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Research article

Higher-order Umbral Differential Problems for ODEs: Theoretical foundations and computational methods

Francesco A. Costabile, Maria I. Gualtieri and Anna Napoli*

Department of Mathematics and Computer Science, University of Calabria, 87036 Rende (CS), Italy

* **Correspondence:** Email: anna.napoli@unical.it; Tel: +39-0984-496488.

Abstract: In this paper, we consider the higher-order Umbral Differential Problem (UDP). It consists of a nonlinear ordinary differential equation of order $r + 1$, $r > 0$, associated with an Umbral Interpolation Problem (UIP) with $r + 1$ conditions. We prove the existence and uniqueness of the solution to the UIP and study the error using Peano's Kernel. This class includes, for example, the classical Initial Value Problem (or Cauchy Problem) and a more recent case, the so-called Bernoulli boundary problem, also referred to as a non-classical boundary problem. We then use a Birkoff-Lagrange collocation method for obtaining numerical solutions. Two new examples of UDPs are presented and analyzed. The first is the Euler Umbral Differential Problem, so named because the solution to the associated UIP is expressed in terms of Euler polynomials. The second example is a higher-order problem whose interpolation conditions are expressed using iterated forward finite differences. These conditions are equivalent to multipoint conditions; therefore, the related UDP is called Multipoint Value Problem. Finally, we present some numerical tests. The results confirm the effectiveness of the proposed method. Conclusions, along with directions for future research, are provided.

Keywords: differential problem; umbral interpolation; linear functional; delta-operator; Euler polynomial
Mathematics Subject Classification: 11B83, 11C99, 65L10

1. Introduction

Ordinary differential equations (ODEs) and fractional differential equations with non-standard boundary conditions have attracted growing interest and are increasingly applied across a wide range of fields, including finance, biology, engineering, physics, and other domains of applied mathematics [2, 3, 10, 19, 24]. In this paper, we consider general boundary conditions expressed by a linear functional and a delta-operator.

We introduce the notion of the Umbral Differential Problem (UDP), namely the application of modern umbral techniques to boundary value problems (BVPs) for ordinary differential equations. This topic is added to those of umbral interpolation [15], umbral quadrature formulas [13], and operator approximation theory by umbral polynomials [14]. In other words, we extend the application of modern umbral calculus techniques to the field of computational mathematics.

Let us consider

i) $X = C^{\infty}[0,1]$, the space of real infinitely differentiable functions on the interval $[0,1]$;

ii) L, a linear functional on X s.t. $L(1) \neq 0$;

iii) $\mathcal{D}$, a delta-operator [16,25] [28, p. 129] on X, i.e. an operator $\mathcal{D}$ such that

$$\mathcal{D}y = \sum_{i=0}^{\infty} b_i \frac{y^{(i)}}{i!}, \quad b_0 = 0,\ b_1 \neq 0,\ b_i \in \mathbb{R}, i \in \mathbb{N}, \quad \forall y \in X;$$

iv) $X_{\mathcal{D}} = \left\{ y \in X \ :\ \mathcal{D}^i y \in X,\ i = 0,\dots,n, \forall n \in \mathbb{N} \right\}$.

Moreover, let $y \in X_{\mathcal{D}}$ and $f\left(x, y(x), y'(x), \dots, y^{(q)}(x)\right)$, $0 \leq q \leq r$, $r \in \mathbb{R}$, $r > 1$, be a function defined on a compact subset $D \subset [0,1] \times \mathbb{R}^{q+1}$ and continuous at least in the interior of the domain of interest. We suppose that f satisfies a uniform Lipschitz condition in $\overline{y} = \left(y, y', \dots, y^{(q)}\right)$, which means that, whatever $\left(x, y_0, \dots, y_q\right)$ and $\left(x, \tilde{y}_0, \dots, \tilde{y}_q\right)$ in D, the following inequality holds

v) $\left| f(x, y_0, \dots, y_q) - f(x, \tilde{y}_0, \dots, \tilde{y}_q) \right| \leq \Lambda \sum_{k=0}^{q} |y_k - \tilde{y}_k|$, with Λ a positive real constant.

Then, we consider the differential problem

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}), \qquad \overline{y} = \left(y, y', y'', \dots, y^{(q)}\right), \quad q \leq r, \quad y \in X, \\ L(\mathcal{D}^i y) = w_i, \quad i = 0, \dots, r, \quad w_i \in \mathbb{R}. \end{cases} \tag{1.1}$$

Definition 1. *Using the notation introduced above, problem* (1.1) *will be referred to as the* Umbral Differential Problem *of order* $r+1$ *associated with* $(L, \mathcal{D})$.

We note that problem (1.1) contains a non-linear differential equation of order $r+1$ and an umbral interpolation problem for $(L, \mathcal{D})$ [15–17]. To the best of our knowledge, the above definition does not appear in the literature. Furthermore, it is well motivated, since certain UDPs have already been introduced and studied in previous works. For example the following ones.

A. Let L be the evaluation functional, i.e.

$$L(g) = g(x_0), \qquad x_0 \in [0,1], \quad g \in X_{\mathcal{D}},$$

and $\mathcal{D} = D_x = \frac{d}{dx}$. Then the UDP (1.1) is the classical IVP

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}), \qquad x \in [0,1] \\ y^{(i)}(x_0) = w_i, \qquad i = 0, \dots, r, \quad x_0 \in [0,1]. \end{cases}$$

This problem has been widely studied in the literature, both from a theoretical and numerical point of view (see [6–9] and references therein).

B. Let $L(g) = \int_0^1 g(x)\,dx$, and $\mathcal{D} = D_x$. In this case problem (1.1) becomes

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}) \\ \int_0^1 y^{(i)}(x)\,dx = \omega_i, \qquad i = 0, \ldots, r-1. \end{cases}$$

This problem is known in the literature as *Bernoulli boundary value problem* or *no classic BVP* (boundary value problem) [10, 21, 22].

In this paper, we give a constructive proof of the existence and uniqueness of solution of the UDP (1.1). The numerical solutions are obtained by a Birkoff-Lagrange collocation method [12]. Two new examples of UDPs are given and studied. Particularly, the paper is organized as follows: in Section 2, we give some preliminary results. In Sections 3, we consider the UDP and discuss some theoretical and computational tools: the existence and uniqueness of the solution and a Birkoff-Lagrange collocation method for the approximate solution. Then, in Section 4, we consider some special cases of the problem under study: the Umbral Euler Differential problem and a Multipoint Differential Problem related to the classic difference operator Δ. The analysis is supported by numerical examples (Section 5), which confirm the theoretical results.

