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Research article

Exploring the fractional Volterra-Fredholm integro-differential equation: An iterative approach

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Abstract: In this paper, we construct a new four-step iterative method and show that our newly designed scheme converges faster than a number of iterative methods. We corroborate our claims by performing numerical experiments. We analyze the strong convergence result to approximate the fixed point of a contractive-like mapping and establish the weak ω^2-stability of our scheme. Further, strong and weak convergence results are incorporated for a generalized α-Reich-Suzuki nonexpansive mapping under some mild assumptions. We also illustrate numerical examples to validate our theoretical claims. Finally, we set forth our scheme to explore a Caputo-type nonlinear fractional Volterra-Fredholm integro-differential equation and a fractional diffusion equation.

Keywords: contractive-like mapping; G_α-RSNEM; fixed point; ω^2-stability; nonlinear fractional Volterra-Fredholm integro-differential equation
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1. Introduction

In past few years, numerical approximation of fixed points for nonlinear operators has become a vital research area because fixed-point theory has come up with indispensable tools for analyzing mathematical functions with underlying structures. Much attention has been paid by researchers to explore fixed points by implementing diverse innovative methodologies and approaches. Fixed-point theory provides a unified framework to explore several significant and consequential real-world problems, including variational inequalities, equilibrium problems, zeros of monotone mappings, ODEs, PDEs, computer simulations, image restoration, mathematical modelling, the mathematics of fractals, etc. Due to its significance, several approaches have been taken to broaden this research field.

A stochastic generalized form of the classical fixed-point theorem is the random fixed-point theorem. Spacek [45] and Hans [18] established random fixed-point theorems involving random contraction mappings on a separable complete metric space. Later, the fixed-point theorems of Banach and Schauder were investigated by Bharucha-Reid [10] in the stochastic form. Following this, random fixed-point theorems involving multi-valued contraction were explored by Itoh [23]. Suzuki [48] reported a generalized version of Banach's contraction principle and the fixed-point results of Nadler in metric spaces. Very recently, Chayut Kongban et al. [25] investigated common random fixed-point theorems of Suzuki type random multi-functions on a Polish space. The well-posedness and stability of fixed-point problems are versatile tools, which are used in various mathematical and engineering applications. The well-posedness is established in the case that the problem has a unique fixed point while the operators involved are multi-valued. For more details and applications of well-posedness and stability, we refer to [54, 55].

On the other hand, one of the vital aspects is to investigate the fixed-points. Figuring out fixed points using direct methods for nonlinear mapping is a challenging task. The fundamental and central fact in fixed-point theory that the fixed point remains unvarying under certain mappings is crucial. In fact, the solution of several mathematical models depends on the fixed point of a convergent iterative procedure. In the history of nonlinear analysis, several iterative methods have been based on the Banach contraction principle for approximating fixed points. However, in the case of a nonexpansive mapping, such a claim becomes valid. Moreover, if a nonexpansive mapping $\mathbb{G}$ has a fixed point, the Picard iterative method fails to estimate it.

Consider a closed convex subset $\mathbb{C} \neq \emptyset$ of a Banach space $\mathbb{B}$. A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ is called a contraction (respectively, nonexpansive), if $\exists \tau \in [0, 1)$ (respectively, $\tau = 1$) that complies with

$$\|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq \tau\|\sigma - \varsigma\|, \forall \sigma, \varsigma \in \mathbb{C},$$

and $F_{\mathbb{G}} =: \{\sigma \in \mathbb{C} : \mathbb{G}(\sigma) = \sigma\}$ denotes the set of fixed points of $\mathbb{G}$. It was Berinde [8] who coined the term weak contraction (WC) or almost contraction (ACM). A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ is referred to as WC if for all $\sigma, \varsigma \in \mathbb{C}$, $\exists \tau \in [0, 1)$ and some constant $\mu \geq 0$ such that

$$\|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq \tau\|\sigma - \varsigma\| + \mu\|\varsigma - \mathbb{G}\sigma\|. \tag{1.1}$$

More precisely, the author established the following theorem to guarantee the existence of a fixed point.

Theorem 1.1. *Let $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ be a mapping satisfying (1.1) and complying with the following relation:*

$$\|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq \tau\|\sigma - \varsigma\| + \mu\|\sigma - \mathbb{G}\sigma\|. \tag{1.2}$$

Then $\mathbb{G}$ has a unique fixed point.

The mapping defined in (1.2) was generalized further by Imoru and Olantiwo [21] by involving a monotonic increasing function which is expressed below:

Definition 1.1. *A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ is called contractive-like, if a strictly increasing continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $\tau \in [0, 1)$ satisfies*

$$\|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq g(\|\sigma - \mathbb{G}\sigma\|) + \tau\|\sigma - \varsigma\|, \forall \sigma, \varsigma \in \mathbb{C}. \tag{1.3}$$

The contractive condition (1.3) is more general and include several contractive conditions, see, [9, 19, 35, 40, 41]. If $gu = \kappa u$, where $\kappa \geq 0$ then (1.3) is identical to (1.2) and if $\kappa u = 0$, then (1.3) turns into

$$\|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq \tau\|\sigma - \varsigma\|, \tau \in [0,1), \forall \sigma, \varsigma \in \mathbb{C}. \tag{1.4}$$

If $\kappa = m\tau, m = (1-\tau)^{-1}, 0 \leq \tau < 1$, we acquire the contractive condition due to Rhoades [41].

In recent years, fixed points for nonexpansive mappings and their generalized versions have received a lot of attention for the reason that numerous concrete and pertinent problems, including convex feasibility and optimization problems, signal processing, control theory, game theory, and fractional integral and differential equations, can be examined by transforming them into fixed-point problems of nonexpansive mappings. It was Browder [11] who determined the fixed point of a nonexpansive mapping for the first time. Later, Goebel and Kirk [14] reported the results of Browder [11] in reflexive Banach spaces. So far, various extensions of nonexpansive mappings have been achieved. One of such worthwhile generalization is from Suzuki [47], which is called Suzuki's generalized nonexpansive mapping, defined by defining Condition (C) as given below:

A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ satisfies Condition (C) if

$$\frac{1}{2}\|\sigma - \mathbb{G}\sigma\| \leq \|\sigma - \varsigma\| \Rightarrow \|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq \|\sigma - \varsigma\|, \forall \sigma, \varsigma \in \mathbb{C}. \tag{1.5}$$

It seems obvious that a nonexpansive mapping obeys Condition (C); however, the converse is not true in general (see [47]). Recently, Pant and Pandey [37] introduced a generalized α-Reich-Suzuki nonexpansive mapping (G_α-RSNEM) by combining a generalized α-nonexpansive mapping and Reich-Suzuki nonexpansive mapping which is defined as follows:

Definition 1.2. *A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ is called a G_α-RSNEM if, for all $\sigma, \varsigma \in \mathbb{C}, \exists \alpha \in [0,1)$,*

$$\frac{1}{2}\|\sigma - \mathbb{G}\sigma\| \leq \|\sigma - \varsigma\| \Rightarrow \|\mathbb{G}\sigma - \mathbb{G}\varsigma\| \leq \alpha\|\sigma - \mathbb{G}\sigma\| + \alpha\|\varsigma - \mathbb{G}\varsigma\| + (1-2\alpha)\|\sigma - \varsigma\|. \tag{1.6}$$

Clearly, the mapping satisfying Condition (C) is identical to that of (1.6) with $\alpha = 0$; however, the converse implication need not be true in general (see [37]).

On the other hand, constructing an iterative scheme to estimate the fixed point is crucial. Some of the commonly used schemes for estimating fixed points are those of Mann [31], Ishikawa [22], Noor [33], S-iteration [2], M-iteration [51], etc. In 2020, Ali and Ali [5] designed a new iterative scheme that converged faster than some key iterative schemes such as [2, 6, 38, 49], which is presented below:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}v_k, \\ v_k = \mathbb{G}w_k, \\ w_k = \mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k], \end{cases} \tag{1.7}$$

where $\{\alpha_k\}_{k=1}^{\infty}$ lies in $(0,1)$. Further, Ali et al. [7] proposed a faster iteration procedure called the D^{**}-iteration process for approximating the fixed point of a G_α-RSNEM, which is expressed as follows:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}^2 v_k, \\ v_k = \mathbb{G}[(1-\alpha_k)\mathbb{G}u_k + \alpha_k \mathbb{G}w_k], \\ w_k = \mathbb{G}[(1-\beta_k)u_k + \beta_k \mathbb{G}u_k], \end{cases} \tag{1.8}$$

where $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ lies in $(0, 1)$. Recently, Hammad et al. [16] designed a four-step iterative method called HR*-iteration to quantify the fixed points of ACM and generalized nonexpansive mappings as follows:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = (1-\alpha_k)v_k + \alpha_k \mathbb{G}v_k, \\ v_k = \mathbb{G}^2 w_k, \\ w_k = \mathbb{G}[(1-\beta_k)x_k + \beta_k \mathbb{G}x_k], \\ x_k = [(1-\gamma_k)u_k + \gamma_k \mathbb{G}u_k], \end{cases} \tag{1.9}$$

where $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}$, and $\{\gamma_k\}_{k=1}^{\infty}$ lie in $(0, 1)$. They claimed, by numerical experiments, that their scheme is faster than that of [49] and others. After this, Ofem et al. [34] examined an A^*-iterative scheme in hyperbolic space as follows:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}v_k, \\ v_k = \mathbb{G}w_k, \\ w_k = \mathbb{G}[(1-\alpha_k)x_k + \alpha_k \mathbb{G}x_k] \\ x_k = \mathbb{G}[(1-\beta_k)u_k + \beta_k \mathbb{G}u_k], \end{cases} \tag{1.10}$$

where $\{\alpha_k\}_{k=1}^{\infty}$ lies in $(0, 1)$. The authors reckoned the fixed point for a contractive-like mapping and also established the stability of proposed scheme. Recently, Hasanen et al. [17] presented an HR-iterative scheme as follows:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}v_k, \\ v_k = \mathbb{G}[(1-\alpha_k)\mathbb{G}w_k + \alpha_k \mathbb{G}w_k], \\ w_k = \mathbb{G}[(1-\beta_k)x_k + \beta_k \mathbb{G}x_k] \\ x_k = \mathbb{G}[(1-\gamma_k)u_k + \gamma_k \mathbb{G}u_k], \end{cases} \tag{1.11}$$

where $\{\alpha_k\}_{k=1}^{\infty}$ lies in $[0, 1]$. The authors reckoned the fixed point and data-dependence for an ACM and also established the stability of the proposed scheme. Very recently, Alam and Rohen [4] introduced a new method for reckoning the fixed point of a contraction mapping as shown below:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}^2 v_k, \\ v_k = \mathbb{G}[(1-\alpha_k)w_k + \alpha_k \mathbb{G}w_k], \\ w_k = \mathbb{G}[(1-\beta_k)x_k + \alpha_k \mathbb{G}x_k], \\ x_k = \mathbb{G}[(1-\gamma_k)u_k + \gamma_k \mathbb{G}u_k], \end{cases} \tag{1.12}$$

for $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}$, and $\{\gamma_k\}_{k=1}^{\infty} \subseteq (0, 1)$. The authors corroborated that their scheme converges faster than those considered in [2, 7, 49] and many more. Very recently, Filali et al. [13] proposed and examined the following four-step iterative scheme:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}[(1-\alpha_k)\mathbb{G}w_k + \alpha_k \mathbb{G}v_k], \\ v_k = \mathbb{G}[(1-\beta_k)w_k + \beta_k \mathbb{G}x_k], \\ w_k = \mathbb{G}^2 x_k, \\ x_k = \mathbb{G}[(1-\gamma_k)u_k + \gamma_k \mathbb{G}u_k], \end{cases} \tag{1.13}$$

for $\{\alpha_k\}_{k=1}^{\infty}$, $\{\beta_k\}_{k=1}^{\infty}$, and $\{\gamma_k\}_{k=1}^{\infty} \subseteq (0,1)$.

