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Research article

Indifference valuation of non-performing loan-backed securities

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Abstract: This paper developed a unified framework for valuing non-performing loan (NPL)-backed securities under regime-switching macroeconomic conditions. The stochastic timing of NPL repayments were modeled by a Cox process with regime-dependent intensity, while investor risk aversion was incorporated through utility indifference pricing. Analytical solutions were obtained in a two-regime setting. Numerical results showed how risk aversion, repayment intensity, and macroeconomic state jointly affected the indifference price of NPL-backed securities.

Keywords: non-performing loans; indifference pricing; regime-switching; Cox process; exponential utility
Mathematics Subject Classification: 91A15, 91G30

1. Introduction

Non-performing loans (NPLs), conventionally defined as loan payments delinquent for 90 days or more, constitute a perennial challenge for financial regulatory authorities [1]. Elevated NPL ratios significantly impair banking sector stability through multiple channels: they degrade asset quality, constrain liquidity provision, diminish profitability margins, and potentially trigger systemic risk contagion [2, 3]. The detrimental effects of NPLs extend to operational efficiency deterioration, constrained intermediation capacity through reduced loan origination, and substantial erosion of profit generation capabilities [4].

Strategic NPL resolution mechanisms can substantially rehabilitate banks' credit portfolios, thereby enhancing cash flow stability and liquidity positions [5]. However, accurate NPL valuation presents considerable methodological challenges, primarily attributable to four factors: (1) market illiquidity and price opacity, (2) uncertainty in collateral recovery rates, (3) pronounced information asymmetries between originating institutions and potential investors [1], and (4) protracted bureaucratic procedures [6]. Furthermore, robust empirical analyses have identified significant regime-dependent dynamics in NPL behavior across different macroeconomic environments, suggesting state-contingent patterns in credit deterioration [7].

The effective disposal of NPLs has emerged as a critical challenge in financial risk management. Drawing on Italy's experience in NPL resolution, a comparative scenario analysis of two primary disposal methods—direct sale and securitization—reveals that securitization significantly reduces costs for originating banks [8]. Under the assumption that portfolio returns follow a lognormal distribution, the value-at-risk (VaR) methodology can be employed to assess the risk-return characteristics of NPL-backed securities. This analysis demonstrates that securitization constitutes the most value-enhancing deleveraging strategy, optimizing outcomes for both banks and investors [9]. These findings suggest that asset securitization represents a comparatively efficient mechanism for NPL resolution.

Unlike standardized traded assets, the NPL market is inherently incomplete, exhibiting three key features: illiquidity, non-replicable cash flows, and pronounced information asymmetry. These structural limitations render conventional risk-neutral pricing models unsuitable. To address this, we adopt a utility indifference pricing framework, which is specifically designed for incomplete markets and explicitly incorporates investor risk aversion into the valuation process [10].

In financial mathematics, transitions between different macroeconomic states are formally characterized as regime switching, which is conventionally modeled using continuous-time, finite-state Markov chains. This framework has been successfully incorporated into various derivative pricing models. Notably, the canonical Black-Scholes option pricing framework has been extended to accommodate regime-switching dynamics for the valuation of American options [11]. Subsequent developments in credit risk modeling have applied this approach to price defaultable options within a reduced-form risk framework [12].

Building upon these theoretical foundations, we employ a finite-state continuous-time Markov chain to model macroeconomic states [13]. This specification proves particularly appropriate given the empirically documented regime-dependent characteristics of both repayment intensities and recovery amounts. The resulting regime-switching model provides a robust framework for characterizing the evolution of repayment risk, thereby establishing a theoretically sound and empirically realistic basis for the valuation of non-performing loans under economic uncertainty.

In this study, we propose a unified valuation framework for NPLs within a regime-switching economic context. Specifically, we employ a Cox process with regime-dependent intensity to model the stochastic payment times and derive the utility indifference price within a Merton-type investment framework featuring exponential utility. Under a two-regime scenario, we provide explicit closed-form solutions for indifference pricing. Numerical illustrations demonstrate the impacts of risk aversion and regime-switching on valuation.

The remainder of the paper is organized as follows. Section 2 formulates the indifference valuation problem for NPLs. Section 3 presents the analytical results. Section 4 offers numerical examples to illustrate the effects of key parameters. Section 5 concludes the study.

