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Research article

Nonlinear oscillation analysis of delay differential equations with mixed neutral terms

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Abstract: This paper investigates novel sufficient conditions for the oscillatory behavior of fourth-order nonlinear differential equations with mixed neutral terms. By employing refined Riccati transformation techniques and advanced analytical approaches, we establish extended criteria that enrich the theoretical understanding of oscillation phenomena within this class of neutral differential equations. The proposed results significantly improve upon previously known conditions in the literature. Moreover, illustrative numerical examples are provided to demonstrate the applicability and sharpness of the obtained criteria. The findings contribute to the broader framework of nonlinear analysis and offer valuable insights into the oscillatory dynamics of functional differential equations with delay and neutral terms.

Keywords: oscillation criteria; nonlinear equations; delay differential equations; mixed neutral terms; canonical form; fourth-order
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this research, we study the oscillatory behavior of a class of fourth-order delay differential equations with mixed neutral terms. The equations under consideration take the following general form:

$$\left(\kappa(\mathrm{s})\left[\mathcal{Z}'''(\mathrm{s})\right]^{\alpha}\right)' + \mathfrak{q}(\mathrm{s})x^{\beta}(b(\mathrm{s})) = 0, \tag{1.1}$$

where

$$\mathcal{Z}(\mathrm{s}) := x(\mathrm{s}) + \hbar_1(\mathrm{s})\, x^{\gamma}(a(\mathrm{s})) - \hbar_2(\mathrm{s})\, x^{\delta}(a(\mathrm{s})), \mathrm{s} \geq \mathrm{s}_0.$$

The analysis in this work is carried out under the following assumptions:

(H1) α, β, γ and δ are quotients of positive odd integers, with $\gamma < 1$, and $\delta > 1$;
(H2) $\hbar_1, \hbar_2, \mathfrak{q} \in C([\mathrm{s}_0, \infty), (0, \infty))$, and $\mathfrak{q}(\mathrm{s}) \neq 0$;
(H3) $a, b \in C^1([\mathrm{s}_0, \infty), \mathbb{R})$ satisfies $b(\mathrm{s}) \leq \mathrm{s}$, $a(\mathrm{s}) \leq \mathrm{s}$, $b'(\mathrm{s}) > 0$ and $\lim_{\mathrm{s}\to\infty} a(\mathrm{s}) = \lim_{\mathrm{s}\to\infty} b(\mathrm{s}) = \infty$;
(H4) $\kappa \in C^1([\mathrm{s}_0, \infty), (0, \infty))$ satisfies

$$\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\kappa^{1/\alpha}(\upsilon)} \mathrm{d}\upsilon \to \infty \text{ as } \mathrm{s} \to \infty. \tag{1.2}$$

In addition, we define the auxiliary functions:

$$R_0(\mathrm{s}) := \int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\kappa^{1/\alpha}(\upsilon)} \mathrm{d}\upsilon \text{ and } R_i(\mathrm{s}) := \int_{\mathrm{s}_0}^{\mathrm{s}} R_{i-1}(\upsilon)\mathrm{d}\upsilon,\ i = 1, 2.$$

A function $x(\mathrm{s})$ is called a solution of the Eq (1.1) if it is continuous and satisfies Eq (1.1) on the interval $[\mathrm{s}_x, \infty)$, for some $\mathrm{s}_x \geqslant \mathrm{s}_0$. We restrict our attention to those solutions $x(\mathrm{s})$ that are continuable, meaning they satisfy the condition

$$\sup\{|x(\mathrm{s})| : \mathrm{s} \geqslant \mathrm{S}\} > 0, \text{ for all } \mathrm{S} > \mathrm{s}_x \geq \mathrm{s}_0.$$

A solution is classified as oscillatory if it is neither eventually positive nor eventually negative; it is referred to as nonoscillatory. Equation (1.1) is said to be in canonical form if condition (1.2) is satisfied.

Neutral differential equations (NDEs) constitute an important class of functional differential equations, distinguished by the presence of derivatives of the unknown function at both the present and past (or delayed) times. This structural complexity makes analyzing their solutions more challenging than for ordinary or standard delay differential equations. The significance of these equations lies in their ability to model systems whose behavior depends on their temporal history, as in various applications in mechanics, engineering, and biology. They are also employed to describe complex physical phenomena, including wave propagation in electrical networks and the stability analysis of nonlinear dynamical systems (see, for example, Brayton [1], Snow [2], Jadlovská [3], Grigorian [4], Aldiaiji et al. [5], and Li et al. [6]).

The analysis of oscillations in differential equations is a fundamental field of applied mathematics, as it constitutes the cornerstone for understanding dynamics characterized by instability and periodic change. Since the beginning of the scientific study of this field, leading mathematicians have established methodological foundations for understanding these complex behaviors. Modern research has focused on developing various analytical techniques, such as comparison methods, Riccati transformations, and integration criteria, which have proven effective in simplifying the study of complex differential equations and deriving precise conditions that guarantee or prevent oscillations. These tools have helped expand the theoretical framework, providing more powerful means for analyzing dynamical systems, and increasing their practical applications in various engineering and natural sciences (see, for example, Gyori and Ladas [7], Zafer [8], Agarwal and Bohner [9], Qaraad et al. [10], and Wu et al. [11]).

The study of the oscillatory behavior of differential equations has witnessed significant progress across different orders. At the second order, new conditions have been established for analyzing the oscillation of delayed and neutral equations using advanced techniques (see, for example, Baculikova and Dzurina [12], Zhang [13], and Li et al. [14]). At the third order, precise criteria have been proposed for treating equations with infinite delay and neutral terms [15, 16]. At the fourth-order level, qualitative contributions have emerged based on innovative approaches that address the theoretical and applied challenges of delay equations [17]. Comprehensive criteria have also been developed for studying multi-delay equations at higher orders, enhancing the theoretical understanding of these complex systems (see, for example, Xing et al. [18], Graef et al. [19], Alnafisah et al. [20], and Batiha et al. [21].

Recent years have witnessed a significant increase in interest in studying the oscillatory behavior of fourth-order nonlinear differential equations, due to their growing importance in applied science and engineering [22, 23]. A number of researchers have addressed specific models of these equations. Jadlovská et al. [24] studied a delay linear equation of the form:

$$x^{(4)}(\mathrm{s}) + \mathfrak{q}(\mathrm{s})x(b(\mathrm{s})) = 0.$$

They were able to define precise conditions that allow the solutions to be classified as oscillatory and non-oscillatory. On the other hand, Grace and Akin [25], Zhang et al. [26], and Masood et al. [27] studied quasilinear equations involving time delays, focusing on equations of the form:

$$\left(\kappa(\mathrm{s})\left[x'''(\mathrm{s})\right]^{\alpha}\right)' + \mathfrak{q}(\mathrm{s})x^{\alpha}(b(\mathrm{s})) = 0.$$

They used various analytical techniques, most notably the direct comparison method, which led to the extraction of new criteria governing the behavior of oscillatory solutions.

Kamo and Usami [28] and Kusano et al. [29] also considered similar equations of the type:

$$\left(\kappa(\mathrm{s})\left[x'''(\mathrm{s})\right]^{\alpha}\right)' + \mathfrak{q}(\mathrm{s})x^{\beta}(b(\mathrm{s})) = 0,$$

with particular emphasis on cases where the parameters of the nonlinear terms change, as they were able to formulate sufficient conditions to guarantee oscillation under these conditions.

In parallel, Bazighifan and Cesarano [30] and Alatwi et al. [31] sought to extend the scope of the analysis to include neutral differential equations, focusing on the equation:

$$\left(\kappa(\mathrm{s})\left[\left(x(\mathrm{s}) + \hbar_1(\mathrm{s})\,x(a(\mathrm{s}))\right)'''\right]^{\alpha}\right)' + \mathfrak{q}(\mathrm{s})x^{\beta}(b(\mathrm{s})) = 0.$$

This helped deepen the understanding of the neutral effect on oscillatory behavior and develop more realistic analytical models.

In this context, Masood et al. [32] presented a study of equations containing nonlinear neutral terms of the form:

$$\left(\kappa(\mathrm{s})\left[\left(x(\mathrm{s}) + \hbar_1(\mathrm{s})\,x^{\gamma}(a(\mathrm{s}))\right)'''\right]^{\alpha}\right)' + \mathfrak{q}(\mathrm{s})x^{\beta}(b(\mathrm{s})) = 0.$$

They employed the Riccati method to establish oscillation criteria, marking a notable advancement in applying this technique to such equations.

Following an extensive review of the existing literature, it becomes evident that the oscillatory behavior of fourth-order neutral differential equations involving mixed nonlinear delay terms has not been adequately investigated. To bridge this gap, the present study develops a unified analytical framework specifically designed for such equations, addressing both the challenges posed by higher-order structures and the intricate effects of multiple neutral delays. Employing a refined Riccati transformation and integral averaging techniques, the paper establishes novel and less restrictive oscillation criteria that significantly generalize and extend previously known results. This methodological advancement not only deepens the theoretical understanding of oscillation phenomena but also broadens the applicability of oscillation theory to more complex functional systems, thereby marking a meaningful contribution to the ongoing development of the field.

The subsequent sections The paper is organized as follows. Section 2 presents preliminary lemmas addressing the monotonic properties of non-oscillatory solutions. In Section 3, oscillation criteria for Eq (1.1) are established using the Riccati transformation and appropriate substitutions, focusing on the structure and eventual positivity of solutions. Section 3 provides numerical examples to illustrate the applicability of the theoretical results. Finally, Section 4 concludes with a summary, remarks, and future research directions.

2. Preliminary results

To develop our main results, we begin by formulating a series of lemmas that investigate the monotonic properties of non-oscillatory solutions associated with the considered class of equations.

Remark 2.1. *Below, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large* s.