Motivated by the research work shown above and recent research trends, we suggest and construct an efficient four-step iterative algorithm as shown below:

$$\begin{cases} u_1 \in \mathbb{C}, \\ u_{k+1} = \mathbb{G}^2 v_k, \\ v_k = \mathbb{G}^2 w_k, \\ w_k = \mathbb{G}^2 x_k, \\ x_k = \mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k], \end{cases} \tag{1.14}$$

where $\{\alpha_k\}_{k=1}^{\infty}$ lies in $(0,1)$. The goal of this paper is accomplished in the following order.

The strong convergence of the proposed scheme for the contractive-like mapping is presented in the second section. The weak ω^2-stability of the iterative method considered here is proved and numerical experiments are performed to compare its rate of convergence with that of some existing methods for contraction mapping. Furthermore, the subsection in this section begins with some definitions and fundamental tools which concludes with strong and weak convergence results for the G_α-RSNEM. The third section is devoted to the implications and significance of our method. We employed our scheme to investigate a Caputo-type nonlinear fractional Volterra-Fredholm integro-differential equation (C-NFVFIDE) and a fractional diffusion model. At the end, concluding remarks and expected future research endeavors are outlined.

2. Convergence and stability results

In this section, we provide the convergence results and weak ω^2-stability and perform numerical experiments for the efficiency of our new proposed scheme.

2.1. Strong convergence and stability for contractive-like mappings

Next, we study strong convergence for exploring the fixed points of a contractive-like mapping (1.3). We also show the weak ω^2-stability of the scheme (1.14) for the mapping considered.

Theorem 2.1. *Let $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ be a closed convex set and $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ satisfies (1.3) with $F_{\mathbb{G}} \neq \emptyset$. Then the sequence $\{u_k\}$ initiated by (1.14) converges to a unique element $u^* \in F_{\mathbb{G}}$.*

Proof. Assume $u^* \in F_{\mathbb{G}}$ and $u \in \mathbb{G}$. Since $\mathbb{G}$ satisfies (1.3) with $F_{\mathbb{G}} \neq \emptyset$,

$$\|\mathbb{G}u - u^*\| = \|\mathbb{G}u^* - \mathbb{G}u\| \leq g(\|u^* - \mathbb{G}u^*\|) + \tau\|u - u^*\| = \tau\|u - u^*\|. \tag{2.1}$$

By (1.14) and (2.1), we obtain

$$\begin{aligned}
\|x_k - u^*\| &= \|\mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\
&\le \tau\|[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\
&\le \tau[(1-\alpha_k)\|u_k - u^*\| + \alpha_k\|\mathbb{G}u_k - u^*\|] \\
&\le \tau[(1-\alpha_k)\|u_k - u^*\| + \alpha_k\tau\|u_k - u^*\|] \\
&= \tau[1-\alpha_k(1-\tau)]\|u_k - u^*\|.
\end{aligned}$$

$$\begin{aligned}
\|w_k - u^*\| &= \|\mathbb{G}^2 x_k - u^*\| \\
&= \|\mathbb{G}(\mathbb{G}x_k) - u^*\| \\
&\le \tau\|\mathbb{G}x_k - u^*\| \\
&\le \tau^2\|x_k - u^*\| \\
&\le \tau^3[1-\alpha_k(1-\tau)]\|u_k - u^*\|.
\end{aligned}$$

$$\begin{aligned}
\|v_k - u^*\| &= \|\mathbb{G}^2 w_k - u^*\| \\
&= \|\mathbb{G}(\mathbb{G}w_k) - u^*\| \\
&\le \tau\|\mathbb{G}w_k - u^*\| \\
&\le \tau^2\|w_k - u^*\| \\
&\le \tau^5[1-\alpha_k(1-\tau)]\|u_k - u^*\|.
\end{aligned}$$

$$\begin{aligned}
\|u_{k+1} - u^*\| &= \|\mathbb{G}^2 v_k - u^*\| \\
&= \|\mathbb{G}(\mathbb{G}v_k) - u^*\| \\
&\le \tau\|\mathbb{G}v_k - u^*\| \\
&\le \tau^2\||v_k - u^*\| \\
&= \tau^7[1-\alpha_k(1-\tau)]\|u_k - u^*\|.
\end{aligned}$$

Since $0 \le \tau < 1$ and $0 < \alpha_k < 1$, thus $1 - \alpha_k(1-\tau) < 1$ and hence $\|u_{k+1} - u^*\| \le \tau^7\|u_k - u^*\|$.

By induction, we can write

$$\|u_{k+1} - u^*\| \le \tau^{7k}\|u_1 - u^*\| \to 0 \text{ as } k \to \infty.$$

Hence, $u_k \to u^* \in F_{\mathbb{G}}$. To ensure the uniqueness of u^*, we take $u_1^*, u_2^* \in \mathbb{C}$ such that $u_1^* \ne u_2^*$ and $u_1^*, u_2^* \in F_{\mathbb{G}}$. Then

$$\begin{aligned}
\|u_1^* - u_2^*\| &= \|\mathbb{G}u_1^* - \mathbb{G}u_2^*\| \\
&\le g(\|u_1^* - \mathbb{G}u_1^*\|) + \tau\|u_1^* - u_2^*\| \qquad (2.2) \\
&= \tau\|u_1^* - u_2^*\|.
\end{aligned}$$

Since $0 \le \tau < 1$, then (2.2) gives $\|u_1^* - u_2^*\| = 0$ and, consequently, $u_1^* = u_2^*$. □

In order to achieve the weak ω^2-stability, the following results are crucial.

Definition 2.1. *[12] Two sequences $\{t_k\}$ and $\{u_k\}$ are known as equivalent if*

$$\lim_{k\to\infty} \|t_k - u_k\| = 0.$$

Definition 2.2. *[50] Let $\{t_k\} \subset \mathbb{C}$ be an arbitrary sequence equivalent to $\{u_k\}$ which is defined by $u_{k+1} = \Lambda(\mathbb{G}, u_k), k \geq 1$ such that $u_k \to u^* \in F_{\mathbb{G}}$. Then $\{u_k\}$ is called weak ω^2-stable with respect to $\mathbb{G}$ if*

$$\lim_{k\to\infty} \|t_{k+1} - \Lambda(\mathbb{G}, t_k)\| = 0 \Leftrightarrow \lim_{k\to\infty} t_k = u^*.$$

Theorem 2.2. *Suppose that $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ is a closed convex set and the mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ satisfies (1.3). If $u^* \in F_{\mathbb{G}}$, then the sequence $\{u_k\}$ initiated by the scheme (1.14) is ω^2-stable with respect to $\mathbb{G}$.*

Proof. Assume that $\{t_k\} \subset \mathbb{C}$ is equivalent to $\{u_k\}$ initiated by the scheme (1.14). Let $\varepsilon_k \in \mathbb{R}^+$ be defined as follows:

$$\begin{cases} t_1 \in \mathbb{C}, \\ r_k = \mathbb{G}[(1-\alpha_k)t_k + \alpha_k \mathbb{G} t_k], \\ p_k = \mathbb{G}^2 r_k, \\ q_k = \mathbb{G}^2 p_k, \\ \varepsilon_k = \|t_{k+1} - \mathbb{G}^2 q_k\|, \end{cases} \tag{2.3}$$

where $\{\alpha_k\}$ lies in $(0, 1)$. To substantiate the ω^2-stability of the scheme (1.14), we find that $\lim_{k\to\infty} \varepsilon_k = 0$ implies $\lim_{k\to\infty} t_k = u^*$. Assume that $\lim_{k\to\infty} \varepsilon_k = 0$. Setting $s_k = (1-\alpha_k)t_k + \alpha_k \mathbb{G} t_k$ and $z_k = (1-\alpha_k)u_k + \alpha_k \mathbb{G} u_k$ and utilizing (1.14) and (2.3), we obtain

$$\begin{aligned} \|z_k - s_k\| &= \|(1-\alpha_k)u_k + \alpha_k \mathbb{G} u_k - [(1-\alpha_k)t_k + \alpha_k \mathbb{G} t_k]\| \\ &\leq (1-\alpha_k)\|u_k - t_k\| + \alpha_k \|\mathbb{G} u_k - \mathbb{G} t_k\| \\ &\leq (1-\alpha_k)\|u_k - t_k\| + g(\|u_k - \mathbb{G} u_k\|) + \tau \|u_k - t_k\| \\ &= [1 - \alpha_k(1-\tau)]\|u_k - t_k\|. \end{aligned} \tag{2.4}$$

