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Research article

On the study of general multifractal analysis of sets using vector-valued functions

Amal Mahjoub[1] **and Najmeddine Attia**[2,*]

[1] Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, Monastir-5000, Tunisia

[2] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia

* **Correspondence:** Email: nattia@kfu.edu.sa; Tel: +966556411269.

Abstract: In this work, we present a general approach to computing the Hewitt-Stromberg dimensions of a set A in a metric space. This technique, originally introduced by Cutler and Tricot and later expanded by Ben Nasr et al., offers a generalization of classical dimension estimates and yields deeper understanding of the scaling behavior of measures in metric spaces. We applied this framework to binomial measures and developed a multifractal one for a function with respect to a gauge function φ. Additionally, we explored the regularity of sets and measures by applying a tailored density theorem, extending existing results and offering novel insights into Moran sets.

Keywords: Hewitt-Stromberg measures; Hewitt-Stromberg dimensions; multifractal formalism; density results
Mathematics Subject Classification: 28A78, 28A80

1. Introduction

The study of multifractal properties of Borel measures has been a central topic in fractal geometry and geometric measure theory since the late 20th century. Multifractal analysis, which seeks to describe the local scaling behavior of measures and their associated singularities, has found applications in a wide range of fields, including dynamical systems, turbulence, image processing, and financial mathematics. During the late 1990s and early 2000s, significant progress was made in developing multifractal formalisms, particularly through the work of Olsen [1], who introduced a comprehensive framework based on multifractal generalizations of the Hausdorff and packing measures. Olsen's formalism provided a unified approach to studying the multifractal spectrum, which characterizes the distribution of singularities in a measure. Olsen's work has since been extensively investigated and expanded upon. For instance, in 2004, Peyriere [2] extended Olsen's formalism to a vector-valued framework, utilizing vectorial multifractal generalizations of the Hausdorff and packing measures. While it is clear that the two most important (and well-known) measures in fractal geometry are the Hausdorff measure and the packing measure, there are nevertheless other interesting fractal measures. Indeed, in 1965, Hewitt and Stromberg introduced a further fractal measure in their classical textbook [3, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example, [4–9]. The Hewitt-Stromberg (H-S) measures are discussed in Edgar's textbook [10] from the late 1990s and have recently received renewed interest in the fractal geometric community. The authors in [5] introduced and studied a multifractal formalism based on the H-S measures. However, we point out that this formalism is parallel to Olsen's multifractal formalism introduced in [1], which is based on the Hausdorff and packing measures.

One of the main problems in multifractal analysis is understanding the multifractal spectrum and the relationship between Rényi dimensions and other multifractal measures. Over the past three decades, there has been significant interest in computing the multifractal spectra of various classes of measures, especially those exhibiting some form of self-similarity (see, for instance, the works in [11–13]). In an effort to develop a general theoretical framework for multifractal analysis of arbitrary measures, Olsen [1] and Pesin [14] proposed defining auxiliary measures in general settings (see also [15, 16] in the context of branching random walks). One of the primary objectives of this paper is to introduce multifractal generalizations of the lower and upper H-S measures. We aim to establish a multifractal formalism for these measures, providing an innovative approach to analyzing multifractal structures. Specifically, we introduce new multifractal lower and upper H-S measures that facilitate the exploration of the multifractal spectrum within a metric space and then extend classical results in this context. Furthermore, we propose a complete description of multifractal analysis of functions in relation to the lower and upper H-S dimensions. By leveraging the newly introduced multifractal measures, we explore the calculation of the multifractal spectra function in a metric space. This approach not only deepens the understanding of multifractal analysis but also offers new tools for investigating the intricate structure of fractal measures.

Let $\alpha \in (0, 1)$, and we define on the interval $[0, 1]$ the binomial measure μ_α with parameter α. In the special case $\alpha = 1/2$, the measure μ_α reduces to a probability measure proportional to the Lebesgue measure on $[0, 1]$. Furthermore, this probability measure exhibits a characteristic self-similar property: For any dyadic subinterval $I \subseteq [0, 1]$, if we divide I into its left and right halves, denoted as $I = (I_L \cup I_R)$, then

$$\mu_\alpha(I_R) = \alpha\mu_\alpha(I).$$

The binomial measure is often regarded as the most straightforward example for understanding multifractal analysis and computing the multifractal spectrum [17–21]. In Section 3.3, we estimate the H-S dimension of the binomial measure using a gauge control function φ. This gauge is essential for controlling the measure and obtaining precise estimates of the binomial measure's multifractal spectrum. By choosing φ appropriately (e.g., with $D_{\_}\varphi = d > 0$), we obtain consistent asymptotics for the lower and upper multifractal H-S functions and the associated Rényi scaling quantities, which allows recovering the multifractal spectrum of the binomial cascade via the Legendre transform. Moreover, this methodology can be applied to various cases, including sets with prescribed digit frequencies, sequences with varying parameters (p_n), and more general multiplicative cascades, thereby extending and refining classical multifractal results ([22–24]).

To investigate the regularity properties of sets and measures, the density theorem serves as a fundamental analytical tool. Over the years, several formulations of density results have been developed for various classes of fractal measures, each tailored to capture different aspects of their local behavior. Notable contributions in this area include the works of [4, 10, 25–30], which provide critical insights into the behavior of density-related properties under a range of geometric and measure-theoretic conditions. In particular, Dai [31] applied these concepts to a class of Moran constructions satisfying the strong separation condition and established the equivalence between Hausdorff and packing measures, despite the potential divergence of their associated dimensions. More generally, density-based characterizations of regularity have been formulated with respect to Hausdorff and packing measures [10, 32], and Hewitt–Stromberg measures [5, 6, 33]. Notably, Tricot et al. [29] demonstrated that subsets of $\mathbb{R}^d$ possess integer Hausdorff and packing dimensions if they exhibit strong regularity. Given a premeasure η on $\mathcal{P}(\mathbb{X})$, where $\mathcal{P}(\mathbb{X})$ is the Borel σ-algebra on $\mathbb{X}$, we denote $\eta^\#$ to be the outer measure constructed using η. We will prove in Section 5 the equivalence of multifractal H-S measures on Moran fractal sets $\mathsf{H}^{q,t}_{\xi,\varphi}$ and $\mathsf{P}^{q,t}_{\xi,\varphi}$ with $\eta^\#$; that is, $\eta^\# \sim \mathsf{H}^{q,t}_{\xi,\varphi} \sim \mathsf{P}^{q,t}_{\xi,\varphi}$. When η is a measure on $\mathcal{P}(\mathbb{X})$, our result was proved for different restrected version of multifractal measures. This leads us to reformulate a tailored versions of the density theorem within the context of our multifractal framework (Lemma 3) related to $\mathsf{H}^{q,t}_{\xi,\varphi}$ and $\mathsf{P}^{q,t}_{\xi,\varphi}$.

This paper is organized as follows. Section 2 introduces the definitions of various multifractal H-S measures and dimensions considered in the study, along with a discussion of their key properties. Section 3 provides estimates for the upper and lower bounds of the H-S dimensions, including an application to the binomial measure. In Section 4, we present a new approach to examining the multifractal formalism of functions. Section 5 establishes density results and demonstrates the equivalence of multifractal H-S measures, with particular emphasis on applications to Moran fractal sets.

2. Preliminaries

In this section, we introduce the multifractal H-S measures, which are central to our analysis. These measures incorporate certain modifications to the standard definitions found in the literature (e.g., [14]) for technical purposes. We begin by introducing the relevant notations and definitions that will be used throughout the paper. Let $(\mathbb{X}, d)$ be a compact metric space. Given a point $x \in \mathbb{X}$ and a radius $r \in (0, \infty)$, we define the open ball centered at x with radius r as:

$$\mathsf{B}(x, r) = \{z \in \mathbb{X} \mid d(x, z) < r\}.$$

Throughout this work, we assume that $\mathbb{X}$ is a compact metric space satisfying the Besicovitch covering property [34, 35]. This property ensures that, for any family of open balls with bounded radii, covering $\mathbb{X}$, one can extract a finite number of subfamilies (called θ-packings) that together cover the space. It is known that Euclidean spaces and ultrametric spaces fulfill this condition. We denote by $\mathcal{B}(\mathbb{X})$ the set of open balls on $\mathbb{X}$ and by $\mathcal{M}(\mathbb{X})$ the set of maps from $\mathcal{B}(\mathbb{X})$ to $[0, +\infty)$. We also define the support of $\mu \in \mathcal{M}(\mathbb{X})$ denoted by $\operatorname{spt}(\mu)$ and given by

$$\operatorname{spt}(\mu) = \mathbb{X} \setminus \bigcup \{\mathsf{B} \in \mathcal{B}(\mathbb{X})\ ;\ \mu(\mathsf{B}) = 0\}.$$

In general metric space, the center x and radius r of a ball are not uniquely determined by the set $\mathsf{B}(x, r)$ so we emphasize a center and a radius given as the constituent of a collection of ordered pairs (x, r) with $x \in \mathbb{X}$ and $r > 0$. Let $\pi = \{(x_i, r)\}_i$ be a collection of $\mathsf{B}(x, r) \cap \mathsf{B}(x', r) = \emptyset$ for all $(x, r) \neq (x', r) \in \pi$. Moreover, π is said to be a (centered) cover of A if $A \subseteq \bigcup_{(x,r)\in\pi} \mathsf{B}(x, r)$. Next, we introduce two functions that play a central role in our analysis:

$$\begin{cases} \tau : \mathbb{X} \times \mathbb{R}_+ & \longrightarrow \mathbb{R}_+, \\ \\ \varphi : \mathbb{R}_+ & \longrightarrow \mathbb{R}, \end{cases}$$

where φ satisfies the following properties:

$$\varphi \text{ is non-decreasing and } \lim_{r\to 0} \varphi(r) = -\infty. \tag{2.1}$$

We consider, for simplicity, the case where $\xi = (\tau, \eta)$ is a pair of functions with $\eta \in \mathcal{M}(\mathbb{X})$.

2.1. Multifractal H-S measures and dimensions

While Hausdorff and packing measures are defined using coverings and packings by family $\{\mathsf{B}(x_i, r_i)\}_i$ with r_i less than a given positive number r, say, the H-S measures are defined using coverings and packings of balls with a fixed radius r. In the following, we will set up, for $t, q \in \mathbb{R}$, the upper and lower multifractal H-S measures denoted, respectively, by $\mathsf{P}^{q,t}_{\xi,\varphi}$ and $\mathsf{H}^{q,t}_{\xi,\varphi}$. Let $A \subseteq \mathbb{X}$ be a nonempty set and $q \in \mathbb{R}$. Let also $\xi = (\tau, \eta)$ be a pair of functions, with $\eta \in \mathcal{M}(\mathbb{X})$. We define

$$\mathcal{T}^q_{\xi,r}(A) = \inf_{(x_i,r)} \left\{ \sum_i \exp\left[-q\tau(x_i, r)\right] \eta(\mathsf{B}(x_i, r)) \right\},$$

$$\mathcal{S}^q_{\xi,r}(A) = \sup_{(x_i,r)} \left\{ \sum_i \exp\left[-q\tau(x_i, r)\right] \eta(\mathsf{B}(x_i, r)) \right\},$$

where the infimum (resp., supremum) is taken over all the constituents $\pi = \{(x_i, r)\}_i$ such that π is a centered covering (resp., packing) of $A \cap \operatorname{spt}(\eta)$. For $t \in \mathbb{R}$, the multifractal H-S pre-measures are defined as follows:

$$\mathsf{L}^{q,t}_{\xi,\varphi}(A) = \liminf_{r\to 0} \mathcal{T}^q_{\xi,r}(A)\, e^{t\varphi(r)} \qquad \text{and} \qquad \mathsf{C}^{q,t}_{\xi,\varphi}(A) = \limsup_{r\to 0} \mathcal{S}^q_{\xi,r}(A)\, e^{t\varphi(r)}.$$

It is clear that the upper multifractal H-S pre-measure $\mathsf{C}^{q,t}_{\xi,\varphi}$ is increasing and $\mathsf{C}^{q,t}_{\xi,\varphi}(\emptyset) = 0$. However it is not countably subadditive. Therefore, we introduce the upper multifractal H-S measure, which we denote by $\mathsf{P}^{q,t}_{\xi,\varphi}$, defined by

$$\mathsf{P}^{q,t}_{\xi,\varphi}(A) = \inf \left\{ \sum_i \mathsf{C}^{q,t}_{\xi,\varphi}(A_i) \;\Big|\; A \subseteq \bigcup_i A_i \,,\; A_i \subseteq \mathbb{X} \right\}.$$

Since the lower multifractal H-S pre-measure $\mathsf{L}^{q,t}_{\xi,\varphi}$ is neither monotone nor countably subadditive, and although it satisfies $\mathsf{L}^{q,t}_{\xi,\varphi}(\emptyset) = 0$, a standard modification is required to construct an outer measure.