2. Model formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting an m-dimensional standard Brownian motion $\boldsymbol{B}(t) = (B_1(t), \ldots, B_m(t))^\top$. In real economic activities, changes in macroeconomic conditions significantly affect the assets and investment income of entities, and the credit quality of defaultable entities. Typically, in line with [14], we model macroeconomic regimes by a continuous-time, finite-state Markov chain $X = \{X(t)\}_{t\geq 0}$ with state space $\mathbb{D} = \{1, \dots, Q\}$ and transition rate matrix

$$\mathbf{A} = \begin{pmatrix} -a_{11} & a_{12} & \cdots & a_{1Q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{Q1} & a_{Q2} & \cdots & -a_{QQ} \end{pmatrix}, \tag{2.1}$$

where $a_{ij} \geqslant 0, i \neq j$ and $a_{ii} = \sum_{i=1, i\neq j}^{Q} a_{ij}$ for $i, j = 1, 2, \cdots, Q$. Let $\mathcal{F}_t^X$ and $\mathcal{F}_t^B$ be the natural filtrations of $X(t)$ and $\boldsymbol{B}(t)$, respectively, and set $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^B$. Assume $X(t)$ and $\boldsymbol{B}(t)$ are independent. For any matrix $\mathbf{C}$, denote its transpose by $\mathbf{C}^\top$.

Consider a bank investing in one risk-free asset and n risky assets over a finite time horizon T. The risk-free rate $r > 0$ is constant. Risky asset prices $S_k(t)$, $k = 1, \dots, n$ are characterized by the following stochastic differential equation

$$\mathrm{d}S_k(t) = S_k(t) \left[\mu_k(t, X(t))\, \mathrm{d}t + \boldsymbol{\sigma}_k(t, X(t))^\top \mathrm{d}\boldsymbol{B}(t)\right], \tag{2.2}$$

where $\boldsymbol{\sigma}_k(t, X(t)) = (\sigma_{k1}(t, X(t)), \dots, \sigma_{km}(t, X(t)))^\top$. Since the price of the risky asset is influenced by changes in macroeconomic conditions, we define $\mu_k(t, X(t))$ and $\sigma_{kj}(t, X(t))$ as functionals of the Markov chain $X(t)$, i.e.,

$$\mu_k(t, X(t)) = \sum_{i=1}^{Q} \mu_k^i \, \mathrm{I}_{\{X(t)=i\}}, \quad \sigma_{kj}(t, X(t)) = \sum_{i=1}^{Q} \sigma_{kj}^i \, \mathrm{I}_{\{X(t)=i\}}, \tag{2.3}$$

where μ_k^i, σ_{kj}^i are constants, and $\mathrm{I}_{\{\cdot\}}$ is the indicator function.

Let $\boldsymbol{\pi}(t) = (\pi_1(t), \dots, \pi_n(t))^\top$ denote the bank's allocation in risky assets at time t. The wealth process $W(t)$ is governed by the following stochastic differential equation

$$\mathrm{d}W(t) = r\big(W(t) - \mathbf{1}^\top \boldsymbol{\pi}(t)\big)\mathrm{d}t + \boldsymbol{\mu}(t, X(t))^\top \boldsymbol{\pi}(t)\mathrm{d}t + \boldsymbol{\pi}(t)^\top \boldsymbol{\sigma}(t, X(t))\, \mathrm{d}\boldsymbol{B}(t), \tag{2.4}$$

where $\boldsymbol{\mu}(t, X(t)) = (\mu_1(t, X(t)), \dots, \mu_n(t, X(t)))^\top$, and

$$\boldsymbol{\sigma}(t, X(t)) = \begin{pmatrix} \sigma_{11}(t, X(t)) & \cdots & \sigma_{1m}(t, X(t)) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(t, X(t)) & \cdots & \sigma_{nm}(t, X(t)) \end{pmatrix}. \tag{2.5}$$

Assume the total value of the bank's non-performing loan pool is normalized to one. Over the finite horizon $[0, T]$, the bank undertakes collection efforts; if unsuccessful, it may liquidate borrower assets via legal proceedings, after which further collection ceases.