Again by (1.14), (2.3), and (2.4), if follows that

$$\begin{aligned} \|x_k - r_k\| &= \|\mathbb{G} z_k - \mathbb{G} s_k]\| \\ &\leq g(\|z_k - \mathbb{G} z_k\|) + \tau \|z_k - s_k\| \\ &= \tau \|z_k - s_k\| \\ &\leq \tau[1 - \alpha_k(1-\tau)]\|u_k - t_k\|. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \|w_k - p_k\| &= \|\mathbb{G}^2 x_k - \mathbb{G}^2 r_k\| \\ &= \|\mathbb{G}(\mathbb{G} x_k) - \mathbb{G}(\mathbb{G} r_k)\| \\ &\leq g(\|\mathbb{G} x_k - \mathbb{G}(\mathbb{G} x_k)\|) + \tau \|\mathbb{G} x_k - \mathbb{G} r_k\| \\ &= \tau \|\mathbb{G} x_k - \mathbb{G} r_k\| \\ &\leq \tau[g(\|x_k - \mathbb{G} x_k\|) + \tau \|x_k - r_k\|] \\ &= \tau^2 \|x_k - r_k\| \\ &\leq \tau^3[1 - \alpha_k(1-\tau)]\|u_k - t_k\|. \end{aligned} \tag{2.6}$$

$$\begin{aligned}
\|v_k - q_k\| &= \|\mathbb{G}^2 w_k - \mathbb{G}^2 p_k\| \\
&= \|\mathbb{G}(\mathbb{G}w_k) - \mathbb{G}(\mathbb{G}p_k)\| \\
&\leq g(\|\mathbb{G}w_k - \mathbb{G}(\mathbb{G}w_k)\|) + \tau\|\mathbb{G}w_k - \mathbb{G}p_k\| \\
&= \tau\|\mathbb{G}w_k - \mathbb{G}p_k\| \\
&\leq \tau[g(\|w_k - \mathbb{G}w_k\|) + \tau\|w_k - p_k\|] \\
&= \tau^2\|w_k - p_k\| \\
&\leq \tau^5[1 - \alpha_k(1 - \tau)]\|u_k - t_k\|.
\end{aligned} \tag{2.7}$$

By recalling the triangle inequality and applying (1.14), (2.3), and (2.7), we obtain

$$\begin{aligned}
\|t_{k+1} - u^*\| &\leq \|t_{k+1} - u_{k+1}\| + \|u_{k+1} - u^*\| \\
&\leq \|t_{k+1} - \mathbb{G}^2 q_k\| + \|\mathbb{G}^2 q_k - u_{k+1}\| + \|u_{k+1} - u^*\| \\
&= \|t_{k+1} - \mathbb{G}^2 q_k\| + \|\mathbb{G}(\mathbb{G}q_k) - \mathbb{G}(\mathbb{G}v_k)\| + \|u_{k+1} - u^*\| \\
&\leq \varepsilon_k + g(\|\mathbb{G}q_k - \mathbb{G}(\mathbb{G}q_k)\|) + \tau\|\mathbb{G}q_k - \mathbb{G}v_k\| + \|u_{k+1} - u^*\| \\
&= \varepsilon_k + \tau\|\mathbb{G}q_k - \mathbb{G}v_k\| + \|u_{k+1} - u^*\| \\
&\leq \varepsilon_k + \tau[g(\|q_k - \mathbb{G}q_k\|) + \tau\|q_k - v_k\|] + \|u_{k+1} - u^*\| \\
&\leq \varepsilon_k + \tau^2\|q_k - v_k\| + \|u_{k+1} - u^*\| \\
&\leq \varepsilon_k + \tau^7[1 - \alpha_k(1 - \tau)]\|u_k - t_k\| + \|u_{k+1} - u^*\|.
\end{aligned} \tag{2.8}$$

It is shown in Theorem 2.1 that $\lim_{k\to\infty} u_k = u^*$, and hence $\lim_{k\to\infty} \|u_{k+1} - u^*\| = 0$. Moreover, the equivalence of the sequences $\{u_k\}$ and $\{t_k\}$ leads to $\lim_{k\to\infty} \|u_k - t_k\| = 0$. Taking the limit on both sides, (2.8) turns into $\lim_{k\to\infty} t_k = u^*$. Thus, the scheme (1.14) is ω^2-stable with respect to $\mathbb{G}$. □

Next, we provide the following examples to establish a comparative analysis of the rate of convergence of the schemes (1.8), (1.12), (1.13), and (1.14). It is manifested that our scheme (1.14) converges faster for a contraction mapping.

Example 2.1. *Let $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ be a contraction mapping defined by*

$$\mathbb{G}(\sigma) = \sqrt{\sigma^2 - 4\sigma + 32}, \forall \sigma \in \mathbb{C} = [1, 100].$$

Consider the sequences $\{\alpha_k\}, \{\beta_k\}$, and $\{\gamma_k\}$ in $(0, 1)$ defined by

$$\alpha_k = \frac{2k}{(5k+1)^2}, \beta_k = \frac{k+2}{(3k^2+1)}, \gamma_k = \frac{k+1}{(k+2)}$$

with the initial guess $\sigma_1 = 45$ and the stop criteria $\|\sigma_n - \sigma\| < 10^{-4}$. The comparison of the number of iterations and the fixed points of different schemes are reported in Table 1 and presented in Figure 1.

Table 1. Comparison of iterations.

Steps	Filali et al.	HR*-iteration scheme	A*-iteration scheme	Hammad et al.	Alam and Rohen	Scheme (1.14)
1	44.6509	43.6956	41.8916	40.2411	39.1502	38.1980
5	25.3614	20.4281	17.3387	12.4784	10.0721	8.6646
10	10.0441	8.2642	8.1036	8.0059	8.0005	8.0000
12	8.4546	8.0286	8.0103	8.0003	8.0000	8.0000
15	8.0359	8.0010	8.0003	8.0000	8.0000	8.0000
18	8.0027	8.0000	8.0000	8.0000	8.0000	8.0000
23	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
25	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000

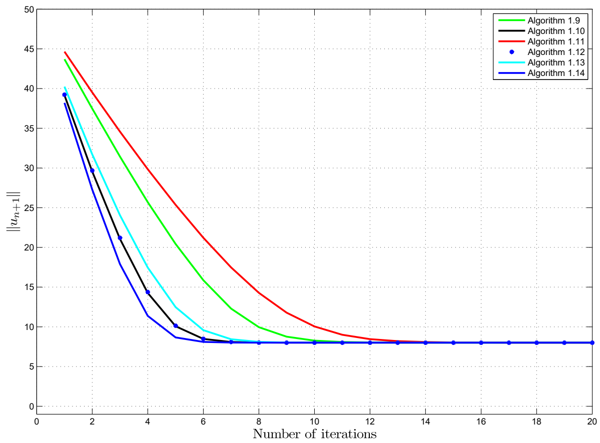

Figure 1. Convergence behavior of the schemes (1.8), (1.12), (1.13), and (1.14).

Example 2.2. *Let* $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ *be a contraction mapping defined by*

$$\mathbb{G}(\sigma) = \frac{3\sigma + 8}{5}, \forall \sigma \in \mathbb{C} = [2, 10].$$

Consider the sequences $\{\alpha_k\}, \{\beta_k\}$*, and* $\{\gamma_k\}$ *in* $(0, 1)$ *defined by*

$$\alpha_k = \frac{2k}{(4k+1)}, \beta_k = \frac{4k^2 + k}{(3k+1)^2}, \gamma_k = \frac{2k}{(7k+3)}$$

with the initial guess $\sigma_1 = 10$ *and the stop criteria* $\|\sigma_n - \sigma\| < 10^{-4}$*. The comparison of the number of iterations and the fixed points of different schemes are reported in Table 2 and presented in Figure 2.*

Table 2. Comparison of iterations.

Steps	Filali et al.	HR*	Alam and Rohen	New scheme
1	7.3124	6.4781	4.6245	4.4379
2	4.6972	4.3697	4.0294	4.0121
3	4.1478	4.0538	4.0014	4.0003
4	4.0315	4.0077	4.0001	4.0000
5	4.0067	4.0011	4.0000	4.0000
6	4.0014	4.0002	4.0000	4.0000
7	4.0003	4.0000	4.0000	4.0000
8	4.0001	4.0000	4.0000	4.0000
9	4.0000	4.0000	4.0000	4.0000
10	4.0000	4.0000	4.0000	4.0000

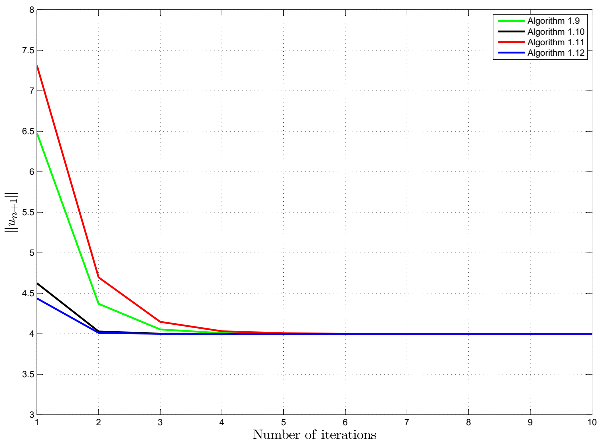

Figure 2. Convergence behavior of the schemes (1.8), (1.12), (1.13), and (1.14).

Next, we give the example below to corroborate the theoretical result in Theorem 2.1 for a contractive-like mapping.

Example 2.3. *Let* $\mathbb{B} = \mathbb{R}$ *and* $\mathbb{C} = [0, 6]$ *with usual norm. Define* $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ *as follows:*

$$\mathbb{G}(u) = \begin{cases} \frac{u}{4}, & u \in [0, 4) \\ \frac{u}{8}, & u \in [4, 6]. \end{cases} \tag{2.9}$$

Evidently, $0 \in F_{\mathbb{G}}$. *The discontinuity of* $\mathbb{G}$ *at* $4 \in [0, 6]$ *ensures that* $\mathbb{G}$ *is not a contraction. Define* $g : [0, \infty) \to [0, \infty)$ *such that* $g(u) = \dfrac{u}{6}$. *Then* g *is strictly increasing continuous mapping with* $g(0) = 0$. *Next, we prove that* $\mathbb{G}$ *satisfies (1.3), i.e.,*

$$||\mathbb{G}u - \mathbb{G}v|| \leq g(||u - \mathbb{G}u||) + \tau||u - v||, \forall u, v \in \mathbb{C}, \tau \in [0, 1). \tag{2.10}$$

