\frac{(q^2-1)(\lambda X_{4,0} + pX_{3,0})\cos(pt)}{(p^2q^2 - p^2 - q^2 + 1)\, p\, (\lambda^2 + p^2)\, (p^2 - q^2)} + \frac{(p^2-1)(qX_{1,0} + \lambda X_{4,0})\cos(qt)}{(p^2q^2 - p^2 - q^2 + 1)\, q\, (p^2 - q^2)\, (\lambda^2 + q^2)}\\
&+\frac{(q^2-1)(pX_{4,0} - \lambda X_{3,0})\sin(pt)}{(p^2q^2 - p^2 - q^2 + 1)\, p\, (\lambda^2 + p^2)\, (p^2 - q^2)} - \frac{(\lambda X_{6,0} + X_{5,0})\cos(t) + (\lambda X_{5,0} - X_{6,0})\sin(t)}{(\lambda^2 + 1)\, (p^2q^2 - p^2 - q^2 + 1)}\\
&+\frac{(p^2-1)(\lambda X_{1,0} - qX_{2,0})\sin(qt)}{(p^2q^2 - p^2 - q^2 + 1)\, q\, (p^2 - q^2)\, (\lambda^2 + q^2)}
\end{aligned} \tag{2.4}$$

of

$$x^{(7)} - \lambda x^{(6)} + Ax^{(5)} - \lambda Ax^{(4)} + Cx^{(3)} - \lambda Cx^{(2)} + (C - A + 1)x' - \lambda(C - A + 1)x = 0,$$

when $\varepsilon \to 0$.

The proof of Theorem 2.1 is established in Section 4 and is based on averaging theory. Refer to Section 3 for further details.

We now present an example to illustrate the effectiveness of the results stated in the following corollary.

Corollary 2.2. *Consider the function* $f(t, x, x', x'', x''', x^{(4)}, x^{(5)}, x^{(6)}, x^{(7)}) = x^{(5)} - sin(t)$*, with the parameters* $p = 2$*,* $q = 3$*, and* $\lambda = \frac{1}{2}$*. Then, the differential Eq* (1.1) *admits one periodic solution* $x(t, \varepsilon)$ *which approaches the periodic solution given by* $\phi_0(t) = -\cos(t)$*, of the equation* $x^{(7)} - \frac{1}{2}x^{(6)} + 14x^{(5)} - 7x^{(4)} + 49x''' - \frac{49}{2}x'' + 36x' - 18x = 0$*, when* $\varepsilon \to 0$*. Note that this periodic solution is unstable. Corollary 2.2 is shown in Section 4.*

Remark 2.3. *In Corollary 2.2, we considered the equation*

$$x^{(7)} - \frac{1}{2}x^{(6)} + 14x^{(5)} - 7x^{(4)} + 49x''' - \frac{49}{2}x'' + 36x' - 18x = x^{(5)} - \sin(t).$$

This nonlinear equation can describe a high-order mechanical oscillator with derivative-based feedback. The presence of the fifth derivative $x^{(5)}$ *on both sides introduces a nonlinear damping effect, while the sinusoidal term*$-\sin(t)$ *on the right-hand side acting as an external periodic forcing that generates multifrequency oscillations. Our analysis predicts the existence of one unstable limit cycle for this equation.*

Let us consider the function $F : \mathbb{R}^7 \to \mathbb{R}^7$, $F = (F_1, F_2, F_3, F_4, F_5, F_6, F_7)^T$ defined by

$$\begin{aligned}
F_1(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} cos(qt)f(t, b)\,dt, & F_2(X_0) &= \frac{-1}{2m\pi}\int_0^{2m\pi} sin(qt)f(t, b)\,dt,\\
F_3(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} cos(pt)f(t, b)\,dt, & F_4(X_0) &= \frac{-1}{2m\pi}\int_0^{2m\pi} sin(pt)f(t, b)\,dt,\\
F_5(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} cos(t)f(t, b)\,dt, & F_6(X_0) &= \frac{-1}{2m\pi}\int_0^{2m\pi} sin(t)f(t, b)\,dt,\\
F_7(X_0) &= \frac{1}{2m\pi}\int_0^{2m\pi} f(t, b)\,dt,
\end{aligned} \tag{2.5}$$

where $X_0 = (X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0}, X_{7,0})^T$ and $b = (B_1, B_2, B_3, B_4, B_5, B_6, B_7)^T$. B_i, $i = \overline{1, 7}$, are given in Appendix B. Then we have the following result.

Theorem 2.4. *Let* $p = \frac{p_1}{p_2}$ *and* $q = \frac{q_1}{q_2}$ *be nonzero rational numbers satisfying the conditions (H1) with* p_1*,* p_2*,* q_1*,* q_2 *as nonzero integers, and* $\lambda = 0$ *in the differential Eq* (1.1)*. Assume that f is a* $2m\pi$*-periodic function satisfying the conditions (H2), where m is the least common multiple of* p_2 *and* q_2*. Then, for each simple zero solution* $X_0^* = (X_{1,0}^*, X_{2,0}^*, X_{3,0}^*, X_{4,0}^*, X_{5,0}^*, X_{6,0}^*, X_{7,0}^*)^T$ *of the system,*

$$F_i(X_0) = 0,\ i = \overline{1, 7} \tag{2.6}$$

satisfying the condition

$$\det\left(\frac{\partial F}{\partial X_0}\bigg|_{X_0=X_0^*}\right) \neq 0. \tag{2.7}$$

Equation (1.1) *admits a* $2m\pi$*-periodic solution* $x(t,\varepsilon)$ *tending to the following periodic solution:*

$$\begin{aligned}\phi(t) = {} & \frac{p^2 X_{1,0}}{(p^2-q^2)\, p^2q^2(p^2-1)}\cos(qt) - \frac{q^2 X_{3,0}}{(p^2-q^2)\, p^2q^2(p^2-1)}\cos(pt) \\ & + \frac{q^2 X_{4,0}}{(p^2-q^2)\, p^2q^2(p^2-1)}\sin(pt) + \frac{p^2 X_{2,0}}{(p^2-q^2)\, p^2q^2(p^2-1)}\sin(qt) \\ & + \frac{X_{6,0}}{(q^2-1)(p^2-1)}\sin(t) - \frac{X_{5,0}}{(q^2-1)(p^2-1)}\cos(t) + \frac{X_{7,0}}{p^2q^2}\end{aligned} \tag{2.8}$$

of

$$x^{(7)} + Ax^{(5)} + Cx''' + (C - A + 1)x' = 0,$$

when $\varepsilon \to 0$*.*

The proof of Theorem 2.4 is established in Section 4 and is based on averaging theory. Refer to Section 3 for further details.

Now, we present examples to illustrate the effectiveness of our results in the following corollary.

Corollary 2.5. *Consider the function* $f(t,x,x',x'',x^{(3)},x^{(4)},x^{(5)},x^{(6)}) = sin(t)+x-x'-x''+x^{(4)}-x^{(5)}+x^{(6)}$ *with* $p = \frac{1}{2}$*,* $q = \frac{1}{3}$*, and* $\lambda = 0$*. Then Eq* (1.1) *admits one periodic solution tending to the periodic solution given by* $\phi_0(t) = -\frac{cos(t)}{4} - \frac{sin(t)}{4}$ *of the equation* $x^{(7)} + \frac{49}{36}x^{(5)} + \frac{7}{18}x''' + \frac{1}{36}x' = 0$*, when* $\varepsilon \to 0$*. Note that this periodic solution is unstable. Corollary 2.5 is shown in Section 4.*

Remark 2.6. *In Corollary 2.5, we analyzed the equation*

$$x^{(7)} + \frac{49}{36}x^{(5)} + \frac{7}{18}x''' + \frac{1}{36}x' = \sin(t) + x - x' - x'' + x^{(4)} + x^{(5)} - x^{(6)}. \tag{2.9}$$

Equation (2.9) *represents a damped multi-degree-of-freedom oscillator, where the combination of derivative terms models stiffness, damping, and higher-order dynamics. The sinusoidal forcing* $\sin(t)$ *generates multifrequency oscillations. The system admits one unstable limit cycle, illustrating the practical relevance of our theoretical results to mechanical and control systems.*