Remark 2.2. *In what follows, we consider only positive solutions, since if* $x(\mathrm{s})$ *is a solution, then* $-x(\mathrm{s})$ *is also a solution.*

Lemma 2.1. *[33] Let* α *be a ratio of two odd positive integers; C and D are constants. Then*

$$Du - Cu^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{D^{\alpha+1}}{C^{\alpha}},\ C > 0. \tag{2.1}$$

Lemma 2.2. *[34] Let C and D be two non-negative real numbers. Then, the following inequality is obtained:*

$$C^{\lambda} + (\lambda - 1) D^{\lambda} - \lambda C D^{\lambda-1} \geq 0 \quad for\ \lambda > 1, \tag{2.2}$$

$$C^{\lambda} - (1 - \lambda) D^{\lambda} - \lambda C D^{\lambda-1} \leq 0 \quad for\ 0 < \lambda < 1, \tag{2.3}$$

where the equality holds if and only if $C = D$*.*

Lemma 2.3. *[35] Let* $\varkappa \in \mathbf{C}^{m}([\mathrm{s}_0, \infty), (0, \infty))$*,* $\varkappa^{(i)}(\mathrm{s}) > 0$ *for* $i = 1, 2, ..., m$*, and* $\varkappa^{(m+1)}(\mathrm{s}) \leq 0$*, eventually. Then, eventually,*

$$\frac{\varkappa(\mathrm{s})}{\varkappa'(\mathrm{s})} \geq \frac{\epsilon \mathrm{s}}{m},$$

for every $\epsilon \in (0, 1)$*.*

Lemma 2.4. *[35] Let $x(\mathrm{s})$ be an eventually positive solution of Eq (1.1). Then, the corresponding function $\mathcal{Z}(\mathrm{s})$ satisfies, for sufficiently large* s, *exactly one of the following:*

$$\begin{aligned}
(\mathrm{C}_1) \quad &: \quad \mathcal{Z}(\mathrm{s}) > 0,\ \mathcal{Z}'(\mathrm{s}) > 0,\ \mathcal{Z}''(\mathrm{s}) > 0,\ \mathcal{Z}'''(\mathrm{s}) > 0,\ (\kappa(\mathrm{s})(\mathcal{Z}'''(\mathrm{s}))^{\alpha})' < 0, \\
(\mathrm{C}_2) \quad &: \quad \mathcal{Z}(\mathrm{s}) > 0,\ \mathcal{Z}'(\mathrm{s}) > 0,\ \mathcal{Z}''(\mathrm{s}) < 0,\ \mathcal{Z}'''(\mathrm{s}) > 0,\ (\kappa(\mathrm{s})(\mathcal{Z}'''(\mathrm{s}))^{\alpha})' < 0,
\end{aligned}$$

for $s \geqslant s_1 \geqslant s_0$.

Lemma 2.5. *Let $x(\mathrm{s})$ be an eventually positive solution of the Eq (1.1) with $\mathcal{Z} \in \mathrm{C}_1$. Then the following properties hold:*
(a) $\mathcal{Z}(\mathrm{s}) \geqslant \kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_2(\mathrm{s})$;
(b) $\frac{\mathcal{Z}''(\mathrm{s})}{R_0(\mathrm{s})}$ *and* $\frac{\mathcal{Z}(\mathrm{s})}{R_2(\mathrm{s})}$ *are decreasing functions*;
(c) $\mathcal{Z}(\mathrm{s}) \geqslant \frac{R_2(\mathrm{s})\mathcal{Z}''(\mathrm{s})}{R_0(\mathrm{s})}$.

Proof. The monotonicity of $\kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})$ implies that

$$\begin{aligned}
\mathcal{Z}''(\mathrm{s}) &\geq \int_{\mathrm{s}_1}^{\mathrm{s}} \kappa^{1/\alpha}(\upsilon)\mathcal{Z}'''(\upsilon)\frac{1}{\kappa^{1/\alpha}(\upsilon)}\mathrm{d}\upsilon \geq \kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})\int_{\mathrm{s}_1}^{\mathrm{s}} \frac{1}{\kappa^{1/\alpha}(\upsilon)}\mathrm{d}\upsilon \\
&\geq \kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_0(\mathrm{s}).
\end{aligned} \tag{2.4}$$

Integrating twice more from s_1 to s, we obtain

$$\mathcal{Z}'(\mathrm{s}) \geq \kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_1(\mathrm{s}), \tag{2.5}$$

and

$$\mathcal{Z}(\mathrm{s}) \geq \kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_2(\mathrm{s}).$$

From (2.4), it follows that

$$\left(\frac{\mathcal{Z}''(\mathrm{s})}{R_0(\mathrm{s})}\right)' = \frac{\kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_0(\mathrm{s}) - \mathcal{Z}''(\mathrm{s})}{\kappa^{1/\alpha}(\mathrm{s})R_0^2(\mathrm{s})} \leq 0.$$

Since $\frac{\mathcal{Z}''(\mathrm{s})}{R_0(\mathrm{s})}$ is decreasing. Consequently,

$$\mathcal{Z}'(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\mathcal{Z}''(\upsilon)}{R_0(\upsilon)}R_0(\upsilon)\,\mathrm{d}\upsilon \geq \frac{\mathcal{Z}''(\mathrm{s})}{R_0(\mathrm{s})}R_1(\mathrm{s}). \tag{2.6}$$

From this we deduce

$$\left(\frac{\mathcal{Z}'(\mathrm{s})}{R_1(\mathrm{s})}\right)' = \frac{\mathcal{Z}''(\mathrm{s})R_1(\mathrm{s}) - R_0(\mathrm{s})\mathcal{Z}'(\mathrm{s})}{R_1^2(\mathrm{s})} \leq 0.$$

Thus, $\frac{\mathcal{Z}'(\mathrm{s})}{R_1(\mathrm{s})}$ is decreasing, and therefore

$$\mathcal{Z}(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\mathcal{Z}'(\upsilon)}{R_1(\upsilon)}R_1(\upsilon)\,\mathrm{d}\upsilon \geq \frac{\mathcal{Z}'(\mathrm{s})}{R_1(\mathrm{s})}R_2(\mathrm{s}). \tag{2.7}$$

Consequently,

$$\left(\frac{\mathcal{Z}(\mathrm{s})}{R_2(\mathrm{s})}\right)' = \frac{\mathcal{Z}'(\mathrm{s})R_2(\mathrm{s}) - R_1(\mathrm{s})\mathcal{Z}(\mathrm{s})}{R_2^2(\mathrm{s})} \leq 0,$$

showing that $\frac{\mathcal{Z}(s)}{R_2(s)}$ is decreasing.
Finally, combining (2.6) and (2.7) yields

$$\mathcal{Z}(s) \geq \frac{R_2(s)\mathcal{Z}''(s)}{R_0(s)}.$$

Thus, the proof is finished. □

3. Main results

In this section, we establish oscillation criteria for solutions of Eq (1.1) using the Riccati transformation with suitable substitutions. The analysis focuses on the equation's structure and eventually positive solutions. Assuming certain functional inequalities hold for large arguments, we proceed under general conditions. We now introduce the following notation:

$$p_1(s) := (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\hbar_1^{\frac{1}{1-\gamma}}(s)\,\hbar^{\frac{\gamma}{\gamma-1}}(s),$$

and

$$p_2(s) := (\delta-1)\delta^{\frac{\delta}{1-\delta}}\hbar_2^{\frac{1}{1-\delta}}(s)\,\hbar^{\frac{\delta}{\delta-1}}(s).$$

Lemma 3.1. *Let x be an eventually positive solution of (1.1), and assume $\hbar \in C([s_0,\infty),(0,\infty))$ such that $\hbar_2(s) \neq 0$ is bounded and*

$$\lim_{s\to\infty}[p_1(s) + p_2(s)] = 0. \tag{3.1}$$

Then, for sufficiently large s, the following holds:
(i) $x(s) \geq c\mathcal{Z}(s)$ *for some* $c \in (0,1)$;
(ii) $(\kappa(s)[\mathcal{Z}'''(s)]^{\alpha})' + c^{\beta}\mathfrak{q}(s)\mathcal{Z}^{\beta}(b(s)) \leq 0$.

Proof. It is clear from the definition of $\mathcal{Z}$ that

$$\mathcal{Z}(s) = x(s) + \left[\hbar(s)x(a(s)) - \hbar_2(s)x^{\delta}(a(s))\right] + [\hbar_1(s)x^{\gamma}(a(s)) - \hbar(s)x(a(s))],$$

or

$$x(s) = \mathcal{Z}(s) - \left[\hbar(s)x(a(s)) - \hbar_2(s)x^{\delta}(a(s))\right] - [\hbar_1(s)x^{\gamma}(a(s)) - \hbar(s)x(a(s))]. \tag{3.2}$$

If we apply the inequality (2.2) with $\lambda = \delta > 1$, $C = \hbar_2^{1/\delta}(s)x(a(s))$, and $D = \left(\frac{1}{\delta}\hbar(s)\hbar_2^{-1/\delta}(s)\right)^{\frac{1}{\delta-1}}$, we get

$$\hbar(s)x(a(s)) - \hbar_2(s)x^{\delta}(a(s)) \leq (\delta-1)\delta^{\frac{\delta}{1-\delta}}\hbar_2^{\frac{1}{1-\delta}}(s)\,\hbar^{\frac{\delta}{\delta-1}}(s) = p_2(s). \tag{3.3}$$

Similarly, if we apply (2.3) with $\lambda = \gamma < 1$, $C = \hbar_1^{1/\delta}(s)x(a(s))$, and $D = \left(\frac{1}{\gamma}\hbar(s)\hbar_1^{-1/\gamma}(s)\right)^{\frac{1}{\gamma-1}}$, we get

$$\hbar_1(s)x^{\gamma}(a(s)) - \hbar(s)x(a(s)) \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\hbar_1^{\frac{1}{1-\gamma}}(s)\,\hbar^{\frac{\gamma}{\gamma-1}}(s) = p_1(s). \tag{3.4}$$

By substituting (3.3) and (3.4) into (3.2), we get

$$x(s) \geq \mathcal{Z}(s) - p_1(s) - p_2(s) = \left(1 - \frac{p_1(s)+p_2(s)}{\mathcal{Z}(s)}\right)\mathcal{Z}(s). \tag{3.5}$$