If $u \in [0, 4)$, *then*

$$||u - \mathbb{G}u|| = \left|u - \frac{u}{4}\right| = \left|\frac{3u}{4}\right| \tag{2.11}$$

and

$$g\Big(\frac{3u}{4}\Big) = \frac{u}{8}. \tag{2.12}$$

If $u \in [4, 6]$*, then*

$$\|u - \mathbb{G}u\| = \Big|u - \frac{u}{8}\Big| = \Big|\frac{7u}{8}\Big| \tag{2.13}$$

and

$$g\Big(\frac{7u}{8}\Big) = \frac{7u}{48}. \tag{2.14}$$

According to the consideration above, the following cases arise.
Case C_1**:** *If* $u, v \in [0, 4)$*, then the first formulation of (2.9) yields*

$$\begin{aligned}
\|\mathbb{G}u - \mathbb{G}v\| &= \Big|\frac{u}{4} - \frac{v}{4}\Big| = \frac{1}{4}|u - v| \\
&\leq \Big|\frac{u}{8}\Big| + \frac{1}{4}|u - v| \\
&= g\Big(\frac{3u}{4}\Big) + \frac{1}{4}|u - v| \\
&= g(\|u - \mathbb{G}u\|) + \frac{1}{4}|u - v|.
\end{aligned}$$

Case C_2**:** *If* $u \in [0, 4)$ *and* $v \in [4, 6]$*, then from (2.9), we can state that*

$$\begin{aligned}
\|\mathbb{G}u - \mathbb{G}v\| &= \Big|\frac{u}{4} - \frac{v}{8}\Big| = \Big|\frac{u}{8} + \frac{u}{8} - \frac{v}{8}\Big| \\
&\leq \Big|\frac{u}{8}\Big| + \frac{1}{8}|u - v| \\
&\leq \Big|\frac{u}{8}\Big| + \frac{1}{4}|u - v| \\
&= g\Big(\frac{3u}{4}\Big) + \frac{1}{4}|u - v| \\
&= g(\|u - \mathbb{G}u\|) + \frac{1}{4}|u - v|.
\end{aligned}$$

Case C_3**:** *If* $u \in [4, 6]$ *and* $v \in [0, 4)$*, then again from (2.9), we obtain*

$$\begin{aligned}
\|\mathbb{G}u - \mathbb{G}v\| &= \Big|\frac{u}{8} - \frac{v}{4}\Big| = \Big|\frac{u}{4} - \frac{u}{8} - \frac{v}{4}\Big| \\
&\leq \Big|\frac{u}{8}\Big| + \frac{1}{4}|u - v| \\
&\leq \Big|\frac{7u}{48}\Big| + \frac{1}{4}|u - v| \\
&= g(\|u - \mathbb{G}u\|) + \frac{1}{4}|u - v|.
\end{aligned}$$

Case C_4**:** *If* $u, v \in [4, 6]$*, then from the second formulation of (2.9), we obtain*

$$\begin{aligned}\|\mathbb{G}u - \mathbb{G}v\| &= \left|\frac{u}{8} - \frac{v}{8}\right| \\ &\leq \left|\frac{7u}{48}\right| + \frac{1}{8}|u - v| \\ &\leq \left|\frac{7u}{48}\right| + \frac{1}{4}|u - v| \\ &= g(\|u - \mathbb{G}u\|) + \frac{1}{4}|u - v|.\end{aligned}$$

Thus, for all possible cases (C_1)*–*(C_4)*,* $\mathbb{G}$ *satisfies (1.3), i.e.,* $\mathbb{G}$ *is a contractive-like mapping with* $\tau = \frac{1}{4}$*,* $g(u) = \frac{u}{6}$*, and* $0 \in F_{\mathbb{G}}$*. Numerically, for* $\alpha_k = \beta_k = \gamma_k = \frac{k}{k+1}, \forall k \in \mathbb{N}$ *for the different initial guesses* $u_0 = 3.9, 2.9, 1.9,$ *and* 0.8*, we have shown in Figure 3 that our scheme (1.14) converges to the unique fixed point 0 utilizing MATLAB R2015a.*

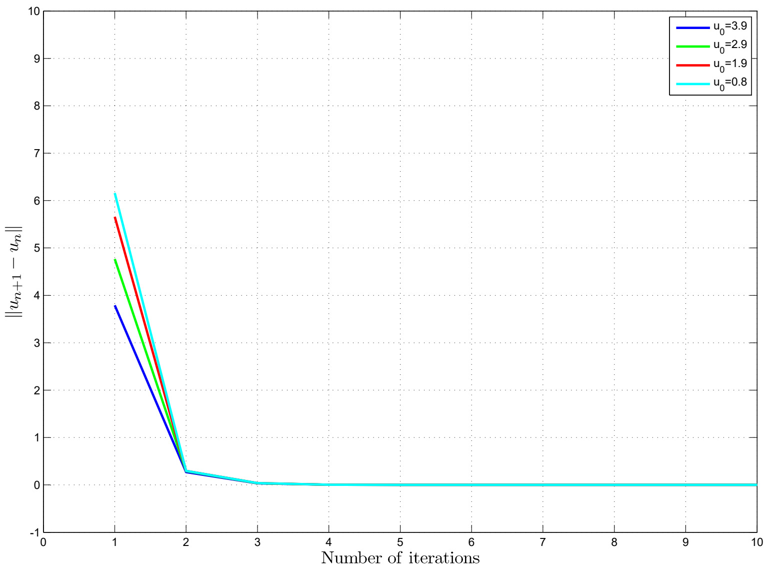

Figure 3. Convergence behavior of the schemes (1.8), (1.12), (1.13), and (1.14).

2.2. Generalized α*-Reich–Suzuki nonexpansive mapping (*G_α*-RSNEM)*

In this subsection, we examine weak and strong convergence for a G_α-RSNEM. To accomplish the goal of this subsection, the following results are crucial.

Definition 2.3. *[37] A Banach space* $\mathbb{B}$ *is referred to as uniformly convex if, for each* $\epsilon \in (0, 2], \exists \delta > 0$ *such that for* $\sigma, \varsigma \in \mathbb{B}$ *with* $\|\sigma\| \leq 1, \|\varsigma\| \leq 1$ *and* $\|\sigma - \varsigma\| > \epsilon \Rightarrow \left\|\frac{\sigma+\varsigma}{2}\right\| < 1 - \delta.$

Definition 2.4. *A Banach space* $\mathbb{B}$ *satisfies Opial's condition if, any sequence* $\{u_k\}$ *in* $\mathbb{B}$ *with* $u_k \rightharpoonup u \in \mathbb{B}$

implies

$$\limsup_{k\to\infty} \|u_k - u\| < \limsup_{k\to\infty} \|u_k - v\|, \forall v \in \mathbb{B} \text{ with } u \neq v.$$

Note that Opial's condition is a useful tool for approximating fixed points in uniformly convex Banach spaces. This guarantees the existence of a unique weak limit in uniformly convex Banach spaces. All Hilbert spaces and l^p spaces $(1 < p < \infty)$ are known to satisfy Opial's condition, while $L^p[0, 2\pi]$ with $1 < p \neq 2$ fails to satisfy it.

Definition 2.5. *Let $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ be a closed convex set. A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ is referred to as demiclosed with respect to $u \in \mathbb{B}$ if $u_k \rightharpoonup u \in \mathbb{C}$ and $\mathbb{G}u_k \to v$ implies $\mathbb{G}u = v$.*

Definition 2.6. *[43] A mapping $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ is known to satisfying Condition (I) if a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for all $c > 0, g(c) > 0$, satisfying*

$$\|u - \mathbb{G}u\| \geq g(d(u, F_{\mathbb{G}})), \forall u \in \mathbb{C},$$

where $d(u, F_{\mathbb{G}}) = \inf_{u^*\in F_{\mathbb{G}}} \|u - u^*\|$.

Let $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ be a closed convex set and $\{u_k\}$ be a bounded sequence in $\mathbb{B}$. For a given $u \in \mathbb{B}$, we write

$$r(u, \{u_n\}) = \limsup_{k\to\infty} \|u - u_k\|.$$

The asymptotic radius of $\{u_k\}$ relative to $\mathbb{C}$ is given as

$$r(\mathbb{C}, \{u_k\}) = \inf\{r(u, \{u_k\}) : u \in \mathbb{C}\},$$

and the asymptotic center $C(\mathbb{C}, \{u_k\})$ of $\{u_k\}$ is given as

$$C(\mathbb{C}, \{u_k\}) = \{u \in \mathbb{C} : r(u, \{u_k\}) = r(\mathbb{C}, \{u_k\})\}.$$

It is known that $C(\mathbb{C}, \{u_k\})$ consists of exactly one element if $\mathbb{B}$ is uniformly convex. The lemma given below from Schu [42] plays a crucial role in achieving the following results.

Lemma 2.1. *Let $\mathbb{B}$ be a uniformly convex Banach space, and the sequence $\{\alpha_k\}$ complies with $0 < u \leq \alpha_k \leq v < 1, \forall k \geq 1$. If, for some $\delta \geq 0$, the sequences $\{u_k\}$ and $\{v_k\}$ in $\mathbb{B}$ satisfy the relations*

$$\limsup_{k\to\infty} \|u_k\| \leq \delta,$$

$$\limsup_{k\to\infty} \|v_k\| \leq \delta,$$

and

$$\limsup_{k\to\infty} \|\alpha_k u_k + (1 - \alpha_k)v_k\| = \delta,$$

then $\lim_{k\to\infty} \|u_k - v_k\| = 0$.

Lemma 2.2. *[36] Let $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ and $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ be a G_α-RSNEM. Then*

$$\|\sigma - \mathbb{G}\varsigma\| \leq \Big(\frac{3 + \tau}{1 - \tau}\Big)\|\sigma - \mathbb{G}\sigma\| + \|\sigma - \varsigma\|, \forall \sigma, \varsigma \in \mathbb{C}, \tau \in (0, 1).$$

Theorem 2.3. *Let $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ be a closed convex set and $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ be a G_α-RSNEM with $F_{\mathbb{G}} \neq \emptyset$. If the sequence $\{u_k\}$ is produced by the scheme (1.14), then $\lim_{k\to\infty} \|u_k - u^*\|$ for each $u^* \in F_{\mathbb{G}}$.*

Proof. Suppose that $\mathbb{G}$ is a G_α-RSNEM with $F_{\mathbb{G}} \neq \emptyset$. Let $u^* \in F_{\mathbb{G}}$ and $u \in \mathbb{C}$. In this case, we have

$$\|\mathbb{G}u - \mathbb{G}u^*\| \leq \|u - u^*\|. \tag{2.15}$$

Now, it follows from scheme (1.14) and relation (2.15) that

$$\begin{aligned} \|x_k - u^*\| &= \|\mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\ &\leq \|[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\ &\leq (1-\alpha_k)\|u_k - u^*\| + \alpha_k\|\mathbb{G}u_k - u^*\| \\ &\leq (1-\alpha_k)\|u_k - u^*\| + \alpha_k\|u_k - u^*\| \\ &= \|u_k - u^*\|. \end{aligned} \tag{2.16}$$