Corollary 2.7. *Consider the function* $f(t,x,x',x'',x''',x^{(4)},x^{(5)},x^{(6)}) = x^2 - 1$ *with* $p = 2$*,* $q = 3$*, and* $\lambda = 0$*. Then Eq* (1.1) *admits eighteen limit cycles tending to the periodic solutions given by*

$$\begin{aligned}
\phi_1(t) &= 0.7916856120\cos(2t) - 7.619194422\sin(3t) + 1.569640374\sin(t) + 0.2446443083,\\
\phi_2(t) &= -0.7916856120\cos(2t) - 7.619194422\sin(3t) + 1.569640374\sin(t) + 0.2446443083,\\
\phi_3(t) &= 0.6812803993\cos(2t) + 4.052234400\sin(3t) + 2.185550991\sin(t) + 0.5511674211,\\
\phi_4(t) &= 0.6812803993\cos(2t) + 4.052234400\sin(3t) + 2.185550991\sin(t) + 0.5511674211,\\
\phi_5(t) &= -0.7916856120\cos(2t) + 7.619194422\sin(3t) - 1.569640374\sin(t) + 0.2446443083,\\
\phi_6(t) &= -0.7916856120\cos(2t) + 7.619194422\sin(3t) - 1.569640374\sin(t) - 0.2446443083,\\
\phi_7(t) &= 0.6812803993\cos(2t) - 4.052234400\sin(3t) + 2.185550991\sin(t) + 0.5511674211,\\
\phi_8(t) &= -0.6812803993\cos(2t) - 4.052234400\sin(3t) - 2.185550991\sin(t) + 0.5511674211,\\
\phi_9(t) &= 2.539731474\cos(3t) + 0.7916856121\cos(2t) + 0.2446443083 - 1.569640374\cos(t),\\
\phi_{10}(t) &= 2.539731474\cos(3t) - 0.7916856121\cos(2t) - 0.2446443083 - 1.569640374\cos(t),\\
\phi_{11}(t) &= -1.350744800\cos(3t) + 0.6812803993\cos(2t) + 0.5511674211 - 2.185550991\cos(t),\\
\phi_{12}(t) &= -1.350744800\cos(3t) - 0.6812803993\cos(2t) - 0.5511674211 - 2.185550991\cos(t),\\
\phi_{13}(t) &= -2.539731474\cos(3t) + 0.7916856121\cos(2t) + 0.2446443083 + 1.569640374\cos(t),\\
\phi_{14}(t) &= -2.539731474\cos(3t) - 0.7916856121\cos(2t) - 0.2446443083 + 1.569640374\cos(t),\\
\phi_{15}(t) &= 1.350744800\cos(3t) + 0.6812803993\cos(2t) + 0.5511674211 + 2.185550991\cos(t),\\
\phi_{16}(t) &= -1.350744800\cos(3t) - 0.6812803993\cos(2t) - 0.5511674211 + 2.185550991\cos(t),\\
\phi_{17}(t) &= 1, \quad \phi_{18}(t) = -1
\end{aligned} \tag{2.10}$$

of the equation $x^{(7)} + 14x^{(5)} + 49x''' + 36x = 0$, *when* $\varepsilon \to 0$. *Note that these periodic solutions are unstable. Corollary 2.7 is shown in Section 4.*

Remark 2.8. *In Corollary 2.7, we studied the autonomous case of Eq* (1.1)*, assuming that* f *is independent of* t*. In this simplified framework, our analysis identifies eighteen limit cycles, reflecting the complex oscillatory nature of the equation. This observation confirms the validity and applicability of Corollary 2.7 in the autonomous setting.*

In the following section, we will provide some information about averaging theory.

3. Fundamental results on averaging theory

We outline the basic result of averaging theory, which will be employed to illustrate the principal findings of this paper.
We examine the bifurcation problem of T–periodic solutions from differential systems of the form

$$\mathbf{x}' = f_0(t,\mathbf{x}) + \varepsilon f_1(t,\mathbf{x}) + \varepsilon^2 f_2(t,\mathbf{x},\varepsilon), \tag{3.1}$$

with $\varepsilon > 0$ being sufficiently small. The functions $f_0, f_1 : \mathbb{R}\times\Omega \mapsto \mathbb{R}^n$ and $f_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \mapsto \mathbb{R}^n$ are C^2 functions that are T–periodic in the first variable. Here, Ω is an open subset of $\mathbb{R}^n$. A key assumption is that the unperturbed system

$$\mathbf{x}' = f_0(t,\mathbf{x}) \tag{3.2}$$

possesses a submanifold of periodic solutions.

We denote by $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ the solution of system (3.2) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. The linearized system of the unperturbed system (3.2) along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ is

$$\mathbf{y}' = D_{\mathbf{x}} f_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{3.3}$$

Let $M_{\mathbf{z}}(t)$ denote the fundamental matrix of the linear differential system (3.3) and let $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ denote the projection of $\mathbb{R}^n$ onto its first k coordinates, i.e., $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Suppose that there exists a k-dimensional submanifold $\mathbf{Z} \subset \mathbf{\Omega}$ consisting of T-periodic solutions of Eq (3.2). The next result provides conditions under which the bifurcation of T-periodic solutions of system (3.1) occurs from the manifold $\mathbf{Z}$.

Theorem 3.1. *Let V be an open and bounded set of $\mathbb{R}^k$ and let $\beta : Cl(V) \to \mathbb{R}^{n-k}$ be a C^2 function. Assume that*

1) $\mathbf{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)) \mid \alpha \in \mathrm{CL}(\mathbf{V})\} \subset \mathbf{\Omega}$, *and for each $\mathbf{z}_\alpha \in \mathbf{Z}$, the solution $\mathbf{x}(\mathbf{t}, \mathbf{z}_\alpha)$ of Eq* (3.2) *is T-periodic.*
2) *For each $\mathbf{z}_\alpha \in \mathbf{Z}$, there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of Eq* (3.3) *such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in its upper-right corner the $k \times (n-k)$ zero matrix, and in its lower-right corner a $(n-k) \times (n-k)$ matrix Δ_α with* $\det(\Delta_\alpha) \neq 0$.

Consider the function $F : \mathrm{Cl}(V) \to \mathbb{R}^k$ defined by

$$F(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) f_1(t, x(t, \mathbf{z}_\alpha))\, \mathbf{dt}\right). \tag{3.4}$$

If there exists $a \in V$ with $F(a) = 0$, and $\det((dF/d\alpha)(a)) \neq 0$, *then there exists a T–periodic solution $\phi(t, \varepsilon)$ of system* (3.1) *such that $\phi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Proof. The proof goes back to [11, 12]; for a shorter proof, see [2]. □

We suppose the existence of an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(\mathbf{V})$, the solution $x(t, \mathbf{z}, \mathbf{0})$ of the unperturbed system (3.2), with initial condition $x(0, \mathbf{z}, \mathbf{0}) = \mathbf{z}$, is T-periodic. The set $\mathrm{Cl}(V)$ is referred to as isochronous for system (3.1), meaning it is entirely composed of periodic trajectories with common period T.

The following result addresses the bifurcation of T-periodic solutions arising from those $x(t, \mathbf{z}, \mathbf{0})$ contained in $\mathrm{Cl}(V)$.