Since $\mathcal{Z}'(\mathrm{s}) > 0$, we find $\mathcal{Z}(\mathrm{s}) \geq c_0$ for some $c_0 > 0$. Therefore, (3.5) leads to

$$x(\mathrm{s}) \geq \left(1 - \frac{p_1(\mathrm{s}) + p_2(\mathrm{s})}{c_0}\right)\mathcal{Z}(\mathrm{s}).$$

In light of (3.1), there exists a constant $c \in (0, 1)$ such that

$$x(\mathrm{s}) \geq c\mathcal{Z}(\mathrm{s}). \tag{3.6}$$

By substituting (3.6) into (1.1), we obtain

$$(\kappa(\mathrm{s})[\mathcal{Z}'''(\mathrm{s})]^{\alpha})' = -\mathfrak{q}(\mathrm{s})x^{\beta}(b(\mathrm{s})) \leq -c^{\beta}\mathfrak{q}(\mathrm{s})\mathcal{Z}^{\beta}(b(\mathrm{s})).$$

Thus, the proof is finished. □

Theorem 3.1. *Let $\beta \geq \alpha$. If there are nondecreasing functions $\phi, \phi_1 \in C^1([s_0, \infty), (0, \infty))$ such that*

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} \left(Lc^{\beta}\phi(\upsilon)\,\mathfrak{q}(\upsilon)\left(\frac{R_2(b(\upsilon))}{R_2(\upsilon)}\right)^{\beta} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\phi'(\upsilon))^{\alpha+1}}{(R_1(\upsilon)\,\phi(\upsilon))^{\alpha}}\right)\mathrm{d}\upsilon = \infty, \tag{3.7}$$

and

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} \left(L_1 c^{\beta}\phi_1(\ell)\int_{\ell}^{\infty}\left(\frac{1}{\kappa(u)}\int_{u}^{\infty}\mathfrak{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u - \frac{\left[\phi_1'(\ell)\right]^2}{4\phi_1(\ell)}\right)\mathrm{d}\ell = \infty, \tag{3.8}$$

hold for every $c, \epsilon \in (0, 1)$, $L, L_1 > 0$. Then (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that Eq (1.1) admits an eventually positive solution $x(\mathrm{s})$. Then, by Lemma 2.4, the corresponding function $\mathcal{Z}$ and its derivatives must satisfy one of the two alternative conditions, namely (C_1) or (C_2).
Suppose that case (C_1) holds. We now introduce an auxiliary function $w(\mathrm{s})$ defined by

$$w(\mathrm{s}) := \phi(\mathrm{s})\frac{\kappa(\mathrm{s})(\mathcal{Z}'''(\mathrm{s}))^{\alpha}}{\mathcal{Z}^{\alpha}(\mathrm{s})} > 0. \tag{3.9}$$

Thus

$$w'(\mathrm{s}) = \phi'(\mathrm{s})\frac{\kappa(\mathrm{s})(\mathcal{Z}'''(\mathrm{s}))^{\alpha}}{\mathcal{Z}^{\alpha}(\mathrm{s})} + \phi(\mathrm{s})\frac{(\kappa(\mathrm{s})(\mathcal{Z}'''(\mathrm{s}))^{\alpha})'}{\mathcal{Z}^{\alpha}(\mathrm{s})} - \alpha\phi(\mathrm{s})\frac{\kappa(\mathrm{s})(\mathcal{Z}'''(\mathrm{s}))^{\alpha}\mathcal{Z}'(\mathrm{s})}{\mathcal{Z}^{\alpha+1}(\mathrm{s})}. \tag{3.10}$$

Using (ii), (3.9), and (3.10), we deduce that

$$w'(\mathrm{s}) \leq -c^{\beta}\phi(\mathrm{s})\,\mathfrak{q}(\mathrm{s})\frac{\mathcal{Z}^{\beta}(b(\mathrm{s}))}{\mathcal{Z}^{\alpha}(\mathrm{s})} + \frac{\phi'(\mathrm{s})}{\phi(\mathrm{s})}w(\mathrm{s}) - \alpha\frac{\mathcal{Z}'(\mathrm{s})}{\mathcal{Z}(\mathrm{s})}w(\mathrm{s}). \tag{3.11}$$

From Lemma 2.5(b), we have that

$$\frac{\mathcal{Z}(b(\mathrm{s}))}{R_2(b(\mathrm{s}))} \geq \frac{\mathcal{Z}(\mathrm{s})}{R_2(\mathrm{s})},$$

and hence,

$$\frac{\mathcal{Z}(b(\mathrm{s}))}{\mathcal{Z}(\mathrm{s})} \geq \frac{R_2(b(\mathrm{s}))}{R_2(\mathrm{s})}. \tag{3.12}$$

It follows from (2.5) that

$$\mathcal{Z}'(\mathrm{s}) \geq \kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_1(\mathrm{s}), \tag{3.13}$$

for every sufficiently large s. Thus, by (3.11)–(3.13), we have

$$\begin{aligned} w'(\mathrm{s}) &\leq -c^{\beta}\phi(\mathrm{s})\,\mathrm{q}(\mathrm{s})\mathcal{Z}^{\beta-\alpha}\frac{\mathcal{Z}^{\beta}(b(\mathrm{s}))}{\mathcal{Z}^{\beta}(\mathrm{s})} + \frac{\phi'(\mathrm{s})}{\phi(\mathrm{s})}w(\mathrm{s}) - \alpha\frac{\kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})R_1(\mathrm{s})}{\mathcal{Z}(\mathrm{s})}w(\mathrm{s}) \\ &= -c^{\beta}\phi(\mathrm{s})\,\mathrm{q}(\mathrm{s})\mathcal{Z}^{\beta-\alpha}(\mathrm{s})\left(\frac{R_2(b(\mathrm{s}))}{R_2(\mathrm{s})}\right)^{\beta} + \frac{\phi'(\mathrm{s})}{\phi(\mathrm{s})}w(\mathrm{s}) - \frac{\alpha R_1(\mathrm{s})}{\phi^{1/\alpha}(\mathrm{s})}w^{(1+\alpha)/\alpha}(\mathrm{s}). \end{aligned} \tag{3.14}$$

Since $\mathcal{Z}'(\mathrm{s}) > 0$, and $\beta \geq \alpha$, there exists an $\mathrm{s}_1 \geq \mathrm{s}_0$ and a constant $L > 0$ such that

$$\mathcal{Z}^{\beta-\alpha}(\mathrm{s}) > L. \tag{3.15}$$

Thus, the inequality (3.14) gives

$$w'(\mathrm{s}) \leq -Lc^{\beta}\phi(\mathrm{s})\,\mathrm{q}(\mathrm{s})\left(\frac{R_2(b(\mathrm{s}))}{R_2(\mathrm{s})}\right)^{\beta} + \frac{\phi'(\mathrm{s})}{\phi(\mathrm{s})}w(\mathrm{s}) - \frac{\alpha R_1(\mathrm{s})}{\phi^{1/\alpha}(\mathrm{s})}w^{(1+\alpha)/\alpha}(\mathrm{s}). \tag{3.16}$$

Using Lemma 2.1, where we define $D = \phi'(\mathrm{s})/\phi(\mathrm{s})$, $C = \alpha R_1(\mathrm{s})/\phi^{1/\alpha}(\mathrm{s})$, and $u(\mathrm{s}) = w(\mathrm{s})$, we find

$$w'(\mathrm{s}) \leq -Lc^{\beta}\phi(\mathrm{s})\,\mathrm{q}(\mathrm{s})\left(\frac{R_2(b(\mathrm{s}))}{R_2(\mathrm{s})}\right)^{\beta} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\phi'(\mathrm{s}))^{\alpha+1}}{(R_1(\mathrm{s})\,\phi(\mathrm{s}))^{\alpha}}.$$

Integrating the above inequality from $\mathrm{s}_2 \geq \mathrm{s}_1$ to s, one arrives at

$$\int_{\mathrm{s}_2}^{\mathrm{s}}\left(Lc^{\beta}\phi(\upsilon)\,\mathrm{q}(\upsilon)\left(\frac{R_2(b(\upsilon))}{R_2(\upsilon)}\right)^{\beta} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\phi'(\upsilon))^{\alpha+1}}{(R_1(\upsilon)\,\phi(\upsilon))^{\alpha}}\right)\mathrm{d}\upsilon \leq w(\mathrm{s}_2),$$

this contradicts (3.27) as $\mathrm{s}\to\infty$.
Consider the case where (C_2) holds. From (ii), we obtain

$$(\kappa(\mathrm{s})\left[\mathcal{Z}'''(\mathrm{s})\right]^{\alpha})' \leq -c^{\beta}\mathrm{q}(\mathrm{s})\mathcal{Z}^{\beta}(b(\mathrm{s})). \tag{3.17}$$

Integrating (3.17) from s to S, we obtain

$$\int_{\mathrm{s}}^{\mathrm{S}}(\kappa(\upsilon)\left[\mathcal{Z}'''(\upsilon)\right]^{\alpha})'\,\mathrm{d}\upsilon \leq -c^{\beta}\int_{\mathrm{s}}^{\mathrm{S}}\mathrm{q}(\upsilon)\mathcal{Z}^{\beta}(b(\upsilon))\mathrm{d}\upsilon.$$

By the Fundamental Theorem of Calculus, it follows that

$$\kappa(\mathrm{S})\left[\mathcal{Z}'''(\mathrm{S})\right]^{\alpha} - \kappa(\mathrm{s})\left[\mathcal{Z}'''(\mathrm{s})\right]^{\alpha} \leq -c^{\beta}\int_{\mathrm{s}}^{\mathrm{S}}\mathrm{q}(\upsilon)\mathcal{Z}^{\beta}(b(\upsilon))\mathrm{d}\upsilon.$$

Letting $\mathrm{S}\to\infty$ gives

$$\lim_{\mathrm{S}\to\infty}\kappa(\mathrm{S})\left[\mathcal{Z}'''(\mathrm{S})\right]^{\alpha} - \kappa(\mathrm{s})\left[\mathcal{Z}'''(\mathrm{s})\right]^{\alpha} \leq -c^{\beta}\int_{\mathrm{s}}^{\infty}\mathrm{q}(\upsilon)\mathcal{Z}^{\beta}(b(\upsilon))\mathrm{d}\upsilon. \tag{3.18}$$