Hence, we modify the definition as follows:

$$\overline{\mathsf{H}}^{q,t}_{\xi,\varphi}(A) = \inf\left\{ \sum_i \mathsf{L}^{q,t}_{\xi,\varphi}(A_i) \;\Big|\, A \subseteq \bigcup_i A_i,\ A_i \text{ is closed in } \mathbb{X}\right\}$$

and

$$\mathsf{H}^{q,t}_{\xi,\varphi}(A) = \sup_{F\subseteq A} \overline{\mathsf{H}}^{q,t}_{\xi,\varphi}(F).$$

The functions $\mathsf{H}^{q,t}_{\xi,\varphi}$ and $\mathsf{P}^{q,t}_{\xi,\varphi}$ are outer measures (in the Carathéodory sense) and $\mathsf{H}^{q,t}_{\xi,\varphi}$ is a metric outer measure, while $\mathsf{P}^{q,t}_{\xi,\varphi}$ does not have this property. For more detailed information, see [5, 36, 37]. An important property of the upper H-S pre-measure, as well as the lower and upper measures, is that

$$\mathsf{P}^{q,t}_{\xi,\varphi} \ \leq\ \mathsf{C}^{q,t}_{\xi,\varphi}.$$

Moreover, since X is a compact metric space satisfying the Besicovitch covering property, there exists an integer $\theta \in \mathbb{N}$ such that

$$\mathsf{H}^{q,t}_{\xi,\varphi} \leq \theta\, \mathsf{P}^{q,t}_{\xi,\varphi}. \tag{2.2}$$

The measure $\mathsf{H}^{q,t}_{\xi,\varphi}$ is of course a multifractal generalization of the lower φ-dimensional H-S measure H^{t}_{φ}, whereas $\mathsf{P}^{q,t}_{\xi,\varphi}$ is a multifractal generalization of the upper φ-dimensional H-S measure P^{t}_{φ}. In fact, it is easily seen that, for $t > 0$, one has

$$\overline{\mathsf{H}}^{q,t}_{\xi,\varphi} = \overline{\mathsf{H}}^{t}_{\varphi}, \qquad \mathsf{H}^{q,t}_{\xi,\varphi} = \mathsf{H}^{t}_{\varphi}, \qquad \text{and} \qquad \mathsf{P}^{q,t}_{\xi,\varphi} = \mathsf{P}^{t}_{\varphi}.$$

Furthermore, if we choose $\varphi(r) = \log r$, and we get the classical lower and upper H-S measures H^t and P^t in their original forms [5]. Our construction extends the concept of outer measures introduced and examined in [38] when considering $\eta = 1$ (see Remark 3). In addition, the measures $\mathsf{H}^{q,t}_{\xi,\varphi}$ and $\mathsf{P}^{q,t}_{\xi,\varphi}$ assign in the usual way a multifractal dimension to subset A of $\mathbb{X}$. These dimensions are represented as $\mathsf{b}^{q}_{\xi,\varphi}(A)$ and $\mathsf{B}^{q}_{\xi,\varphi}(A)$, respectively. More precisely, one has

$$\begin{aligned}
\mathsf{b}^{q}_{\xi,\varphi}(A) &= \inf\left\{t\in\mathbb{R} \ \mid \mathsf{H}^{q,t}_{\xi,\varphi}(A)=0\right\} = \sup\left\{t\in\mathbb{R} \ \mid \mathsf{H}^{q,t}_{\xi,\varphi}(A)=\infty\right\},\\
\mathsf{B}^{q}_{\xi,\varphi}(A) &= \inf\left\{t\in\mathbb{R} \ \mid \mathsf{P}^{q,t}_{\xi,\varphi}(A)=0\right\} = \sup\left\{t\in\mathbb{R} \ \mid \mathsf{P}^{q,t}_{\xi,\varphi}(A)=\infty\right\}.
\end{aligned}$$

The pre-measures $\mathsf{C}^{q,t}_{\xi,\varphi}$ and $\mathsf{L}^{q,t}_{\xi,\varphi}$ also assign in the usual way a multifractal dimension to each subset A of $\mathbb{X}$. They are denoted by $\Lambda^{q}_{\xi,\varphi}(A)$ and $\Theta^{q}_{\xi,\varphi}(A)$ where

$$\begin{aligned}
\Theta^{q}_{\xi,\varphi}(A) &= \inf\left\{t\in\mathbb{R} \ \mid \mathsf{L}^{q,t}_{\xi,\varphi}(A)=0\right\} = \sup\left\{t\in\mathbb{R} \ \mid \mathsf{L}^{q,t}_{\xi,\varphi}(A)=\infty\right\},\\
\Lambda^{q}_{\xi,\varphi}(A) &= \inf\left\{t\in\mathbb{R} \ \mid \mathsf{C}^{q,t}_{\xi,\varphi}(A)=0\right\} = \sup\left\{t\in\mathbb{R} \ \mid \mathsf{C}^{q,t}_{\xi,\varphi}(A)=\infty\right\}.
\end{aligned}$$

In the same way as for the measure, if $\eta(\mathsf{B}) = 1$ for all $\mathsf{B} \in \mathcal{B}(\mathbb{X})$ or $q = 0$, then $\mathsf{B}^{q}_{\xi,\varphi}(A)$ will be denoted by $\mathsf{B}^{q}_{\tau,\varphi}(A)$ and $\mathsf{B}_{\eta,\varphi}(A)$, respectively, and $\mathsf{b}^{q}_{\xi,\varphi}(A)$ will be denoted by $\mathsf{b}^{q}_{\tau,\varphi}(A)$ and $\mathsf{b}_{\eta,\varphi}(A)$, respectively. The notations for the H-S-functions are summarized in the table below (Table 1).

Table 1. H-S measures and their dimensions.

H-S-Functions	$\eta(\mathsf{B}) = 1$	$q = 0$	$\eta(\mathsf{B}) = 1$ and $q = 0$
$\mathsf{H}^{q,t}_{\xi,\varphi}$	$\mathsf{H}^{t}_{\tau,\varphi}$	$\mathsf{H}^{t}_{\eta,\varphi}$	H^{t}_{φ}
$\mathsf{P}^{q,t}_{\xi,\varphi}$	$\mathsf{P}^{q,t}_{\tau,\varphi}$	$\mathsf{P}^{t}_{\eta,\varphi}$	P^{t}_{φ}
$\mathsf{b}^{q,t}_{\xi,\varphi}$	$\mathsf{b}^{q,t}_{\tau,\varphi}$	$\mathsf{b}^{t}_{\eta,\varphi}$	$\underline{\dim}_{\varphi}$
$\mathsf{B}^{q,t}_{\xi,\varphi}$	$\mathsf{B}^{q,t}_{\tau,\varphi}$	$\mathsf{B}^{t}_{\eta,\varphi}$	$\overline{\dim}_{\varphi}$

Remark 1. *In the special case where $q = 0$, $\eta = 1$, and $\varphi(r) = \log r$, we come back to the classical definitions of lower and upper Hewitt–Stromberg measures H^t and P^t, and the classical lower and upper Hewitt–Stromberg dimensions $\underline{\dim}$ and $\overline{\dim}$ (see [36]). In particular, we get*

$$\overline{\mathsf{H}}^{q,t}_{\xi,\varphi} = \overline{\mathsf{H}}^{t}, \qquad \mathsf{H}^{q,t}_{\xi,\varphi} = \mathsf{H}^{t}, \qquad \mathsf{P}^{q,t}_{\xi,\varphi} = \mathsf{P}^{t}$$

and

$$\mathsf{b}^{q}_{\xi,\varphi}(A) = \underline{\dim}\,(A), \quad \textit{and} \quad \mathsf{B}^{q}_{\xi,\varphi}(A) = \overline{\dim}\,(A).$$

As a direct consequence of (2.2), the dimensions defined above satisfy the following conditions:

$$\mathsf{b}^{q}_{\xi,\varphi}(A) \le \mathsf{B}^{q}_{\xi,\varphi}(A) \le \Lambda^{q}_{\xi,\varphi}(A).$$

Furthermore, we define the mutual multifractal H-S dimension functions $\mathsf{b}_{\xi,\varphi}$, $\mathsf{B}_{\xi,\varphi}$, and $\Lambda_{\xi,\varphi} : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ by

$$\mathsf{b}_{\xi,\varphi}(q) := \mathsf{b}^{q}_{\xi,\varphi}(\operatorname{spt}(\eta)), \quad \mathsf{B}_{\xi,\varphi}(q) := \mathsf{B}^{q}_{\xi,\varphi}(\operatorname{spt}(\eta)), \quad \text{and} \quad \Lambda_{\xi,\varphi}(q) := \Lambda^{q}_{\xi,\varphi}(\operatorname{spt}(\eta)).$$

In fact, it is easily seen that when $\eta \equiv 1$, the mutual multifractal H-S dimension functions $\mathsf{b}_{\xi,\varphi}$, $\mathsf{B}_{\xi,\varphi}$, and $\Lambda_{\xi,\varphi}$ will be the multifractal H-S functions $\mathsf{b}_{\tau,\varphi}$, $\mathsf{B}_{\tau,\varphi}$, and $\Delta_{\tau,\varphi}$. Moreover, it is well known [36, 37] that $\Lambda_{\xi,\varphi}$ and $\mathsf{B}_{\xi,\varphi}$ are convex.

2.2. Construction of Moran sets

Moran sets can be seen as a generalization of self-similar sets, where different classes of similarity mappings are applied at each level with varying translations. This variability leads to a loss of ergodic properties in Moran sets. Moran sets play a fundamental role in constructing both examples and counterexamples across various mathematical disciplines, including number theory, ergodic theory, and multifractal analysis (see, for instance, [40–42]). Their structured yet flexible nature allows researchers to develop intricate sets with controlled geometric and measure-theoretic properties. These sets serve as essential tools for exploring phenomena such as irregular distributions, fractal dimensions, and complex dynamical behaviors, making them invaluable in theoretical investigations and practical applications alike. These sets will be utilized in Section 5 to examine the equivalence of the measures explored in this paper, under the strong separation condition.

The construction is based on a positive sequence of integers $\{n_k\}_{k\ge1}$ and a family of positive numbers $\{c_{k,j} : 1 \le j \le n_k,\ k \ge 1\}$ such that

$$\sum_{j=1}^{n_k} c_{k,j} \le 1, \qquad \mathbf{c}_k = (c_{k,1}, c_{k,2}, \ldots, c_{k,n_k}), \tag{2.3}$$

for all $k \in \mathbb{N}$. Let $m \leq k$ be two positive integers, and we define

$$\Sigma_{m,k} = \Big\{(i_m, i_{m+1}, \ldots, i_k) \ \text{ such that } \quad 1 \leq i_j \leq n_j, m \leq j \leq k\Big\}$$

and when $m = 1$,

$$\Sigma_k = \Sigma_{1,k} = \Big\{(i_1, i_2, \ldots, i_k) \ \text{ such that } \quad 1 \leq i_j \leq n_j, m \leq j \leq k\Big\}.$$

We also set

$$\Sigma_0 = \emptyset \qquad \text{and} \qquad \Sigma = \bigcup_{k \geq 0} \Sigma_k.$$

For $k, N \in \mathbb{N}$, such that $k + 1 \leq N$, we consider $\sigma = (i_1, i_2, \ldots, i_k) \in \Sigma_k$, $j = (j_{k+1}, j_{k+2}, \ldots, j_m) \in \Sigma_{k+1,m}$, and we define

$$\sigma * j = (i_1, i_2, \ldots, i_k, j_{k+1}, j_{k+2}, \ldots, j_m).$$

Definition 1. *Let $(\mathbb{X}, d)$ be a complete metric space, and let I be a compact subset of $\mathbb{X}$ such that* $\mathrm{int}(I) \neq \emptyset$ *(we can assume that the diameter of I, denoted by $|I|$, is 1). We will say that the family $\mathcal{F} = \{I_\sigma \mid \sigma \in \Sigma\}$ has the Moran structure if the following condition holds:*

(1) For every sequence $(i_1, i_2, \ldots, i_k) \in \Sigma_k$, the set $I_{i_1 i_2 \ldots i_k}$ is similar to I. That is, there exists a similarity transformation

$$S_{i_1 i_2 \ldots i_k} : \mathbb{X} \rightarrow \mathbb{X} \quad \textit{such that} \quad S_{i_1 i_2 \ldots i_k}(I) = I_{i_1 i_2 \ldots i_k},$$

where we assume that $I_\emptyset = I$.

(2) $\forall\, k \geq 1$, $(i_1, i_2, \ldots, i_{k-1}) \in \Sigma_{k-1}$, $I_{i_1 i_2 \ldots i_k}(i_k \in \{1, 2, \ldots, n_k\})$ are subsets of $I_{i_1 i_2 \ldots i_{k-1}}$ and

$$I^\circ_{i_1 i_2 \ldots i_{k-1}, i_k} \cap I^\circ_{i_1 i_2 \ldots i_{k-1}, i'_k} = \emptyset, \qquad 1 \leq i_k < i'_k \leq n_k,$$

where I° denotes the interior of I.