Again, incorporating (2.15) and (2.16) into the scheme (1.14), we obtain

$$\begin{aligned} \|w_k - u^*\| &= \|\mathbb{G}^2 x_k - u^*\| \\ &= \|\mathbb{G}(\mathbb{G}x_k) - u^*\| \\ &\leq \|\mathbb{G}x_k - u^*\| \\ &\leq \|x_k - u^*\| \\ &\leq \|u_k - u^*\|. \end{aligned} \tag{2.17}$$

Like wise, we can write

$$\begin{aligned} \|v_k - u^*\| &= \|\mathbb{G}^2 w_k - u^*\| \\ &= \|\mathbb{G}(\mathbb{G}w_k) - u^*\| \\ &\leq \|\mathbb{G}w_k - u^*\| \\ &\leq \|w_k - u^*\| \\ &\leq \|u_k - u^*\|. \end{aligned} \tag{2.18}$$

Finally, it follows from (2.15), (2.18), and (1.14) that

$$\begin{aligned} \|u_{k+1} - u^*\| &= \|\mathbb{G}^2 v_k - u^*\| \\ &= \|\mathbb{G}(\mathbb{G}v_k) - u^*\| \\ &\leq \|\mathbb{G}v_k - u^*\| \\ &\leq \|v_k - u^*\| \\ &\leq \|u_k - u^*\|. \end{aligned} \tag{2.19}$$

Thus $\{\|u_k - u^*\|\}$ is a decreasing sequence bounded below, and hence for each $u^* \in F_{\mathbb{G}}$, $\lim_{k\to\infty} \|u_k - u^*\|$ exists. □

Theorem 2.4. *Suppose that all the assumptions of Theorem 2.3 are fulfilled. Then $F_{\mathbb{G}} \neq \emptyset$ if and only if the sequence $\{u_k\}$ produced by the scheme (1.14) is bounded and $\lim_{k\to\infty} \|\mathbb{G}u_k - u_k\| = 0$.*

Proof. It is evident from Theorem 2.3 that $\{u_k\}$ is bounded and that for each $u^* \in F_{\mathbb{G}}$, $\lim_{k\to\infty} \|u_k - u^*\|$ exists. We can then assume

$$\lim_{k\to\infty} \|u_k - u^*\| = \vartheta. \tag{2.20}$$

From (2.16) and (2.20), we acquire the following relation:

$$\limsup_{k\to\infty} \|x_k - u^*\| \le \limsup_{k\to\infty} \|u_k - u^*\| = \vartheta. \tag{2.21}$$

Similarly, (2.15) and (2.20) yields

$$\limsup_{k\to\infty} \|\mathbb{G}u_k - u^*\| \le \limsup_{k\to\infty} \|u_k - u^*\| = \vartheta. \tag{2.22}$$

Further, from (1.14), we conclude that

$$\begin{aligned} \|u_{k+1} - u^*\| &= \|\mathbb{G}^2 v_k - u^*\| \\ &= \|\mathbb{G}(\mathbb{G}v_k) - u^*\| \\ &\le \|\mathbb{G}v_k - u^*\| \\ &\le \|v_k - u^*\| \\ &= \|\mathbb{G}(\mathbb{G}w_k) - u^*\| \\ &\le \|\mathbb{G}w_k - u^*\| \\ &\le \|w_k - u^*\| \\ &= \|\mathbb{G}^2 x_k - u^*\| \\ &= \|\mathbb{G}(\mathbb{G}x_k) - u^*\| \\ &\le \|\mathbb{G}x_k - u^*\| \\ &\le \|x_k - u^*\|, \end{aligned} \tag{2.23}$$

which leads to

$$\vartheta \le \liminf_{k\to\infty} \|x_k - u^*\|. \tag{2.24}$$

One can deduce from (2.21) and (2.24) that

$$\begin{aligned} \vartheta &= \lim_{k\to\infty} \|x_k - u^*\| \\ &= \lim_{k\to\infty} \|\mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\ &\le \lim_{k\to\infty} \|(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k - u^*\| \\ &= \lim_{k\to\infty} \|(1-\alpha_k)(u_k - u^*) + \alpha_k(\mathbb{G}u_k - u^*)\| \\ &\le \lim_{k\to\infty} ((1-\alpha_k)\|u_k - u^*\| + \alpha_k\|\mathbb{G}u_k - u^*\|) \\ &\le \lim_{k\to\infty} ((1-\alpha_k)\|u_k - u^*\| + \alpha_k\|u_k - u^*\|) \\ &\le \vartheta. \end{aligned} \tag{2.25}$$

Consequently,

$$\lim_{k\to\infty} \|(1-\alpha_k)(u_k - u^*) + \alpha_k(\mathbb{G}u_k - u^*)\| = \vartheta. \tag{2.26}$$

Bringing (2.20), (2.22), (2.26), and Lemma 2.1 into play, we find that

$$\lim_{k\to\infty} \|u_k - \mathbb{G}u_k\| = 0. \tag{2.27}$$

Contrarily, suppose that $\{u_k\}$ is bounded and $\lim_{k\to\infty} \|u_k - \mathbb{G}u_k\| = 0$. Assuming that $u^* \in C(\mathbb{C}, \{u_k\})$ and recalling Lemma 2.2, we obtain

$$\begin{aligned} r(\mathbb{G}u^*, \{u_k\}) &= \limsup_{k\to\infty} \|u_k - \mathbb{G}u^*\| \\ &\le \Big(\frac{3+\tau}{1-\tau}\Big) \limsup_{k\to\infty} \|\mathbb{G}u_k - u_k\| + \limsup_{k\to\infty} \|u_k - u^*\| \\ &= \limsup_{k\to\infty} \|u_k - u^*\| = r(u^*, \{u_k\}), \end{aligned}$$

which implies $\mathbb{G}u^* \in C(\mathbb{C}, \{u_k\})$, and the uniform convexity of $\mathbb{B}$ guarantees that the set $C(\mathbb{C}, \{u_k\})$ is a singleton. Therefore, $\mathbb{G}u^* = u^*$, and hence $F_{\mathbb{G}} \neq \emptyset$. □

Theorem 2.5. *Suppose that $\mathbb{C}, \mathbb{B}$, and $\mathbb{G}$ are identical, as in Theorem 2.3, with $F_{\mathbb{G}} \neq \emptyset$. If $\mathbb{G}$ satisfies Opial's condition and $\{u_k\}$ is produced by the scheme (1.14), then $u_k \rightharpoonup u \in F_{\mathbb{G}}$.*

Proof. Under the assumption that $F_{\mathbb{G}} \neq \emptyset$, it is proven in Theorems 2.3 and 2.4 that $\lim_{k\to\infty} \|\mathbb{G}u_k - u^*\|$ exists and $\lim_{k\to\infty} \|u_k - \mathbb{G}u_k\| = 0$. In order to furnish the proof, it is enough to show that $\{u_k\}$ has a unique weak subsequential limit in $F_{\mathbb{G}}$. Suppose to the contrary that $\{u_k\}$ has two subsequences $\{u_{k_i}\}$ and $\{u_{k_j}\}$ such that $u_{k_i} \rightharpoonup u$ and $u_{k_j} \rightharpoonup v$. Consequently, by Theorem 2.4, $(I - \mathbb{G})$ is demiclosed at 0. Consequently, $(I - \mathbb{G})u = 0$ and hence $\mathbb{G}u = u$. In the same manner, we get $\mathbb{G}v = v$. To determine the uniqueness, assume that $u \neq v$. Then, by taking advantage of Opial's condition, we obtain

$$\begin{aligned} \lim_{k\to\infty} \|u_k - u\| &= \lim_{k_i\to\infty} \|u_{k_i} - u\| \\ &< \lim_{k_i\to\infty} \|u_{k_i} - v\| \\ &= \lim_{k\to\infty} \|u_k - v\| \\ &= \lim_{k_j\to\infty} \|u_{k_j} - v\| \\ &< \lim_{k_j\to\infty} \|u_{k_j} - u\| \\ &= \lim_{k\to\infty} \|u_k - u\|, \end{aligned}$$

which is a contradiction; thus $u = v$. Hence, $u_k \rightharpoonup u \in F_{\mathbb{G}}$. □

Theorem 2.6. *Suppose that $\mathbb{C}, \mathbb{B}$, and $\mathbb{G}$ are identical, as in Theorem 2.3, with $F_{\mathbb{G}} \neq \emptyset$. The sequence $\{u_k\}$ produced by the scheme (1.14) converges to an element in $F_{\mathbb{G}}$ if and only if $\liminf_{k\to\infty} d(u_k, F_{\mathbb{G}}) = 0$, where $d(u_k, F_{\mathbb{G}}) = \inf\{\|u_k - u^*\| : u^* \in F_{\mathbb{G}}\}$.*

Proof. Proof of the direct part is straightforward. Contrarily, assume that $\liminf_{k\to\infty} d(u_k, F_{\mathbb{G}}) = 0$ and $u^* \in F_{\mathbb{G}}$. It is evident from Theorem 2.3 that for each $u^* \in F_{\mathbb{G}}$, $\lim_{k\to\infty} \|u_k - u^*\|$ exists. In order to reach the conclusion, it is sufficient to substantiate that $\{u_k\}$ is Cauchy in $\mathbb{C}$. Seeing that $\lim_{k\to\infty} d(u_k, F_{\mathbb{G}}) = 0$, then for each $\epsilon > 0$, $\exists\vartheta_0 \in \mathbb{N}$ such that

$$d(u_k, F_{\mathbb{G}}) < \frac{\epsilon}{2},$$

$$\inf\{\|u_k - u^*\| : u^* \in F_{\mathbb{G}}\} < \frac{\epsilon}{2}, \forall k \geq \vartheta_0.$$

In particular, $\inf\{\|u_{\vartheta_0} - u^*\| : u^* \in F_{\mathbb{G}}\} < \frac{\epsilon}{2}$. Thus, $u^* \in F_{\mathbb{G}}$ exists such that

$$\|u_{\vartheta_0} - u^*\| < \frac{\epsilon}{2}.$$

Thus, for $t, r \geq \vartheta_0$, we can express

$$\begin{aligned}\|u_{k+1} - u_k\| &\leq \|u_{k+1} - u^*\| + \|u_k - u^*\| \\ &\leq \|u_{\vartheta_0} - u^*\| + \|u_{\vartheta_0} - u^*\| \\ &= 2\|u_{\vartheta_0} - u^*\| < \epsilon.\end{aligned}$$