2. Preliminaries

To make the paper self-contained, some preliminary results are needed, both on umbral techniques and on BVPs.

2.1. Some basic umbral calculus techniques

With the above notation and hypotheses, we consider the problem of determining a polynomial $P_n(x)$ of degree exactly n, with $n \in \mathbb{N}$, such that

$$L(\mathcal{D}^i P_n) = w_i, \quad i = 0, \ldots, r. \tag{2.1}$$

This problem is called *umbral interpolation problem* (UIP) for $(L, \mathcal{D})$.

Proposition 1. *The UIP* (2.1) *has the unique solution*

$$P_n(t) = -\frac{1}{G}\begin{vmatrix} 0 & 1 \quad t \quad t^2 \quad \cdots \quad t^n \\ w_0 & \\ \vdots & L(\mathcal{D}^i t^j) \\ \vdots & \\ w_n & \end{vmatrix}, \tag{2.2}$$

where $G = det\left(L(\mathcal{D}^i t^j)\right), i, j = 0, \ldots, n.$

Proof. Setting $L_i = L(\mathcal{D}^i)$, $i = 0, \dots, n$, we have $n + 1$ linearly independent linear functionals. Then the proof follows by the general linear interpolation theory [23, p. 35]. □

Remark 1. *We note that if* $f \equiv 0$ *in the UDP* (1.1)*, the solution is the umbral interpolant* (2.2).

Proposition 2. *If* $y \in X_{\mathcal{D}}$ *and* $L(\mathcal{D}^i y) = w_i$, $i = 0, \dots, r$, *then*

$$y(x) = P_r[y](x) + \int_a^b K_r(x,t)\, y^{(r+1)}(t)\, dt, \qquad \forall x \in [a,b], \tag{2.3}$$

with $P_r[y](x) \equiv P_r(x)$ *defined as in* (2.2)*, i.e. it satisfies* $L(\mathcal{D}^i y) = L(\mathcal{D}^i P_r[y])$, $i = 0, \dots, r$, *and*

$$K_r(x,t) = \frac{1}{r!}\left[(x-t)_+^r - P_r[(\cdot - t)_+^r](x)\right]. \tag{2.4}$$

Proof. The proof follows taking into account that the operator $P_r[y]$ reproduces exactly the powers x^j, $j = 0, \dots, r$. Peano Kernel Theorem [23, p.70] is also used. □

Corollary 1. *From Proposition 2 we have* $L(\mathcal{D}^i K_r) = 0,\ i = 0, \dots, r.$

From [11, pp. 163–169] the following three theorems can be derived.

Theorem 1. *Let* L *and* $\mathcal{D}$ *be a linear functional and a delta-operator as in ii) and iii), respectively. There exists a unique polynomial sequence* $\{S_n\}_{n \in \mathbb{N}}$ *(*S_n *a polynomial of degree exactly* n*) such that*

a) $\mathcal{D} S_n(x) = n S_{n-1}$, $n \geq 1$;

b) $L\left(\mathcal{D}^i S_n\right) = n!\, \delta_{n,i}$, $\quad i = 0, \dots, n$, *where* $\delta_{n,i}$ *is the Kroneker symbol;*

c) the polynomial sequence $\{S_n\}_{n \in \mathbb{N}}$ *is a basis for* $\mathcal{P}_n$*, the set of polynomials of degree at most* n*.*

Condition *b*) of the previous theorem allows the representation of polynomials in the basis $\{S_n\}_{n \in \mathbb{N}}$.

Theorem 2. *For every polynomial* $q(x) \in \mathcal{P}_n$ *the following identity holds:*

$$q_n(x) = \sum_{i=0}^{n} L\left(\mathcal{D}^i q_n\right) \frac{S_i(x)}{i!}, \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

Particularly, for $P_n(x)$ *we get*

$$P_n(x) = \sum_{i=0}^{n} w_i \frac{S_i(x)}{i!} = \sum_{i=0}^{n} L\left(\mathcal{D}^i P_n\right) \frac{S_i(x)}{i!}, \quad \forall x \in \mathbb{R}.$$

The extension of Theorem 2 to functions $y \in X_{\mathcal{D}}$ is possible.

Theorem 3. *For all* $y \in X_{\mathcal{D}}$ *the following identity holds:*

$$y(x) = P_r[y](x) + R_r[y](x), \qquad \forall x \in [0,1],$$

where

$$P_r[y](x) = \sum_{i=0}^{r} L\left(\mathcal{D}^i y\right) \frac{S_i(x)}{i!},$$

$$R_r[y](x) = \int_0^1 K_r(x,t)\, y^{(r+1)}(t)\, dt,$$

and $K_r(x,t)$ *is as in* (2.4).

2.2. Relationship between BVPs and Fredholm integral equations

In [12] an interesting equivalence has been proved between BVPs and Fredholm integral equations.

Theorem 4. *[12] Let $y \in X$ and $f : [0,1] \times \mathbb{R}^q \to \mathbb{R}$. If L_i, $i = 0, \dots, r$, are $r+1$ l.i. linear functionals on $X_{\mathcal{D}}$, then the BVP*

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}), & \overline{y} = \left(y, y', y'', \dots, y^{(q)}\right), \\ L_i(y) = w_i, & i = 0, \dots, r, \end{cases}$$

is equivalent to the Fredholm integral equation

$$y(x) = Q_r[y](x) + \int_0^1 K_r(x,t) f(t, \overline{y})\, dt, \tag{2.5}$$

where $Q_r[y](x)$ is the interpolating polynomial satisfying $L_i\left(Q_r[y]\right) = w_i$, $i = 0, \dots, r$, and $K_r(x,t)$ is the related Peano Kernel

$$K_r(x,t) = \frac{1}{r!}\left[(x-t)_+^r - Q_r[(\cdot - t)_+^r](x)\right].$$

Assuming $L_i(P) = L\left(\mathcal{D}^i P\right)$, the polynomial $Q_r[y](x)$ is the umbral interpolating polynomial for $(L, \mathcal{D})$, i.e.

$$Q_r[y](x) = \sum_{i=0}^{r} L\left(\mathcal{D}^i Q_r\right) \frac{S_i(x)}{i!}.$$

3. The Umbral Differential Problem: Theoretical and computational tools

In this section, we show that the UDP (1.1) admits a unique solution that can be computed numerically. At this stage, we do not aim at providing the most general existence and uniqueness proof, nor at developing the most efficient numerical scheme. Our purpose is simply to demonstrate the feasibility and solvability of this new BVP, leaving questions of optimization to future research.