Theorem 3.2. *(Perturbations of an isochronous set) We assume that there exists an open and bounded set V with* $\mathrm{Cl}(V) \subset \Omega$ *such that for each* $\mathbf{z} \in \mathrm{Cl}(\mathbf{V})$, *the solution $x(t, \mathbf{z})$ is T–periodic, and then we consider the function $F :$* $\mathrm{Cl}(V) \to \mathbb{R}^n$:

$$F(\mathbf{z}) = \frac{1}{T}\int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}) f_1(t, \mathbf{x}(t, \mathbf{z}))dt. \tag{3.5}$$

If there exists $a \in V$ with $F(a) = 0$, and

$$\det\left(\left(\frac{dF}{d\mathbf{z}}\right)(a)\right) \neq 0, \tag{3.6}$$

then there exists a T–periodic solution $\phi(t, \varepsilon)$ of system (3.1) *such that $\phi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Proof. This follows immediately from Theorem (3.1), taking $k = n$. □

Theorem 3.3. *Under the hypothesis of Theorem* (3.2)*, condition* (3.6) *guarantees the existence and uniqueness of a T–periodic solution of system* (3.1) *when ε is small, in such a way that* $\mathbf{x}(0,\varepsilon) \to \alpha$ *as* $\varepsilon \to 0$*. Furthermore, if the real parts of all eigenvalues of the matrix* $((dF/d\mathbf{z})(\alpha))$ *are negative, then* $\mathbf{x}(t,\varepsilon)$ *is stable. Conversely, if at least one eigenvalue has a positive real part, then the periodic solution* $\mathbf{x}(t,\varepsilon)$ *is unstable.*

4. Proofs of the results

Proof of Theorem 2.1. Assume that $x = x_1$, $x_2 = x_1'$, $x_3 = x_1''$, $x_4 = x_1'''$, $x_5 = x_1^{(4)}$, $x_6 = x_1^{(5)}$, and $x_7 = x_1^{(6)}$. The differential Eq (1.1) is thus equivalent to the following first-order system:

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = x_4, \\ x_4' = x_5, \\ x_5' = x_6, \\ x_6' = x_7, \\ x_7' = \lambda x_7 - Ax_6 + \lambda Ax_5 - Cx_4 + \lambda Cx_3 - (C-A+1)x_2 + \lambda(C-A+1)x_1 + \varepsilon f(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7). \end{cases} \tag{4.1}$$

Clearly, for $\varepsilon = 0$, system (4.1) has a unique simple singular point at the origin. The associated eigenvalues are $\pm i$, $\pm ip$, $\pm iq$, and λ. Now, we express system (4.1) in such a way that the linear part at the origin is represented in its real Jordan form. To this end, we introduce the linear change of variables:

$$X = Bx, \qquad X,\ x \in \mathbb{R}^7, \tag{4.2}$$

where $X = (X_1, X_2, \ldots, X_7)^T$, $x = (x_1, x_2, \ldots, x_7)^T$, and the matrix B is given by

$$B = \begin{pmatrix} 0 & -\lambda p^2 & p^2 & -\lambda(p^2+1) & p^2+1 & -\lambda & 1 \\ -\lambda p^2 q & p^2 q & -\lambda q(p^2+1) & q(p^2+1) & -\lambda q & q & 0 \\ 0 & -q^2\lambda & q^2 & -\lambda(q^2+1) & q^2+1 & -\lambda & 1 \\ -\lambda q^2 p & p^2 q & -\lambda p(q^2+1) & p(q^2+1) & -\lambda p & p & 0 \\ 0 & -\lambda q^2 p^2 & p^2 q^2 & -\lambda(p^2+q^2) & p^2+q^2 & -\lambda & 1 \\ -\lambda p^2 q^2 & p^2 q^2 & -\lambda(p^2+q^2) & p^2+q^2 & -\lambda & 1 & 0 \\ p^2 q^2 & 0 & p^2q^2+p^2+q^2 & 0 & p^2+q^2+1 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

We obtain the system

$$\begin{cases} X_1' = -qX_2 + \varepsilon G(t,X), \\ X_2' = qX_1, \\ X_3' = -pX_4 + \varepsilon G(t,X), \\ X_4' = pX_3, \\ X_5' = -X_6 + \varepsilon G(t,X), \\ X_6' = X_5 + \varepsilon G(t,X), \\ X_7' = \lambda X_7 + \varepsilon G(t,X), \end{cases} \tag{4.4}$$

where $G(t, X) = f(t, c)$, with $c = (C_1, C_2, C_3, C_4, C_5, C_6, C_7)^T$. C_i, $i = \overline{1, 7}$, are given in Appendix A. Note that the linear part of system (4.4) at the origin is in real Jordan normal form. Moreover, the change of variables given in (4.2) is well-defined whenever p and q are nonzero rational numbers satisfying conditions (H1), since the determinant of the transformation matrix is

$$\det(B) = -(\lambda^2 + 1)(q-1)^2(q+1)^2(\lambda^2 + q^2)(p-1)^2(p+1)^2(\lambda^2 + p^2)(p-q)^2(p+q)^2 pq.$$

We now apply Theorem 3.1 to system (4.4). Note that this system can be rewritten in the form of system (3.1), by taking

$$\mathbf{x} = (X_1, X_2, X_3, X_4, X_5, X_6, X_7)^T, \qquad f_0(t, \mathbf{x}) = (-qX_2,\ qX_1,\ -pX_4,\ pX_3,\ -X_6,\ X_5,\ \lambda X_7)^T,$$

$$f_1(t, \mathbf{x}) = (G(t, X),\ 0,\ G(t, X),\ 0,\ G(t, X),\ 0,\ G(t, X))^T, \qquad f_2(t, \mathbf{x}) = (0, 0, 0, 0, 0, 0, 0)^T.$$

Let $x(t, X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0}, X_{7,0}, \varepsilon)$ denote the solution of system (4.4) with initial condition

$$x(0, X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0}, X_{7,0}, \varepsilon) = (X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0}, X_{7,0}).$$

It is clear that the unperturbed system (4.4) with $\varepsilon = 0$ has a center at the origin in the subspace $(X_1, X_2, X_3, X_4, X_5, X_6)$. The corresponding periodic solutions are given by

$$x(t, X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0}, 0) = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t), X_7(t)),$$

where

$$X(t) = \begin{pmatrix} \cos(qt)X_{1,0} - \sin(qt)X_{2,0} \\ \sin(qt)X_{1,0} + \cos(qt)X_{2,0} \\ \cos(pt)X_{3,0} - \sin(pt)X_{4,0} \\ \sin(pt)X_{3,0} + \cos(pt)X_{4,0} \\ \cos(t)X_{5,0} - \sin(t)X_{6,0} \\ \sin(t)X_{5,0} + \cos(t)X_{6,0} \\ 0 \end{pmatrix}. \tag{4.5}$$

Note that all these orbits are periodic and share the same period $2m\pi$, where m is defined in the statement of Theorem 2.4.

For our system, the set V and α in Theorem 3.1 are given by

$$V = \left\{ (X_1, X_2, X_3, X_4, X_5, X_6, 0)^T \in \mathbb{R}^7 \,\middle|\, 0 < \sum_{i=1}^{6} X_i^2 < \rho \right\},$$

for some arbitrary $\rho > 0$ and $\alpha = (X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0})$ belonging to V.

The unperturbed system (4.4) has the fundamental matrix $M(t)$ over the periodic solution (4.16), which is

$$M(t) = M_z(t) = \begin{pmatrix} \cos(qt) & -sin(qt) & 0 & 0 & 0 & 0 & 0 \\ \sin(qt) & \cos(qt) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & -\sin(pt) & 0 & 0 & 0 \\ 0 & 0 & \sin(pt) & \cos(pt) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(t) & -\sin(t) & 0 \\ 0 & 0 & 0 & 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

Its inverse matrix of $M(t)$ is

$$M^{-1}(t) = \begin{pmatrix} \cos(qt) & sin(qt) & 0 & 0 & 0 & 0 & 0 \\ -\sin(qt) & \cos(qt) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & \sin(pt) & 0 & 0 & 0 \\ 0 & 0 & -\sin(pt) & \cos(pt) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(t) & \sin(t) & 0 \\ 0 & 0 & 0 & 0 & -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\lambda t} \end{pmatrix}.$$

On the other hand, a simple calculation gives

$$M^{-1}(0) - M^{-1}(2k\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-e^{-2\lambda m\pi} \end{pmatrix}.$$

We have $1 - e^{-2\lambda m\pi} \neq 0$, because $\lambda m \neq 0$. We have shown that all the hypotheses of Theorem 2.1 are satisfied. We now compute the function $F(\alpha)$ as given in Theorem 2.1. Equation $F(\alpha) = 0$ becomes

$$F(X_0) = (0, 0, 0, 0, 0, 0)^T, \tag{4.6}$$

where $F = (F_1, F_2, F_3, F_4, F_5, F_6)^T$, and the functions F_i, $i = 1, \ldots, 6$, are defined in (2.1). According to Theorem 3.1, if there exists

$$\alpha = X_0^* = (X_{1,0}^*, X_{2,0}^*, X_{3,0}^*, X_{4,0}^*, X_{5,0}^*, X_{6,0}^*)^T \in \mathbb{R}^6 \setminus \{(0, 0, 0, 0, 0, 0)^T\} \subset V,$$

such that $F_i(\alpha) = 0$ for all $i = 1, \ldots, 6$, and the condition (2.3) holds, then the differential Eq (1.1) admits a periodic solution $x(t, \varepsilon)$ that tends to the solution $x_0(t)$ of the linear equation

$$x^{(7)} - \lambda x^{(6)} + Ax^{(5)} - \lambda Ax^{(4)} + Cx^{(3)} - \lambda Cx'' + (C - A + 1)x' - \lambda(C - A + 1)x = 0, \tag{4.7}$$

as $\varepsilon \to 0$, given in (2.4). All such solutions are periodic with period $2m\pi$. This concludes the proof of Theorem 2.1. □