(3) For every $k \geq 1$ and $1 \leq j \leq n_k$, if $\sigma = (i_1, i_2, \ldots, i_{k-1}, j) \in \Sigma_k$, we set

$$0 < c_{k,j} = \frac{|I_{i_1 i_2 \ldots i_{k-1} j}|}{|I_{i_1 i_2 \ldots i_{k-1}}|} < 1, \quad k \geq 2.$$

Assume that $\mathcal{F}$ is a collection of subsets of I with a Moran structure. We define the Moran set $E = \bigcap_{k \geq 1} \bigcup_{\sigma \in \Sigma_k} I_\sigma$ as a set determined by $\mathcal{F}$, and we refer to $\mathcal{F}_k = \{I_\sigma : \sigma \in \Sigma_k\}$ as the k-order fundamental sets of E. The set I is called the original set of E. We assume that $\lim_{k \to \infty} \max_{\sigma \in \Sigma_k} |I_\sigma| = 0$. For all sequences $w = (i_1, i_2, \ldots, i_k, \ldots) \in \Sigma$, we use the abbreviation $w|_k$ to denote the first k elements of the sequence.

$$I_k(w) = I_{w|_k} = I_{i_1 i_2 \ldots i_k}, \qquad c_k(w) = c_{k,i_k}. \tag{2.4}$$

We assume that E satisfies the strong separation condition (**SSC**). Specifically, let $I_{\sigma*1}, I_{\sigma*2}, \ldots, I_{\sigma*n_{k+1}}$ be the $(k+1)$-order fundamental subset of $I_\sigma \in \mathcal{F}$. We will say that I_σ satisfies the (**SSC**) if there exists a sequence of positive real numbers $(\delta_k)_k$ such that, for all $i \neq j$, the following condition holds:

$$\mathrm{dist}(I_{\sigma*i}, I_{\sigma*j}) \geq \delta_k |I_\sigma|,$$

where $\delta = \inf_k \delta_k \in (0, 1)$.

Remark 2. *Consider the special case when, for all $k \geq 1$, $c_{k,1} = c_{k,2} = \cdots = c_{k,n_k} = c_k$. Then the set E will be called a homogeneous Moran set. Let $I_\sigma(x)$ denote the unique subset of level k containing $x \in E$, where $\sigma \in \Sigma_k$. In this case, $|I_\sigma(x)| = \prod_{j=1}^{k} c_j$, which implies that $I_\sigma(x) \subset \mathsf{B}(x, r)$ for $\prod_{j=1}^{k} c_j < r \leq \prod_{j=1}^{k-1} c_j$.*

3. Estimation of multifractal dimensions

3.1. Bouligand-Minkowski's dimensions and multifractal Rényi dimensions

In this paragraph, we investigate the relationship between the lower and upper multifractal H-S functions $\mathsf{b}_{\xi,\varphi}$ and $\mathsf{B}_{\xi,\varphi}$, and the multifractal box dimension. These results extend the classical Bouligand-Minkowski dimensions and provide a framework for analyzing the multifractal structure of sets and measures.

Proposition 1. *Let $A \subseteq \mathbb{X}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfy* (2.1)*. Then, for all $q \in \mathbb{R}$, we have*

$$\Theta^q_{\xi,\varphi}(A) = \liminf_{r\to 0} \frac{\log \mathcal{T}^q_{\xi,r}(A)}{-\varphi(r)} \qquad \textit{and} \qquad \Lambda^q_{\xi,\varphi}(A) = \limsup_{r\to 0} \frac{\log \mathcal{S}^q_{\xi,r}(A)}{-\varphi(r)}.$$

Proof. Suppose that

$$\liminf_{r\to 0} \frac{\log \mathcal{T}^q_{\xi,r}(A)}{-\varphi(r)} < \Theta^q_{\xi,\varphi}(A) - \epsilon,$$

for some $\varepsilon > 0$. Then for any $\delta > 0$, there exists $0 < r \leq \delta$ such that $\mathcal{T}^q_{\xi,r}(A)\, e^{\left(\Theta^q_{\xi,\varphi}(A)-\varepsilon\right)\varphi(r)} < 1$. Therefore,

$$\infty \;=\; \mathsf{L}^{q,\Theta^q_{\xi,\varphi}(A)-\varepsilon}_{\xi,\varphi}(A) \leq 1,$$

which is a contradiction. Then

$$\liminf_{r\to 0} \frac{\log \mathcal{T}^q_{\xi,r}(A)}{-\varphi(r)} \geq \Theta^q_{\xi,\varphi}(A) - \varepsilon.$$

Now, suppose that

$$\liminf_{r\to 0} \frac{\log \mathcal{T}^q_{\xi,r}(A)}{-\varphi(r)} > \Theta^q_{\xi,\varphi}(A) + \varepsilon,$$

for some $\varepsilon > 0$. Then, there exists $\delta > 0$ such that for all $0 < r \leq \delta$, $\mathcal{T}^q_{\xi,r}(A)\, e^{(\Theta^q_{\xi,\varphi}(A)+\varepsilon)\varphi(r)} > 1$. Therefore, we obtain

$$0 \;=\; \mathsf{L}^{q,\Theta^q_{\xi,\varphi}(A)+\varepsilon}_{\xi,\varphi}(A) > 1,$$

which is a contradiction. Then

$$\liminf_{r\to 0} \frac{\log \mathcal{T}^q_{\xi,r}(A)}{-\varphi(r)} \leq \Theta^q_{\xi,\varphi}(A) + \varepsilon,$$

for any $\varepsilon > 0$, as required. The proof of the second is similar.

□

Remark 3. *Consider the special case where* $\mathbb{X} = \mathbb{R}^d$ *(*$d \geq 1$*) and, for any* $x \in \mathbb{R}^d$ *and* $r > 0$*, the function* $\tau(x, r)$ *is given by*

$$\tau(x, r) = -\log\left(\prod_{i=1}^{d} \nu_i(\mathsf{B}(x, r))\right),$$

where $\nu_i \in \mathcal{M}(\mathbb{R}^d)$*. Let* $\varphi : (0, 1) \to (-\infty, 0)$ *be a non-decreasing function. For* $\eta = 1$*, we obtain results proved in [38] and for* $\varphi(r) = \log r$*, we obtain the setting studied in [39]. Furthermore, when* $d = 1$*, we find the classical formalism [1, 43–45].*

Example 1. *In this example, we consider the simplest case of Proposition 1, that is,* $q = 0$ *and* $\eta = 1$ *so that we get*

$$\Theta^q_{\xi,\varphi} := \underline{\dim}^{\varphi}_B \quad \text{and} \quad \Lambda^q_{\xi,\varphi} := \overline{\dim}^{\varphi}_B,$$

where $\underline{\dim}^{\varphi}_B$ *and* $\overline{\dim}^{\varphi}_B$ *generalize the lower and upper box-dimension* $\underline{\dim}_B$ *and* $\overline{\dim}_B$*, respectively [32, 46]. Take* φ *as in Proposition 1 and assume that*

$$-\frac{\ln \gamma_1 n}{a} \leq \varphi\Big(\frac{1}{2n^2}\Big) \leq \varphi\Big(\frac{1}{n^2}\Big) \leq -\frac{\ln \gamma_2 n}{a} \tag{3.1}$$

where $a > 0$ *and* $\gamma_1, \gamma_2 > 0$*. We define, for* $n \geq 1$*, the set* $A_n = \{0\} \bigcup \{1/k, \quad k \leq n\}$ *and*

$$K = \bigcup_n A_n = \Big\{0\Big\} \bigcup \Big\{1/n, \quad n \in \mathbb{N}\Big\}.$$

This set has been previously studied in the context of box-counting dimensions [8, 32]. We will now prove that the generalized lower box dimension $\Theta^0_{\xi,\varphi}(K) = a$*. For* $n \geq 2$ *and* $\frac{1}{2n^2} \leq r_n < \frac{1}{n^2}$*, note that*

$$\mathcal{T}^q_{\xi,r_k}(A_n) \geq n + 1$$

and then, for $t = a$*, we obtain using* (3.1)

$$\mathcal{T}^0_{\xi,r_n}(K)e^{t\varphi(r_n)} \geq \mathcal{T}^q_{\xi,r_k}(A_n)e^{t\varphi(r_n)} \geq (n + 1)e^{-\ln(\gamma_1 n)}.$$

It follows that $\mathsf{L}^a_{\xi,\varphi}(K) \geq 1/\gamma_1 > 0$*. Thereby,*

$$\Theta^0_{\xi,\varphi}(K) \geq \Theta^0_{\xi,\varphi}(A_n) \geq a.$$

In the other hand, we have $\mathcal{S}_{\xi,\varphi}(A_n) = n + 1$ *and then*

$$\mathcal{S}^q_{\xi,r_k}(A_n)e^{t\varphi(r_n)} \leq (n + 1)e^{-\ln(\gamma_2 n)}.$$

It follows that $\mathsf{C}^a_{\xi,\varphi}(K) = 1/\gamma_2 < \infty$*. Thereby,* $\Lambda^0_{\xi,\varphi}(A_n) \leq a$ *and then*

$$\Lambda^0_{\xi,\varphi}(K) = \sup_n \Lambda^0_{\xi,\varphi}(A_n) \leq a.$$

Finally, one has

$$\Theta^0_{\xi,\varphi}(K) = \Lambda^0_{\xi,\varphi}(K) = a.$$

Now, we introduce the mutual multifractal generalization of the L^q-dimensions, also called the Renyi dimensions, based on integral representations. Let A be a compact subset of $\mathbb{X}$ and η be a compactly supported Borel measure. For all $q \geq 0$, we denote

$$I_\xi^q(r) = \int_K \exp\left[-q\tau(x,r)\right] d\eta(x),$$

where $K := A \cap \operatorname{spt}(\eta) \neq \emptyset$.

Proposition 2. *Let A be a compact subset of $\mathbb{X}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies* (2.1)*. For all $q \geq 0$, one has*

$$\Theta_{\xi,\varphi}^q(A) = \liminf_{r\to 0} \frac{\log I_\xi^q(r)}{-\varphi(r)} \qquad \textit{and} \qquad \Lambda_{\xi,\varphi}^q(A) = \limsup_{r\to 0} \frac{\log I_\xi^q(r)}{-\varphi(r)}.$$

Proof. Let $\left(\mathsf{B}(x_i,r)\right)_i$ be a packing of $A \cap \operatorname{spt}(\eta)$. One has

$$\begin{aligned}
\sum_i e^{-q\tau(x_i,r)}\, \eta(\mathsf{B}(x_i,r)) &= \sum_i e^{-q\tau(x_i,r)} \int_{\mathsf{B}(x_i,r)} d\eta(x) \\
&\leq \int_{\cup_i \mathsf{B}(x_i,2r)} e^{-q\tau(x_i,2r)}\, d\eta(x) \\
&\leq \int_K e^{-q\tau(x_i,2r)}\, d\eta(x).
\end{aligned}$$

Thus $\mathcal{T}_\xi^q(r) \geq I_\xi^q(2r)$. On the other hand, for every $r > 0$, we can apply Besicovitch's covering theorem [47,48] to $\left(\mathsf{B}(x_i,2r)\right)_i$ to get θ finite or countable sub-families $\{\mathsf{B}(x_{1j},2r)\}_j, \ldots, \{\mathsf{B}(x_{\theta j},2r)\}_j$, such that $\{\mathsf{B}(x_{ij},2r)\}_j$ is a packing of $A \cap \operatorname{spt}(\eta)$ and

$$F \subseteq \bigcup_{i=1}^{\theta} \bigcup_j \mathsf{B}(x_{ij},2r).$$

It holds that

$$\begin{aligned}
\int_K e^{-q\tau(x_i,r)}\, d\eta(x) &\leq \sum_{i=1}^{\theta} \sum_j \int_{\mathsf{B}(x_i,2r)} e^{-q\tau(x_i,r)}\, d\eta(x) \\
&\leq \sum_{i=1}^{\theta} \sum_j e^{-q\tau(x_i,2r)}\, \eta(\mathsf{B}(x_i,2r)),
\end{aligned}$$

which implies that $\mathcal{T}_\xi^q(r) \geq I_\xi^q(2r)$. □

Remark 4. *Let ν and η be two probability measures on $\mathbb{R}^d$. Setting $\varphi(r) = \log r$ and $\tau(x,r) = -\log \nu(\mathsf{B}(x,r))$, we obtain for $q > 0$*

$$\Lambda_{\nu,\eta}(q) = \limsup_{r\to 0} \frac{1}{\log r} \log \int_K \nu(\mathsf{B}(x,r))^q\, d\eta(x).$$

It is closely related to the Renyi dimension in its integral version (see [14]) and if η is a Gibbs measure for the measure ν, then there exists a measure η on $\operatorname{spt}(\eta)$ *and constants $C > 0$ and $t_q \in \mathbb{R}$ such that for every $x \in \operatorname{spt}(\eta)$ and every $0 < r < \lambda$*

$$C^{-1}\nu(\mathsf{B}(x,r))^q(2r)^{t_q} \leq \eta(\mathsf{B}(x,r)) \leq C\nu(\mathsf{B}(x,r))^q(2r)^{t_q}.$$

$\Lambda_{\nu,\eta}$ represents the C_ν function of Olsen's multifractal formalism [49]. This quantity appears as a generalization of the upper q-spectral dimension defined in [50]. For $q \geq 0$,

$$D_q(\nu) = \limsup_{r\to 0} \frac{1}{q\log r} \log \int_K \nu(\mathsf{B}(x,r))^q \, d\nu(x).$$