3.1. Existence and uniqueness of solution

The following theorem establishes conditions under which a solution to the UDP (1.1) exists and is the only solution.

Theorem 5. *(Existence and uniqueness) Let $y(x) \in X_{\mathcal{D}}$, f be a real function, defined on a compact subset $D \subset [0,1] \times \mathbb{R}^{q+1}$, continuous at least at the interior of the domain of interest. Moreover, suppose that f satisfies a Lipschitz condition with respect to y, with constant Λ. Let*

$$H_r = \max_{0 \le i \le q} \max_{0 \le x,t \le 1} \left| \frac{d^i}{dx^i} K_r(x,t) \right|, \tag{3.1}$$

being $K_r(x,t)$ the Peano's kernel as in Proposition 2,

$$Q = \max_{(t, \overline{y}(t)) \in D} \left| f(t, \overline{y}(t)) \right|, \tag{3.2}$$

and suppose that

$$(q+1) H_r \Lambda < 1. \tag{3.3}$$

Then the UDP (1.1) *has a unique solution.*

Proof. Assume $L_i(y) = w_i = L\left(\mathcal{D}^i y\right)$. From Theorem 4, the UDP (1.1) is equivalent to the Fredholm integral equation

$$y(x) = P_r[y](x) + \int_0^1 K_r(x,t) f(t, \overline{y}(t))\, dt, \tag{3.4}$$

where

$$P_r[y](t) = \sum_{i=0}^{r} L\left(\mathcal{D}^i y\right) \frac{S_i(x)}{i!},$$

and

$$K_r(x,t) = \frac{1}{r!}\left[(x-t)_+^r - P_r[(\cdot - t)_+^r](x)\right].$$

We consider the sequence of functions

$$\begin{cases} y_0(x) = P_r[y](x) \\ y_m(x) = P_r[y](x) + \int_0^1 K_r(x,t) f(t, \overline{y}_{m-1}(t))\, dt, \quad m \geq 1. \end{cases}$$

Taking into account (3.1) and (3.2), we have

$$\left|y_1^{(s)}(x) - y_0^{(s)}(x)\right| = \left|\int_0^1 \frac{d^s}{dx^s} K_r(x,t) f(t, \overline{y}_0(t)) dt\right| \leq Q H_r, \qquad s = 0, \ldots, q.$$

For the convergence of the sequence $\{y_m\}_{m\in\mathbb{N}}$ we observe first that, by (3.1) and the lipschitzianity of f, we obtain

$$\begin{aligned} \left|y_{m+1}^{(s)}(x) - y_m^{(s)}(x)\right| &\leq \int_0^1 \left|\frac{d^s}{dx^s} K_r(x,t)\right| \left|f(t, \overline{y}_m(t)) - f(t, \overline{y}_{m-1}(t))\right| dt \\ &\leq H_r \Lambda \sum_{i=0}^{q} \left|y_m^{(i)}(t) - y_{m-1}^{(i)}(t)\right|, \qquad s = 0, \ldots, q. \end{aligned}$$

Therefore, if we consider the norm

$$\|y\| = \max_{0\leq t\leq 1} \sum_{i=0}^{q} \left|y^{(i)}(t)\right|,$$

we have

$$\|y_{m+1} - y_m\| \leq (q+1) H_r \Lambda \, \|y_m - y_{m-1}\| \leq \Omega^m Q H_r \tag{3.5}$$

where $\Omega = (q+1) H_r \Lambda$.

From (3.3) and (3.5), the sequence $\{y_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence, hence it converges uniformly. Then, in view of the Lebesgue's theorem on the passage to the limit under the integral sign, we have that the limit of the sequence $\{y_m\}_{m\in\mathbb{N}}$ is a solution of (3.4).

For the uniqueness we observe that, if there were two distinct solutions $y(x)$ and $z(x)$, we would have

$$|y(x) - z(x)| \leq \int_0^1 |K_r(x,t)||f(t, \overline{y}(t)) - f(t, z^*(t))| dt \leq H_r \Lambda \int_0^1 \sum_{i=0}^{q} \left|y^{(i)}(t) - z^{(i)}(t)\right| dt.$$

Hence, $\|y - z\| \leq \Omega \|y - z\|$. From (3.3), the thesis follows.

□

Remark 2. *For specific cases, it is possible to compute an upper bound on H_r, as we will see in what follows.*

3.2. Computational procedure

To obtain a procedure for computing a numerical solution, we draw inspiration from the so-called Birkhoff–Lagrange collocation method, which has been extensively studied and applied, for instance in [12, 20]. We suppose that the hypotheses of Theorem 5 are satisfied, that is, there exists the solution $y(x)$ of problem (1.1). Moreover, suppose that $y(x) \in X_{\mathcal{D}}$ and let m be a positive integer. It is well-known that, if x_i, $i = 0, \dots, m$, are $m + 1$ distinct points in $[0, 1]$, then the Lagrange identity

$$y^{(r+1)}(x) = \sum_{i=0}^{m} l_i(x) y^{(r+1)}(x_i) + R_m[y](x), \tag{3.6}$$

holds, where $l_i(x)$ are the fundamental Lagrange polynomials

$$l_i(x) = \frac{\omega_m(x)}{(x - x_i)\omega'_m(x_i)}, \qquad \omega_m(x) = (x - x_0) \cdots (x - x_m),$$

and

$$R_m[y](x) = \frac{1}{(m+1)!}\omega_m(x) y^{(r+m+1)}(\xi), \qquad \xi \in [0, 1]$$

is the remainder. Assuming $y^{(r+1)}(x) = f(x, \overline{y})$, and substituting (3.6) in (3.4), we get

$$y(x) = P_r[y](x) + \int_0^1 K_r(x, t) \sum_{i=0}^{m} l_i(t) y^{(r+1)}(x_i)\, dt + \int_0^1 K_r(x, t) R_m[y](x)\, dt.$$

By standard calculations we obtain

$$y(x) = P_r[y](x) + \sum_{i=0}^{m} p_{r,i}(x) f(x_i, \overline{y}(x_i)) + T_{m,r}[y](x), \tag{3.7}$$

where

$$p_{r,i}(x) = \int_0^1 K_r(x, t) l_i(t)\, dt,$$

and

$$T_{m,r}[y](x) = \int_0^1 K_r(x, t) R_m[y](x)\, dt.$$

Therefore, neglecting the error term, we can consider the implicit polynomial

$$y_{r,m}(x) = P_r[y](x) + \sum_{i=0}^{m} p_{r,i}(x) f(x_i, \overline{y}_{r,m}(x_i)), \tag{3.8}$$

where $\overline{y}_{r,m}(x_i) = \left(y_{r,m}(x_i), y'_{r,m}(x_i), \dots, y^{(q)}_{r,m}(x_i)\right)$. We can prove (see [20]) that (3.8) is a Birkoff-Lagrange collocation polynomial for the differential equation in (1.1) on the points x_k, $k = 0, \dots, m$, that is,

$$y^{(r+1)}(x_k) = f\left(x_k, \overline{y}(x_k)\right), \qquad k = 0, \dots, m.$$