Since $(\kappa(\mathrm{s})\,[\mathcal{Z}'''(\mathrm{s})]^{\alpha})' \leq 0$, then $\kappa(\mathrm{s})\,[\mathcal{Z}'''(\mathrm{s})]^{\alpha}$ is positive and monotonically decreasing, and thus

$$\lim_{\mathrm{S}\to\infty} \kappa(\mathrm{S})\,[\mathcal{Z}'''(\mathrm{S})]^{\alpha} = 0.$$

Substituting this into (3.18) yields

$$-\kappa(\mathrm{s})\,[\mathcal{Z}'''(\mathrm{s})]^{\alpha} \leq -c^{\beta}\int_{\mathrm{s}}^{\infty} \mathrm{q}(\upsilon)\mathcal{Z}^{\beta}(b(\upsilon))\mathrm{d}\upsilon,$$

or equivalently,

$$\kappa(\mathrm{s})\,[\mathcal{Z}'''(\mathrm{s})]^{\alpha} \geq c^{\beta}\int_{\mathrm{s}}^{\infty} \mathrm{q}(\upsilon)\mathcal{Z}^{\beta}(b(\upsilon))\mathrm{d}\upsilon. \tag{3.19}$$

As $\mathcal{Z}(\mathrm{s}) > 0$, $\mathcal{Z}'(\mathrm{s}) > 0$, and $\mathcal{Z}''(\mathrm{s}) < 0$, Lemma 2.3 implies that $\mathcal{Z}(\mathrm{s}) \geq \epsilon \mathrm{s}\mathcal{Z}'(\mathrm{s})$ for all $\epsilon \in (0,1)$. Integrating (3.19) from $b(\mathrm{s})$ to s, we get

$$\frac{\mathcal{Z}(b(\mathrm{s}))}{\mathcal{Z}(\mathrm{s})} \geq \left(\frac{b(\mathrm{s})}{\mathrm{s}}\right)^{1/\epsilon}.$$

Therefore, (3.19) becomes

$$\kappa(\mathrm{s})\,[\mathcal{Z}'''(\mathrm{s})]^{\alpha} \geq c^{\beta}\int_{\mathrm{s}}^{\infty} \mathrm{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathcal{Z}^{\beta}(\upsilon)\,\mathrm{d}\upsilon.$$

Since $\mathcal{Z}'(\mathrm{s}) > 0$, then

$$\kappa(\mathrm{s})\,[\mathcal{Z}'''(\mathrm{s})]^{\alpha} \geq c^{\beta}\mathcal{Z}^{\beta}(\mathrm{s})\int_{\mathrm{s}}^{\infty} \mathrm{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon,$$

or equivalently

$$\mathcal{Z}'''(\mathrm{s}) \geq c^{\beta/\alpha}\mathcal{Z}^{\beta/\alpha}(\mathrm{s})\left(\frac{1}{\kappa(\mathrm{s})}\int_{\mathrm{s}}^{\infty} \mathrm{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}. \tag{3.20}$$

By integrating (3.20) from s to ∞, it follows that

$$\mathcal{Z}''(\mathrm{s}) \leq -c^{\beta/\alpha}\mathcal{Z}^{\beta/\alpha}(\mathrm{s})\int_{\mathrm{s}}^{\infty}\left(\frac{1}{\kappa(u)}\int_{u}^{\infty} \mathrm{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u. \tag{3.21}$$

Now, define

$$F(\mathrm{s}) := \phi_1(\mathrm{s})\,\frac{\mathcal{Z}'(\mathrm{s})}{\mathcal{Z}(\mathrm{s})}. \tag{3.22}$$

Then, $F(\mathrm{s}) \geq 0$ for $\mathrm{s}\geq\mathrm{s}_1\geq\mathrm{s}_0$ and

$$\begin{aligned} F' &= \phi_1'(\mathrm{s})\,\frac{\mathcal{Z}'(\mathrm{s})}{\mathcal{Z}(\mathrm{s})} + \phi_1(\mathrm{s})\,\frac{\mathcal{Z}''(\mathrm{s})}{\mathcal{Z}(\mathrm{s})} - \phi_1(\mathrm{s})\,\frac{[\mathcal{Z}'(\mathrm{s})]^2}{\mathcal{Z}^2(\mathrm{s})} \\ &= \phi_1(\mathrm{s})\,\frac{\mathcal{Z}''(\mathrm{s})}{\mathcal{Z}(\mathrm{s})} + \frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})}F(\mathrm{s}) - \frac{1}{\phi_1(\mathrm{s})}F^2(\mathrm{s}). \end{aligned}$$

Hence, by (3.21), we get

$$F'(\mathrm{s}) \leq -c^{\beta/\alpha}\phi_1(\mathrm{s})\,\mathcal{Z}^{\beta/\gamma-1}(\mathrm{s})\int_{\mathrm{s}}^{\infty}\left(\frac{1}{\kappa(u)}\int_{u}^{\infty} \mathrm{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u$$

$$+\frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})}F(\mathrm{s})-\frac{1}{\phi_1(\mathrm{s})}F^2(\mathrm{s}). \tag{3.23}$$

Because $\mathcal{Z}'(\mathrm{s})>0$ and $\beta\geq\alpha$, there exist constants $L_1>0$ and $\mathrm{s}_2\geq\mathrm{s}_1$ such that

$$\mathcal{Z}^{\beta/\alpha-1}(\mathrm{s})\geq L_1. \tag{3.24}$$

Substituting (3.24) into (3.23), we have

$$\begin{aligned} F'(\mathrm{s}) &\leq -L_1c^{\beta/\alpha}\phi_1(\mathrm{s})\int_{\mathrm{s}}^{\infty}\left(\frac{1}{\kappa(u)}\int_u^{\infty}\mathfrak{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u \\ &\quad+\frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})}F(\mathrm{s})-\frac{1}{\phi_1(\mathrm{s})}F^2(\mathrm{s}). \end{aligned} \tag{3.25}$$

Using Lemma 2.1 with $D=\phi_1'(\mathrm{s})/\phi_1(\mathrm{s})$, $C=1/\phi_1(\mathrm{s})$, and $u(\mathrm{s})=F(\mathrm{s})$, we obtain

$$\frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})}F(\mathrm{s})-\frac{1}{\phi_1(\mathrm{s})}F^2(\mathrm{s})\leq\frac{\left[\phi_1'(\mathrm{s})\right]^2}{4\phi_1(\mathrm{s})}.$$

Consequently, (3.25) leads to

$$F'(\mathrm{s})\leq -L_1c^{\beta/\alpha}\phi_1(\mathrm{s})\int_{\mathrm{s}}^{\infty}\left(\frac{1}{\kappa(u)}\int_u^{\infty}\mathfrak{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u+\frac{\phi_1^2(\mathrm{s})}{4\phi_1(\mathrm{s})}. \tag{3.26}$$

By integrating (3.26) from s_2 to s, we get

$$\int_{\mathrm{s}_2}^{\mathrm{s}}\left(L_1c^{\beta/\alpha}\phi_1(\ell)\int_{\ell}^{\infty}\left(\frac{1}{\kappa(u)}\int_u^{\infty}\mathfrak{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\beta/\epsilon}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u-\frac{\left[\phi_1'(\ell)\right]^2}{4\phi_1(\ell)}\right)\mathrm{d}\ell\leq F(\mathrm{s}_2),$$

which contradicts (3.8) as $\mathrm{s}\to\infty$. This completes the proof. □

Theorem 3.2. *Let $\beta\geq\alpha$. If there is a nondecreasing functions $\phi_2\in C^1([s_0,\infty),(0,\infty))$ such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left(c^{\beta}\phi_2(\upsilon)\,\mathfrak{q}(\upsilon)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left[\phi'(\upsilon)\right]^{\alpha+1}}{\left(L_2b'(\upsilon)R_1(b(\upsilon))\phi_2(\upsilon)\right)^{\alpha}}\right)\mathrm{d}\upsilon=\infty, \tag{3.27}$$

and (3.8) holds, for every $c\in(0,1)$, $L_2>0$. Then (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that Eq (1.1) admits an eventually positive solution $x(\mathrm{s})$. Then, by Lemma 2.4, the corresponding function $\mathcal{Z}(\mathrm{s})$ and its derivatives must satisfy one of the two alternative conditions, namely (C_1) or (C_2).
Suppose that case (C_1) holds. Now, define a function $w_1(\mathrm{s})$ by

$$w_1(\mathrm{s}):=\phi_2(\mathrm{s})\frac{\kappa(\mathrm{s})\left(\mathcal{Z}'''(\mathrm{s})\right)^{\alpha}}{\mathcal{Z}^{\beta}(b(\mathrm{s}))}>0,\ \mathrm{s}\geq\mathrm{s}_1. \tag{3.28}$$

Thus,

$$w_1'(\mathrm{s}) = \phi_2'(\mathrm{s})\frac{\kappa(\mathrm{s})\left(\mathcal{Z}'''(\mathrm{s})\right)^{\alpha}}{\mathcal{Z}^{\beta}(b(\mathrm{s}))}+\phi_2(\mathrm{s})\frac{\left(\kappa(\mathrm{s})\left(\mathcal{Z}'''(\mathrm{s})\right)^{\alpha}\right)'}{\mathcal{Z}^{\beta}(b(\mathrm{s}))}$$

$$-\beta\phi_2(\mathrm{s})\,b'(\mathrm{s})\,\frac{\kappa(\mathrm{s})\,(\mathcal{Z}'''(\mathrm{s}))^{\alpha}\cdot\mathcal{Z}'(b(\mathrm{s}))}{\mathcal{Z}^{\beta+1}(b(\mathrm{s}))}. \tag{3.29}$$

We see from (ii), (3.28), and (3.29) that

$$w_1'(\mathrm{s})\le\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})-\beta b'(\mathrm{s})\,\frac{\mathcal{Z}'(b(\mathrm{s}))}{\mathcal{Z}(b(\mathrm{s}))}w_1(\mathrm{s}). \tag{3.30}$$