Proof of Corollary 2.2. Consider the seventh-order nonlinear differential equation

$$x^{(7)} - \tfrac{1}{2}x^{(6)} + 14x^{(5)} - 7x^{(4)} + 49x^{(3)} - \tfrac{49}{2}x'' + 36x' - 18x = x^{(5)} - \sin(t), \tag{4.8}$$

which corresponds to the parameters $p = 2$, $q = 3$, and $\lambda = \frac{1}{2}$, with a perturbation term given by $f(t, x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_6 - \sin(t)$. The functions $F_i = F_i(X_0)$ from Theorem 2.1 are

$$\begin{aligned} F_1 &= -\tfrac{81}{1480}X_{1,0} + \tfrac{243}{740}X_{2,0}, \\ F_2 &= -\tfrac{243}{740}X_{1,0} - \tfrac{81}{1480}X_{2,0}, \\ F_3 &= -\tfrac{64}{255}X_{4,0} + \tfrac{16}{255}X_{3,0}, \\ F_4 &= \tfrac{16}{255}X_{4,0} + \tfrac{64}{255}X_{3,0}, \\ F_5 &= -\tfrac{1}{120}X_{5,0} + \tfrac{1}{60}X_{6,0}, \\ F_6 &= \tfrac{1}{2} - \tfrac{1}{60}X_{5,0} - \tfrac{1}{120}X_{6,0}. \end{aligned}$$

Solving the system $F_i = 0$, for $i = 1, \dots, 6$, yields a solution

$$X_0^* = (0, 0, 0, 0, 24, 12)^T.$$

The determinant of the Jacobian matrix of the system $(F_1, F_2, F_3, F_4, F_5, F_6)^T$ at this point is

$$\det\left(\frac{\partial F}{\partial X_0}\right)\Bigg|_{X_0=(0,0,0,0,24,12)^T} = 2.575516693 \times 10^{-6} \neq 0.$$

Therefore, by Theorem 2.1, for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small, the perturbed differential Eq (4.8) has a periodic solution $x(t, \varepsilon)$, which converges to the solution $\phi_0(t)$ of the unperturbed differential system as $\varepsilon \to 0$.
Substituting the solution X_0^* into the solution (2.4) of the unperturbed system yields the solution

$$\phi_0(t) = -\cos(t),$$

which satisfies

$$x^{(7)} - \frac{1}{2}x^{(6)} + 14x^{(5)} - 7x^{(4)} + 49x''' - \tfrac{49}{2}x'' + 36x' - 18x = 0. \tag{4.9}$$

Now, we analyze the stability of the corresponding periodic solution. The Jacobian matrix of the system $(F_1, F_2, F_3, F_4, F_5, F_6)^T$ at the point $(0, 0, 0, 0, 24, 12)^T$ is

$$J = \begin{pmatrix} -\frac{81}{1480} & \frac{243}{740} & 0 & 0 & 0 & 0 \\ -\frac{243}{740} & -\frac{81}{1480} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{16}{255} & -\frac{64}{255} & 0 & 0 \\ 0 & 0 & \frac{64}{255} & \frac{16}{255} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{120} & \frac{1}{60} \\ 0 & 0 & 0 & 0 & -\frac{1}{60} & -\frac{1}{120} \end{pmatrix}.$$

The corresponding eigenvalues are

$$\begin{aligned} \mu_1 &= -\tfrac{1}{120} - \tfrac{i}{60}, \quad \mu_2 = -\tfrac{1}{120} + \tfrac{i}{60}, \\ \mu_3 &= -\tfrac{81}{1480} - \tfrac{243i}{740}, \quad \mu_4 = -\tfrac{81}{1480} + \tfrac{243i}{740}, \\ \mu_5 &= \tfrac{16}{255} - \tfrac{64i}{255}, \quad \mu_6 = \tfrac{16}{255} + \tfrac{64i}{255}. \end{aligned}$$

We observe that μ_5 and μ_6 have positive real parts. Therefore, by Theorem 3.3, the corresponding periodic solution is unstable. □

Proof of Theorem 2.4. We want to study the periodic solutions of the class of seventh-order differential equations

$$x^{(7)} + Ax^{(5)} + Cx^{(3)} + (C - A + 1)x' = \varepsilon f(t, x, x', x'', x''', x^{(4)}, x^{(5)}, x^{(6)}). \tag{4.10}$$

This is the case where $\lambda = 0$, $p = \frac{p_1}{p_2}$, and $q = \frac{q_1}{q_2}$ are nonzero rational numbers, satisfying the conditions (H1) as the differential Eq (1.1). Assume that $x = x_1$, $x_2 = x_1'$, $x_3 = x_1''$, $x_4 = x_1'''$, $x_5 = x_1^{(4)}$,

$x_6 = x_1^{(5)}$, and $x_7 = x_1^{(6)}$. Then the differential Eq (4.10) is equivalent to the following first-order differential system:

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = x_4, \\ x_4' = x_5, \\ x_5' = x_6, \\ x_6' = x_7, \\ x_7' = -Ax_6 - Cx_5 - (C - A + 1)x_2 + \varepsilon f(t, x_1, x_2, x_3, x_4, x_5, x_6, x_7). \end{cases} \tag{4.11}$$

Clearly, for $\varepsilon = 0$, system (4.11) admits a simple singular point at the origin, with eigenvalues $\pm i$, $\pm ip$, $\pm iq$, and 0. Now, we express system (4.11) such that its linear part in $(0, 0, 0, 0, 0, 0, 0)^T$ is represented in the real Jordan form. Using the linear transformation

$$X = Bx, \qquad X,\ x \in \mathbb{R}^7, \tag{4.12}$$

where $X = (X_1, X_2, \ldots, X_7)^T$, $x = (x_1, x_2, \ldots, x_7)^T$, and the matrix B is given by

$$B = \begin{pmatrix} 0 & 0 & p^2 & 0 & p^2+1 & 0 & 1 \\ 0 & p^2 q & 0 & q(p^2+1) & 0 & q & 0 \\ 0 & 0 & q^2 & 0 & q^2+1 & 0 & 1 \\ 0 & q^2 p & 0 & p(q^2+1) & 0 & p & 0 \\ 0 & 0 & p^2q^2 & 0 & p^2+q^2 & 0 & 1 \\ 0 & p^2q^2 & 0 & p^2+q^2 & 0 & 1 & 0 \\ p^2q^2 & 0 & p^2q^2+p^2+q^2 & 0 & p^2+q^2+1 & 0 & 1 \end{pmatrix}, \tag{4.13}$$

we obtain the system

$$\begin{cases} X_1' = -qX_2 + \varepsilon G(t, X), \\ X_2' = qX_1, \\ X_3' = -pX_4 + \varepsilon G(t, X), \\ X_4' = pX_3, \\ X_5' = -X_6 + \varepsilon G(t, X), \\ X_6' = X_5, \\ X_7' = \varepsilon G(t, X), \end{cases} \tag{4.14}$$

where $G(t, X) = f(t, b)$, $b = (B_1, B_2, B_3, B_4, B_5, B_6, B_7)^T$. and B_i, $i = \overline{1, 7}$ are given in Appendix B. Note that the linear part of system (4.14) at the origin is in real Jordan form. The transformation (4.12) is valid for p, q satisfying the conditions (H1), since the determinant of matrix B is

$$\det(B) = -p^3q^3(p^4q^2 - p^2q^4 - p^4 + q^4 + p^2 - q^2)^2.$$

We apply Theorem 3.2 to system (4.14), which can be written in the form of system (3.1), with

$$\mathbf{x} = (X_1, X_2, X_3, X_4, X_5, X_6, X_7)^T, \qquad f_0(t, \mathbf{x}) = (-qX_2,\ qX_1,\ -pX_4,\ pX_3,\ -X_6,\ X_5,\ 0)^T,$$

$$f_1(t, \mathbf{x}) = (G(t, X),\ 0,\ G(t, X),\ 0,\ G(t, X),\ 0,\ G(t, X))^T, \qquad f_2(t, \mathbf{x}) = (0, 0, 0, 0, 0, 0, 0)^T.$$