To calculate the numerical solution $y_{r,m}(x)$, we need the values $y_i^{(s)} = y_{r,m}^{(s)}(x_i)$, $i = 0, \ldots, m$, $s = 0, \ldots, q$. Therefore, we solve the following system:

$$y_k^{(s)} = P_r^{(s)}[y](x_k) + \sum_{i=1}^{m} p_{r,i}^{(s)}(x_k) f(x_i, \overline{y}_i), \quad s = 0, \ldots, q, \; k = 0, \ldots, m. \tag{3.9}$$

System (3.9) can be written in the form

$$Y - AF(Y) = B, \tag{3.10}$$

where

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_q \end{pmatrix}, \qquad A_s = \begin{pmatrix} \tilde{a}_{0,0}^{(s)} & \cdots & \tilde{a}_{0,m}^{(s)} \\ \vdots & & \vdots \\ \tilde{a}_{m,0}^{(s)} & \cdots & \tilde{a}_{m,m}^{(s)} \end{pmatrix},$$

with

$$\tilde{a}_{i,j}^{(s)} = p_{r,j}^{(s)}(x_i), \quad i, j = 0, \ldots, m, \; s = 0, \ldots, q,$$

$$Y = (\overline{Y}_0, \ldots, \overline{Y}_q)^T, \qquad \overline{Y}_s = \left(y_0^{(s)}, \ldots, y_m^{(s)}\right),$$

$$F(Y) = (\underbrace{F_m, \ldots, F_m}_{q+1})^T, \qquad F_m = (f_0, \ldots, f_m)^T, \qquad f_i = f(x_i, \overline{y}_i),$$

$$B = \left(C_0, \ldots, C_q\right)^T, \qquad C_s = \left(P_r^{(s)}[y](x_0), \ldots, P_r^{(s)}[y](x_m)\right).$$

If we set $G(Y) = AF(Y) + B$, then system (3.10) can be solved by the iterative method

$$Y_m^{[\nu+1]} = G\left(Y_m^{[\nu]}\right), \qquad \nu \geq 0, \tag{3.11}$$

with a fixed $Y_m^{[0]} = (\alpha_0, \ldots, \alpha_0) \in \mathbb{R}^{(m+1)(q+1)}$. If $\Lambda \|A\|_\infty < 1$, where Λ is the Lipschitz constant, then G is contractive. Hence system (3.10) has a unique solution. Moreover, if Y is the exact solution, we have

$$\left\|Y_m^{[\nu+1]} - Y\right\|_\infty \leq \frac{T^\nu}{1 - T} \left\|Y_m^{[1]} - Y_m^{[0]}\right\|_\infty,$$

where $T = \Lambda \|A\|_\infty$.

Summarizing, the proposed method consists of the following steps:

- determine the interpolating polynomial $P_r[y](x)$ satisfying the boundary conditions;
- consider the Fredholm integral equation equivalent to the given UDP;
- approximate $y^{(r+1)}(x)$ by Lagrange interpolating polynomial on a set of fixed nodal points;
- compute the elements of matrix A and solve system (3.10);
- obtain polynomial $y_{r,m}(x)$.

4. Examples

Here we give two new examples of UDP of order $r + 1$.

4.1. Umbral Euler Differential Problem of order $r + 1$

If we set

$$L(g) = \frac{g(0) + g(1)}{2}, \qquad \mathcal{D} = D_x, \tag{4.1}$$

then we consider the UDP

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}(x)) \\ \dfrac{y^{(i)}(1) + y^{(i)}(0)}{2} = w_i, \quad i = 0, \ldots, r, \ \ w_i \in \mathbb{R}. \end{cases} \tag{4.2}$$

We call (4.2) the *Umbral Euler Differential Problem*, since the functional L as in (4.1) generates Euler polynomials [11, p.133].

To our knowledge, it has not been studied in the literature. Our study follows Theorem 4, i.e. identity (3.4). Hence, first of all, we need the solution of the Euler interpolation problem

$$\frac{y^{(i)}(1) + y^{(i)}(0)}{2} = w_i, \quad i = 0, \ldots, r, \ \ w_i \in \mathbb{R}. \tag{4.3}$$

4.1.1. Euler interpolation problem

For the solution of the Euler interpolation problem (4.3) the following propositions holds.

Proposition 3. *(Umbral Euler interpolation).* [16,21] *Let* $y \in C^r[0, 1]$. *The interpolation problem* (4.3) *has the unique solution*

$$P_r[y](x) = \sum_{i=0}^{r} w_i \frac{E_i(x)}{i!}, \tag{4.4}$$

where $E_i(x)$ *is the Euler polynomial of degree* i *as defined in [11, p.123].*

Proposition 4. [16] *Let* $y \in C^{r+1}[0, 1]$. *The following identity holds:*

$$y(x) = P_r[y](x) + \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} \frac{E_i(x)}{2} \int_0^1 G_{r,i}(x, t) y^{(r+1)}(t)\, dt, \tag{4.5}$$

with $P_r[y]$ *as in* (4.4) *and*

$$G_{r,i}(x, t) = \begin{cases} (-1)^{r-i} t^{r-i} & 0 \le t \le x, \\ -(1 - t)^{r-i} & x < t \le 1. \end{cases} \tag{4.6}$$

Proposition 5. *With the above hypothesis and notation, if*

$$\Omega_r = \sup_{0 \le x \le 1} |y^{(r+1)}(x)|,$$

then

$$|y(x) - P_r[y](x)| \le= \frac{\Omega_r}{3\pi^{r-1}} \sum_{i=0}^{r} \frac{\pi^i}{i!}, \qquad \forall x \in [0, 1].$$

Proof. Let $\widetilde{E}_i(x)$ be the Euler polynomial of degree i as defined by Jordan [26]. From the inequality $|\widetilde{E}_i(x)| \le \frac{2}{3\pi^{i-1}}$, $0 \le x \le 1$ [26, p.303], we get

$$|E_i(x)| = \left|i!\,\widetilde{E}_i(x)\right| \le \frac{2i!}{3\pi^{i-1}}, \qquad 0 \le x \le 1. \tag{4.7}$$

Hence, the proof is a direct consequence of Proposition 4. □

Proposition 4 and Proposition 5 suggest to consider the convergence of $P_r[y]$, that is, the expansion of a real function in Euler polynomials (see [5, 18] and references therein).