In particular, in the case $\nu = \eta$, one has

$$\Lambda_{\nu,\nu}(q) = qD_q(\nu).$$

Proposition 3. *For $A \subseteq \mathbb{X}$ and $q \in \mathbb{R}$, we have*

$$\mathsf{b}^q_{\xi,\varphi}(A) = \sup_{F\subseteq A} \inf \left\{ \sup \Theta^q_{\xi,\varphi}(A_i) \mid \quad F \subseteq \bigcup_i A_i \ \ \textit{and} \ \ A_i \subseteq \mathbb{X} \right\}$$

and

$$\mathsf{B}^q_{\xi,\varphi}(A) = \inf \left\{ \sup \Lambda^q_{\xi,\varphi}(A_i) \mid \quad A \subseteq \bigcup_i A_i \ \ \textit{and} \ \ A_i \subseteq \mathbb{X} \right\}.$$

Proof. Suppose

$$\Gamma := \sup_{F\subseteq A} \inf \left\{ \sup \Theta^q_{\xi,\varphi}(A_i) \mid \quad F \subseteq \bigcup_i A_i \ \ \text{and} \ \ A_i \subseteq \mathbb{X} \right\}.$$

Assume that $\Gamma < \mathsf{b}^q_{\xi,\varphi}(A)$ and take $t \in (\Gamma\,,\ \mathsf{b}^q_{\xi,\varphi}(A))$. Then, for all $F \subseteq A$, there exists $(A_i)_i$ of the bounded subset of F such that

$$F \subseteq \bigcup_i A_i \qquad \text{and} \qquad \sup_i \Theta^q_{\xi,\varphi}(A_i) < t.$$

Therefore, $\mathsf{L}^{q,t}_{\xi,\varphi}(A_i) = 0$, which implies that $\overline{\mathsf{H}}^{q,t}_{\xi,\varphi}(F) = 0$. This gives that $\mathsf{H}^{q,t}_{\xi,\varphi}(F) = 0$. This is a contradiction. Now assume that $\mathsf{b}^q_{\xi,\varphi}(A) < \Gamma$. Then, for all $t \in (\mathsf{b}^q_{\xi,\varphi}(A)\ ,\ \Gamma)$, we have $\mathsf{H}^{q,t}_{\xi,\varphi}(A) = 0$. It follows from this that $\overline{\mathsf{H}}^{q,t}_{\xi,\varphi}(F) = 0$ for all subsets $F \subseteq A$. Consequently, there exists a family of bounded subsets $(A_i)_i$ of F such that for every $n \geq 0$,

$$F \subseteq \bigcup_i A_i \qquad \text{and} \qquad \sup_i \mathsf{L}^q_{\xi,\varphi}(A_i) < \infty.$$

Then $\sup_i \Theta^q_{\xi,\varphi}(A_i) \leq t$ for any i and $\Gamma \leq t$. This is a contradiction.

□

3.2. Estimation of the H-S dimensions

In this section, we establish upper and lower bounds for the H-S dimensions of a Borel set $A \subseteq \mathbb{X}$ with respect to a measure $\eta \in \mathcal{M}(\mathbb{X})$. These results generalize classical dimension estimates, such as those of Billingsley [51] and Tricot [23], and provide a framework for analyzing the fine-scale structure of sets and measures in a wide range of settings. It will prove convenient to use the following notations: If $\eta \in \mathcal{M}(\mathbb{X})$, one considers the outer measure $\eta^{\#}$ on $\mathbb{X}$ associated with η by

$$\overline{\eta}_r(A) = \mathcal{T}^0_{\xi,r}(A), \quad \overline{\eta}^0(A) = \mathsf{L}^0_{\eta,\varphi}(A), \qquad \overline{\eta}(A) = \overline{\mathcal{H}}^0_{\eta,\varphi}(A), \quad \text{and} \quad \eta^{\#}(A) = \mathsf{H}^0_{\eta,\varphi}(A), \tag{3.2}$$

and the essential supremum of a function ψ is given by

$$\operatorname*{ess\,sup}_{x\in A,\eta^{\sharp}} \psi(x) = \inf\{t \in \mathbb{R}; \eta^{\sharp}(A \cap \{\psi > t\}) = 0\}.$$

Theorem 1. *Let $A \subseteq \mathbb{X}$ be a Borel set. Let $\eta \in \mathcal{M}(\mathbb{X})$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies* (2.1).

(1) Assume that $\mathsf{B}_{\eta,\varphi}(A) \leq 0$, and then

$$\underline{\dim}_{\varphi}(A) \leq \sup_{x\in A} \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)} \quad \textit{and} \quad \overline{\dim}_{\varphi}(A) \leq \sup_{x\in A} \limsup_{r\to 0} \frac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)}.$$

(2) Assume that $\eta^{\sharp}(A) > 0$ and then

$$\underline{\dim}_{\varphi}(A) \geq \operatorname*{ess\,sup}_{x\in A} \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)} \quad \textit{and} \quad \overline{\dim}_{\varphi}(A) \geq \operatorname*{ess\,sup}_{x\in A} \limsup_{r\to 0} \frac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)}.$$

Proof. (1) For $x \in A$, we take $s > \sup_{x\in A} \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)}$ and $\epsilon > 0$. Under our hypothesis $\overline{\dim}_{\eta,\varphi}(A) \leq 0$, we have $\mathsf{P}^{\epsilon}_{\eta,\varphi}(A) = 0$. Therefore, we can choose a partition A_i of the set A such that

$$\sum_i \mathsf{C}^{\epsilon/2}_{\eta,\varphi}(A_i) < 1 \quad \text{and} \quad \sum_i \mathsf{C}^{\epsilon}_{\eta,\varphi}(A_i) = 0.$$

Let $i \in \mathbb{N}$ and choose $F \subseteq A_i$. For $0 < \delta$ and $x \in F$, we can find $\beta_x \geq 2$ and $\frac{\delta}{\beta_x} < r_x < \delta$ such that

$$\eta(\mathsf{B}(x, r_x)) \geq e^{s\varphi(r_x)}.$$

Let $\{\mathsf{B}(x_i, r_x)\}_{x\in F}$ cover $\overline{F}$, where $\overline{F}$ denotes the closure of F. So, we can choose a finite subset $I \in \mathbb{N}$ such that the family $\{\mathsf{B}(x_i, r_{x_i})\}_{i\in I}$ is a centered γ-covering of F. Take $\beta = \max_{i\in I} \beta_{x_i}$, and then from (2.1) and for all $i \in I$, one has

$$\eta(\mathsf{B}(x, \delta)) \geq \eta(\mathsf{B}(x, r_{x_i})) \geq e^{s\varphi(r_{x_i})}.$$

Since $\{\mathsf{B}(x_i, \delta)\}_{i\in I}$ is a centered covering of F, using Besicovitch's covering theorem, we can construct θ finite or countable sub-families $\{\mathsf{B}(x_{1j}, \delta)\}_j, \ldots, \{\mathsf{B}(x_{\zeta j}, \delta)\}_j$, such that $\{\mathsf{B}(x_{ij}, \delta)\}_j$ is a packing of F:

$$F \subseteq \bigcup_{i=1}^{\theta} \bigcup_j \mathsf{B}(x_{ij}, \delta) \quad \text{and} \quad \eta(\mathsf{B}(x_{ij}, \delta)) \geq e^{s\varphi(\frac{\delta}{\beta})}.$$

Then, one has

$$\mathcal{T}_{\frac{\delta}{\beta}}(A)\, e^{(s+\varepsilon)\varphi(\delta/\beta)} \le \sum_{ij} e^{(s+\varepsilon)\varphi(\delta/\beta)} \le \sum_{ij} \eta(\mathsf{B}(x_{ij},\delta))e^{\varepsilon\varphi(\delta)} \le \theta\, \mathcal{S}_{\eta,\delta}(A)\, e^{\varepsilon\varphi(\delta)}.$$

This clearly implies that $\mathsf{L}_{\varphi}^{s+\varepsilon}(A_i) < \theta\, \mathsf{C}_{\eta}^{\epsilon}(A_i) = 0$ and then $\overline{\mathsf{H}}_{\varphi}^{s+\varepsilon}(A_i) = 0$. It follows that $\underline{\dim}_{\varphi}(A_i) \le s + \varepsilon$, for all $\varepsilon > 0$. Finally, we get $\underline{\dim}_{\varphi}(A) \le s$.

Now, we will prove the second assertion. Let $s > \sup_{x\in A} \limsup_{r\to 0} \dfrac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)}$ and $\epsilon > 0$. Under our hypothesis $\overline{\dim}_{\eta,\varphi}(A) \le 0$, we have $\mathsf{P}_{\eta,\varphi}^{\epsilon}(A) = 0$. Therefore, we can choose a partition A_i of the set A such that

$$\sum_i \mathsf{C}_{\eta,\varphi}^{\epsilon/2}(A_i) < 1 \quad \text{and} \quad \sum_i \mathsf{C}_{\eta,\varphi}^{\epsilon}(A_i) = 0.$$

For all $x \in A$, we can choose $\delta > 0$ such that, for all $0 < r < \delta$, we have $\eta(\mathsf{B}(x,r)) \ge e^{s\varphi(r)}$. Consider the set

$$A(k) = \Big\{x \in A;\ \ \forall\, \delta \le 1/k,\ \ \eta(\mathsf{B}(x,r)) \ge e^{s\varphi(r)}\Big\}.$$

Fix $k \in \mathbb{N}$ and $0 < r < \min(\delta, 1/k)$. Let $\{\mathsf{B}(x_i,r)\}_i$ be a centered packing of $A_i \cap A(k)$. Therefore, we have

$$\mathcal{S}_{\delta}(A_i \cap A(k))e^{(s+\varepsilon)\varphi(\delta)} \le \sum_{ij} \eta(\mathsf{B}(x_{ij},\delta))e^{\varepsilon\varphi(\delta)} \le \mathcal{S}_{\eta,\delta}(A_i)e^{\varepsilon\varphi(\delta)},$$

and then $\mathsf{C}_{\varphi}^{s+\varepsilon}(A_i \cap A(k)) < \mathsf{C}_{\eta,\varphi}^{\epsilon}(A_i) = 0$. It follows that

$$\mathsf{P}_{\varphi}^{s+\varepsilon}(A(k)) \le \sum_i \mathsf{C}_{\varphi}^{s+\varepsilon}(A_i \cap A(k)) < \sum_i \mathsf{C}_{\eta,\varphi}^{\epsilon}(A_i) = 0$$

and since $A = \bigcup_k A(k)$, we obtain $\overline{\dim}_{\varphi}(A) \le s$.

(2) Take $s < \operatorname*{ess\,sup}_{x\in A} \liminf_{r\to 0} \dfrac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)}$. Consider the set

$$K = \Big\{x \in A \mid \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)} > s\Big\}.$$

It is clear that $\eta^{\sharp}(K) > 0$. For each $x \in A$, we can find $\delta_0 > 0$ such that for each $0 < r < \delta_0$, one has

$$\eta(\mathsf{B}(x,r)) \le e^{s\varphi(r)}.$$

Now, let $(K_i)_i$ be a countable partition of K. Consider the set

$$K_{i_p} := \Big\{x \in K_i \mid \delta < \frac{1}{p},\ \ \eta(\mathsf{B}(x,r)) \le e^{s\varphi(r)}\Big\}.$$

Fix $p \in \mathbb{N}$ and E is a subset of K_{i_p}. Then, if $\left\{\mathsf{B}(x_i, r)\right\}_{i\in\{1,\dots,\mathcal{T}_\delta(E)\}}$ is a centered covering of E with $0 < r < \min(\delta_0, 1/p)$, one has

$$\mathcal{T}_{\eta,r}(E) \leq \sum_i \eta(\mathsf{B}(x_i, r)) \leq e^{s\varphi(r)}\, \mathcal{T}_r(E),$$

from which it follows that $\mathsf{L}^0_{\eta,\varphi}(E) \leq \mathsf{L}^s_\varphi(E) \leq \mathsf{L}^s_\varphi(K_{i_p})$ and $\overline{\mathsf{H}}^0_{\eta,\varphi}(K_{i_p}) \leq 2^{-s}\, \overline{\mathsf{H}}^s(K_{i_p})$. Since $K_i = \cup_p K_{i_p}$ for all i and $\eta^\#(K) > 0$, by making $\delta_0 \to 0$, we obtain

$$0 < \eta^\sharp(K_{i_p}) \leq \sum_j \sum_p \overline{\mathsf{H}}^0_\eta(K_{i_p}) \leq 2^{-s} \sum_j \sum_p \overline{\mathsf{H}}^s(K_{i_p}).$$

Therefore, $0 < \mathsf{H}^s_\varphi(K) < \mathsf{H}^s_\varphi(E)$, which implies

$$\underline{\dim}_\varphi(A) \geq s \qquad \text{for all} \qquad s < \operatorname*{ess\,sup}_{x\in A} \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)}.$$