We now focus on analyzing the periodic solutions of system (4.14) in the specific case where $\varepsilon = 0$. The corresponding periodic solutions are as follows:

$$X(t) = \begin{pmatrix} \cos(qt)X_{1,0} - \sin(qt)X_{2,0} \\ \sin(qt)X_{1,0} + \cos(qt)X_{2,0} \\ \cos(pt)X_{3,0} - \sin(pt)X_{4,0} \\ \sin(pt)X_{3,0} + \cos(pt)X_{4,0} \\ \cos(t)X_{5,0} - \sin(t)X_{6,0} \\ \sin(t)X_{5,0} + \cos(t)X_{6,0} \\ X_{7,0} \end{pmatrix}. \tag{4.15}$$

This set of periodic orbits forms a seven-dimensional family, all sharing the same period $2m\pi$, where m is defined in Theorem 2.1. To determine the periodic solutions of our Eq (4.10), we compute the zeros $a = X_0^* = (X_{1,0}^*, X_{2,0}^*, X_{3,0}^*, X_{4,0}^*, X_{5,0}^*, X_{6,0}^*, X_{7,0}^*)^T$ of the system $F(\mathbf{z}) = \mathbf{0}$, where $F(\mathbf{z})$ is given by Eq (3.5). The fundamental matrix of the unperturbed system (4.14) over the periodic solution is

$$M(t) = \begin{pmatrix} \cos(qt) & -\sin(qt) & 0 & 0 & 0 & 0 & 0 \\ \sin(qt) & \cos(qt) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & -\sin(pt) & 0 & 0 & 0 \\ 0 & 0 & \sin(pt) & \cos(pt) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(t) & -\sin(t) & 0 \\ 0 & 0 & 0 & 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse of the matrix $M(t)$ is

$$M^{-1}(t) = \begin{pmatrix} \cos(qt) & \sin(qt) & 0 & 0 & 0 & 0 & 0 \\ \sin(qt) & \cos(qt) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(pt) & -\sin(pt) & 0 & 0 & 0 \\ 0 & 0 & \sin(pt) & \cos(pt) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(t) & -\sin(t) & 0 \\ 0 & 0 & 0 & 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculating the function $F(\mathbf{z})$ given in Theorem 2.4, the system $F(\mathbf{z}) = \mathbf{0}$ is written as

$$F_i(X_0) = 0, i = \overline{1,7}, \tag{4.16}$$

in which F_i, for $i = \overline{1,7}$, are given in (2.5). If the condition (2.7) is satisfied, the simple zeros X_0^* of system (4.6) with respect to the coordinates $X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}, X_{6,0}$, and $X_{7,0}$ give periodic solutions of system (4.14) for $\varepsilon > 0$ small enough. Thus, Eq (4.10) has a periodic solution $x(t,\varepsilon)$ tending to a solution $\phi_0(t)$ of

$$x^{(7)} + Ax^{(5)} + Cx^{(3)} + (C - A + 1)x' = 0,$$

as $\varepsilon \to 0$, with all orbits having period $2m\pi$. □

Proof of Corollary 2.5. Consider the differential equation

$$x^{(7)} + \frac{49}{36}x^{(5)} + \frac{7}{18}x^{(3)} + \frac{1}{36}x' = \sin(t) + x - x' - x'' + x^{(4)} - x^{(5)} + x^{(6)}, \tag{4.17}$$

which corresponds to the parameters $p = \frac{1}{2}$, $q = \frac{1}{3}$, $\lambda = 0$, and

$$f(t, x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \sin(t) + x_1 - x_2 - x_3 + x_5 - x_6 + x_7.$$

The functions $F_i = F_i(X_0)$, for $i = \overline{1,7}$, from Theorem 2.4 are

$$\begin{aligned}
F_1 &= -\frac{409}{10}X_{1,0} - \frac{123}{10}X_{2,0},\\
F_2 &= \frac{123}{10}X_{1,0} - \frac{409}{10}X_{2,0},\\
F_3 &= \frac{249}{10}X_{3,0} + \frac{51}{5}X_{4,0},\\
F_4 &= -\frac{51}{5}X_{3,0},\\
F_5 &= -\frac{3}{2}X_{5,0} - \frac{3}{2}X_{6,0},\\
F_6 &= \frac{3}{2}X_{5,0} - \frac{3}{2}X_{6,0} - \frac{1}{2},\\
F_7 &= 36X_{7,0}.
\end{aligned}$$

The system $F_1 = F_2 = F_3 = F_4 = F_5 = F_6 = F_7 = 0$ admits the solution

$$X_0^* = \left(0, 0, 0, 0, \frac{1}{6}, -\frac{1}{6}, 0\right)^T.$$

The Jacobian determinant (2.7) evaluated at this solution is

$$\det\left(\frac{\partial F}{\partial X_0}\right)\bigg|_{X_0=\left(0,0,0,0,\frac{1}{6},-\frac{1}{6},0\right)^T} = 3.074425697 \times 10^7.$$

Thus, for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small, there exists a periodic solution $x(t, \varepsilon)$ of the seventh-order differential Eq (4.17). Substituting the solution X_0^* into (2.8) yields the function

$$\phi_0(t) = -\frac{\cos(t)}{4} - \frac{\sin(t)}{4},$$

which satisfies the equation

$$x^{(7)} + \frac{49}{36}x^{(5)} + \frac{7}{18}x^{(3)} + \frac{1}{36}x' = \sin(t) + x_1 - x_2 - x_3 + x_5 - x_6 + x_7,$$

in the limit as $\varepsilon \to 0$. Now, we study the stability of the periodic solution discussed above. The

Jacobian matrix at the point $\left(0,0,0,0,\frac{1}{6},-\frac{1}{6},0\right)^T$ is

$$\begin{pmatrix} -40.9 & -12.3 & 0 & 0 & 0 & 0 & 0 \\ 12.3 & -40.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24.9 & 10.2 & 0 & 0 & 0 \\ 0 & 0 & -10.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.5 & -1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\begin{aligned} &\mu_1 = -40.9 + 12.3i, \quad \mu_2 = -40.9 - 12.3i, \\ &\mu_3 = 19.588024205745, \quad \mu_4 = 5.31119757942552, \\ &\mu_5 = -1.5 + 1.5i, \quad \mu_6 = -1.5 - 1.5i, \quad \mu_7 = 36. \end{aligned}$$

We observe that at least one eigenvalue has a positive real part (in this case, μ_3 and μ_4). Therefore, according to Theorem 3.3, the associated periodic solution is unstable. □

Proof of Corollary 2.7. Consider the differential equation

$$x^{(7)} + 14x^{(5)} + 49x^{(3)} + 36x = x^2 - 1, \tag{4.18}$$

which corresponds to the parameters $p = 1$, $q = 3$, $\lambda = 0$, and $f(t, x_1, x_2, x_3, x_4, x_5, x_6, x_7) = x_1^2 - 1$. The functions $F_i = F_i(X_0)$, $i = \overline{1,7}$, from Theorem 2.4 are

$$\begin{aligned} F_1 &= \tfrac{1}{2880}X_{4,0}X_{6,0} - \tfrac{1}{2880}X_{5,0}X_{3,0} - \tfrac{1}{12960}X_{1,0}X_{7,0}, \\ F_2 &= -\tfrac{1}{2880}X_{4,0}X_{5,0} - \tfrac{1}{2880}X_{3,0}X_{6,0} - \tfrac{1}{12960}X_{2,0}X_{7,0}, \\ F_3 &= \tfrac{1}{2304}X_{5,0}^2 - \tfrac{1}{2304}X_{6,0}^2 + \tfrac{1}{2160}X_{3,0}X_{7,0} + \tfrac{1}{17280}X_{1,0}X_{5,0} + \tfrac{1}{17280}X_{2,0}X_{6,0}, \\ F_4 &= \tfrac{(15X_{6,0}+X_{2,0})X_{5,0}}{17280} + \tfrac{1}{2160}X_{7,0}X_{4,0} - \tfrac{1}{17280}X_{1,0}X_{6,0}, \\ F_5 &= \tfrac{(-15X_{6,0}-X_{2,0})X_{4,0}}{43200} + \tfrac{(-50X_{7,0}-15X_{3,0})X_{5,0}}{43200} - \tfrac{1}{43200}X_{1,0}X_{3,0}, \\ F_6 &= \tfrac{(-15X_{5,0}-X_{2,0})X_{4,0}}{43200} + \tfrac{(-50X_{7,0}+15X_{3,0})X_{6,0}}{43200} - \tfrac{1}{43200}X_{2,0}X_{3,0}, \\ F_7 &= \tfrac{1}{259200}X_1^2 + \tfrac{1}{259200}X_{2,0}^2 + \tfrac{1}{7200}X_{3,0}^2 + \tfrac{1}{7200}X_{4,0}^2 + \tfrac{1}{1152}X_{5,0}^2 + \tfrac{1}{1152}X_{6,0}^2 + \tfrac{1}{1296}X_{7,0}^2 - 1. \end{aligned}$$