Definition 2. *[18] A real entire function $f(x)$ belongs to the class $M(h)$, $h > 0$, if there exists a positive number $p < h$ such that*

$$f^{(n)}(0) = O(p^n), \qquad n \to \infty.$$

Concerning this class of functions we have the following result.

Lemma 1. *[18, 30] If $f(x)$ belongs to $M(h)$, then there exists a positive number $p < h$ such that*

$$f^{(n)}(x) = O(p^n), \qquad \forall x \in [0, 1].$$

We note explicitly that if f belongs to the class $M(h)$, then it is of exponential type less than h.

Proposition 6. *(Convergence) If $y(x) \in M(\pi)$, then*

$$\lim_{r\to\infty} P_r[y](x) = y(x), \qquad \forall x \in [0, 1]$$

or, equivalently,

$$\sum_{i=0}^{\infty} \frac{y^{(i)}(0) + y^{(i)}(1)}{2} \frac{E_i(x)}{i!} = y(x), \qquad \forall x \in [0, 1].$$

Proof. The proof follows from Proposition 5 and Lemma 1. □

After Proposition 6 we can say that a function f belonging to the class $M(\pi)$ has an expansion in Euler polynomials [4].

4.1.2. Numerical solution of the Euler UDP

Taking into account (4.5) and (4.6), after calculations, the Birkoff-Lagrange collocation polynomial for the Euler UDP becomes

$$y_{r,m}(x) = P_r[y](x) + \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} \frac{E_i(x)}{2} \sum_{j=0}^{m} f(x_j, \overline{y}_{r,m}(x_j)) \sum_{k=0}^{r-i} \Phi_{i,k,j}(x), \tag{4.8}$$

where

$$\Phi_{i,k,j}(x) = (-1)^k \left[\alpha_{k,j}(x) x^{r-i-k} + \beta_{k,j}(x)(x-1)^{r-i-k}\right] \prod_{l=0}^{k-1} (r - i - l),$$

being $\alpha_{k,j}(x) = \underbrace{\int_0^x \cdots \int_0^x}_{k+1} l_j(t)dt \cdots dt$ and $\beta_{k,j}(x) = \underbrace{\int_1^x \cdots \int_1^x}_{k+1} l_j(t)dt \cdots dt$.

It can be written in the form (3.8), with

$$p_{r,j}(x) = \frac{1}{2r!} \sum_{i=0}^{r} \binom{r}{i} E_i(x) \sum_{k=0}^{r-i} \Phi_{i,k,j}(x).$$

To obtain the values $\overline{y}_{r,m}(x_j)$, $j = 0, \ldots, m$, we solve the nonlinear system

$$y_{r,m}^{(s)}(x_\ell) = P_r^{(s)}[y](x_\ell) + \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} \frac{E_i^{(s)}(x_\ell)}{2} \sum_{j=0}^{m} f(x_j, \overline{y}_{r,m}(x_j)) \sum_{k=0}^{r-i} \Phi_{i,k,j}^{(s)}(x_\ell),$$

$s = 0, \ldots, q,\ \ell = 0, \ldots, m.$

In order to calculate $\alpha_{k,j}(x_\ell)$ and $\beta_{k,j}(x_\ell)$ we use the algorithm proposed in [9] for the computation of

$$\underbrace{\int_a^x \cdots \int_a^x}_{k} q_{m,i}(t)\, dt \cdots dt \tag{4.9}$$

where $q_{0,0}(t) = 1$, $q_{m,i}(t) = (t - x_1) \cdots (t - x_{i-1})(t - x_{i+1}) \cdots (t - x_m)$, $i = 1, 2, \ldots m$.

We set $g_{0,1,a}^{(i)}(x) = x - a$ and

$$g_{s,j,a}^{(i)}(x) = \underbrace{\int_a^x \cdots \int_a^x}_{j} \left(t - z_1^{(i)}\right)\left(t - z_2^{(i)}\right) \cdots \left(t - z_s^{(i)}\right) dt \cdots dt$$

where

$$z_j^{(i)} = \begin{cases} x_j & \text{if} \quad j < i \\ x_{j+1} & \text{if} \quad j \geq i \end{cases} \qquad j = 1, \ldots, m-1\,.$$

Then the integral (4.9) can be computed by the recurrence formula

$$g_{s,j,a}^{(i)}(x) = \left(x - z_s^{(i)}\right) g_{s-1,j,a}^{(i)}(x) - j g_{s-1,j+1,a}^{(i)}(x)\,.$$

Remark 3. *In the Euler case,*

$$H_r = \max_{0 \leq m \leq q} \max_{0 \leq x,t \leq 1} \left| \frac{d^m}{dx^m} K_r(x,t) \right| < \frac{\pi}{3}.$$

In fact, from (4.7) *and* $E_k^{(m)}(x) = \dfrac{k!}{(k-m)!} E_{k-m}(x)$*, we get*

$$\max_{0 \leq m \leq q} \max_{0 \leq x,t \leq 1} \left| \frac{d^m}{dx^m} K_r(x,t) \right| < \max_{0 \leq m \leq q} \frac{\pi}{3r!} \sum_{k=m}^{r} \binom{r}{k} \frac{k!}{\pi^{k-m}} = \max_{0 \leq m \leq q} \frac{1}{3\pi^{r-m-1}} \sum_{j=0}^{r-m} \frac{\pi^j}{j!} = \frac{\pi}{3}.$$

4.2. A Multipoint Differential Problem of order $r+1$

Let L be the evaluation functional, that is,

$$L(g) = g(0), \qquad \forall g \in X.$$

Let $\mathcal{D}$ be the delta-operator

$$\mathcal{D}\, g(x) = \frac{g(x+h) - g(x)}{h} = \frac{\Delta g(x)}{h}, \qquad h > 0, \tag{4.10}$$

being Δ the difference finite operator [26].

In this case the UDP (1.1) becomes

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}) \\ \dfrac{\Delta^i y(0)}{h^i} = w_i, \quad i = 0, \dots, r, \quad w_i \in \mathbb{R}, \end{cases} \tag{4.11}$$

where the delta-operator (4.10) is evaluated at $x = 0$.