Applying (2.5), we get

$$\mathcal{Z}'(b(\mathrm{s}))\ge\kappa^{1/\alpha}(b(\mathrm{s}))\mathcal{Z}'''(b(\mathrm{s}))\,R_1(b(\mathrm{s})), \tag{3.31}$$

holds for sufficiently large s. By substituting (3.31) into (3.30), we derive

$$\begin{aligned} w_1'(\mathrm{s}) &\le \frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})-\alpha b'(\mathrm{s})\,\frac{\kappa^{1/\alpha}(b(\mathrm{s}))\mathcal{Z}'''(b(\mathrm{s}))\,R_1(b(\mathrm{s}))}{\mathcal{Z}(b(\mathrm{s}))}w_1(\mathrm{s}) \\ &= -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\alpha b'(\mathrm{s})\,R_1(b(\mathrm{s}))\,\frac{\kappa^{1/\alpha}(b(\mathrm{s}))\mathcal{Z}'''(b(\mathrm{s}))}{\mathcal{Z}(b(\mathrm{s}))}w_1(\mathrm{s}). \end{aligned}$$

Since $(\kappa(\mathrm{s})\,(\mathcal{Z}'''(\mathrm{s}))^{\alpha})'<0$, then

$$\kappa^{1/\alpha}(\mathrm{s})\,\mathcal{Z}'''(\mathrm{s})\le\kappa^{1/\alpha}(b(\mathrm{s}))\,\mathcal{Z}'''(b(\mathrm{s})).$$

Then

$$\begin{aligned} w_1'(\mathrm{s}) &\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\alpha b'(\mathrm{s})\,R_1(b(\mathrm{s}))\,\frac{\kappa^{1/\alpha}(\mathrm{s})\,\mathcal{Z}'''(\mathrm{s})}{\mathcal{Z}(b(\mathrm{s}))}w_1(\mathrm{s}) \\ &= -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\frac{\alpha b'(\mathrm{s})\,R_1(b(\mathrm{s}))}{\phi_2^{1/\alpha}(\mathrm{s})}\left[\mathcal{Z}(b(\mathrm{s}))\right]^{\frac{\beta-\alpha}{\alpha}}w_1^{\frac{\alpha+1}{\alpha}}(\mathrm{s}). \end{aligned} \tag{3.32}$$

Since $\mathcal{Z}'>0$ and $\beta\ge\alpha$, we know there exist constants $L_2>0$ and $\mathrm{s}_2\ge\mathrm{s}_1$ such that

$$\left[\mathcal{Z}(b(\mathrm{s}))\right]^{\frac{\beta-\alpha}{\alpha}}\ge L_2,\ \ \mathrm{s}\ge\mathrm{s}_2. \tag{3.33}$$

Thus, the inequality (3.32) gives

$$w_1'(\mathrm{s})\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\frac{L_2\alpha b'(\mathrm{s})\,R_1(b(\mathrm{s}))}{\phi_2^{1/\alpha}(\mathrm{s})}w_1^{\frac{\alpha+1}{\alpha}}(\mathrm{s}). \tag{3.34}$$

By applying Lemma 2.1, with the substitutions $D=\phi_2'(\mathrm{s})/\phi_2(\mathrm{s})$, $C=L_2\alpha b'(\mathrm{s})\,R_1(b(\mathrm{s}))/\phi_2^{1/\alpha}(\mathrm{s})$, and $u(\mathrm{s})=w_1(\mathrm{s})$, we get

$$w_1'(\mathrm{s})\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\left[\phi_2'(\mathrm{s})\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(L_2 b'(\mathrm{s})\,R_1(b(\mathrm{s}))\,\phi_2(\mathrm{s})\right)^{\alpha}}. \tag{3.35}$$

Integrating (3.35) from $\mathrm{s}_3\ge\mathrm{s}_2$ to s, one arrives at

$$\int_{\mathrm{s}_3}^{\mathrm{s}}\left(c^{\beta}\phi_2(\upsilon)\,\mathrm{q}(\upsilon)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left[\phi'(\upsilon)\right]^{\alpha+1}}{\left(L_2 b'(\upsilon)\,R_1(b(\upsilon))\,\phi_2(\upsilon)\right)^{\alpha}}\right)\mathrm{d}\upsilon\le w(\mathrm{s}_3),$$

this contradicts (3.27) as $\mathrm{s}\to\infty$.
Suppose that case (C_2) holds, as shown in the proof of (3.8) in Theorem 3.1.
This completes the proof. □

Theorem 3.3. *Let $0<\beta<\alpha$. If there is a nondecreasing function $\phi_1, \phi_2 \in C^1([s_0,\infty),(0,\infty))$ such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left(c^{\beta}\phi_2(\upsilon)\,\mathrm{q}(\upsilon)-\frac{1}{(\beta+1)^{\beta+1}}\frac{\left[\phi_2'(\upsilon)\right]^{\beta+1}}{(L_3 b'(\upsilon)R_1(\upsilon)\phi_2(\upsilon))^{\beta}}\right)\mathrm{d}\upsilon=\infty, \tag{3.36}$$

and

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left[\widetilde{L}\phi_1(\ell)(\ell-\mathrm{s}_0)^{\frac{\beta-\gamma}{\gamma}}\int_{\ell}^{\infty}\left(\frac{1}{\kappa(u)}\int_u^{\infty}\mathrm{q}(\upsilon)\left(\frac{b(\upsilon)}{\upsilon}\right)^{\frac{\beta}{\epsilon}}\mathrm{d}\upsilon\right)^{1/\alpha}\mathrm{d}u-\frac{\left(\phi_1'(\ell)\right)^2}{4\phi_1(\ell)}\right]\mathrm{d}\ell=\infty, \tag{3.37}$$

hold for some $c,\epsilon\in(0,1)$, $L_3>0$, $\widetilde{L}=c^{\beta/\alpha}L_3^{\beta/\gamma-1}$. Then (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that Eq (1.1) admits an eventually positive solution $x(\mathrm{s})$. Then, by Lemma 2.4, the corresponding function $\mathcal{Z}$ and its derivatives must satisfy one of the two alternative conditions, namely (C_1) or (C_2).

Case where (C_1) is satisfied, as shown in the proof of (3.27) in Theorem 3.2. The function $w_1(\mathrm{s})$ is defined in (3.28), then (3.29) holds. By using (ii), (3.28), and (3.29), we conclude that

$$w_1'(\mathrm{s})\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\beta b'(\mathrm{s})\frac{\mathcal{Z}'(b(\mathrm{s}))}{\mathcal{Z}(b(\mathrm{s}))}w_1(\mathrm{s}).$$

By using (2.5), we see that

$$\begin{aligned} w_1'(\mathrm{s}) &\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\beta b'(\mathrm{s})\,\kappa^{1/\alpha}(\mathrm{s})R_1(\mathrm{s})\left[\mathcal{Z}'''(\mathrm{s})\right]^{\frac{\beta-\alpha}{\beta}}\frac{\left[\mathcal{Z}'''(\mathrm{s})\right]^{\alpha/\beta}}{\mathcal{Z}(b(\mathrm{s}))}w_1(\mathrm{s}) \\ &= -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\frac{\beta b'(\mathrm{s})\,\kappa^{1/\alpha}(\mathrm{s})R_1(\mathrm{s})}{(\phi_2(\mathrm{s})\,\kappa(\mathrm{s}))^{1/\beta}}\left[\mathcal{Z}'''(\mathrm{s})\right]^{\frac{\beta-\alpha}{\beta}}w_1^{(\beta+1)/\beta}(\mathrm{s}) \\ &= -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\frac{\beta b'(\mathrm{s})R_1(\mathrm{s})}{\phi_2^{1/\beta}(\mathrm{s})}\left[\kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})\right]^{\frac{\beta-\alpha}{\beta}}w_1^{(\beta+1)/\beta}(\mathrm{s}). \end{aligned}$$

Note that $0<\beta<\alpha$ and (C_1) holds. Since $\kappa'(\mathrm{s})\ge 0$, we deduce that $\mathcal{Z}^{(4)}(\mathrm{s})\le 0$; this readily infers that $\mathcal{Z}'''(\mathrm{s})$ is nonincreasing. Also, $\left[\kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})\right]^{\frac{\beta-\alpha}{\beta}}$ is increasing. Then there are $L_3>0$ and $\mathrm{s}_2\ge\mathrm{s}_1$ such that

$$\left[\kappa^{1/\alpha}(\mathrm{s})\mathcal{Z}'''(\mathrm{s})\right]^{\frac{\beta-\alpha}{\beta}}\ge L_3,\ \ \mathrm{s}\ge\mathrm{s}_2. \tag{3.38}$$

By using (3.38), it follows that

$$w_1'(\mathrm{s})\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{\phi_2'(\mathrm{s})}{\phi_2(\mathrm{s})}w_1(\mathrm{s})-\frac{\beta L_3 b'(\mathrm{s})R_1(\mathrm{s})}{\phi_2^{1/\beta}(\mathrm{s})}w_1^{\frac{\beta+1}{\beta}}(\mathrm{s}). \tag{3.39}$$

By applying Lemma 2.1, with the substitutions: $D=\phi_2'(\mathrm{s})/\phi_2(\mathrm{s})$, $C=\beta L_3 b'(\mathrm{s})R_1(\mathrm{s})/\phi_2^{1/\beta}(\mathrm{s})$, and $u(\mathrm{s})=w_1(\mathrm{s})$. From (3.39), we get

$$w_1'(\mathrm{s})\le -c^{\beta}\phi_2(\mathrm{s})\,\mathrm{q}(\mathrm{s})+\frac{1}{(\beta+1)^{\beta+1}}\frac{\left[\phi_2'(\mathrm{s})\right]^{\beta+1}}{(L_3 b'(\mathrm{s})R_1(\mathrm{s})\phi_2(\mathrm{s}))^{\beta}}. \tag{3.40}$$

The integration of (3.40) within the interval $[\mathrm{s}_3, \mathrm{s}]$, yields

$$\int_{\mathrm{s}_3}^{\mathrm{s}} \left(c^{\beta} \phi_2(\upsilon)\, \mathrm{q}(\upsilon) - \frac{1}{(\beta+1)^{\beta+1}} \frac{\left[\phi_2'(\upsilon)\right]^{\beta+1}}{(L_3 b'(\upsilon) R_1(\upsilon) \phi_2(\upsilon))^{\beta}} \right) \mathrm{d}\upsilon \le w(\mathrm{s}_2),$$

which contradicts (3.36) as $\mathrm{s} \to \infty$.