Thus, $\{u_k\}$ is Cauchy in $\mathbb{C}$ and $\mathbb{C}$ is closed, and therefore $\exists v \in \mathbb{C}$ such that $u_k \to v$ as $k \to \infty$. Further, $\lim_{k\to\infty} d(u_k, F_{\mathbb{G}}) = 0$ yields $d(v, F_{\mathbb{G}}) = 0$; accordingly, $v \in F_{\mathbb{G}}$. □

Theorem 2.7. *Suppose that $\mathbb{C}, \mathbb{B}$, and $\mathbb{G}$ are identical, as in Theorem 2.3, with $F_{\mathbb{G}} \neq \emptyset$. If $\emptyset \neq \mathbb{C} \subseteq \mathbb{B}$ is a convex compact set. If the sequence $\{u_k\}$ is produced by the scheme (1.14) then $u_k \to t \in F_{\mathbb{G}}$.*

Proof. From Theorem 2.4, we obtain $\lim_{k\to\infty} \|u_k - \mathbb{G}u_k\| = 0$, and $\{u_k\}$, being a sequence in compact subset $\mathbb{C}$, has a strongly convergent subsequence, say $\{u_{k_i}\}$, such that $u_{k_i} \to t$ as $k_i \to \infty$. By employing Lemma 2.2, we obtain

$$\|u_{k_i} - \mathbb{G}t\| \leq \Big(\frac{3+\tau}{1-\tau}\Big)\|u_{k_i} - \mathbb{G}u_{k_i}\| + \|u_{k_i} - t\|.$$

As $i \to \infty$, we have $u_{k_i} \to \mathbb{G}t$, which turns into $t = \mathbb{G}t$. Hence, $t \in F_{\mathbb{G}}$ and the existence of $\lim_{k\to\infty} \|u_k - t\|$ by Theorem 2.3 leads to $\lim_{k\to\infty} u_k = t$. □

Example 2.4. *Let $\mathbb{B} = \mathbb{R}$ with $\|\cdot\| = |\cdot|$ and $\mathbb{C} = [2, 6]$. Define $\mathbb{G} : \mathbb{C} \to \mathbb{C}$ as follows:*

$$\mathbb{G}(\sigma) = \begin{cases} \frac{\sigma+8}{4}, & \sigma < 6, \\ 2, & \sigma = 6. \end{cases} \tag{2.28}$$

Here, we demonstrate that $\mathbb{G}$ does not satisfy Condition (C). To achieve this, we choose $\sigma = 5$ and $\varsigma = 6$, and thus

$$\frac{1}{2}\|\sigma - \mathbb{G}\sigma\| = \frac{1}{2}\Big|5 - \frac{\sigma+8}{4}\Big| = \frac{1}{2}\Big|\frac{12-\sigma}{4}\Big| = \frac{7}{8} < 1 = \|\sigma - \varsigma\|.$$

However,

$$\|\mathbb{G}\sigma - \mathbb{G}\varsigma\| = |\mathbb{G}(5) - \mathbb{G}(6)| = \Big|\frac{13}{4} - 2\Big| = \frac{5}{4} > 1 = \|\sigma - \varsigma\|.$$

Thus, $\mathbb{G}$ does not satisfy Condition (C). Next, we prove that $\mathbb{G}$ is a G_α-RSNEM with $\alpha = \frac{1}{2}$. Corresponding to the mapping $\mathbb{G}$, the following cases arise.

Case C_1**:** *If $\sigma, \varsigma < 6$, then*

$$\begin{aligned}
&\alpha\|\sigma - \mathbb{G}\sigma\| + \alpha\|\varsigma - \mathbb{G}\varsigma\| + (1-2\alpha)\|\sigma - \varsigma\| \\
&= \frac{1}{2}\Big|\sigma - \Big(\frac{\sigma+8}{4}\Big)\Big| + \frac{1}{2}\Big|\varsigma - \Big(\frac{\varsigma+8}{4}\Big)\Big| \\
&= \frac{1}{2}\Big|\frac{3\sigma-8}{4}\Big| + \frac{1}{2}\Big|\frac{3\varsigma-8}{4}\Big| \\
&\geq \frac{1}{2}\Big|\frac{3(\sigma-\varsigma)}{4}\Big| \\
&= \frac{3}{8}|\sigma-\varsigma| \\
&\geq \frac{2}{8}|\sigma-\varsigma| = \|\mathbb{G}\sigma - \mathbb{G}\varsigma\|.
\end{aligned}$$

Case C_2**:** *If $\sigma < 6, \varsigma = 6$, then*

$$\begin{aligned}
&\alpha\|\sigma - \mathbb{G}\sigma\| + \alpha\|\varsigma - \mathbb{G}\varsigma\| + (1-2\alpha)\|\sigma - \varsigma\| \\
&= \frac{1}{2}\Big|\sigma - \Big(\frac{\sigma+8}{4}\Big)\Big| + \frac{1}{2}|6-2| \\
&= \Big|\frac{3\sigma-8}{8}\Big| + 2 \\
&\geq \Big|\frac{3\sigma+8}{8}\Big| \\
&\geq \Big|\frac{3\sigma}{8}\Big| \\
&\geq \Big|\frac{2\sigma}{8}\Big| = \|\mathbb{G}\sigma - \mathbb{G}\varsigma\|.
\end{aligned}$$

Case C_3**:** *If $\sigma = 6, \varsigma < 6$, then*

$$\begin{aligned}
&\alpha\|\sigma - \mathbb{G}\sigma\| + \alpha\|\varsigma - \mathbb{G}\varsigma\| + (1-2\alpha)\|\sigma - \varsigma\| \\
&= \frac{1}{2}|6-2| + \frac{1}{2}\Big|\varsigma - \Big(\frac{\varsigma+8}{4}\Big)\Big| \\
&= 2 + \Big|\frac{3\varsigma-8}{8}\Big| \\
&\geq \Big|\frac{3\varsigma+8}{8}\Big| \\
&\geq \Big|\frac{3\varsigma}{8}\Big| \\
&\geq \Big|\frac{2\varsigma}{8}\Big| = \|\mathbb{G}\sigma - \mathbb{G}\varsigma\|.
\end{aligned}$$

Case C_4**:** *If $\sigma = 6, \varsigma = 6$, then*

$$\begin{aligned}
&\alpha\|\sigma - \mathbb{G}\sigma\| + \alpha\|\varsigma - \mathbb{G}\varsigma\| + (1-2\alpha)\|\sigma - \varsigma\| \\
&= \frac{1}{2}|6-2| + \frac{1}{2}|6-2| = 4 > 0 = \|\mathbb{G}\sigma - \mathbb{G}\varsigma\|.
\end{aligned}$$

Thus for all possible cases (C_1)–(C_4) *and for* $\alpha = \frac{1}{2}$, $\mathbb{G}$ *satisfies*

$$\frac{1}{2}\|\sigma - \mathbb{G}\sigma\| \leq \|\sigma - \varsigma\| \Rightarrow \alpha\|\sigma - \mathbb{G}\sigma\| + \alpha\|\varsigma - \mathbb{G}\varsigma\| + (1 - 2\alpha)\|\sigma - \varsigma\|.$$

Hence, $\mathbb{G}$ *is a* G_α*-RSNEM with* $\sigma^* = \frac{8}{3} \in F_{\mathbb{G}}$. *Next, by considering* $\alpha_k = \beta_k = \gamma_k = \frac{k}{k+1}, \forall k \in \mathbb{N}$. *The convergence of our scheme (1.14) and a comparison with (1.8), (1.12), and (1.13) is shown in Figure 4.*

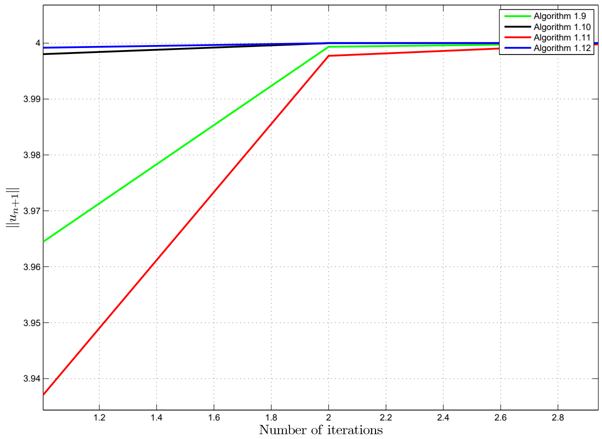

Figure 4. Convergence behavior of the schemes (1.8), (1.12), (1.13), and (1.14) for a G_α-RSNEM.

3. Application

In this section, we explore some applications by implementing our designed scheme. In the first subsection, we employ our scheme (1.14) to investigate a Caputo-type nonlinear fractional Volterra–Fredholm integro-differential equation, and the second subsection is devoted to the study of a fractional diffusion model of oxygen.

3.1. Caputo-type nonlinear fractional Volterra–Fredholm integro-differential equation

It is known that fractional calculus is pursued from pure mathematics in the mid 19th century. However, its considerable and worthwhile applications have been reported in diverse interdisciplinary fields including physics and engineering. Fractional derivatives are extended form of ordinary derivatives which are used to describe the functioning patterns of several physical phenomena. Fractional differential equations (FDEs) have been investigated widely as these equations are useful for tackling the real-world problems appearing in different domains including engineering, mechanics, economics, biology, etc., [28,32,46,52,53]. It is a burdensome task to examine the models of nonlinear FDEs using analytical methods, [39]. But these days, several multi-disciplinary problems can be tackled by using fixes point approaches. Iterative techniques based on fixed points have been used to explore several classes of FDEs; for more details, see, [1,3,15,20,24,29]. The fractional derivative of $\mathbb{G}u$ is given as

$$ {}^{\gamma}\mathcal{D}_u^{\kappa}\mathbb{G}(u) = \frac{1}{\Gamma(\kappa-\tau)}\int_a^u \mathcal{G}^{(\tau)}(s)(u-s)^{n-\kappa-1}ds, \tau-1<\kappa<\tau. $$

Now, we employ the scheme (1.14) to investigate the following C-NFVFIDE:

$$ {}^{\gamma}\mathcal{D}_t^{\kappa}\zeta(u) = \rho(u)\zeta(u) + \xi(u) + \int_0^u \mathcal{G}_1(u,s)\psi_1(\zeta(s))ds + \int_0^1 \mathcal{G}_2(u,s)\psi_2(\zeta(s))ds, \tag{3.1} $$

$$ \zeta^i(0) = \delta_i, 1 = 0, 1, 2, \cdots, k-1. \tag{3.2} $$

Here, ${}^{\gamma}\mathcal{D}_t^{\kappa}$ indicates a Caputo-fractional derivative of order $k-1<\kappa\leq k$ regarding t, and $\zeta:[0,1]\to\mathbb{R}$ is a continuous function which is to be estimated for each $i=1,2$; $\mathcal{G}_i:[0,1]\times[0,1]\to\mathbb{R}$ and $\psi_i:\mathbb{R}\to\mathbb{R}$ are continuous and L_{ψ_i}-Lipschitz continuous, respectively. The solution set of the C-NFVFIDE (3.1) and (3.2) is denoted as Σ.