It is known [23, p.51] that

$$\Delta^i y(0) = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} y(jh), \qquad i = 0, \dots, r.$$

Then, $\frac{\Delta^i y(0)}{h^i} = w_i$, $i = 0, \dots, r$, is equivalent to

$$y(ih) = v_i, \qquad v_i = \sum_{j=0}^{i} h^j \binom{i}{j} w_j, \qquad i = 0, \dots, r.$$

Substituting in (4.11), we get

$$\begin{cases} y^{(r+1)}(x) = f(x, \overline{y}) \\ y(ih) = v_i, \quad i = 0, \dots, r. \end{cases} \tag{4.12}$$

This is a multipoint higher-order differential problem [1].

It is known that

$$\lim_{h \to 0} \frac{\Delta^i y(x_0)}{h^i} = y^{(i)}(x_0), \qquad i = 0, \dots, r. \tag{4.13}$$

Therefore we can say that for $h \to 0$, the UDP (4.11) becomes the classic higher-order Cauchy differential problem, or initial value problem.

Definition 3. *The UDP* (4.11) *is called the higher-order Multipoint Umbral Differential Problem or degenerate Cauchy problem.*

Observe that, from (4.13), we have

$$\Delta^i y(x_0) \approx y^{(i)}(x_0), \qquad i = 0, \dots, r.$$

This leads to several numerical approximations for the UDP (4.12), which we plan to address in a future paper.

4.2.1. The equivalent Fredholm integral equation

In order to give numerical solutions to problem (4.12) by (3.8), we consider the equivalent integral equation. We note that the interpolation problem related to (4.12) is the known interpolation on equidistant points [27, 29]. In this case the interpolating polynomial, in the Newton form, is

$$P_{r,h}[y](x) = \sum_{i=0}^{r} \frac{\Delta_h^i y(0)}{i! h^i} (x)_{i,h}, \tag{4.14}$$

where $(x)_{i,h} = x(x-h)\cdots(x-ih)$.

Proposition 7. *Let $y \in X$. The following identity holds:*

$$y(x) = P_{r,h}[y](x) + \frac{1}{r!} \sum_{i=0}^{r} \gamma_{r,i}(x) \int_{ih}^{x} (ih-t)^r y^{(r+1)}(t)\, dt, \tag{4.15}$$

with $P_r[y]$ as in (4.14)*, and*

$$\gamma_{r,i}(x) = \sum_{j=i}^{r} \binom{j}{i} \frac{(-1)^{j-i}}{h^j j!} (x)_{j,h}. \tag{4.16}$$

Proof. From (2.3),

$$y(x) = P_{r,h}[y](x) + \int_0^1 K_r(x,t)\, y^{(r+1)}(t)\, dt, \qquad \forall x \in [0,1],$$

with

$$\begin{aligned} K_r(x,t) &= \frac{1}{r!}\left[(x-t)_+^r - P_{r,h}[(\cdot - t)_+^r](x)\right] \\ &= \frac{1}{r!}\left[(x-t)_+^r - \sum_{i=0}^{r}\sum_{j=i}^{r} \binom{j}{i} \frac{(-1)^{i-j}}{h^j j!} (x)_j (jh-t)_+^r\right] \\ &= \frac{1}{r!}\left[(x-t)_+^r - \sum_{i=0}^{r} \gamma_{r,i}(x)(ih-t)_+^r\right], \end{aligned}$$

and $\gamma_{r,i}(x)$ as in (4.16). Taking into account that

$$(x-t)^r = \sum_{i=0}^{r} \gamma_{r,i}(x)(ih-t)^r,$$

we get

$$y(x) = P_{r,h}[y](x) + \frac{1}{r!} \sum_{i=0}^{r} \gamma_{r,i}(x) \int_0^1 G_{r,i}(x,t) y^{(r+1)}(t)\, dt, \qquad \forall x \in [0,1],$$

with

$$G_{r,i}(x,t) = \begin{cases} (ih-t)^r - (ih-t)_+^r & 0 \le t \le x, \\ -(ih-t)_+^r & x \le t \le 1. \end{cases}$$

After calculations we obtain relation (4.15).

□

4.2.2. Numerical solution of UDP (4.11)

For the numerical solution of UDP (4.11) we proceed as with the Euler UDP. In this case, the polynomial (3.8) becomes

$$y_{r,m}(x) = P_r[y](x) + \frac{1}{r!}\sum_{i=0}^{r}\gamma_{r,i}(x)\sum_{j=0}^{m} f(x_j, \overline{y}_{r,m}(x_j))\sum_{k=0}^{r}\widetilde{\Phi}_{k,i,j}(x), \tag{4.17}$$

with $\gamma_{r,i}(x)$ as in (4.16), and

$$\widetilde{\Phi}_{k,i,j}(x) = \sigma_{k,i,j}(x)\,(ih - x)^{r-k}\prod_{l=0}^{k-1}(r - l),$$

being $\sigma_{k,i,j}(x) = \underbrace{\int_{ih}^{x}\cdots\int_{ih}^{x}}_{k+1} l_j(t)dt\cdots dt.$

An alternative form of $y_{r,m}(x)$ is

$$y_{r,m}(x) = P_r[y](x) + \frac{1}{r!}\sum_{j=0}^{m}\widetilde{q}_{r,i,j}(x)f(x_j, \overline{y}_{r,m}(x_j)), \tag{4.18}$$

with

$$\widetilde{q}_{r,i,j}(x) = \sum_{i=0}^{r}\gamma_{r,i}(x)\sum_{k=0}^{r}\widetilde{\Phi}_{k,i,j}(x).$$

For the calculation of $\sigma_{k,i,j}(x)$ we can use the algorithm in [9].

5. Numerical examples

The method described in the previous sections is now employed to obtain numerical approximations, serving to validate the theoretical results through some test problems. As the true solutions are known, we consider the error functions

$$Err_{r,m}(x) = \left|y(x) - y_{r,m}(x)\right|,$$

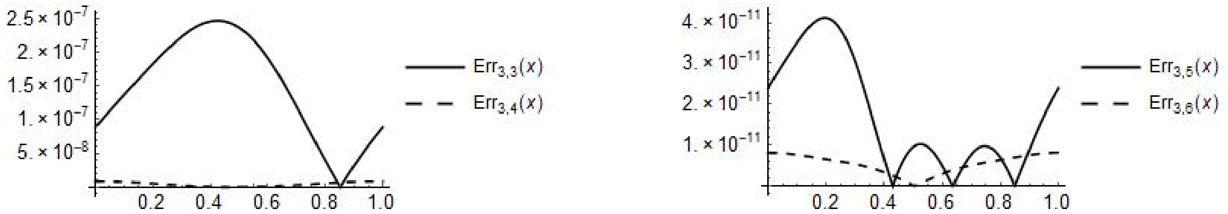
with $y_{r,m}(x)$ given by (4.8) and (4.18); m is the number of collocation points. Chebyshev points were used. Almost the same results are obtained using equally spaced nodes. All computations presented in this work were performed with Mathematica (Wolfram Research, Version 10).