Now, we will prove the second assertion. Let $s < \operatorname*{ess\,sup}_{x\in A} \limsup_{r\to 0} \dfrac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)}$ and set

$$K = \left\{x \in A \ \mid \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x, r))}{\varphi(r)} > s \right\}.$$

We have $\eta^\sharp(K) > 0$, so there exists a subset E of K such that for all $x \in E$ and all $\varepsilon > 0$, we can find a positive real number $0 < r_x < \varepsilon$ such that

$$\eta(\mathsf{B}(x, r_x)) \leq e^{s\varphi(r_x)}.$$

Let $\{\mathsf{B}(x_i, r_x)\}_{x\in F}$ ε-cover $\overline{E}$. So, we can choose a finite subset $I \in \mathbb{N}$ such that the family $\{\mathsf{B}(x_i, r_{x_i})\}_{i\in I}$ is a centered ε-covering of E. Take $\delta = \max_{i\in I} r_{x_i}$, and then for all $i \in I$, one has that $\{\mathsf{B}(x_i, \delta)\}_{i\in I}$ is a centered covering of F. Using Besicovitch's covering theorem, we can construct θ finite or countable sub-families $\{\mathsf{B}(x_{1j}, \delta)\}_j, \dots, \{\mathsf{B}(x_{\theta j}, \delta)\}_j$, such that $\{\mathsf{B}(x_{ij}, \delta)\}_j$ is a packing of F,

$$F \subseteq \bigcup_{i=1}^{\theta} \bigcup_j \mathsf{B}(x_{ij}, \delta), \quad \text{and} \quad \eta(\mathsf{B}(x_{ij}, \delta)) \leq e^{s\varphi(\delta)}.$$

Then one has $\mathcal{T}_{\eta,\delta}(E) \leq \sum_{i,j} \eta(\mathsf{B}(x_{ij}, \delta)) \leq \theta\, \mathcal{S}_\delta(A) e^{s\varphi(\delta)}$. This implies that

$$\mathsf{L}^0_\eta(E) \leq \theta\, \mathsf{C}^s_\varphi(E) \leq \theta\, \mathsf{C}^s_\varphi(K) \quad \text{and} \quad \overline{\mathsf{H}}^0_\eta(K) \leq 2^{-s}\, C\, \theta\, \mathsf{C}^s(K).$$

Hence, if $K = \bigcup_i K_i$, then

$$0 < \eta^\sharp(K) \leq \sum_i \overline{\mathsf{H}}^0_\eta(K_i) \leq 2^{-s}\, C\, \theta \sum_i \mathsf{C}^s_\varphi(K_i).$$

Thus, $\mathsf{P}^s_\varphi(E) > 0$. Finally, we get $\overline{\dim}_\varphi(A) \geq s$.

□

Corollary 1. *Let $A \subseteq \mathbb{X}$ be a Borel set. Let η be a finite Borel measure in $\mathbb{X}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies* (2.1). *Assume that $\eta(A) > 0$ and*

$$\liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)} = \limsup_{r\to 0} \frac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)},$$

for all $x \in A$. Then $\underline{\dim}_\varphi(A) = \overline{\dim}_\varphi(A)$.

3.3. Application: Binomial measure

In this section, we will consider a special case when τ and η are two functions defined by using binomial measures [52]. We will start with the definitions. Let $n \in \mathbb{N}$ and define

$$\mathcal{F}_n = \{I = [k/2^n, (k+1)/2^n[0 \leqslant k < 2^n\},$$

the family of dyadic intervals of the nth generation contained in the interval $[0, 1)$. Moreover, if $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}^n$, we denote by $I_{\epsilon_1\ldots\epsilon_n}$ the dyadic interval of the nth generation; that is,

$$I_{\epsilon_1\ldots\epsilon_n} = \left[\sum_{i=1}^{n} \frac{\epsilon_i}{2^i}, \sum_{i=1}^{n} \frac{\epsilon_i}{2^i} + \frac{1}{2^n}\right[.$$

For $I = I_{\epsilon_1\ldots\epsilon_n}$ and $J = I_{\epsilon_{n+1}\ldots\epsilon_{n+p}}$, we will denote their concatenation as $IJ = I_{\epsilon_1\ldots\epsilon_{n+p}}$. Furthermore, for any $x \in [0, 1)$, we denote by $I_n(x)$ the unique element of $\mathcal{F}_n$ to which x belongs, and $|I_n(x)| = 2^{-n}$ is its length. Let $0 < p < 1$ and let η be a binomial measure on $[0, 1)$ with parameter p, defined by

$$\eta(I_{\epsilon_1\ldots\epsilon_{n-1}1}) = p\,\eta(I_{\epsilon_1\ldots\epsilon_{n-1}}) \qquad \text{and} \qquad \eta(I_{\epsilon_1\ldots\epsilon_{n-1}0}) = (1-p)\,\eta(I_{\epsilon_1\ldots\epsilon_{n-1}}). \tag{3.3}$$

Hence the binomial measure is a probability measure which is defined via a recursive construction ([18,52,53]). This recursive definition implies that the sequence $(\epsilon_n)_{n\geq 1}$ can be interpreted as a sequence of independent binomial random variables with parameter p. Specifically, for each $i \geq 1$,

$$\eta(\{\epsilon_i = 1\}) = p \quad \text{and} \quad \eta(\{\epsilon_i = 0\}) = 1 - p.$$

Using the independence of the random variables $\epsilon_1, \epsilon_2, \ldots$, the measure η of a dyadic interval $I_{\epsilon_1\ldots\epsilon_n}$ can be defined by

$$\eta(I_{\epsilon_1\ldots\epsilon_n}) = p^{s_n}(1-p)^{n-s_n}$$

where $s_n = \epsilon_1 + \cdots \epsilon_n$ counts the number of 1s in the sequence $\epsilon_1 \ldots \epsilon_n$. In particular, when $p = 1/2$, the measure η coincides with the Lebesgue measure on $[0, 1)$. For $x \in [0, 1)$, we denote by $I_n(x)$ the n-th-level set containing x. We introduce the sequence of random variables X_n defined by

$$X_n(x) = -\log_2\left(\frac{\eta(I_n(x))}{\eta(I_{n-1}(x))}\right) \quad \text{and} \quad \frac{S_nX(x)}{n} := \frac{X_1(x) + \cdots + X_n(x)}{n} = -\frac{\log \eta(I_n(x))}{n \log 2}.$$

In this section, we take $\mathbb{X} = [0, 1)$ and

$$\tau(\mathsf{B}) = -\log \eta(\mathsf{B}), \qquad \forall\, \mathsf{B} \in \mathcal{B}(\mathbb{X}). \tag{3.4}$$

Let φ be a function that satisfies (2.1) and define the lower and upper order of φ, respectively, by

$$\underline{D}^\varphi = \liminf_{r\to 0} \frac{\varphi(r)}{\log r} \quad \text{and} \quad \overline{D}^\varphi = \limsup_{r\to 0} \frac{\varphi(r)}{\log r}.$$

We will denote by D^φ the common value if the limit exists and it will be called the order of φ.

Lemma 1. *Let* $A = \text{spt}(\eta)$ *and* $\xi = (\tau, \eta)$*, where* τ *is define by* (3.4). *Then, for all* $q \in \mathbb{R}$*, we have*

$$\Theta_{\xi,\varphi}(q-1) \le \frac{1}{d_2}\log_2(p^q + (1-p)^q)$$

for all φ *satisfying* $\overline{D}^{\varphi} \ge d_2 > 0$. *Furthermore,*

$$\Lambda_{\xi,\varphi}(q-1) \ge \frac{1}{d_1}\log_2(p^q + (1-p)^q)$$

for all φ *satisfying* $\underline{D}^{\varphi} \le d_1$*, for some* $d_1 > 0$.

Proof. First observe, for all $n \ge 1$, that

$$\mathcal{S}^{q-1}_{\xi,2^{-n}}(A) \;=\; \mathcal{T}^{q-1}_{\xi,2^{-n}}(A) = \sum_{I \in \mathcal{F}_n} \eta(I)^q,$$

still with the convention $0^q = 0$. In addition, using the binomial formula, we obtain $\mathcal{S}^{q-1}_{\xi,2^{-n}}(A) = \mathcal{T}^{q-1}_{\xi,2^{-n}}(A) = \left(p^q + (1-p)^q\right)^n$. Assume that $\overline{D}^{\varphi} \ge d_2$ then for n big enough, we have $\varphi(2^{-n}) \ge \log(2^{-nd_2})$. Using Proposition 1, we get, for all $q \in \mathbb{R}$,

$$\begin{aligned}
\Theta_{\xi,\varphi}(q-1) &= \liminf_{r\to 0} \frac{\log \mathcal{S}^{q-1}_{\xi,r}(A)}{-\varphi(r)} \\
&\le \liminf_{n\to\infty} -\frac{1}{\varphi(2^{-n})}\log\left(p^q + (1-p)^q\right)^n \\
&\le \frac{1}{d_2}\log_2(p^q + (1-p)^q).
\end{aligned}$$

Similarly, assume that $\underline{D}^{\varphi} \le d_1$ and then for n big enough, we have $\varphi(2^{-n}) \le \log(2^{-nd_1})$. Using Proposition 1, we get, for all $q \in \mathbb{R}$,

$$\begin{aligned}
\Lambda_{\xi,\varphi}(q-1) &= \limsup_{r\to 0} \frac{\log \mathcal{T}^{q-1}_{\xi,r}(A)}{-\varphi(r)} \\
&\ge \limsup_{n\to\infty} -\frac{1}{\varphi(2^{-n})}\log\left(p^q + (1-p)^q\right)^n \\
&\ge \frac{1}{d_1}\log_2(p^q + (1-p)^q).
\end{aligned}$$

□

In the particular case where the function φ satisfies $D^{\varphi} = d$, we obtain

$$\Lambda_{\xi,\varphi}(q-1) = \Theta_{\xi,\varphi}(q-1) = \frac{1}{d}\log_2(p^q + (1-p)^q). \tag{3.5}$$

(1) Let $A_p \subseteq [0,1)$ be the set of sequences with a prescribed limiting frequency p of the digit 1. Using the strong law of large numbers, we know that s_n/n converges $d\eta$-almost surely (a.s.) to p. Hence, $\eta(A_p) = 1$. In addition, for all φ such that $D^{\varphi} = d$, we have

$$\lim_{n\to\infty} -\frac{\log \eta(I_n(x))}{n(d-\epsilon)\log 2} \le \lim_{r\to 0} \frac{\log \eta(B(x,r))}{\varphi(r)} \le \lim_{n\to\infty} -\frac{\log \eta(I_n(x))}{n(d+\epsilon)\log 2}, \tag{3.6}$$

for all fixed $\epsilon > 0$. Note that

$$-\frac{\log \eta(I_n(x))}{n \log 2} = -\frac{s_n}{n} \log_2 p - (1 - \frac{s_n}{n}) \log_2(1 - p).$$

It follows, since $s_n/n \to p \; d\eta$ a.s., that

$$\lim_{n\to\infty} \frac{\log \eta(I_n(x))}{n \log 2} = -p \log_2 p - (1 - p) \log_2(1 - p) := h(p). \tag{3.7}$$

Thus, from (3.6), we have

$$\frac{h(p)}{d - \epsilon} \leq \lim_{r\to 0} \frac{\log \eta(B(x, r)}{\varphi(r)} \leq \frac{h(p)}{d + \epsilon}.$$

Taking $\epsilon \to 0$, we obtain $\dim_\eta(x) := \lim_{r\to 0} \dfrac{\log \eta(B(x, r)}{\varphi(r)} = \dfrac{h(p)}{d}$. In addition, from (3.5) for $q = 1$, one has $\mathsf{B}_{\eta,\varphi}(A_p) \leq \Lambda_{\eta,\varphi}(A_p) \leq \Lambda_{\xi,\varphi}(0) = 0$. By Theorem 1, the H-S dimensions of the set A are given by

$$\underline{\dim}_\varphi(A_p) = \overline{\dim}_\varphi(A_p) = \frac{h(p)}{d} := h_d(p).$$

(2) Now , we consider the set $E(\beta)$ of point $x \in [0, 1]$ such that $\dim_\eta(x) = \beta$, for $\beta \in \mathbb{R}$. Using (3.6), we get $\eta(E(h_d(p)) = 1$ and then $\underline{\dim}_\varphi(E(h_d(p)) = \overline{\dim}_\varphi(E(h_d(p)) = h_d(p)$ by Theorem 1. In addition, from (3.5), we have

$$\Lambda_{\xi,\varphi}(q - 1) = \frac{1}{d} \log_2(p^q + (1 - p)^q).$$

Now, we consider the case when $\beta \neq h_d(p)$. Let η_θ be binomial measure with parameter $\theta \in (0, 1)$ and then

$$\begin{aligned}\lim_{n\to\infty} \frac{\log \eta(I_n(x))}{\log |I_n(x)|} &= \lim_{n\to\infty} -\frac{s_n}{n} \log_2 p - (1 - \frac{s_n}{n}) \log_2(1 - p) \\ &= h(\theta, p) := -\theta \log_2 p - (1 - \theta) \log_2(1 - p) \;\; d\eta_\theta \quad \text{a.s.}\end{aligned}$$

Thus from (3.6), we have $\dim_\eta(x) = \frac{h(\theta,p)}{d} \; d\eta_\theta$ a.s.

$$\begin{aligned}\eta_\theta(I_n(x)) &= \eta(I_n(x))^q e^{\Lambda_{\xi,\varphi}(q-1)\varphi(2^{-n})} \\ &\approx (p^{s_n}(1 - p)^{n-s_n})^q \,(p^q + (1 - p)^q)^{-n} \\ &\approx \left(\frac{p^q}{p^q + (1 - p)^q}\right)^{s_n} \left(\frac{(1 - p)^q}{p^q + (1 - p)^q}\right)^{n-s_n}.\end{aligned}$$

Thus, η_θ is a binomial measure with parameter θ, where

$$\theta = \frac{p^q}{p^q + (1 - p)^q},$$

and then we obtain $\beta = h_d(\theta)$. Moreover, we have $\beta = -\Lambda'_{\xi,\varphi}(q-1)$ which implies that

$$\dim m_\theta = -q\Lambda'_{\xi,\varphi}(q-1) + \Lambda_{\xi,\varphi}(q-1).$$

Using Theorem 1, we obtain that

$$\underline{\dim}(E(\beta)) = \overline{\dim}(E(\beta)) = -q\Lambda'_{\xi,\varphi}(q-1) + \Lambda_{\xi,\varphi}(q-1),$$

for all $\beta = \Lambda'_{\xi,\varphi}(q-1)$, where negative dimension means that the set is empty.