The system $F_1 = F_2 = F_3 = F_4 = F_5 = F_6 = F_7 = 0$ possesses eighteen solutions:

$$\begin{aligned} X_1^* &= (0,\ 342.8637490,\ 47.50113672,\ 0,\ 0,\ 14.12676337,\ 8.807195100), \\ X_2^* &= (0,\ 342.8637490,\ -47.50113672,\ 0,\ 0,\ 14.12676337,\ 8.807195100), \\ X_3^* &= (0,\ -182.350548,\ 40.87682395,\ 0,\ 0,\ 19.66995892,\ 19.84202716), \\ X_4^* &= (0,\ -182.350548,\ -40.87682395,\ 0,\ 0,\ 19.66995892,\ 19.84202716), \\ X_5^* &= (0,\ -342.8637490,\ -47.50113672,\ 0,\ 0,\ -14.12676337,\ 8.807195100), \\ X_6^* &= (0,\ -342.8637490,\ -47.50113672,\ 0,\ 0,\ -14.12676337,\ -8.807195100), \end{aligned}$$

$$X_7^* = (0,\ 182.3505480,\ 40.87682395,\ 0,\ 0,\ -19.66995892,\ 19.84202716),$$
$$X_8^* = (0,\ 182.3505480,\ -40.87682395,\ 0,\ 0,\ 19.66995892,\ 19.84202716),$$
$$X_9^* = (-342.8637490,\ 0,\ 47.50113672,\ 0,\ 14.12676337,\ 0,\ 8.807195100),$$
$$X_{10}^* = (-342.8637490,\ 0,\ -47.50113672,\ 0,\ 14.12676337,\ 0,\ -8.807195100),$$
$$X_{11}^* = (182.350548,\ 0,\ -40.87682395,\ 0,\ 19.66995892,\ 0,\ 19.84202716),$$
$$X_{12}^* = (182.350548,\ 0,\ 40.87682395,\ 0,\ 19.66995892,\ 0,\ -19.84202716),$$
$$X_{13}^* = (342.8637490,\ 0,\ 47.50113672,\ 0,\ -14.12676337,\ 0,\ 8.807195100),$$
$$X_{14}^* = (342.8637490,\ 0,\ -47.50113672,\ 0,\ -14.12676337,\ 0,\ -8.807195100),$$
$$X_{15}^* = (-182.350548,\ 0,\ -40.87682395,\ 0,\ -19.66995892,\ 0,\ 19.84202716),$$
$$X_{16}^* = (182.350548,\ 0,\ 40.87682395,\ 0,\ -19.66995892,\ 0,\ -19.84202716),$$
$$X_{17}^* = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 36),$$
$$X_{18}^* = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -36).$$

The Jacobian determinant (2.7) evaluated at the eighteen equilibrium points X_i^*, for $i = 1, \dots, 18$, takes the following values, respectively:

$$-4.767150139 \times 10^{24},\ -2.960048789 \times 10^{24},\ -6.118039333 \times 10^{23},\ 4.402762881 \times 10^{-26},$$
$$-2.960048789 \times 10^{24},\ 4.767150139 \times 10^{-24},\ -6.118039333 \times 10^{23},\ 6.377630181 \times 10^{-15},$$
$$-2.960048789 \times 10^{24},\ 2.960048789 \times 10^{-24},\ -6.118039496 \times 10^{-23},\ 6.118039496 \times 10^{-23},$$
$$-2.960048789 \times 10^{-24},\ 2.960048789 \times 10^{-24},\ -6.118039496 \times 10^{-23},\ -6.257351482 \times 10^{-15},$$
$$2.067271461 \times 10^{-13},\ -2.067271461 \times 10^{-13}.$$

By substituting these values into Eq (2.8), we obtain, for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small, that the seventh-order differential Eq (4.18) admits eighteen periodic solutions tending to the periodic solutions ϕ_i, $i = 1, \dots, 18$, given by (2.10).

Now, we study the stability of the periodic solution discussed above.

Since the determinant of the Jacobian matrix evaluated at the points X_1^*, X_2^*, X_3^*, X_5^*, X_7^*, X_9^*, X_{11}^*, X_{13}^*, X_{15}^*, X_{16}^*, and X_{18}^* is negative, the corresponding solutions ϕ_1, ϕ_2, ϕ_3, ϕ_5, ϕ_7, ϕ_9, ϕ_{11}, ϕ_{13}, ϕ_{15}, ϕ_{16}, and ϕ_{18} are unstable.

Since the determinant of the Jacobian matrix evaluated at the points $X_4^*, X_6^*, X_8^*, X_{10}^*, X_{12}^*, X_{14}^*$, and X_{17}^* is positive, it is necessary to compute the eigenvalues of their corresponding Jacobian matrices applying Theorem 3.3. The eigenvalues of the Jacobian matrix at X_4^* are

$$\mu_1 = -0.00900077514812283, \quad \mu_2 = 0.000142801923586169,$$
$$\mu_3 = 0.00774110873553667, \quad \mu_4 = 0.0184656759718059,$$
$$\mu_5 = -0.0173231798938589, \quad \mu_6 = -0.0000256310472512487,$$
$$\mu_7 = -5.39695840339848 \times 10^{-10}.$$

The eigenvalues of the Jacobian matrix at X_6^* are

$$\begin{aligned}
\mu_1 &= 0.0101393836708586 + 0.0118680618818181\,i,\\
\mu_2 &= 0.0101393836708586 - 0.0118680618818181\,i,\\
\mu_3 &= 0.00301035814238277,\\
\mu_4 &= -0.0134532728275767 + 0.0160142831043595\,i,\\
\mu_5 &= -0.0134532728275767 - 0.0160142831043595\,i,\\
\mu_6 &= 0.00361742017514634,\\
\mu_7 &= 4.10711581169977 \times 10^{-12}.
\end{aligned}$$

The eigenvalues of the Jacobian matrix at X_8^* are

$$\begin{aligned}
\mu_1 &= -0.000818280650725262,\\
\mu_2 &= -0.000149291919137369 + 0.0147706009789729\,i,\\
\mu_3 &= -0.000149291919137369 - 0.0147706009789729\,i,\\
\mu_4 &= 0.0270288660879672,\\
\mu_5 &= -0.000778141856117547 + 0.00732498942670811\,i,\\
\mu_6 &= -0.000778141856117547 - 0.00732498942670811\,i,\\
\mu_7 &= -0.0243557178847322.
\end{aligned}$$

The eigenvalues of the Jacobian matrix in X_{10}^* and X_{14}^* are

$$\begin{aligned}
\mu_1 &= -0.0222234046679042,\\
\mu_2 &= 0.0127517025950769 + 0.0127597471154981\,i,\\
\mu_3 &= 0.0127517025950769 - 0.0127597471154981\,i,\\
\mu_4 &= 0.00641777450185045,\\
\mu_5 &= -1.53020247284969 \times 10^{-13},\\
\mu_6 &= -0.00484888750787348 + 0.0198312069155036\,i,\\
\mu_7 &= -0.00484888750787348 - 0.0198312069155036\,i.
\end{aligned}$$