The following results from Mamoud et al. [30] are crucial to achieve the desired goal.

Lemma 3.1. *For some $\zeta_0(u)\in C([0,1],\mathbb{R})$, $\zeta(u)\in\Sigma$, if it solves the following integral equation:*

$$ \begin{aligned} \zeta(u) = \zeta_0(u) &+ \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\rho(s)\zeta(s)ds + \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\xi(s)ds \\ &+ \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\Big(\int_0^s \mathcal{G}_1(s,\tau)\psi_1(\zeta(\tau))d\tau + \int_0^1 \mathcal{G}_2(s,\tau)\psi_2(\zeta(\tau))d\tau\Big)ds. \end{aligned} \tag{3.3} $$

Theorem 3.1. *[30] Suppose that the following assumptions hold.*

(A_1) For any $\zeta_1,\zeta_2\in C([0,1],\mathbb{R})$, $\exists L_{\psi_1}$, and L_{ψ_2} such that

$$ |\psi_1(\zeta_1(u)) - \psi_1(\zeta_2(u))| \leq L_{\psi_1}|\zeta_1-\zeta_2|, $$

and

$$ |\psi_2(\zeta_1(u)) - \psi_2(\zeta_2(u))| \leq L_{\psi_2}|\zeta_1-\zeta_2|. $$

(A_2) The positive continuous functions $T_1^*, T_2^* \in \{(x,y)\in\mathbb{R}^2 : 0\leq y\leq x\leq 1\}$ exist such that

$$ T_1^* = \max_{u,s\in[0,1]}\int_0^u |\mathcal{G}_1(u,s)|ds < \infty, $$

and

$$ T_2^* = \max_{u,s\in[0,1]}\int_0^u |\mathcal{G}_2(u,s)|ds < \infty. $$

(A_3) $\rho,\xi:[0,1]\to\mathbb{R}$ are continuous.
(A_4) If $\Big(\frac{\|\rho\|_\infty + T_1^*L_{\psi_1} + T_2^*L_{\psi_2}}{\Gamma(\kappa+1)}\Big) < 1$.

Then the C-NFVFIDE (3.1) and (3.2) admits a unique solution.

Next, we find the approximate solution of the C-NFVFIDE (3.1) and (3.2) by applying the scheme (1.14).

Theorem 3.2. *Let $C([0,1],\mathbb{R})$ be a Banach space with the norm $\|f-g\|_\infty = \max_{u\in[0,1]} |f(u)-g(u)|$ and $\mathbb{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is defined as*

$$\mathbb{G}\zeta(u) = \zeta_0(u) + \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\rho(s)\zeta(s)ds + \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\xi(s)ds$$
$$+ \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\Big(\int_0^s \mathcal{G}_1(s,\tau)\psi_1(\zeta(\tau))d\tau + \int_0^1 \mathcal{G}_2(s,\tau)\psi_2(\zeta(\tau))d\tau\Big)ds.$$

If assumptions (A_1)–(A_4) hold and the sequence $\{\zeta_k\}$ produced by (1.14) satisfies $\zeta_0 = \sum_{i=0}^{k-1}\zeta^i(0)\frac{u^i}{i!}$, then $\zeta_k \to u^* \in \Sigma$.

Proof. Clearly, the C-NFVFIDE (3.1)–(3.2) satisfies assumptions (A_1)–(A_4). If $\zeta^* \in C([0,1],\mathbb{R})$ is a fixed point of $\mathbb{G}$, then $\zeta^* \in \Sigma$. Next, we prove that the scheme (1.14) converges to ζ^*.

$$|\mathbb{G}\zeta(u) - \mathbb{G}\zeta^*(u)| \leq \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}|\rho(s)||\zeta(s)-\zeta^*(s)|ds$$
$$+ \frac{1}{\Gamma(\kappa)}\int_0^u (u-s)^{\kappa-1}\Big(\int_0^s |\mathcal{G}_1(s,\tau)||\psi_1(\zeta(\tau)) - \psi_1(\zeta^*(\tau))|d\tau$$
$$+ \int_0^1 |\mathcal{G}_2(s,\tau)||\psi_2(\zeta(\tau)) - \psi_2(\zeta^*(\tau))|d\tau\Big)ds$$
$$\leq \Big(\frac{\|\rho\|_\infty}{\Gamma(\kappa+1)} + \frac{T_1^*L_{\psi_1}}{\Gamma(\kappa+1)} + \frac{T_2^*L_{\psi_2}}{\Gamma(\kappa+1)}\Big)|\zeta(u)-\zeta^*(u)|,$$

which turns into

$$|\mathbb{G}\zeta(u) - \mathbb{G}\zeta^*(u)| \leq \Big(\frac{\|\rho\|_\infty + T_1^*L_{\psi_1} + T_2^*L_{\psi_2}}{\Gamma(\kappa+1)}\Big)|\zeta(u)-\zeta^*(u)|. \tag{3.4}$$

Assumption (A_4) shows that $\Big(\frac{\|\rho\|_\infty + T_1^*L_{\psi_1} + T_2^*L_{\psi_2}}{\Gamma(\kappa+1)}\Big) < 1$, assuming that $\tau = \Big(\frac{\|\rho\|_\infty + T_1^*L_{\psi_1} + T_2^*L_{\psi_2}}{\Gamma(\kappa+1)}\Big)$. Then for any strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, one can express (3.4) as

$$|\mathbb{G}\zeta(u) - \mathbb{G}\zeta^*(u)| \leq g(\|\zeta(u) - \mathbb{G}\zeta(u)\|) + \tau\|\zeta(u)-\zeta^*(u)\|. \tag{3.5}$$

Thus, $\mathbb{G}$ satisfies (1.4) and hence, by Theorem 2.1, $\zeta_k \to \zeta^* \in \Sigma$. □

3.2. Fractional diffusion equation

Oxygen plays a vital role for the functioning of humans by providing energy to every cell. Humans take air directly from the atmosphere through the lungs, and the air is supplied to the tissue. The rate of oxygen consumption diffuses from the air into the blood in the lungs is not same as in a real situation [27]. The subsequent stages of transporting oxygen transport to tissue have been noted by investigating hemoglobin-oxygen kinetics. Oxygen mixes with hemoglobin in large amounts and in the blood plasma in small amounts, and oxygenated blood is transported through the arteries to the capillaries. The theory of transportation of energy was brought forward by Krogh [26], known as the Krogh tissue cylinder. In his theory, he revealed that transportation of oxygen in tissue occurs due to oxygen's tension (PO_2). The mathematicians Erlang and Krogh jointly derived a mathematical model involving differential equations to express a diffusion model. The solution of the mathematical model expresses the oxygen tension in the tissue as a function of its spatial position within the tissue cylinder.

Our aim here is to employ our efficient four-step iterative method (1.14) to study the fractional diffusion model developed by Srivastava and Rai [44]. The model involving the concentration of oxygen $C(r, z, u)$ and the rate of consumption per volume of tissue $K(r, z, u)$ is shown below:

$$\frac{\partial^{\zeta} C}{\partial u^{\zeta}} - \tau \frac{\partial^{\varrho} C}{\partial u^{\varrho}} = \nabla(d \cdot \nabla C) - K, \quad \zeta, \varrho \in (0, 1] \tag{3.6}$$

where $\frac{\partial^{\zeta} C}{\partial u^{\zeta}}$ is a fractional order derivative for $0 < \zeta < 1$ representing the sub-diffusion process, and d is the diffusion coefficient of oxygen. The net diffusion of oxygen to tissue is $\frac{\partial^{\zeta} C}{\partial u^{\zeta}} - \tau \frac{\partial^{\varrho} C}{\partial u^{\varrho}}$, where τ expresses the time lag in the concentration of oxygen C along the z-axis.

Equation (3.6) can equivalently be presented by the following relation:

$$C(r, z, u) = C(r, z, 0)\Big(1 - \tau \frac{u^{\zeta - \varrho}}{\Gamma(\zeta - \varrho + 1)}\Big) + \tau \mathcal{D}_u^{-(\zeta - \varrho)} C + \mathcal{D}_u^{-\zeta}(\nabla(d \cdot \nabla C)) - \mathcal{D}_u^{-\zeta} K. \tag{3.7}$$

Relation (3.7) can be re-written as

$$\Phi(r, z, u) = \Phi(r, z, 0)\Big(1 - \tau \frac{u^{\zeta - \varrho}}{\Gamma(\zeta - \varrho + 1)}\Big) + \tau \mathcal{D}_u^{-(\zeta - \varrho)} \Phi + \mathcal{D}_u^{-\zeta}(\nabla(d \cdot \nabla \Phi) - K). \tag{3.8}$$

Equivalently,

$$\Phi(r, z, u) = \mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, \Phi(s), K) ds, \tag{3.9}$$

where
$\mathcal{K}(\Phi_0) = \Phi(r, z, 0)\Big[1 - \tau \frac{u^{\zeta - \varrho}}{\Gamma(\zeta - \varrho + 1)}\Big]$ and $\mathbb{H}(s, \Phi, K) = \tau \frac{\partial^{\varrho} \Phi}{\partial u^{\varrho}} + (\nabla(d \cdot \nabla \Phi) - K)$.
We define the integral operator $\mathbb{G}$ as

$$\mathbb{G}\Phi(r, z, u) = \mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, \Phi(s), K) ds \tag{3.10}$$

and the norm on $\mathbb{J} = ([0, U], \mathbb{R})$ is given by

$$\|\Phi\| = \sup_{u \in [0, U]} \{|\Phi(u)| : \Phi \in \mathbb{J}\}.$$

The theorem given below is crucial in obtaining our main result in this section.

Lemma 3.2. *Suppose that the following assumptions are fulfilled.*

(c_1) There is a constant $L_{\mathbb{H}} > 0$ *that satisfies*

$$(\forall \Phi_1, \Phi_2 \in \mathbb{J}, u \in [0, U]), \quad |\mathbb{H}(u, \Phi_1(u), K) - \mathbb{H}(u, \Phi_2(u), K)| \leq L_{\mathbb{H}} |\Phi_1 - \Phi_2|;$$

(c_2) $\frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} < 1$.