Case where (C_2) is satisfied, as shown in the proof of (3.8) in Theorem 3.1. The function $F(\mathrm{s})$ is given in (3.22), and therefore (3.23) holds, which can be written as follows:

$$\begin{aligned} F'(\mathrm{s}) &\le -c^{\beta/\alpha} \phi_1(\mathrm{s})\, \mathcal{Z}^{\beta/\gamma-1}(\mathrm{s}) \int_{\mathrm{s}}^{\infty} \left(\frac{1}{\kappa(u)} \int_u^{\infty} \mathrm{q}(\upsilon) \left(\frac{b(\upsilon)}{\upsilon} \right)^{\beta/\epsilon} \mathrm{d}\upsilon \right)^{1/\alpha} \mathrm{d}u \\ &\quad + \frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})} F(\mathrm{s}) - \frac{1}{\phi_1(\mathrm{s})} F^2(\mathrm{s}). \end{aligned} \tag{3.41}$$

Since $\mathcal{Z}'' < 0$, it follows that $\mathcal{Z}'$ is decreasing. Consequently, we have

$$\mathcal{Z}(\mathrm{s}) = \int_{\mathrm{s}_1}^{\mathrm{s}} \mathcal{Z}'(\upsilon)\, \mathrm{d}\upsilon \le \mathcal{Z}'(\mathrm{s}_1)(\mathrm{s}-\mathrm{s}_1) = L_4(\mathrm{s}-\mathrm{s}_1),\ L_4 := \mathcal{Z}'(\mathrm{s}_1) > 0. \tag{3.42}$$

From (3.42), combined with the fact that $0 < \beta < \alpha$, implying $0 < \beta/\alpha < 1$, we conclude

$$\mathcal{Z}^{\beta/\alpha-1}(\mathrm{s}) \ge L_4^{\beta/\alpha-1}(\mathrm{s}-\mathrm{s}_1)^{\beta/\alpha-1}. \tag{3.43}$$

Hence, the inequality (3.41) becomes

$$\begin{aligned} F'(\mathrm{s}) &\le -c^{\beta/\alpha} L_4^{\beta/\alpha-1} \phi_1(\mathrm{s})(\mathrm{s}-\mathrm{s}_1)^{\beta/\alpha-1} \int_{\mathrm{s}}^{\infty} \left(\frac{1}{\kappa(u)} \int_u^{\infty} \mathrm{q}(\upsilon) \left(\frac{b(\upsilon)}{\upsilon} \right)^{\beta/\epsilon} \mathrm{d}\upsilon \right)^{1/\alpha} \mathrm{d}u \\ &\quad + \frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})} F(\mathrm{s}) - \frac{1}{\phi_1(\mathrm{s})} F^2(\mathrm{s}). \end{aligned} \tag{3.44}$$

Applying Lemma 2.1 with $C = \phi_1'(\mathrm{s})/\phi_1(\mathrm{s})$, $D = 1/\phi_1(\mathrm{s})$, and $u(\mathrm{s}) = F(\mathrm{s})$, we can deduce that

$$\frac{\phi_1'(\mathrm{s})}{\phi_1(\mathrm{s})} F(\mathrm{s}) - \frac{1}{\phi_1(\mathrm{s})} F^2(\mathrm{s}) \le \frac{\left(\phi_1'(\mathrm{s})\right)^2}{4\phi_1(\mathrm{s})}.$$

Consequently, (3.44) leads to

$$F'(\mathrm{s}) \le -c^{\beta/\alpha} L_4^{\beta/\alpha-1} \phi_1(\mathrm{s})(\mathrm{s}-\mathrm{s}_1)^{\beta/\gamma-1} \int_{\mathrm{s}}^{\infty} \left(\frac{1}{\kappa(u)} \int_u^{\infty} \mathrm{q}(\upsilon) \left(\frac{b(\upsilon)}{\upsilon} \right)^{\beta/\epsilon} \mathrm{d}\upsilon \right)^{1/\alpha} \mathrm{d}u + \frac{\left(\phi_1'(\mathrm{s})\right)^2}{4\phi_1(\mathrm{s})}. \tag{3.45}$$

By integrating (3.45) from s_2 to s, we deduce

$$\int_{\mathrm{s}_2}^{\mathrm{s}} \left[\widetilde{L} \phi_1(\ell)(\ell-\mathrm{s}_1)^{\beta/\alpha-1} \int_{\ell}^{\infty} \left(\frac{1}{\kappa(u)} \int_u^{\infty} \mathrm{q}(\upsilon) \left(\frac{b(\upsilon)}{\upsilon} \right)^{\beta/\epsilon} \mathrm{d}\upsilon \right)^{1/\alpha} \mathrm{d}u + \frac{\left(\phi_1'(\ell)\right)^2}{4\phi_1(\ell)} \right] \mathrm{d}\ell \le F(\mathrm{s}_2),$$

which contradicts (3.8) as $\mathrm{s} \to \infty$. Thus, we have completed the proof. □

Remark 3.1. *Theorem 3.3 extends the results of Theorems 3.1 and 3.2 by covering the case* $0 < \beta < \alpha$*, providing more general oscillation conditions than those of the previous two theorems.*

Remark 3.2. *It is worth noting that Eq (1.1) analyzed in this study is a generalization of several equations previously studied in the literature. For example, if* $\alpha = \beta = 1$, $\hbar_1(\mathrm{s}) = \hbar_2(\mathrm{s}) = 0$, *and* $(\mathrm{s}) = 1$, *Eq (1.1) reduces to the linear model taken in [24]. In the case of* $\alpha = \beta$, *with* $\hbar_1(\mathrm{s}) = \hbar_2(\mathrm{s}) = 0$, *it matches the quasi-linear models studied in [26]. If only* $\hbar_1(\mathrm{s}) = \hbar_2(\mathrm{s}) = 0$, *it reverts to the form used in [28, 29]. If both conditions are met,* $\gamma = 1$ *and* $\hbar_2(\mathrm{s}) = 0$, *the equation takes the form shown in [30, 31], while if only* $\hbar_2(\mathrm{s}) = 0$, *it takes the form shown in [32].*

4. Examples

In this section, two illustrative examples are presented that support the validity of the theoretical results and highlight their relevance in the context of the study.

Example 4.1. *Consider the NDE given by:*

$$\left(\mathrm{s}^{-1}\left(x(\mathrm{s}) + \frac{1}{\mathrm{s}}x^{1/3}\left(\frac{1}{2}\mathrm{s}\right) - x^3\left(\frac{1}{2}\mathrm{s}\right)\right)'''\right)' + \frac{\mathfrak{q}_0}{\mathrm{s}^5}x\left(\frac{1}{3}\mathrm{s}\right) = 0,\ \mathrm{s} \geq 1. \tag{4.1}$$

Here, the parameters are defined as:

$$\begin{aligned} \alpha &= \beta = 1,\ \gamma = \frac{1}{3},\ \delta = 3,\ a(\mathrm{s}) = \frac{1}{2}\mathrm{s},\ b(\mathrm{s}) = \frac{1}{3}\mathrm{s}, \\ \kappa(\mathrm{s}) &= \mathrm{s},\ \hbar(\mathrm{s}) = \hbar_1(\mathrm{s}) = \frac{1}{\mathrm{s}},\ \hbar_2(\mathrm{s}) = 1,\ \textit{and}\ \mathfrak{q}(\mathrm{s}) = \frac{\mathfrak{q}_0}{\mathrm{s}^5}. \end{aligned}$$

Now, we calculate

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(\upsilon)}\mathrm{d}\upsilon = \int_1^{\infty} \upsilon \mathrm{d}\upsilon = \infty.$$

Next, we find

$$R_0(\mathrm{s}) = \frac{\mathrm{s}^2}{2},\ R_1(\mathrm{s}) = \frac{\mathrm{s}^3}{6},\ R_2(\mathrm{s}) = \frac{\mathrm{s}^4}{24},$$

$$p_1(\mathrm{s}) = (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\hbar_1^{\frac{1}{1-\gamma}}(\mathrm{s})\hbar^{\frac{\gamma}{\gamma-1}}(\mathrm{s}) = \frac{2}{3\sqrt{3}}\frac{1}{\mathrm{s}},$$

and

$$p_2(\mathrm{s}) = (\delta-1)\delta^{\frac{\delta}{1-\delta}}\hbar_2^{\frac{1}{1-\delta}}(\mathrm{s})\hbar^{\frac{\delta}{\delta-1}}(\mathrm{s}) = \frac{2}{3\sqrt{3}}\frac{1}{\mathrm{s}^{3/2}}.$$

Hence,

$$\lim_{\mathrm{s}\to\infty}[p_1(\mathrm{s}) + p_2(\mathrm{s})] = \lim_{\mathrm{s}\to\infty}\left[\frac{2}{3\sqrt{3}}\cdot\frac{1}{\mathrm{s}} + \frac{2}{3\sqrt{3}}\cdot\frac{1}{\mathrm{s}^{3/2}}\right] = 0,$$

which confirms the validity of condition (3.1). The asymptotic behavior of $p_1(\mathrm{s}) + p_2(\mathrm{s})$ *is illustrated in Figure 1:*