The eigenvalues of the Jacobian matrix at X_{12}^* are

$$\begin{aligned}
\mu_1 &= 0.00297237133833268,\\
\mu_2 &= 0.000562672704093976 + 0.0300104705246154\,i,\\
\mu_3 &= 0.000562672704093976 - 0.0300104705246154\,i,\\
\mu_4 &= -0.0336012645375206,\\
\mu_5 &= 0.0372189428624418,\\
\mu_6 &= -0.00771539507580955,\\
\mu_7 &= 2.36773771160715 \times 10^{-12}.
\end{aligned}$$

The eigenvalues of the Jacobian matrix at X_{17}^* are

$$\mu_1 = -0.00277777777800000, \quad \mu_2 = -0.00277777777800000,$$
$$\mu_3 = 0.0166666666700000, \quad \mu_4 = 0.0166666666700000,$$
$$\mu_5 = -0.0416666666700000, \quad \mu_6 = -0.0416666666700000,$$
$$\mu_7 = 0.0555555555600000.$$

We observe that, in each case, at least one eigenvalue has a positive real part. Therefore, according to Theorem 3.3, the corresponding periodic solutions $\phi_4, \phi_6, \phi_8, \phi_{10}, \phi_{12}, \phi_{14}$, and ϕ_{17} are unstable. □

5. Conclusions

In this work, we have addressed the complex problem of identifying and analyzing periodic solutions in a class of perturbed seventh-order non-autonomous differential equations that arise in modeling damped multi-frequency oscillations. By using first-order averaging theory, we successfully reduced the dimensionality of the equation, making the detection of limit cycles more tractable. The analytical conditions obtained provide clear criteria for the existence of isolated periodic solutions emanating from the equilibrium, which are otherwise difficult to detect in such high-order equations. Moreover, the local stability of these limit cycles was investigated through a detailed Jacobian analysis of the averaged system, allowing us to classify the nature of the bifurcating periodic solutions. These findings not only demonstrate the effectiveness of averaging techniques in the context of high-order nonlinear dynamics but also offer valuable theoretical tools for future studies on complex oscillatory behavior in mechanical and control systems.

Overall, this research contributes to a deeper understanding of the dynamics of high-dimensional nonlinear equations and opens pathways for their application in the design and analysis of real-world engineering systems exhibiting multi-frequency behavior. For the class of seventh-order non-autonomous differential equations considered in this paper, the averaging theory applied up to the first order allows the detection of at most one small-amplitude limit cycle bifurcating from the trivial solution. In contrast, in the autonomous case, an explicit example has been identified exhibiting up to eighteen limit cycles, illustrating the rich and complex oscillatory dynamics that such equations can display. In general, higher-order perturbations or additional nonlinearities could give rise to several limit cycles. However, establishing a precise upper bound for the number of limit cycles in high-dimensional equations remains an open and challenging problem, since the complexity of the averaged functions grows rapidly with the equation's order. Determining whether this theoretical upper bound can be attained requires a deeper bifurcation analysis or numerical continuation, which goes beyond the scope of the present work.

Use of Generative-AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. C. E. Berrehail, A. Makhlouf, Periodic solutions for a class of perturbed sixth-order autonomous differential equations, *Arab J. Math. Sci.*, **31** (2022), 22–32. https://doi.org/10.1108/AJMS-02-2022-0045
2. A. Buica, J. P. Françoise, J. Llibre, Periodic solutions of nonlinear periodic differential systems with a small parameter, *Commun. Pur. Appl. Anal.*, **6** (2006), 103–111. https://doi.org/10.3934/CPAA.2007.6.103
3. M. R. Cândido, J. Llibre, D. D. Novaes, Persistence of periodic solutions for higher order perturbed differential systems via Lyapunov–Schmidt reduction, *Nonlinearity*, **30** (2017). https://doi.org/10.1088/1361-6544/aa7e95
4. C. Christopher, C. Li, J. Torregrosa, *Limit cycles of differential equations*, Birkhäuser, Basel, 2007. https://doi.org/10.1007/978-3-7643-8410-4_12
5. M. Inc, A. Akgül, Numerical solution of seventh-order boundary value problems by a novel method, *Abstr. Appl. Anal.*, 2014, 745287. https://doi.org/10.1155/2014/745287
6. S. Jiang, L. Wang, Z. Chen, Higher-order averaging and stability analysis of time-periodic nonlinear control systems, *Nonlinear Dynam.*, **97** (2019), 2637–2654.
7. P. Kalyani, M. N. Lemma, Solutions of seventh-order boundary value problems using ninth degree spline functions and comparison with eighth degree spline solutions, *J. Appl. Math. Phys.*, **4** (2016), 249–261. https://doi.org/10.4236/jamp.2016.42032
8. P. Kalyani, M. N. Lemma, D. B. Feyisa, Spline solution of linear seventh-order boundary value problems using tenth-degree functions and comparison with different degrees of spline solutions, *Int. J. Basic Sci. Appl. Comput.*, **2** (2019), 13–19.
9. M. Khazaei, Y. Karamipour, Numerical solution of the seventh-order boundary value problems using B-spline method, *arXiv Preprint*, 2021. https://doi.org/10.48550/arXiv.2109.06030
10. M. Han, L. Sheng, Bifurcation of limit cycles in piecewise smooth systems via Melnikov function, *J. Appl. Anal. Comput.*, **5** (2015), 809–815. https://doi.org/10.11948/2015061
11. I. G. Malkin, *Some problems of the theory of nonlinear oscillations*, US Atomic Energy Commission, Technical Information Service, **1** (1959).
12. M. Roseau, *Vibrations non linéaires et théorie de la stabilité*, 1966.
13. J. A. Sanders, F. Verhulst, J. Murdock, *Averaging methods in nonlinear dynamical systems*, New York: Springer, **59** (2007).
14. S. M. E. Sayed, D. Kaya, An application of the ADM to seventh-order Sawada-Kotara equations, *Appl. Math. Comput.*, **157** (2004), 93–101. https://doi.org/10.1016/j.amc.2003.08.104

15. H. Séka, A. R. Kouassi, A new seventh order Runge–Kutta family: Comparison with the method of Butcher and presentation of a calculation software, *Math. Comput. Sci.*, **4** (2019), 68–75. https://doi.org/10.11648/j.mcs.20190403.12
16. Z. Shi, F. Li, Numerical solution of high-order differential equations by using periodized Shannon wavelets, *Appl. Math. Model.*, **38** (2014), 2235–2248. https://doi.org/10.1016/j.apm.2013.10.030
17. A. A. Soliman, A numerical simulation and explicit solutions of KdV–Burgers' and Lax's seventh-order KdV equations, *Chaos Soliton. Fract.*, **29** (2006), 294–302. https://doi.org/10.1016/j.chaos.2005.08.054
18. A. E. Tabet, A. Makhlouf, Periodic solutions for a class of seventh-order differential equations, *Palest. J. Math.*, **13** (2024), 278–284. Available from: `https://pjm.ppu.edu/sites/default/files/papers/PJM_13%284%29_2024_278_to_284.pdf`.
19. R. P. Tewarson, A seventh-order numerical method for solving boundary value nonlinear ordinary differential equations, *Int. J. Numer. Meth. Eng.*, **18** (1982), 1313–1320. https://doi.org/10.1002/nme.1620180904
20. F. Verhulst, *Nonlinear differential equations and dynamical systems*, 2 Eds., Berlin, Heidelberg: Springer, 1996. https://doi.org/10.1007/978-3-642-61453-8

Appendix A

We provide below the detailed expressions of the coefficients C_i, $i = \overline{1,7}$, appearing in Theorem 2.1.