(3) Consider a sequence $(p_n)_n$ of real numbers such that $p_n \in (0,1)$ for all $n \geq 1$. We define the measure m in the following way:

$$m(I_{\epsilon_1\ldots\epsilon_{n-1}1}) = p_n\, m(I_{\epsilon_1\ldots\epsilon_{n-1}}) \qquad \text{and} \qquad m(I_{\epsilon_1\ldots\epsilon_{n-1}0}) = (1-p_n)\,(I_{\epsilon_1\ldots\epsilon_{n-1}}).$$

Clearly, the random variables ϵ_n are independent and verify

$$m(\{\epsilon_n = 1\}) = p_n \quad \text{and} \quad m(\{\epsilon_n = 0\}) = 1 - p_n.$$

Notice that the random variables X_n are independent and bounded in L^2. The strong law of large numbers ensures that the sequence

$$\frac{S_nX(x) - \mathbb{E}[S_nX(x)]}{n} \longrightarrow 0 \quad \text{a.s.}$$

dm almost every $x \in [0,1)$. Hence,

$$\liminf_{n\to\infty} -\frac{\log m(I_n(x))}{n\log 2} = \liminf_{n\to\infty} \mathbb{E}\Big(\frac{S_nX(x)}{n}\Big) = \liminf_{n\to\infty} h(p_n),$$

and

$$\limsup_{n\to\infty} -\frac{\log m(I_n(x))}{n\log 2} = \limsup_{n\to\infty} \mathbb{E}\Big(\frac{S_nX(x)}{n}\Big) = \limsup_{n\to\infty} h(p_n),$$

dm almost every $x \in [0,1)$. By Theorem 1, for $A = \operatorname{spt}(\eta)$, we have

$$\underline{\dim}(A) = \liminf_{n\to\infty} h(p_n) \quad \text{and} \quad \overline{\dim}(A) = \limsup_{n\to\infty} h(p_n).$$

4. The multifractal formalism

As is known, the multifractal formalism aims at giving expressions of the dimension of the level sets of local Hölder exponents of functions in terms of the Legendre transform of some free "energy" function. In this section, we apply the main result of Theorem 1 to study the multifractal formalism of a function τ with respect to ζ. This approach gives a more flexible and general way to study local regularity than traditional methods based on Hausdorff or packing measures. This involves characterizing the local variations in roughness by assessing the distribution of Hölder singularities at small scales. Specifically, this heterogeneity can be described through the lower and upper ζ-local dimensions of τ at a point $x \in \mathbb{X}$ defined, respectively, as

$$\underline{\alpha}_{\tau,\varphi}(x) = \liminf_{r\to 0} \frac{\tau(x,r)}{-\zeta(r)} \qquad \text{and} \qquad \overline{\alpha}_{\tau,\varphi}(x) = \limsup_{r\to 0} \frac{\tau(x,r)}{-\zeta(r)}.$$

We refer to the common value as the ζ-local dimension of τ at the point x, and denote it by $\alpha_{\tau,\varphi}(x)$. For $\alpha, \beta \geq 0$, we define the level sets

$$\underline{X}_{\tau,\varphi}(\alpha) = \left\{x \in \mathbb{X} \;\middle|\; \underline{\alpha}_{\tau,\varphi}(x) \geq \alpha \right\} \qquad \text{and} \qquad \overline{X}_{\tau,\varphi}(\alpha) = \left\{x \in \mathbb{X} \;\middle|\; \overline{\alpha}_{\tau,\varphi}(x) \leq \alpha \right\}.$$

We will be interested in the set

$$X_{\tau,\varphi}(\alpha, \beta) = \underline{X}_{\tau,\varphi}(\alpha) \cap \overline{X}_{\tau,\varphi}(\beta) \qquad \text{and} \qquad X_{\tau,\varphi}(\alpha) = X_{\tau,\varphi}(\alpha, \alpha).$$

In this paragraph, we will state our main results concerning the estimation of the lower and upper H-S dimensions of the set $X_{\tau,\varphi}(\alpha)$ using the Legendre transform of the multifractal H-S functions, where the Legendre transform$f^* : \mathbb{R} \to [-\infty, +\infty]$ of a real-valued function f is defined by $f^*(\alpha) = \inf\{q\alpha + f(q), \;\; q \in \mathbb{R}\}$. Let us define

$$\underline{\alpha} = \sup_{q>0} -\frac{\mathsf{b}_{\tau,\varphi}(q)}{q} \qquad \text{and} \qquad \overline{\alpha} = \inf_{q<0} -\frac{\mathsf{b}_{\tau,\varphi}(q)}{q}.$$

Theorem 2 and Proposition 4 below extend a previous result given in [54].

Theorem 2. *For any $\alpha \in (\underline{\alpha}, \overline{\alpha})$, we have*

$$\overline{\dim}_{\varphi}\Big(\overline{X}_{\tau,\varphi}(\alpha)\Big) \;\leq\; \mathsf{B}^*_{\tau,\varphi}(\alpha) \qquad \textit{and} \qquad \underline{\dim}_{\varphi}\Big(\overline{X}_{\tau,\varphi}(\alpha)\Big) \;\leq\; \mathsf{b}^*_{\tau,\varphi}(\alpha).$$

A negative dimension means that $\overline{X}_{\tau,\varphi}(\alpha)$ is empty.

Given that the direct computation of the multifractal spectrum for a given function is typically intractable, the following theorem provides a sufficient condition for obtaining a rigorous lower bound.

Theorem 3. *Let $q \in \mathbb{R}$ and ζ satisfy* (2.1). *Assume that $\mathsf{H}^{q,\Lambda_{\tau,\varphi}(q)}_{\tau,\varphi}(\mathbb{X}) > 0$. Then*

$$\underline{\dim}_{\varphi}\Big(X_{\tau,\varphi}(\Lambda'_l(q), \Lambda'_r(q))\Big) \geq \begin{cases} -q\,\Lambda'_l(q) + \Lambda_l(q) & \textit{if } q \geq 0, \\ \\ -q\,\Lambda'_r(q) + \Lambda_r(q) & \textit{if } q \leq 0. \end{cases}$$

In particular, if $\Lambda_{\tau,\varphi}$ is differentiable at q and $\alpha = -\Lambda'_{\tau,\varphi}(q)$, one has $\mathsf{b}_{\tau,\varphi}(q) = \mathsf{B}_{\tau,\varphi}(q) = \Lambda_{\tau,\varphi}(q)$ and

$$\underline{\dim}_{\varphi}\left(X_{\tau,\varphi}(\alpha)\right) = \overline{\dim}_{\varphi}\left(X_{\tau,\varphi}(\alpha)\right) = \Lambda^*_{\tau,\varphi}(\alpha).$$

To establish the lower bound in multifractal analysis, let us begin by proving a key lemma that will guide us through the process. This lemma offers fundamental insights into the support of the measure $\eta^\sharp$.

Lemma 2. *Let $f(t) = \mathsf{B}^t_{\xi,\varphi}(A)$ and assume that $f(0) = 0$ and $\eta^\sharp(\operatorname{spt}(\eta)) > 0$. Then one has*

$$\eta^\sharp\Big(\Big(X_{\tau,\varphi}(-f'_-(0), -f'_+(0))\Big)^c\Big) = 0,$$

where f'_-, f'_+ represent the left and right derivatives of the function f, respectively.

Proof. We start by proving that

$$\eta^{\sharp}\Big(\Big\{x \in \mathbb{X} \ \mid \ \limsup_{r\to 0} \frac{\tau(x,\ r)}{-\varphi(r)} > -f'_{-}(0)\Big\}\Big) = 0.$$

Given $\delta > -f'_{-}(0)$, there exist δ' and t such that they satisfy $\delta > \delta' > -f'_{-}(0)$ and $\varphi(-t) < \delta' t$. It is clear that $\mathsf{P}^{-t,\delta' t}_{\xi,\varphi}(\mathbb{X}) = 0$. We can choose a countable partition $\mathbb{X} = \bigcup_j E_j$ such that

$$\sum_j \mathsf{C}^{-t,\delta' t}_{\varkappa,\mu}(E_j) \le 1 \qquad \text{and} \qquad \mathsf{C}^{-t,\delta t}_{\varkappa,\mu}(E_j) = 0.$$

Define the set

$$K_\delta = \left\{x \in \mathbb{X} \ \Big| \ \limsup_{\varepsilon\to 0} \frac{\tau(x,r)}{-\varphi(r)} > \delta\right\}.$$

If $x \in K_\delta\gamma$, then for all $\gamma > 0$, there exists $r \le \gamma$ such that $e^{-\tau(x,r)} \le e^{\delta\zeta(\varepsilon)}$. Let F be a subset of K_δ and set $F_j = F \cap E_j$. For $\varepsilon > 0$, for all j, one can find a Besicovitch γ-cover $\{\mathsf{B}(x_{j,k}, r_{j,k})\}$ of F_j such that

$$e^{-\tau(x_{jk},r_{jk})} \le e^{\delta\varphi(r_{jk})}.$$

We have

$$\begin{aligned}
\mathcal{T}_\eta(F_j) \le \sum_k \eta(\mathsf{B}(x_{jk}, r_{jk}) &= \sum_k e^{-t\tau(x_{jk},r_{jk}))}\, e^{t\tau(x_{jk},r_{jk})}\, \eta(\mathsf{B}(x_{jk}, r_{jk}) \\
&\le \sum_k e^{-t\tau(x_{jk},r_{jk}))}\, e^{\delta\varphi(r_{jk})}\, \eta(\mathsf{B}(x_{jk}, r_{jk}) \\
&= \mathcal{S}^t_\delta(F_j) \sum_k e^{\delta\varphi(r_{jk})},
\end{aligned}$$

which, together with the Besicovitch property, implies $\overline{\eta}_\varepsilon(F_j) \le \mathsf{C}^{t,\delta t}_{\xi,\varphi,\varepsilon}(E_j)$. Therefore, $\overline{\eta}(F_j) \le \mathsf{C}^{t,\delta t}_{\xi,\varphi}(E_j)$. This implies $\eta^{\sharp}(F) = 0$, and hence $\eta^{\sharp}(K_\delta) = 0$. In the same way, one proves that

$$\eta^{\sharp}\Big(\Big\{x \in \mathbb{X} \ \mid \ \liminf_{\delta\to 0} \frac{\tau(x,\ \delta))}{-\varphi(r)} < -f'_{+}(0)\Big\}\Big) = 0.$$

□

Proposition 4. *Let $f(t) = \mathsf{B}_{\xi,\eta}(t)$ and assume that $f(0) = 0$ and that $\eta^{\sharp}(\operatorname{spt}(\eta)) > 0$. Then for all $x \in \mathrm{X}_\varkappa(f'_{-}(0), f'_{+}(0))$, one has*

$$\underline{\dim}_\varphi\Big(\mathrm{X}_\varkappa(f'_{-}(0), f'_{+}(0))\Big) \ge \inf_x \liminf_{r\to 0} \frac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)}$$

and

$$\overline{\dim}_\varphi\Big(\mathrm{X}_{\tau,\varphi}(f'_{-}(0), f'_{+}(0))\Big) \ge \inf_x \limsup_{r\to 0} \frac{\log \eta(\mathsf{B}(x,e))}{\varphi(r)}.$$

Proof. This follows immediately from Theorem 1 and Proposition 2. □

Now, we are able to give the proof of Theorems 2 and 3. For $q \in \mathbb{E}$ and $t \in \mathbb{R}$, we take

$$\eta(\mathsf{B}(x,r)) = \exp\Big[-q\tau(x,r) + \Lambda_{\tau,\varphi}(q)\,\varphi(r)\Big].$$