Example 1. *Consider the following Euler UDP*

$$\begin{cases} y^{(4)}(x) = \sin x + \sin^2 x - (y'')^2, & x \in [0, 1] \\ \frac{y(1)+y(0)}{2} = \frac{\sin(1)}{2}, \quad \frac{y'(1)+y'(0)}{2} = \frac{1+\cos(1)}{2}, \\ \frac{y''(1)+y''(0)}{2} = -\frac{\sin(1)}{2}, \quad \frac{y'''(1)+y'''(0)}{2} = -\frac{1+\cos(1)}{2}. \end{cases} \tag{5.1}$$

The analytical solution is $y(x) = \sin x$.

Figure 1 shows the graphs of the error functions for $m = 3, 4$ (Figure 1a) and the graph of the error functions for $m = 5, 6$ (Figure 1b). On the horizontal axis we report the values of x, and on the vertical axis the corresponding values of $Err_{r,m}(x)$, for different m and $r = 3$.



(a) $m = 3$ (solid line) $m = 4$ (dashed line) **(b)** $m = 5$ (solid line) $m = 6$ (dashed line)

Figure 1. Error functions-Problem (5.1).

It can be observed that the function error $Err_{r,m}(x)$ decreases with increasing m. Figure 2 illustrates the decay of the absolute error as m grows, displayed on a $\log_{10}$ scale. On the horizontal axis are the values of m, with $m = 2, ..., 7$, while on the vertical axis the corresponding values of $Log_{10}(Err_{r,m}(x))$.

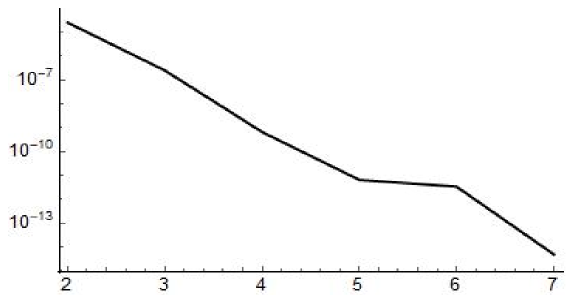


Figure 2. $\log_{10}$ plot against m-Problem (5.1).

Table 1 presents the maximum error and the CPU time (in seconds) required to execute the entire algorithm for various values of m.

Table 1. Maximum error-Problem (5.1).

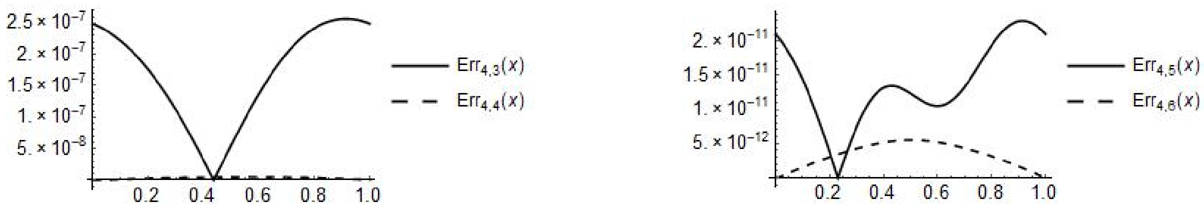
m	$\|\|Err_{3,m}\|\|_\infty$	CPU time
2	$2.48 \cdot 10^{-05}$	1.02
3	$2.46 \cdot 10^{-07}$	1.80
4	$6.48 \cdot 10^{-10}$	5.10
5	$6.34 \cdot 10^{-12}$	8.71
6	$3.33 \cdot 10^{-12}$	10.20
7	$4.77 \cdot 10^{-15}$	19.50

Example 2. *Consider the following fifth order Euler UDP*

$$\begin{cases} y^{(5)}(x) = y^2(x)\, e^x, & x \in [0, 1] \\ \dfrac{y^{(i)}(1) + y^{(i)}(0)}{2} = \dfrac{e+1}{2}, & i = 0, \ldots, 3. \end{cases} \tag{5.2}$$

The analytical solution is $y(x) = e^x$.

Figure 3 shows the graphs of the error functions for $m = 3, 4$ *(Figure 3a) and the graph of the error functions for* $m = 5, 6$ *(Figure 3b).*



(a) $m = 3$ (solid line) $m = 4$ (dashed line) **(b)** $m = 5$ (solid line) $m = 6$ (dashed line)

Figure 3. Error functions-Problem (5.2).

Figure 4 presents the plot of the absolute error as m increases, on a $\log_{10}$ *scale.*

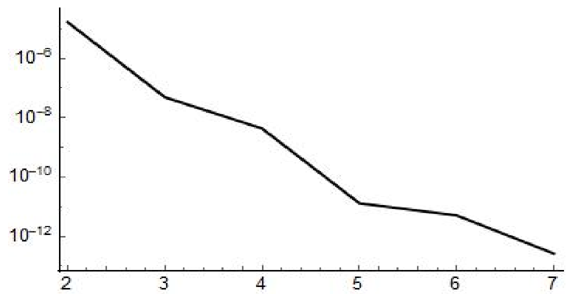


Figure 4. Log_{10} plot-Problem (5.2).

Table 2 presents the maximum error for different values of m.

Table 2. Maximum error-Problem (5.2).

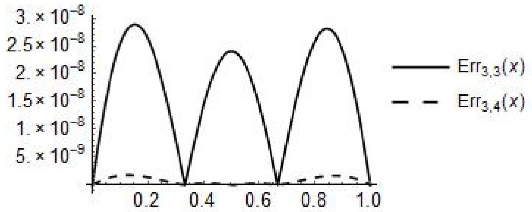
m	$\|\|Err_{4,m}\|\|_\infty$
2	$1.68 \cdot 10^{-05}$
3	$4.96 \cdot 10^{-08}$
4	$4.37 \cdot 10^{-09}$
5	$1.32 \cdot 10^{-11}$
6	$5.16 \cdot 10^{-12}$
7	$2.66 \cdot 10^{-13}$

Example 3. *Consider the following UDP*

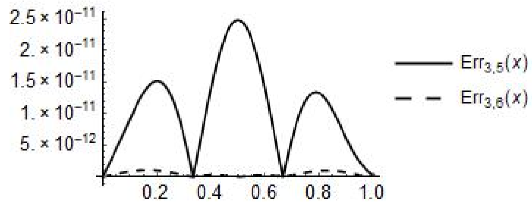
$$\begin{cases} y^{(4)}(x) = \sin x + \sin^2 x - (y'')^2, & x \in [0, 1] \\ y(0) = 0, \ \ \Delta_h y(0) = \sin \frac{1}{3}, \ \ \Delta_h^2 y(0) = \sin \frac{2}{3} - 2 \sin \frac{1}{3}, \\ \Delta_h^3 y(0) = \sin 1 + 3 \sin \frac{1}{3} - 3 \sin \frac{2}{3}. \end{cases} \tag{5.3}$$

$h = \frac{1}{3}$. *The analytical solution is* $y(x) = \sin x$.