$$
\begin{aligned}
C_1 = & \frac{-(q^2-1)(\lambda X_{4,0}+pX_{3,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)\,p\,(\lambda^2+p^2)\,(p^2-q^2)} + \frac{(p^2-1)(qX_{1,0}+\lambda X_{4,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)\,q\,(p^2-q^2)\,(\lambda^2+q^2)} \\
& + \frac{(q^2-1)(pX_{4,0}-\lambda X_{3,0})\sin(pt)}{(p^2q^2-p^2-q^2+1)\,p\,(\lambda^2+p^2)\,(p^2-q^2)} - \frac{(\lambda X_{6,0}+X_{5,0})\cos(t)+(\lambda X_{5,0}-X_{6,0})\sin(t)}{(\lambda^2+1)\,(p^2q^2-p^2-q^2+1)} \\
& + \frac{(p^2-1)(\lambda X_{1,0}-qX_{2,0})\sin(qt)}{(p^2q^2-p^2-q^2+1)\,q\,(p^2-q^2)\,(\lambda^2+q^2)}, \\
C_2 = & \frac{(q^2-1)(pX_{4,0}-\lambda X_{4,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{(p^2-1)(\lambda X_{1,0}-qX_{2,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} \\
& + \frac{(q^2-1)(\lambda X_{4,0}+pX_{3,0})\sin(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} - \frac{(p^2-1)(qX_{1,0}+\lambda X_{2,0})\sin(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} \\
& - \frac{(\lambda X_{5,0}-X_{6,0})\cos(t)-(\lambda X_{6,0}+X_{5,0})\sin(t)}{(\lambda^2+1)(p^2q^2-p^2-q^2+1)}, \\
C_3 = & \frac{p(q^2-1)(\lambda X_{4,0}+pX_{3,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} - \frac{q(p^2-1)(qX_{1,0}+\lambda X_{2,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} \\
& - \frac{p(q^2-1)(pX_{4,0}-\lambda X_{3,0})\sin(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{(\lambda X_{6,0}+X_{5,0})\cos(t)+(\lambda X_{5,0}-X_{6,0})\sin(t)}{(\lambda^2+1)(p^2q^2-p^2-q^2+1)} \\
& - \frac{q(p^2-1)(\lambda X_{1,0}-qX_{2,0})\sin(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)}, \\
C_4 = & -\frac{p^2(q^2-1)(pX_{4,0}-\lambda X_{3,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} - \frac{q^2(p^2-1)(\lambda X_{1,0}-qX_{2,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} \\
& - \frac{p^2(q^2-1)(\lambda X_{4,0}+pX_{3,0})\sin(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{(\lambda X_{5,0}-X_{6,0})\cos(t)-(\lambda X_{6,0}+X_{5,0})\sin(t)}{(\lambda^2+1)(p^2q^2-p^2-q^2+1)} \\
& - \frac{q^2(p^2-1)(qX_{1,0}+\lambda X_{2,0})\sin(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)},
\end{aligned}
$$

$$C_5 = -\frac{p^3(q^2-1)(\lambda X_{4,0}+pX_{3,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{q^3(p^2-1)(qX_{0,1}+\lambda X_{2,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} + \frac{p^3(q^2-1)(pX_{4,0}-\lambda X_{3,0})\sin(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{q^3(p^2-1)(\lambda X_{1,0}-qX_{2,0})\sin(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} - \frac{(\lambda X_{6,0}+X_{5,0})\cos(t)+(\lambda X_{5,0}-X_{6,0})\sin(t)}{(\lambda^2+1)(p^2q^2-p^2-q^2+1)},$$

$$C_6 = \frac{p^4(q^2-1)(pX_{4,0}-\lambda X_{3,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{q^4(p^2-1)(\lambda X_{1,0}-qX_{2,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} + \frac{p^4(q^2-1)(\lambda X_{4,0}+pX_{3,0})\sin(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} - \frac{q^4(p^2-1)(qX_{1,0}+\lambda X_{2,0})\sin(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} - \frac{(\lambda X_{5,0}-X_{6,0})\cos(t)-(\lambda X_{6,0}+X_{5,0})\sin(t)}{(\lambda^2+1)(p^2q^2-p^2-q^2+1)},$$

$$C_7 = -\frac{q^5(p^2-1)(qX_{1,0}+\lambda X_{2,0})\cos(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} + \frac{p^5(q^2-1)(\lambda X_{4,0}+pX_{3,0})\cos(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \frac{(\lambda X_{0,6}+X_{0,5})\cos(t)+(\lambda X_{0,5}-X_{0,6})\sin(t)}{(\lambda^2+1)(p^2q^2-p^2-q^2+1)} - \frac{q^5(p^2-1)(\lambda X_{0,1}-qX_{0,2})\sin(qt)}{(p^2q^2-p^2-q^2+1)(p^2-q^2)(\lambda^2+q^2)} - \frac{p^5(q^2-1)(X_{0,4}p-\lambda X_{0,3})\sin(pt)}{(p^2q^2-p^2-q^2+1)(\lambda^2+p^2)(p^2-q^2)} + \cos(t)X_{7,0}.$$

Appendix B

We provide below the detailed expressions of the coefficients B_i, $i = \overline{1,7}$, appearing in Theorem 3.1.

$$B_1 = \frac{p^2X_{1,0}\cos(qt)-q^2X_{3,0}\cos(pt)}{(p^2-q^2)\,q^2p^2\,(p^2-1)} + \frac{q^2X_{4,0}\sin(pt)+p^2X_{2,0}\sin(qt)}{(p^2-q^2)\,p^2\,q^2\,(p^2-1)} - \frac{\cos(t)\,X_{5,0}-\sin(t)\,X_{6,0}}{(q^2-1)\,(p^2-1)} + \frac{X_{7,0}}{q^2\,p^2},$$

$$B_2 = \frac{(1-p^2)(qX_{2,0}\cos(qt)+pX_{1,0}\sin(qt))}{(p^2q^2-p^2-q^2+1)\,q\,p\,(p^2-q^2)} - \frac{(1-q^2)(X_{4,0}\cos(pt)+X_{2,0}\sin(pt))}{(p^2q^2-p^2-q^2+1)\,p\,(p^2-q^2)} + \frac{\sin(t)\,X_{5,0}+\cos(t)\,X_{6,0}}{(p^2q^2-p^2-q^2+1)},$$

$$B_3 = \frac{X_{3,0}\cos(pt)-X_{1,0}\cos(qt)}{(p^2-q^2)\,(p^2-1)} - \frac{X_{4,0}\sin(pt)}{(p^2-q^2)\,(p^2-1)} + \frac{X_{2,0}\sin(qt)}{(p^2-q^2)\,(q^2-1)} + \frac{\cos(t)\,X_{5,0}}{(p^2-1)\,(q^2-1)} - \frac{\sin(t)\,X_{6,0}}{(p^2-1)\,(q^2-1)},$$

$$B_4 = \frac{q\,(p^2-1)\,X_{2,0}\,\cos(qt)+(1-q^2)\,p\,X_{4,0}\,\cos(pt)}{(p^2-q^2)\,(p^2q^2-p^2-q^2+1)} + \frac{q\,(p^2-1)\,X_{1,0}\,\sin(qt)+(1-q^2)\,p\,X_{3,0}\,\sin(pt)}{(p^2-q^2)\,(p^2q^2-p^2-q^2+1)} - \frac{\sin(t)\,X_{5,0}+\cos(t)\,X_{6,0}}{p^2q^2-p^2-q^2+1},$$

$$B_5 = -\frac{p^2\,X_{3,0}\,\cos(pt)}{(p^2-q^2)\,(p^2-1)} + \frac{p^2\,(q^2-1)\,X_{4,0}\,\sin(pt)+q^2\,X_{1,0}\,\cos(qt)}{(p^2-q^2)\,(q^2-1)} - \frac{q^2\,X_{2,0}\,\sin(qt)+\cos(t)\,X_{5,0}-\sin(t)\,X_{6,0}}{(p^2-q^2)\,(q^2-1)},$$

$$B_6 = \frac{q^3\,(1-p^2)\,X_{2,0}\,\cos(qt)+p^3\,(q^2-1)\,X_{4,0}\cos(pt)}{(p^2-q^2)\,(p^2q^2-p^2-q^2+1)} + \frac{q^3\,(1-p^2)\,X_{1,0}\,\sin(qt)+p^3\,(q^2-1)\,X_{3,0}\,\sin(pt)}{(p^2-q^2)\,(p^2q^2-p^2-q^2+1)} + \frac{\sin(t)\,X_{5,0}+\cos(t)\,X_{6,0}}{p^2q^2-p^2-q^2+1},$$

$$B_7 = \frac{p^4(q^2-1)X_{3,0}\cos(pt)}{(p^2-q^2)(p^2-1)} - \frac{p^4(1-p^2)X_{1,0}\cos(qt)}{(p^2-q^2)(q^2-1)} + \frac{p^4X_{4,0}\sin(pt)}{(p^2-q^2)(p^2-1)} + \frac{q^4X_{2,0}\sin(qt)}{(p^2-q^2)(q^2-1)} + \frac{\cos(t)X_{5,0}-\sin(t)X_{6,0}}{(p^2-1)(q^2-1)}.$$