By a simple calculation, we get $\Lambda_{\tau,\varphi}(t) = \Lambda_{\tau,\varphi}(q+t) - \Lambda_{\tau,\varphi}(q)$ and if $x \in \overline{\mathrm{X}}_{\tau,\varphi}(\alpha)$, we have

$$\limsup_{r\to 0} \frac{\log \eta(\mathsf{B}(x,r))}{\varphi(r)} = \limsup_{r\to 0} \; -q\,\frac{\tau(x,r)}{\varphi(r)} + \Lambda_{\tau,\varphi}(q) \leq q\alpha + \Lambda_{\tau,\varphi}(q).$$

So, by Theorem 1 (1), one gets

$$\overline{\dim}_{\varphi}(\mathrm{X}_{\tau,\varphi}(\alpha)) \leq \overline{\dim}_{\varphi}(\overline{\mathrm{X}}_{\tau,\varphi}(\alpha)) \leq q\alpha + \mathsf{B}_{\tau,\varphi}(q).$$

We now turn to the proof of the lower bound. If, moreover, we suppose that $\mathsf{H}^{q,\Lambda_{\tau,\varphi}(q)}_{\tau,\varphi}(\mathbb{X}) > 0$, then from the construction of the measure $\eta^\sharp$, we have $\eta^\sharp(\operatorname{spt}(\eta)) > 0$. By taking Lemma 2 into consideration, we get

$$\eta^\sharp\Big(\mathrm{X}_{\tau,\varphi}(-\Lambda'_{\tau,\varphi})\Big) > 0.$$

Then, it follows from Theorem 1 (2) that

$$\underline{\dim}_{\varphi}\Big(\mathrm{X}_{\tau,\varphi}(\alpha)\Big) \geq q\alpha + \Lambda_{\tau,\varphi}(q).$$

Example 2. *In this example, we will study the validity of our multifractal formalism for quasi-Bernoulli measures. The notations are the same as in Application 3.3. Let μ and ν be two probability measures on $[0,1)$ such that μ is a quasi-Bernoulli measure. We say that the probability measure μ is a quasi-Bernoulli measure on $[0,1)$, satisfying*

$$\forall (n,p) \in \mathbb{N}^2, \forall I \in \mathcal{F}_n, \forall J \in \mathcal{F}_p, \quad \frac{1}{C}\mu(I)\mu(J) \leqslant \mu(IJ) \leqslant C\mu(I)\mu(J), \tag{4.1}$$

where C is a positive constant independent of n, p, I, and J and we write $\mu(IJ) \approx \mu(I)\mu(J)$. Now, we consider the functions τ and φ as

$$\begin{cases} \tau(x,r) = -\log \mu(\mathsf{B}(x,r)) \\ \\ \varphi(r) = \log(|I_n|), \end{cases}$$

where $2^{-n-1} \leq r < 2^{-n}$ and we consider the set

$$\mathrm{X}_{\tau,\varphi}(\alpha) = \left\{ x \in [0,1) \;\mid\; \lim_{n\to+\infty} \frac{\log \mu\,(I_n(x))}{\log |I_n(x)|} = \alpha \right\}$$

and the following function $\Lambda_{\tau,\varphi}$ defined as follows $\forall q \in \mathbb{R}$,

$$\Lambda_{\tau,\varphi}(q) = \limsup_{n\to+\infty} \frac{1}{n\log 2} \log\left(\sum_{I\in\mathcal{F}_n} \mu(I)^q\right).$$

Choose, as an application of our results, $\eta = \mu(\mathsf{B}(x,r))^q 2^{-n\Lambda_{\tau,\varphi}(q)}$. *From Proposition 1, it is easy to compute*

$$\Lambda_{\xi,\varphi}(x) \approx \limsup_{n\to\infty} \frac{1}{n}\log_2\left(2^{-n\Lambda_{\tau,\varphi}(q)}\sum_{I\in\mathcal{F}_n}\mu(I)^{t+q}\right). \tag{4.2}$$

Then $\Lambda_{\xi,\varphi}(x) = \Lambda_{\tau,\varphi}(x+q) - \Lambda_{\tau,\varphi}(q)$. *It is clear that* $\Lambda_{\xi,\varphi}(0) = 0$, *and when* $\Lambda_{\xi,\varphi}$ *is differentiable at the point q, we have* $\Lambda'_{\xi,\varphi}(0) = \Lambda'_{\tau,\varphi}(q)$. *We have*

$$\lim_{n\to\infty}\frac{\log\eta(I_n(x))}{\log|I_n(x)|} = q\lim_{n\to\infty}\frac{\log\mu(I_n(x))}{\log|I_n(x)|} + \Lambda_{\tau,\varphi}(q).$$

Take $\alpha = -\Lambda'_{\tau,\varphi}(q)$ *and since* $\mathsf{H}^{q,\Lambda_{\tau,\varphi}(q)}_{\tau,\varphi}(\mathrm{X}_{\tau,\varphi}(\alpha)) > 0$, *then it results from Theorem 1 that*

$$\underline{\dim}\,(\mathrm{X}_{\tau,\varphi}(\alpha)) = \overline{\dim}\,(\mathrm{X}_{\tau,\varphi}(\alpha)) = \Lambda^*_{\tau,\varphi}(\alpha).$$

The main idea in Example 2 is to consider the function η (not a measure). However, one still needs to prove that

$$\mathsf{H}^{q,\Lambda_{\tau,\varphi}(q)}_{\tau,\varphi}(\mathrm{X}_{\tau,\varphi}(\alpha)) > 0. \tag{4.3}$$

The set $\mathrm{X}_{\tau,\varphi}(\alpha)$ has already been studied by several authors. In particular, it follows from [55, 56] that for every $q \in \mathbb{R}$, there exists a probability measure ν_q and a constant $C_q > 0$ such that

$$\forall n \geq 1,\ \forall I \in \mathcal{F}_n, \qquad \frac{1}{C_q}\mu(I)^q|I|^{\Lambda_{\tau,\varphi}(q)} \leqslant \nu_q(I) \leqslant C_q\,\mu(I)^q|I|^{\Lambda_{\tau,\varphi}(q)}. \tag{4.4}$$

Moreover, this measure is supported on the set $\mathrm{X}_{\tau,\varphi}(\alpha)$, which implies (4.3). In addition, one has

$$\lim_{n\to\infty}\frac{\log\nu_q(I_n(x))}{\log|I_n(x)|} = \Lambda^*_{\tau,\varphi}(\alpha).$$

The conclusion then follows by applying the mass distribution principle. This corresponds to our case when we take $\eta = \nu_q$.

Remark 5. *The function* $\Lambda_{\xi,\varphi}(x)$ *given in* (4.2) *is closely related to the large deviation principle, which can be used to determine the dimension of the measure* $\eta^\#$. *Indeed, take* $\eta = \nu_q$ *as defined in* (4.4), *and define, for* $q \in \mathbb{R}$ *and* $n \geq 1$, *the sequence of probability measures* $\nu_{q,n}$ *by*

$$\nu_{q,n}(B) = \nu_q\Big(\{x \in [0,1) : \tfrac{1}{n}\log\nu_q(I_n(x)) \in B\}\Big). \tag{4.5}$$

We also introduce the associated logarithmic moment generating function:

$$L_n(q,s) = \frac{1}{n}\log\int_{[0,1)}\nu_q(I_n(x))^s\,d\nu_q(x), \qquad s \in \mathbb{R}.$$

One can easily check that

$$L_q(s) := \limsup_{n\to\infty} L_n(q,s)$$

$$= \limsup_{n\to\infty} \frac{1}{n} \log_2 \left(2^{-n(1+s)\Lambda_{\tau,\varphi}(q)} \sum_{I\in\mathcal{F}_n} \mu(I)^{q(1+s)} \right)$$
$$= \Lambda_{\tau,\varphi}(q(1+s)) - (1+s)\Lambda_{\tau,\varphi}(q).$$

For $\epsilon > 0$, consider the deviation set

$$A_{q,\epsilon} = \left\{\alpha \in \mathbb{R} : \left|\alpha + L_q'(0)\right| \geq \epsilon\right\}.$$

By the large deviation principle, one obtains

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_{q,n}(A_{q,\epsilon}) \leq \sup_{\alpha\in A_{q,\epsilon}} L_q^*(\alpha).$$

Moreover, we have

$$L_q'(0) = -\Lambda_{\tau,\varphi}^*(-\Lambda_{\tau,\varphi}'(q)), \qquad L_q^*(L_q'(0)) = 0 = \max L_q^*.$$

Since L_q is differentiable at 0, it follows that $L_q^*(\alpha) < L_q^*(L_q'(0))$ for all $\alpha \neq L_q'(0)$. Indeed, suppose that $L_q^*(\alpha) = 0$. By the definition of the Legendre transform and the fact that $L_q(0) = 0$, we have

$$\forall s \in \mathbb{R}, \qquad L_q(s) \geq L_q(0) + s\alpha.$$

But since L_q is convex and differentiable at 0, this forces $\alpha = L_q'(0)$. Hence,

$$\gamma_{q,\epsilon} := \sup_{\alpha\in A_{q,\epsilon}} L_q^*(\alpha) < 0.$$

Consequently, for n sufficiently large,

$$\nu_q\left(\left\{x \in [0,1) : \tfrac{1}{n} \log \nu_q(I_n(x)) \in A_{q,\epsilon}\right\}\right) \leq e^{n\gamma_{q,\epsilon}/2}.$$

By the Borel–Cantelli lemma, it follows that

$$\frac{1}{n} \log \nu_q(I_n(x)) \in B(\Lambda^*(-\Lambda'(q)), \epsilon) \quad \textit{for all n large enough.}$$

Letting $\epsilon \to 0$ along a countable sequence yields the desired conclusion.

5. Density results and equivalence of $\eta^{\#}$ on Moran sets

In this section, we consider η as a premeasure on $\mathcal{P}(\mathbb{X})$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies (2.1). For $q, t \in \mathbb{R}$, we write

$$\Psi_{\xi,\varphi}^{q,t}(x,r) = \exp\left[q\tau(x,r)\right] \eta(\mathsf{B}(x,r)) e^{-t\varphi(r)}. \tag{5.1}$$

First, we will reformulate a density theorem (Lemma 3) related to the upper and lower multifractal H-S measures, respectively, $\mathsf{P}_{\xi,\varphi}^{q,t}$ and $\mathsf{H}_{\xi,\varphi}^{q,t}$, which will be applied to study the equivalence of multifractal H-S measures on Moran fractal sets. We define the upper and lower densities of $\Psi_{\xi,\varphi}^{q,t}$ at $x \in \mathbb{X}$ by

$$\overline{d}_{\Psi}^{q,t}(x) = \limsup_{r\to 0} \Psi_{\xi,\varphi}^{q,t}(x,r) \qquad \text{and} \qquad \underline{d}_{\Psi}^{q,t}(x) = \liminf_{r\to 0} \Psi_{\xi,\varphi}^{q,t}(x,r). \tag{5.2}$$

Lemma 3. *Let η be a premeasure on $\mathcal{P}(\mathbb{X})$, $q,t \in \mathbb{R}$, and $\Psi^{q,t}_{\xi,\varphi}$ be the function defined in* (5.1). *Assume that $\Psi^{q,t}_{\xi,\varphi}$ is blanketed and $\mathsf{P}^{q,t}_{\tau,\varphi}(E) < \infty$, and then for all Borel subsets E of $\mathbb{X}$, we have*

$$\mathcal{H}^{q,t}_{\tau,\varphi}(E) \inf_{x\in E} \underline{d}^{q,t}_{\Psi}(x) \leq \eta^{\#}(E) \leq \mathcal{H}^{q,t}_{\tau,\varphi}(E) \sup_{x\in E} \overline{d}^{q,t}_{\Psi}(x). \tag{5.3}$$

$$\mathsf{P}^{q,t}_{\tau,\varphi}(E) \inf_{x\in E} \underline{d}^{q,t}_{\xi,\varphi}(x) \leq \eta^{\#}(E) \leq \mathsf{P}^{q,t}_{\tau,\varphi}(E) \sup_{x\in E} \overline{d}^{q,t}_{\tau,\varphi}(x), \tag{5.4}$$

where $\eta^{\#}(E) = \mathcal{H}^{0}_{\eta,\varphi}(E)$ as defined in (3.2).

The inequalities (5.3) and (5.4) follow directly from the multifractal density theorems previously established for the generalized Hausdorff measure $\mathsf{H}^{q,t}_{\tau,\varphi}$ and packing measure $\mathsf{P}^{q,t}_{\tau,\varphi}$ [17]. These results are natural extensions of classical density theorems originally proven for scalar measures with respect to standard Hausdorff and packing measures (see [10, 25, 27]) (we refer also the reader to [25, 57] for a different version of this result). Now, recall the construction of the Moran set defined in Section 2.2. The following result follows from standard arguments and can be found in various forms such as [1, 4, 6, 26].