Figure 5 shows the graphs of the error functions for m $= 3, 4$ *(Figure 5a) and the graph of the error functions for m* $= 5, 6$ *(Figure 5b).*



(a) $m = 3$ (solid line) $m = 4$ (dashed line)



(b) $m = 5$ (solid line) $m = 6$ (dashed line)

Figure 5. Error functions-Problem (5.3).

Figure 6 provides a graphical representation of $\log_{10}(error)$ *versus m, highlighting their relationship.*

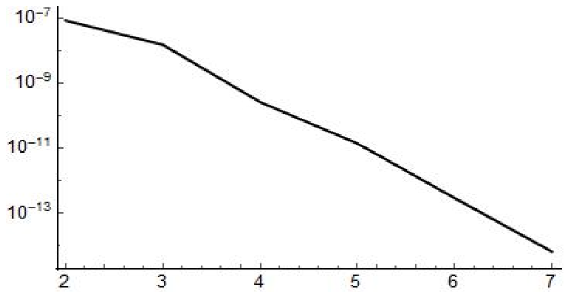


Figure 6. Log_{10} plot-Problem (5.3).

Table 3 lists the maximum error for different m values.

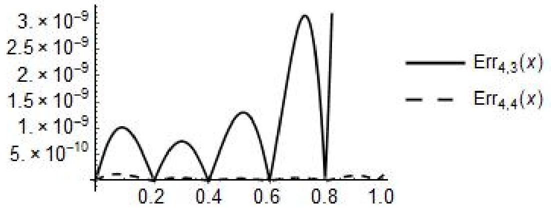
Table 3. Maximum error-Problem (5.3).

m	$\|\|Err_{3,m}\|\|_\infty$
2	$8.46 \cdot 10^{-08}$
3	$1.52 \cdot 10^{-08}$
4	$2.63 \cdot 10^{-10}$
5	$1.40 \cdot 10^{-11}$
6	$2.89 \cdot 10^{-13}$
7	$6.55 \cdot 10^{-15}$

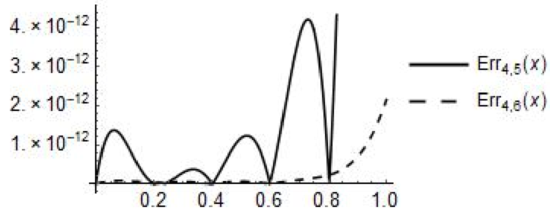
Example 4. *Consider the following fifth order UDP*

$$\begin{cases} y^{(5)}(x) = y^2(x)\, e^x, \qquad x \in [0, 1] \\ y(0) = 1,\ \ \Delta_h y(0) = e^{\frac{1}{5}} - 1,\ \ \Delta_h^2 y(0) = 1 - 2e^{\frac{1}{5}} + e^{\frac{2}{5}}, \\ \Delta_h^3 y(0) = -1 + 3e^{\frac{1}{5}} - 3e^{\frac{2}{5}} + e^{\frac{3}{5}},\ \Delta_h^4 y(0) = 1 - 4e^{\frac{1}{5}} + 6e^{\frac{2}{5}} - 4e^{\frac{3}{5}} + e^{\frac{4}{5}}, \\ \Delta_h^5 y(0) = e - 1 + 5e^{\frac{1}{5}} - 10e^{\frac{2}{5}} + 10e^{\frac{3}{5}} - 5e^{\frac{4}{5}}. \end{cases} \tag{5.4}$$

The analytical solution is $y(x) = e^x$. As in the previous examples, the following figures show the graphs of the error functions for m $= 3, 4$ (Figure 7a), for m $= 5, 6$ (Figure 7b) and the variation of $\log_{10}(error)$ *with respect to m (Figure 8).*



(a) $m = 3$ (solid line) $m = 4$ (dashed line)



(b) $m = 5$ (solid line) $m = 6$ (dashed line)

Figure 7. Error functions-Problem (5.4).

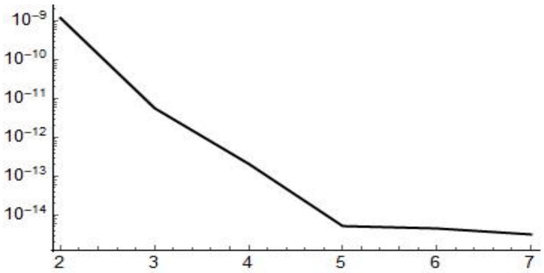


Figure 8. Log_{10} plot-Problem (5.4).

Table 4 reports the maximum error for several choices of m.

Table 4. Maximum error-Problem (5.4).

m	$\|\|Err_{4,m}\|\|_\infty$
2	$1.19 \cdot 10^{-09}$
3	$5.68 \cdot 10^{-12}$
4	$2.09 \cdot 10^{-13}$
5	$3.18 \cdot 10^{-15}$
6	$5.19 \cdot 10^{-15}$
7	$4.88 \cdot 10^{-16}$

6. Conclusions

We introduced the higher-order UDP, that is, a nonlinear ordinary differential equation of higher order associated with an umbral interpolation problem of the same order to that of the differential equation. We established the existence and uniqueness of the solution through constructive Picard-type iterations. A numerical procedure for practical calculations based on Birkhoff-Lagrange interpolation is proposed. Two new examples of UDPs are presented and analyzed: the so-called Euler umbral problem and a higher-order multipoint differential problem. The latter problem is also referred to as the degenerate Cauchy problem since, as $h \to 0$, the conditions tend to the initial conditions. Some numerical examples are provided, showing promising results. Further theoretical and computational developments are possible. On the other hand, a natural direction would be to consider the problem in appropriate functional spaces and to study more general conditions for the existence of the solution. Numerical implementation using finite difference methods is also a viable approach. Naturally, new explicit examples are desirable.

Author contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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