Distinct from standard loans, NPL repayments feature both stochastic timing and amounts. Let $\tau_{k,l}$ denote the l-th repayment time and Y_k the associated repayment amount of borrower k for $k = 1, 2, \cdots, K$. Specifically, $\tau_{k,l}$ is the l-th jump time of a Cox process $N_k(t)$ with intensity $\lambda_k(t, X(t))$. This means that we start with a standard Poisson process $\bar{N}_k(t)$ independent of $\lambda_k(t, X(t))$ and construct $N_k(t)$ by

$$N_k(t) = \bar{N}_k(\Lambda_k(t)), \quad \Lambda_k(t) = \int_0^t \lambda_k(s, X(s))\, \mathrm{d}s. \tag{2.6}$$

Therefore, the l-th repayment time $\tau_{k,l}$ from borrower k can be characterized as

$$\tau_{k,l} = \inf\{t > 0 : N_k(t) \geq l\}.$$

Assumption 2.1. *The Poisson processes $\bar{N}_1(t), \bar{N}_2(t), \cdots, \bar{N}_K(t)$ are independent.*

Using Assumption 2.1, the Cox processes $N_1(t), N_2(t), \cdots, N_K(t)$ are independent given $\mathcal{F}_t^X$.

Remark 2.1. *The repayment intensity refers to the number of repayment events occurring per unit of time. In real-world economic activities, repayment counts exhibit additivity. To align the model with empirical observations, it is necessary to assume that Assumption 2.1 holds, thereby ensuring that the repayment intensity within the model also satisfies the additive property.*

Considering that the repayment times and amounts are influenced by changes in macroeconomic conditions, we define the repayment intensity $\lambda_k(t, X(t))$ as a functional of the Markov chain $X(t)$, i.e.,

$$\lambda_k(t, X(t)) = \sum_{i=1}^{Q} \lambda_{k,i} \mathrm{I}_{\{X(t)=i\}} \text{ for } k = 1, 2, \cdots, K, \tag{2.7}$$

where $\lambda_{k,i}$ is a non-negative constant. This implies that the repayment intensity varies across different macroeconomic states.

Similarly, let $f_k(y, t, X(t))$ denote the probability density of the non-negative repayment $Y_k(t)$, defined as

$$f_k(y, t, X(t)) = \sum_{i=1}^{Q} f_{k,i}(y)\, \mathrm{I}_{\{X(t)=i\}} \text{ for } k = 1, 2, \cdots, K, \tag{2.8}$$

where $f_{k,i}(y)$ is the density in state i. Then, the cumulative cash flow from borrower k up to time t is

$$\mathrm{d}L_k(t) = Y_k(t)\, \mathrm{d}N_k(t). \tag{2.9}$$

Conditional on $\mathcal{F}_T$, the jump size $Y_k(t)$ (with density $f_k(y, t, X(t))$) is independent of $L_k(t-)$, $L_k(T) - L_k(t)$, and of $N_k(t)$.

Thus, the accumulated cash flow $L(t)$ that the bank receives from all borrowers in the pool at time t is

$$\mathrm{d}L(t) = \sum_{k=1}^{K} Y_k(t)\, \mathrm{d}N_k(t). \tag{2.10}$$

$\{L(t)\}_{0\leq t\leq T}$ and the wealth process $\{W(t)\}_{0\leq t\leq T}$ are assumed conditionally independent given $\mathcal{F}_T^X$.

By the property of conditional independence and identical distribution, following [15, pp.44-45], we can rewrite the process $L(t)$ in Eq (2.10) as a compound Cox process, as stated in Lemma 2.1.

Lemma 2.1. *The cash flow $L(t)$ that the bank receives from all borrowers at time t can be rewritten as*

$$\mathrm{d}L(t) = Y(t)\, \mathrm{d}N(t), \tag{2.11}$$

where $N(t)$ is a Cox process with intensity process $\lambda(t, X(t)) = \sum_{i=1}^{Q} \lambda_i\, \mathrm{I}_{\{X(t)=i\}}$ and $\lambda_i = \sum_{k=1}^{K} \lambda_{k,i}$. The density function of $Y(t)$ is $f(y, t, X(t)) = \sum_{i=1}^{Q} f_i(y)\, \mathrm{I}_{\{X(t)=i\}}$ with $f_i(y) = \sum_{k=1}^{K} \frac{\lambda_{k,i}}{\lambda_i} f_{k,i}(y)$.