Lemma 4. *We take a Moran set $E \subseteq I$ and a premeasure η with* $\mathrm{spt}(\eta) \subseteq E$. *Suppose that E satisfies condition (**SSC**). Then, there exist positive constants γ_i for $1 \leq i \leq 4$, which depend on δ, q, t, such that the following inequalities hold for any $h(i) \in E$,*

$$\gamma_1 \underline{\lim}_{n\to\infty} \Psi^{q,t}_{\xi,\varphi}(I_n(i)) \leq \underline{\lim}_{r\to 0} \Psi^{q,t}_{\xi,\varphi}(h(i), r) \leq \gamma_2 \underline{\lim}_{n\to\infty} \Psi^{q,t}_{\xi,\varphi}(I_n(i)), \tag{5.5}$$

$$\gamma_3 \overline{\lim}_{n\to\infty} \Psi^{q,t}_{\xi,\varphi}(I_n(i)) \leq \overline{\lim}_{r\to 0} \Psi^{q,t}_{\xi,\varphi}(h(i), r) \leq \gamma_4 \overline{\lim}_{n\to\infty} \Psi^{q,t}_{\xi,\varphi}(I_n(i)). \tag{5.6}$$

The next result extends a previous result given in [54].

Theorem 4. *Suppose $E \subseteq I$ is a Moran set that satisfies the (**SSC**) condition. Let η be a premeasure on $\mathcal{P}(\mathbb{X})$, with the support contained within E. Assume that*

$$0 < \lim_{n\to+\infty} \Psi^{q,\alpha}_{\xi,\varphi}(I_n(i)) < \infty, \qquad \textit{for any } i \in D. \tag{5.7}$$

Then, $\eta^{\#} \llcorner A \sim \mathcal{H}^{q,\alpha}_{\tau,\varphi} \llcorner A \sim \mathsf{P}^{q,\alpha}_{\tau,\varphi} \llcorner A$.

Proof. Assume the Eq (5.7) holds and set

$$\Omega_n = \left\{ i \in \Sigma \ \mid \ \overline{\lim}_{r\to 0} \Psi^{q,\alpha}_{\xi,\varphi}(h(i), r)) < n \right\} \nearrow_n \Sigma.$$

Let $B \subseteq E$ such that $\eta^{\#}(B) = 0$. Then, there exists a sequence of open sets $\{G_k\}_k$, such that $B \subseteq G_k$ and $\eta^{\#}(G_k) \leq \dfrac{1}{k}$, for all k. Denote $\eta^{\#}_k(.) = \eta^{\#}_{\llcorner G_k}$. Using Lemma 3, we deduce, for $k, n \in \mathbb{N}$ and $x \in B$, that

$$\mathsf{P}^{q,t}_{\tau,\varphi}(B \cap h(\Omega_n)) \quad \leq \quad \eta^{\#}_k(B) \sup_{x\in B\cap h(\Omega_n)} \left\{ \overline{\lim}_{r\to 0} \Psi^{q,\alpha}_{\xi,\varphi}(x, r) \right\}$$

$$\leq \quad \eta^{\#}(G_k) \sup_{x\in B\cap h(\Omega_n)} \Big\{\overline{\lim_{r\to 0}}\Psi^{q,\alpha}_{\xi,\varphi}(x,r)\Big\} \leq \frac{n}{k}.$$

Let $k \to +\infty$ to get $\mathsf{P}^{q,t}_{\tau,\varphi}(B\cap h(\Omega_n)) = 0$, for any $n \in \mathbb{N}$. It follows that

$$\mathsf{P}^{q,t}_{\tau,\varphi}(B) \leq c \sum_{n\geq 1} \mathsf{P}^{q,t}_{\tau,\varphi}(B\cap h(\Omega_n)) = 0.$$

On the other hand, assume that $\mathsf{P}^{q,t}_{\tau,\varphi}(B) = 0$, and consider the set

$$\Omega'_n = \Big\{ i \in D \ \mid \ \overline{\lim_{r\to 0}}\Psi^{q,\alpha}_{\xi,\varphi}(h(i),r) > \frac{1}{n} \Big\}.$$

Using Lemma 3 again, one obtains

$$\begin{aligned} 0 = \mathsf{P}^{q,t}_{\tau,\varphi}(B\cap h(\Omega'_n)) \quad &\geq \quad \eta^{\#}(B\cap h(\Omega'_n)) \inf_{x\in B\cap h(\Omega'_n)} \Big\{\overline{\lim_{r\to 0}}\Psi^{q,\alpha}_{\xi,\varphi}(x,r)\Big\} \\ &\geq \quad \frac{\eta^{\#}(B\cap h(\Omega'_n))}{n}. \end{aligned}$$

It follows, for all $n \in \mathbb{N}$, that $\eta^{\#}(B\cap h(\Omega'_n)) = 0$. Since $\eta^{\#}\Big(h(\cup_{n\geq 1}\Omega'_n)\Big) = 0$, we get $\eta^{\#}(B) = 0$, as required. □

Remark 6. *One can describe the strong regularity of a subset of $\mathbb{R}^d$ with respect to our function similar to Tricot's work [27], which was further generalized in [1, 6, 26].*

Example 3. *In this example, we consider $\mathbb{X} = [0, 1]$ and and study a special case in which the condition of all the numbers $c_{i_1,i_2,\ldots,i_n}$ depend only on the length and the last variable, that is,*

$$c_{i_1,i_2,\ldots,i_n} = c_{n,i_n}.$$

We define the pressure function π and the function η defined, respectively, by

$$\pi(t) = \lim_{n\to+\infty} \pi_n(t) \quad \textit{where} \quad \pi_n(t) = \frac{1}{n}\log\Bigg(\sum_{(i_1,i_2,\ldots,i_n)\in D_n} \prod_{j=1}^{n} c^t_{j,i_j}\Bigg), \tag{5.8}$$

and

$$\eta(i_1 i_2 \ldots i_n) = \prod_{j=1}^{n} \frac{c^t_{j,i_j}}{Z^t_j}, \tag{5.9}$$

where $Z^t_n = \sum_j c^t_{n,j}$ for any $(i_1, i_2, \ldots, i_n) \in \Sigma_n$, $n \in \mathbb{N}$. It is not difficult to see that π is strictly decreasing and continuous. Moreover, we have

$$\eta(i_1 i_2 \ldots i_n) = |I_n(w)|^t\Big(\prod_{j=1}^{n} Z^t_j\Big)^{-1}.$$

Since, for all n, we have

$$\sum_{(i_1,i_2,\ldots i_n)\in D_n} \prod_{j=1}^{n} c^t_{j,i_j} = \prod_{j=1}^{n} Z^t_j$$

and then

$$\lim_{n\to+\infty} \frac{\eta(i_1 i_2 \dots i_n)}{|I_n(w)|^\gamma} = \lim_{n\to+\infty} |I_n(w)|^{t-\gamma}\Big(e^{-\log \prod_{j=1}^n Z_j^t}\Big) = \lim_{n\to+\infty} |I_n(w)|^{t-\gamma}\Big(e^{-n\pi_n(t)}\Big).$$

Now, we define $\varphi(r) = \log r$ *and* $\tau(x, r)) = -\log \nu(\mathsf{B}(x, r))$*, where the measure* ν *is defined on cylinder* I_σ *for* $\sigma \in D_k$*, by* $\nu(I_\sigma) = \left(\prod_{j=1}^k n_j\right)^{-1}$ *, and we define the function* β *by*

$$\prod_{j=1}^{k} n_j^q \, c_{j,i_j}^{-\beta(q)} = 1. \tag{5.10}$$

Hence, for $\sigma \in \Sigma_k$*, we have*

$$\nu(I_\sigma)^q = \left(\prod_{j=1}^{k} n_j\right)^{-q} = \prod_{j=1}^{k} c_{j,i_j}^{\beta(q)} = |I_\sigma|^{\beta(q)} .$$

Therefore, for all $w \in D$*, we have* $\lim_{n\to+\infty} \Psi_{\xi,\varphi}^{q,t}(x, r) = \lim_{n\to+\infty} \eta(I_\sigma)|I_\sigma|^{-(\beta(q)+t)}$*. Suppose that* $t + \beta(q) := -\alpha$ *is the unique number such that*

$$\sum_{(i_1,i_2,\dots,i_n)\in\Sigma_n} |I_{i_1 i_2 \dots i_n}|^{-\alpha} = 1.$$

It is clear that $\pi(-\alpha) = 0$*, and then*

$$\lim_{n\to+\infty} \Psi_{\xi,\varphi}^{q,t}(x, r) = +\infty \quad \text{if} \quad \gamma + \beta(q) + \alpha > 0$$

and

$$\lim_{n\to+\infty} \Psi_{\xi,\varphi}^{q,t}(x, r) = 0 \qquad \text{if} \quad \gamma + \beta(q) + \alpha < 0.$$

We assume that

$$0 < \lim_{n\to+\infty} \Psi_{\xi,\varphi}^{q,t}(x, r) < \infty$$

and then from Corollary 1, we have

$$\eta^{\#} \llcorner E \sim \mathcal{H}_\varphi^{q,\alpha} \llcorner E \sim \mathsf{P}_\varphi^{q,\alpha} \llcorner E.$$

Example 4. *Let* $A = \{a, b\}$ *be a two-letter alphabet, and* A^* *be the free monoid generated by A. Let F be the homomorphism on* A^**, defined by* $F(a) = ab$ *and* $F(b) = a$*. It is easy to see that* $F^n(a) = F^{n-1}(a)F^{n-2}(a)$*. We denote by* $|F^n(a)|$ *the length of the word* $F^n(a)$*, thus*

$$F^n(a) = s_1 s_2 \cdots s_{|F^n(a)|}, \quad s_i \in A.$$

Therefore, as $n \to +\infty$*, we get the infinite sequence*

$$\omega = \lim_{n\to+\infty} F^n(a) = s_1 s_2 s_3 \cdots s_n \cdots \in \{a, b\}^{\mathbb{N}}$$

which is called the Fibonacci sequence. For any $n \geq 1$*, write* $\omega_n = \omega|_n = s_1 s_2 \cdots s_n$*. We denote by* $|\omega_n|_a$ *the number of the occurrence of the letter a in* ω_n*, and* $|\omega_n|_b$ *the number of the occurrence of b. Then* $|\omega_n|_a + |\omega_n|_b = n$*. It follows that* $\lim_{n\to+\infty} \frac{|\omega_n|_a}{n} = \gamma$*, where* $\gamma^2 + \gamma = 1$.

Let $0 < r_a < \frac{1}{2}, 0 < r_b < \frac{1}{3}, r_a, r_b \in \mathbb{R}$*. In the Moran construction above, let*

$$n_k = \begin{cases} 2, & \text{if } s_k = a, \\ 3, & \text{if } s_k = b, \end{cases}$$

$$c_{k_j} = c_k = \begin{cases} r_a, & \text{if } s_k = a \\ r_b, & \text{if } s_k = b \end{cases}, \quad 1 \le j \le n_k.$$

Then we construct the homogeneous Moran set relating to the Fibonacci sequence and denote it by $E := E(\omega) = (I, \{n_k\}, \{c_k\})$*. By the construction of* E*, we have*

$$|I_\sigma| = r_a^{|\omega_k|_a} r_b^{|\omega_k|_b}, \quad \forall \sigma \in \Sigma_k,$$

and define a measure η *supported by* E *such that for any* $k \ge 1$ *and* $\sigma_0 \in \Sigma_k$*,*

$$\eta(I_{\sigma_0}) = \frac{\left|I_{\sigma_0}\right|^\alpha}{\sum_{\sigma \in D_k} |I_\sigma|^\alpha}, \quad \text{for all } \alpha \in \mathbb{R}.$$

Let $\alpha = \frac{-\log 2 - \gamma \log 3}{\log r_a + \gamma \log r_b}$*, where* $\gamma^2 + \gamma = 1$*. It is clear that there exists a positive constant* c *such that*

$$\lim_{n\to+\infty} \frac{\nu_\alpha(I_n(w))}{|I_n(w)|^s} = c \lim_{n\to+\infty} |I_n(w)|^{\alpha - s} \quad \text{for all} \quad s \in \mathbb{R}.$$

Now for $|I_\sigma| = r$*, we consider the functions* $\tau(x, r)) = -\log r$ *and* $\varphi(r) = \log r$*. Then*

$$\lim_{n\to+\infty} \Psi_{\xi,\varphi}^{q,t}(x, r) = \lim_{n\to+\infty} \eta(I_\sigma)|I_\sigma|^{-(q+t)} = \begin{cases} 0 & \text{if } t + s + q < 0, \\ \infty & \text{if } t + s + q > 0. \end{cases}$$

In particular, for $q = 0$*, assume that*

$$0 < \lim_{n\to+\infty} \Psi_{\xi,\varphi}^{q,s}(x, r) < \infty$$

and then from Corollary 1, we have

$$\eta^{\#} \llcorner E \sim \mathcal{H}_\varphi^s \llcorner E \sim \mathsf{P}_\varphi^s \llcorner E.$$

6. Conclusions

In this work, we introduced a general multifractal framework based on the Hewitt–Stromberg measures and their vector-valued extensions. The proposed approach unifies and extends classical Hausdorff and packing formalisms, providing new tools for studying the local regularity of measures and functions. Applications to binomial and Moran sets confirmed the consistency of the model and the validity of the associated multifractal formalism through Legendre transforms.

Author contributions

All authors contributed equally to the preparation of this manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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