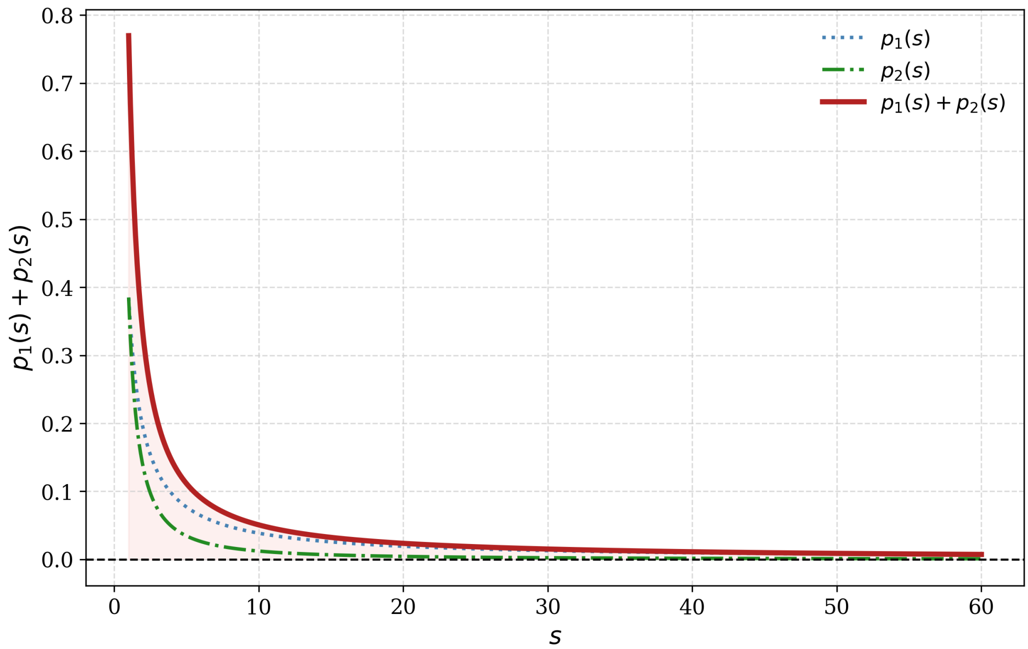

Figure 1. Illustrates the asymptotic behavior of $p_1(s) + p_2(s)$.

To further examine the oscillatory behavior, we employ various test functions and apply the corresponding conditions:

1. *Using* $\phi(\mathrm{s}) = \mathrm{s}^4$ *and applying condition (3.7):*

$$\limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \left(Lc\upsilon^4 \cdot \frac{\mathfrak{q}_0}{\upsilon^5} \cdot \left(\frac{1}{3^4}\right) - \frac{1}{2^2} \cdot \frac{4^2\left(\upsilon^3\right)^2}{\frac{\upsilon^3}{6}\upsilon^4} \right) \mathrm{d}\upsilon = \limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \left(Lc\mathfrak{q}_0 \cdot \left(\frac{1}{3^4}\right) - 24 \right) \cdot \frac{1}{\upsilon} \mathrm{d}\upsilon,$$

which diverges to infinity if

$$\mathfrak{q}_0 > \frac{1944}{Lc},\ L > 0 \textit{ and } c \in (0,1). \tag{4.2}$$

2. *Using* $\phi_2(\mathrm{s}) = \mathrm{s}^4$ *and applying condition (3.27):*

$$\limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \left(c\upsilon^4 \cdot \frac{\mathfrak{q}_0}{\upsilon^5} - \frac{1}{2^2} \cdot \frac{4^2\left[\upsilon^3\right]^2}{L_2\frac{1}{3} \cdot \frac{\upsilon^3}{6(3)^3} \cdot \upsilon^4} \right) \mathrm{d}\upsilon = \limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \left(c\mathfrak{q}_0 - \frac{2^3 \cdot 3^5}{L_2} \right) \frac{1}{\upsilon} \mathrm{d}\upsilon,$$

which diverges provided that

$$\mathfrak{q}_0 > \frac{1944}{cL_2},\ L_2 > 0 \textit{ and } c \in (0,1). \tag{4.3}$$

3. *Using* $\phi_1(\mathrm{s}) = \mathrm{s}$ *and applying condition (3.8):*

$$\begin{aligned} &\limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \left(L_1 c\ell \int_\ell^\infty \left(u \int_u^\infty \frac{\mathfrak{q}_0}{\upsilon^5} \left(\frac{\upsilon}{3\upsilon}\right)^{1/\epsilon} \mathrm{d}\upsilon \right) \mathrm{d}u - \frac{1}{4\ell} \right) \mathrm{d}\ell \\ = \ &\limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \left(\frac{L_1 c}{3^{1/\epsilon}} \ell \int_\ell^\infty \frac{\mathfrak{q}_0}{4u^3} \mathrm{d}u - \frac{1}{4\ell} \right) \mathrm{d}\ell \end{aligned}$$

$$= \limsup_{s\to\infty} \int_1^s \left(\frac{L_1 c \mathfrak{q}_0}{8 \cdot 3^{1/\epsilon}} - \frac{1}{4} \right) \cdot \frac{1}{\ell} d\ell,$$

which diverges if:

$$\mathfrak{q}_0 > \frac{2}{L_1 c} \cdot 3^{1/\epsilon},\ L_1 > 0 \text{ and } c, \epsilon \in (0,1). \tag{4.4}$$

Therefore, in accordance with Theorems 3.1 and 3.2, if inequalities (4.2)–(4.4) are satisfied, then Eq (4.1) exhibits oscillatory behavior.

Figure 2 presents the oscillatory region of Eq (4.1), with conditions (4.2)–(4.4) satisfied. The shaded intersection indicates the parameter set in which the solutions exhibit oscillatory behavior. Parameters are chosen as $L = L_1 = 1$, $\epsilon = 0.25$, *and* $c \in (0, 1)$.

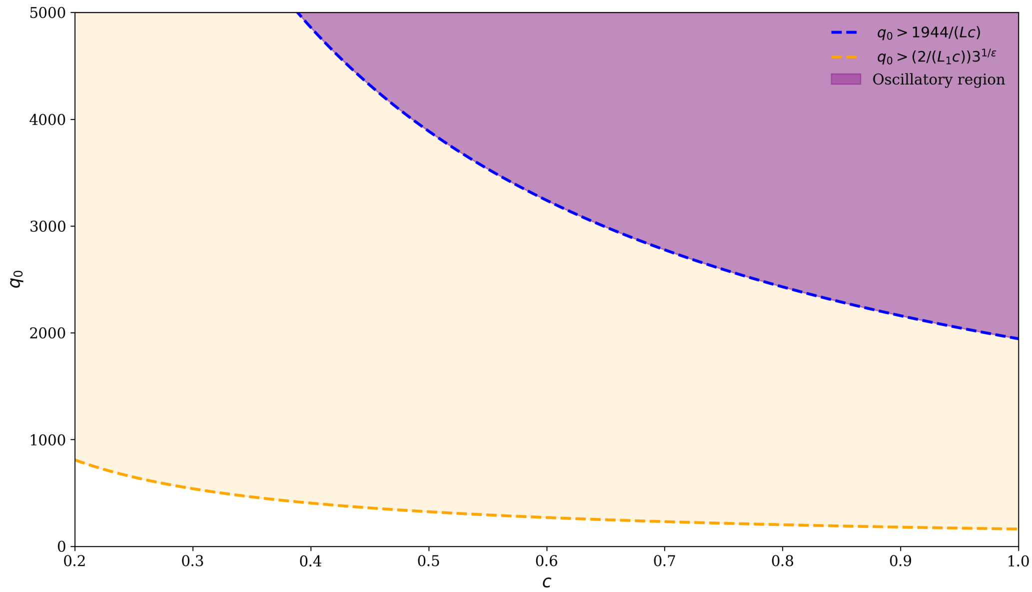

Figure 2. Oscillatory region of Eq (4.1) satisfying conditions (4.2)–(4.4).

Example 4.2. *Consider the NDE given by*

$$\left(s^{-1/3} \left(\left(x(s) + \frac{1}{s} x^{1/5} \left(\frac{1}{2} s \right) - x^5 \left(\frac{1}{2} s \right) \right)''' \right)^{1/3} \right)' + \frac{\mathfrak{q}_0}{s^{7/3}} x^{1/3} \left(\frac{1}{2} s \right) = 0,\ s \geq 1, \tag{4.5}$$

for $s \geq 1$, *and* $b_0 > 0$. *The associated parameters are specified as follows:*

$$\begin{aligned} \alpha &= \beta = \frac{1}{3},\ \gamma = \frac{1}{5},\ \delta = 5,\ a(s) = \frac{1}{2}s,\ b(s) = \frac{1}{2}s, \\ \kappa(s) &= s^{-1/3},\ \hbar(s) = \hbar_1(s) = \frac{1}{s},\ \hbar_2(s) = 1,\ \text{and } \mathfrak{q}(s) = \frac{\mathfrak{q}_0}{s^{7/3}}. \end{aligned}$$

Next, we find

$$R_0(s) = \frac{s^2}{2},\ R_1(s) = \frac{s^3}{6},\ R_2(s) = \frac{s^4}{24}.$$

Moreover,

$$p_1(s) = \frac{4}{5^{5/4} s} \text{ and } p_2(s) = \frac{4}{5^{5/4} s^{5/4}}.$$

Therefore

$$\lim_{s\to\infty}\left[p_1(s)+p_2(s)\right]=\lim_{s\to\infty}\left[\frac{4}{5^{5/4}}\frac{1}{s}+\frac{4}{5^{5/4}}\frac{1}{s^{5/4}}\right]=0,$$

which implies that condition (3.1) is satisfied. The asymptotic behavior of $p_1(s)+p_2(s)$ *is illustrated in Figure 3:*