Then, the fractional diffusion model (3.6) admits a unique solution. Now, we are in a position to present the main result of this section.

Theorem 3.3. *Assume that the conditions (c_1) and (c_2) of Lemma 3.2 hold. Let $\{\alpha_k\}$ lies in (0,1) such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then the fractional diffusion model (3.6) admits a unique solution u^* and the sequence $\{u_k\}$ produced by the iterative scheme (1.14) converges to u^*.*

Proof. Let $\{u_k\}$ be a sequence generated by the iterative scheme (1.14) for the operator $\mathbb{G} : \mathbb{J} \to \mathbb{J}$ defined by

$$\mathbb{G}\Phi(u) = \mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, \Phi(s), K)ds.$$

We shall show that $\{u_k\}$ converges to u^* as $n \to \infty$. Taking the assumptions of the Lemma 3.2 into play, it follows from the last formulation of (1.14) and (3.10) that

$$\begin{aligned}
\|x_k - u^*\| &= \|\mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\
&\leq \|\mathbb{G}[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - \mathbb{G}u^*\| \\
&\leq \max_{u\in[0,U]} |\mathbb{G}[(1-\alpha_k)u_k(u) + \alpha_k \mathbb{G}u_k(u)] - \mathbb{G}u^*(u)| \\
&\leq \max_{u\in[0,U]} \Big| \mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, [(1-\alpha_k)u_k(s) + \alpha_k \mathbb{G}u_k(s)], K)ds \\
&\quad - \mathcal{K}(\Phi_0) - \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, u^*, K)ds \Big| \\
&\leq \frac{1}{\Gamma(\zeta)} \max_{u\in[0,U]} \Big| \int_0^u \mathbb{H}(s, [(1-\alpha_k)u_k(s) + \alpha_k \mathbb{G}u_k(s)], K) - \mathbb{H}(s, u^*(s), K) \Big| ds \\
&\leq \frac{L_{\mathbb{H}}}{\Gamma(\zeta)} \max_{u\in[0,U]} \int_0^u |[(1-\alpha_k)u_k(s) + \alpha_k \mathbb{G}u_k(s)] - u^*(s)| ds \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} \|[(1-\alpha_k)u_k + \alpha_k \mathbb{G}u_k] - u^*\| \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} [(1-\alpha_k)\|u_k - u^*\| + \alpha_k \|\mathbb{G}u_k - u^*\|] \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} [(1-\alpha_k)\|u_k - u^*\| + \alpha_k \max_{u\in[0,U]} |\mathbb{G}u_k(u) - \mathbb{G}u^*(u)|] \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} \Big[(1-\alpha_k)\|u_k - u^*\| + \alpha_k \max_{u\in[0,U]} \Big| \mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, \mathbb{G}u_k(s), K)ds \\
&\quad - \mathcal{K}(\Phi_0) - \frac{1}{\Gamma(\zeta)} \int_0^u \mathbb{H}(s, u^*(s), K)ds \Big| \Big] \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} \Big[(1-\alpha_k)\|u_k - u^*\| + \frac{\alpha_k}{\Gamma(\zeta)} \max_{u\in[0,U]} \Big| \int_0^u \mathbb{H}(s, \mathbb{G}u_k(s), K) \\
&\quad - \mathbb{H}(s, u^*(s), K) \Big| ds \Big] \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} [(1-\alpha_k)\|u_k - u^*\| + \frac{\alpha_k L_{\mathbb{H}}}{\Gamma(\zeta)} \max_{u\in[0,U]} \int_0^u |\mathbb{G}u_k(s) - u^*(s)| ds] \\
&\leq \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} [(1-\alpha_k)\|u_k - u^*\| + \frac{\alpha_k L_{\mathbb{H}} U}{\Gamma(\zeta)} \|u_k - u^*|] \\
&= \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)} \Big[1 - \alpha_k \Big(1 - \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)}\Big)\Big] \|u_k - u^*\|.
\end{aligned}$$

Again it follows from the second formulation of the scheme (1.14), relation (3.10) and utilizing the assumptions of the Lemma 3.2, we obtain

$$\begin{aligned}
\|w_k - u^*\| &= \|\mathbb{G}^2 x_k - u^*\| \\
&\leq \|\mathbb{G}(\mathbb{G}x_k) - \mathbb{G}u^*\Big\| \\
&\leq \max_{u\in[0,U]} |\mathbb{G}(\mathbb{G}x_k)(s) - \mathbb{G}u^*(s)| \\
&\leq \max_{u\in[0,U]} \Big|\mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)}\int_0^u \mathbb{H}(s, \mathbb{G}(\mathbb{G}x_k)(s)), K)ds \\
&\quad - \mathcal{K}(\Phi_0) - \frac{1}{\Gamma(\zeta)}\int_0^u \mathbb{H}(s, u^*(s), K)ds\Big| \\
&\leq \frac{1}{\Gamma(\zeta)} \max_{u\in[0,U]} \Big| \int_0^u \mathbb{H}(s, \mathbb{G}(\mathbb{G}x_k)(s), K) - \mathbb{H}(s, u^*(s), K)\Big| ds \\
&\leq \frac{L_{\mathbb{H}}}{\Gamma(\zeta)} \max_{u\in[0,U]} \int_0^u |\mathbb{G}x_k(s) - u^*(s)| ds \\
&\leq \frac{L_{\mathbb{H}}U}{\Gamma(\zeta)} \|\mathbb{G}x_k - u^*\| \\
&\leq \frac{L_{\mathbb{H}}U}{\Gamma(\zeta)} \|\mathbb{G}x_k - \mathbb{G}u^*\| \\
&\leq \frac{L_{\mathbb{H}}U}{\Gamma(\zeta)} \max_{u\in[0,U]} |\mathbb{G}x_k(s) - \mathbb{G}u^*(s)| \\
&\leq \frac{L_{\mathbb{H}}U}{\Gamma(\zeta)} \max_{u\in[0,U]} \Big|\mathcal{K}(\Phi_0) + \frac{1}{\Gamma(\zeta)}\int_0^u \mathbb{H}(s, \mathbb{G}x_k(s), K)ds \\
&\quad - \mathcal{K}(\Phi_0) - \frac{1}{\Gamma(\zeta)}\int_0^u \mathbb{H}(s, u^*(s), K)ds\Big| \\
&\leq \frac{L_{\mathbb{H}}U}{\Gamma^2(\zeta)} \max_{u\in[0,U]} \Big| \int_0^u \mathbb{H}(s, \mathbb{G}x_k(s), K) - \mathbb{H}(s, u^*(s), K)\Big| ds \\
&\leq \frac{L_{\mathbb{H}}^2 U}{\Gamma^2(\zeta)} \max_{u\in[0,U]} \int_0^u |x_k(s) - u^*(s)| ds \\
&\leq \frac{L_{\mathbb{H}}^2 U^2}{\Gamma^2(\zeta)} \|x_k - u^*\| \\
&\leq \frac{L_{\mathbb{H}}^3 U^3}{\Gamma^3(\zeta)} \Big[1 - \alpha_k\Big(1 - \frac{L_{\mathbb{H}}U}{\Gamma(\zeta)}\Big)\Big] \|u_k - u^*\|.
\end{aligned}$$

In the same fashion, we can obtain

$$\begin{aligned}
\|v_k - u^*\| &= \|\mathbb{G}^2 w_k - u^*\| \\
&\leq \frac{L_{\mathbb{H}}^2 U^2}{\Gamma^2(\zeta)} \|w_k - u^*\| \\
&\leq \frac{L_{\mathbb{H}}^5 U^5}{\Gamma^5(\zeta)} \Big[1 - \alpha_k\Big(1 - \frac{L_{\mathbb{H}}U}{\Gamma(\zeta)}\Big)\Big] \|u_k - u^*\|.
\end{aligned} \tag{3.11}$$

Finally, the first relation of (1.14) along with (3.10) and the assumptions of the Lemma 3.2, (3.11) turns into

$$\begin{aligned} \|u_{k+1} - u^*\| &= \|\mathbb{G}^2 v_k - u^*\Big\| \\ &\le \frac{L_{\mathbb{H}}^2 U^2}{\Gamma^2(\zeta)} \|v_k - u^*\| \\ &\le \frac{L_{\mathbb{H}}^7 U^7}{\Gamma^7(\zeta)} \Big[1 - \alpha_k \Big(1 - \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)}\Big)\Big] \|u_k - u^*\|. \end{aligned} \tag{3.12}$$

The assumption (c_2) guarantees that $\frac{L_{\mathbb{H}}^7 U^7}{\Gamma^7(\zeta)} < 1$, and hence (3.12) turns into

$$\|u_{k+1} - u^*\| \le \Big[1 - \alpha_k \Big(1 - \frac{L_{\mathbb{H}} U}{\Gamma(\zeta}\Big)\Big] \|u_k - u^*\|. \tag{3.13}$$

By induction, we obtain

$$\|u_{k+1} - u^*\| \le \|u_0 - u^*\| \prod_{z=0}^{k} \Big[1 - \alpha_z \Big(1 - \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)}\Big)\Big]. \tag{3.14}$$

Recalling the fact $1 - s \le e^{-s}$ for all $0 \le s \le 1$, we obtain

$$\|u_{k+1} - u^*\| \le \|u_0 - u^*\| \prod_{z=0}^{k} e^{-\left(1 - \frac{L_{\mathbb{H}} U}{\Gamma(\zeta)}\right) \sum_{z=0}^{k} \alpha_z}. \tag{3.15}$$

Taking the limit $k \to \infty$, we obtain $\lim_{m \to \infty} \|u_k - u^*\| = 0$. This completes the proof. □

4. Conclusions

In this study, a new iterative scheme (1.14) is designed and utilized to determine the fixed point of a contractive-like mapping and a G_α-RSNEM. The weak ω^2-stability of the scheme (1.14) is also shown. Some weak and strong convergence results of (1.14) for the G_α-RSNEM are also reported. The theoretical outcomes are also validated by illustrative examples. Finally, a C-NFVFIDE and a fractional diffusion model are studied by employing our scheme (1.14). Further enhancement of the proposed scheme and the development of more efficient and easily implementable iterative schemes for approximating fixed points are appreciable. The study of random fixed-point theorems involving random contractive-like mapping is a worthy and open research direction.

Author contributions

D. Filali: Funding, writing–review and editing, supervision; M. Dilshad: Conceptualization, writing–review and editing; M. Akram: Conceptualization, writing–original draft preparation, writing review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

Authors declare no conflicts of interest in this paper.

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