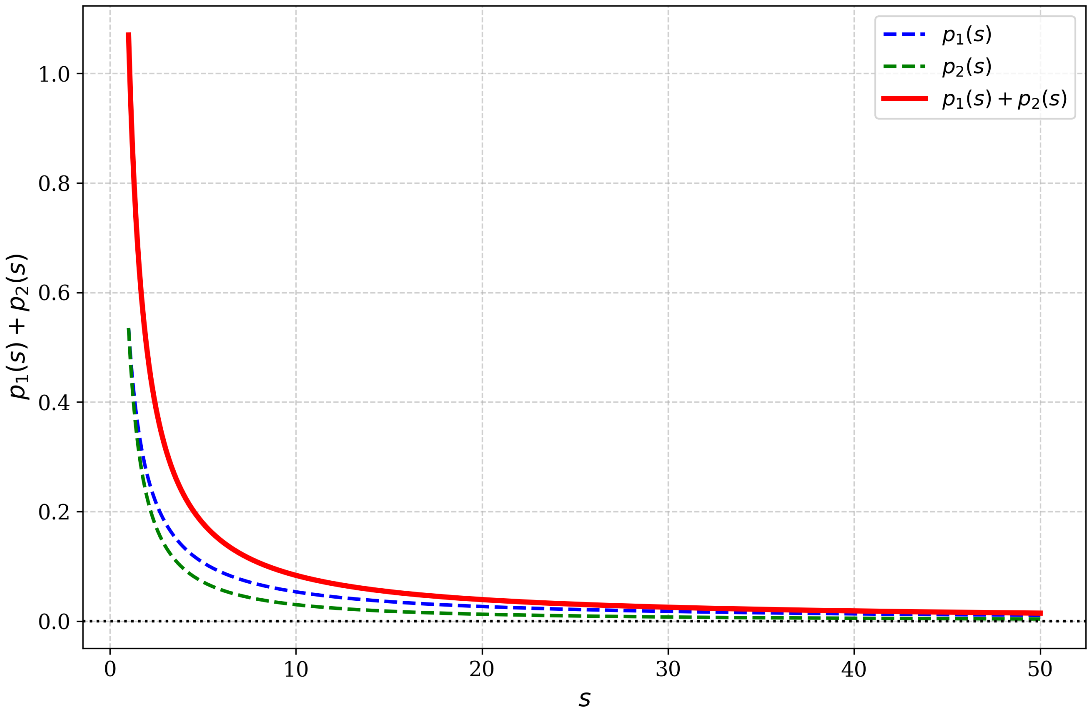

Figure 3. Illustrates the asymptotic behavior of $p_1(s)+p_2(s)$.

To further examine the oscillatory behavior, we employ various test functions and apply the corresponding conditions:

1. *Using* $\phi(s)=s^{4/3}$ *and applying condition (3.7):*

$$\limsup_{s\to\infty}\int_1^s\left(Lc^{1/3}\upsilon^{4/3}\cdot\frac{q_0}{\upsilon^{7/3}}\cdot\left(\frac{1}{2^4}\right)^{1/3}-\frac{1}{\left(\frac{4}{3}\right)^{4/3}}\cdot\frac{\left(\frac{4}{3}\upsilon^{\frac{1}{3}}\right)^{\frac{4}{3}}}{\left(\frac{\upsilon^3}{6}\cdot\upsilon^{4/3}\right)^{1/3}}\right)d\upsilon$$

$$=\limsup_{s\to\infty}\int_1^s\left(\frac{c^{1/3}Lq_0}{2^{4/3}}-6^{1/3}\right)\cdot\frac{1}{\upsilon}d\upsilon,$$

which diverges to infinity if:

$$q_0>\left(\frac{96}{cL}\right)^{1/3},\ L>0 \text{ and } c\in(0,1). \tag{4.6}$$

2. *Using* $\phi_2(s)=s^{4/3}$ *and applying condition (3.27):*

$$\limsup_{s\to\infty}\int_{s_0}^s\left(c^{1/3}\upsilon^{4/3}\cdot\frac{q_0}{\upsilon^{7/3}}-\frac{1}{\left(\frac{4}{3}\right)^{4/3}}\cdot\frac{\left[\frac{4}{3}\cdot\upsilon^{1/3}\right]^{4/3}}{\left(L_2\frac{1}{2}\cdot\frac{\upsilon^3}{6(2^3)}\cdot\upsilon^{4/3}\right)^{1/3}}\right)d\upsilon$$

$$= \limsup_{s\to\infty} \int_{s_0}^{s} \left(c^{1/3} \mathfrak{q}_0 - \frac{2^{5/3} 3^{1/3}}{L_2^{1/3}} \right) \frac{1}{\upsilon} d\upsilon,$$

which diverges if:

$$\mathfrak{q}_0 > \left(\frac{96}{cL_2} \right)^{1/3}, \ L_2 > 0 \textit{ and } c \in (0,1). \tag{4.7}$$

3. *Using* $\phi_1(s) = s$ *and applying condition (3.8):*

$$\limsup_{s\to\infty} \int_{1}^{s} \left(cL_1 \ell \int_{\ell}^{\infty} \left(u^{1/3} \int_{u}^{\infty} \frac{\mathfrak{q}_0}{\upsilon^{7/3}} \cdot \left(\frac{\frac{1}{3}\upsilon}{\upsilon} \right)^{1/3\epsilon_2} d\upsilon \right)^3 du - \frac{1}{4\ell} \right) d\ell$$

$$= \limsup_{s\to\infty} \int_{1}^{s} \left(L_1 \ell \int_{\ell}^{\infty} \left(\frac{3c^{1/3}\mathfrak{q}_0}{4} \cdot \left(\frac{1}{3} \right)^{1/3\epsilon_2} \right)^3 \cdot \frac{1}{u^3} du - \frac{1}{4\ell} \right) d\ell$$

$$= \limsup_{s\to\infty} \int_{1}^{s} \left(\frac{1}{2} \left(\frac{3c^{1/3}\mathfrak{q}_0 \sigma^{1/3\epsilon_2}}{4} \cdot \left(\frac{1}{3} \right)^{1/3\epsilon_2} \right)^3 - \frac{1}{4} \right) \frac{1}{\ell} d\ell,$$

which is satisfied provided that

$$\mathfrak{q}_0 > \left(\frac{32}{27cL_1^3} \cdot 3^{1/\epsilon_2} \right)^{1/3}, \ L_1 > 0, \textit{ and } c, \epsilon \in (0,1). \tag{4.8}$$

Hence, in accordance with Theorems 3.1 and 3.2, if inequalities (4.6)–(4.8) are satisfied, then the solution of Eq (4.5) exhibits oscillatory behavior.

Figure 4 illustrates the oscillatory region of Eq (4.5) where the conditions (4.6)–(4.8) are satisfied. The shaded intersection represents the parameter domain in which the solutions of Eq (4.5) exhibit oscillatory behavior. The parameters are $L = L_1 = 1$, $\epsilon = 0.25$, *and* $c \in (0,1)$.

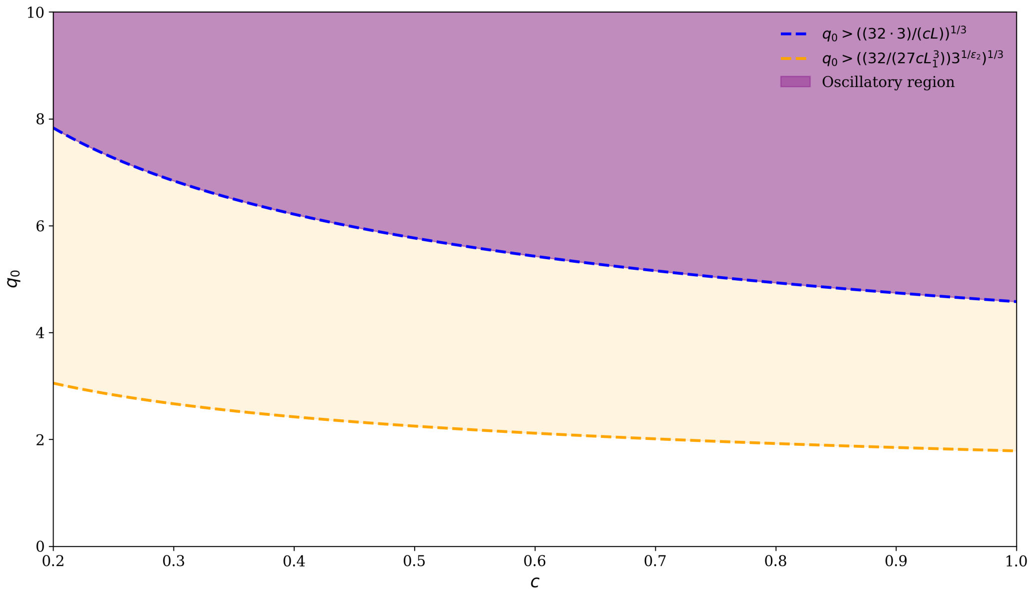

Figure 4. Oscillatory region of Eq (4.5) satisfying conditions (4.6)–(4.8).

Remark 4.1. *The results in [28–30] are not applicable to Examples 4.1 and 4.2, as they do not cover cases involving a mixed neutral term. Accordingly, our results provide a clear extension that encompasses these specific cases.*

5. Conclusions

In this paper, new oscillation criteria have been established for a class of fourth-order nonlinear neutral differential equations involving mixed neutral terms. The main contribution of the study lies in deriving novel and more general sufficient conditions that extend and refine existing results in the literature. The employed methodology, based on a generalized Riccati transformation, allows the conversion of the original fourth-order neutral differential equation into an equivalent system of first-order differential inequalities, providing an efficient analytical framework to examine the oscillatory nature of solutions. These results not only broaden the theoretical understanding of neutral-type equations but also introduce flexible techniques applicable to a wider range of functional differential systems.

For future research, it would be of interest to relax some of the current assumptions, particularly the condition (3.1), to develop more comprehensive oscillation criteria. Furthermore, extending the present results to odd-order equations and exploring alternative neutral structures such as

$$\mathcal{Z}(\mathrm{s}) := x(\mathrm{s}) + \hbar_1(\mathrm{s})\, x^{\gamma}(a_1(\mathrm{s})) + \hbar_2(\mathrm{s})\, x^{\delta}(a_2(\mathrm{s}))$$

may provide deeper insights into the dynamics of nonlinear systems with neutral terms.

Author contributions

Fahd Masood: Methodology, investigation, writing-original draft preparation, writing-review and editing, supervision; Mohammed N. Alshehri: Methodology, investigation; Iambo Loredana Florentina: Methodology, investigation, writing-original draft preparation; A. F. Aljohani: Methodology, investigation, writing-original draft preparation, writing-review and editing; Omar Bazighifan: Methodology, investigation, writing-review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no competing interests.

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