Assume the bank has exponential utility

$$U(w) = -e^{-\gamma w}$$

with the absolute risk aversion coefficient $\gamma > 0$.

The bank securitizes a proportion $0 < u \le 1$ of NPLs and sells them, so that $v = 1 - u$ is retained. Let $p^u(t)$ denote the indifference value at time t. The bank receives fraction v of cumulative cash flow $L(T)$, with initial wealth $W(t) = w + p^u(t)$ evolving as in Eq (2.4). The maximal expected utility is

$$\begin{aligned} &V(t, w + p^u(t), l, i; v) \\ &= \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(W(T) + v\Pi(L(T))) \mid W(t) = w + p^u(t), L(t) = l, X(t) = i\right], \end{aligned} \tag{2.12}$$

where $\Pi(L(T))$ is the present value of the cumulative cash flow $L(T)$ collected from non-performing loan borrowers.

If the bank does not securitize, its wealth process starts at $W(t) = w$ and is governed by Eq (2.4). Then, the bank's maximum expected utility is given by

$$V(t, w, l, i; 1) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(W(T) + \Pi(L(T))) \mid W(t) = w, L(t) = l, X(t) = i\right]. \tag{2.13}$$

From the principle of indifference valuation, the indifference price $p^u(t)$ of the non-performing loan-backed securities satisfies

$$V(t, w + p^u(t), l, i; v) = V(t, w, l, i; 1). \tag{2.14}$$

3. Analytical results

In this section, we present an analytical solution for the indifference price $p^u(t)$ in Eq (2.14). Define $g(l; v) = v\Pi(l)$, and consider

$$V(t, w, l, i; v) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(W(T) + g(L(T); v)) \mid W(t) = w, L(t) = l, X(t) = i\right]. \tag{3.1}$$

Theorem 3.1. *Suppose $g(l; v)$ is bounded. For any $(t, w, l, i) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{D}$,*

$$V(t, w, l, i; v) = -e^{-\gamma w e^{r(T-t)}} e^{-\beta_i (T-t)} \mathbb{E}\left[e^{-\gamma g(L(T); v)} \mid L(t) = l, X(t) = i\right], \tag{3.2}$$

where $\beta_i = \boldsymbol{\alpha}_i^\top (\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i^\top)^{-1} \boldsymbol{\alpha}_i - \frac{1}{2} \sum_{j=1}^m \left((\boldsymbol{\sigma}_j^i)^\top (\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i^\top)^{-1} \boldsymbol{\alpha}_i\right)^2$, $\boldsymbol{\alpha}_i = (\mu_1^i - r, \ldots, \mu_n^i - r)^\top$, $\boldsymbol{\sigma}_j^i = (\sigma_{1j}^i, \ldots, \sigma_{nj}^i)^\top$, *and* $\boldsymbol{\sigma}_i = (\sigma_{kj}^i)_{n \times m}$.

Proof. To find the value function Eq (3.1), we solve the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \sup_{\pi \in \mathcal{A}} \Bigg\{ & V_t(t, w, l, i; v) + \left(rw + \sum_{k=1}^n \pi_k (\mu_k^i - r)\right) V_w(t, w, l, i; v) \\ & + \frac{1}{2} \sum_{j=1}^m \left(\sum_{k=1}^n \pi_k \sigma_{kj}^i\right)^2 V_{ww}(t, w, l, i; v) + \sum_{j=1}^Q a_{ij} V(t, w, l, j; v) \end{aligned}$$

$$+ \lambda_i \mathbb{E}\left[V(t,w,l+Y(t),i;v) - V(t,w,l,i;v)\right]\Bigg\} = 0, \tag{3.3}$$

subject to

$$V(T,w,l,i;v) = -e^{-\gamma(w+g(l;v))}. \tag{3.4}$$

Conjecture that

$$V(t,w,l,i;v) = -e^{-\gamma w A(t)} h(t,l,i;v), \tag{3.5}$$

with the boundary condition

$$A(T) = 1 \text{ and } h(T,l,i;v) = e^{-\gamma g(l;v)}. \tag{3.6}$$

Substituting Eq (3.5) into Eq (3.3) yields

$$\begin{aligned}
\sup_{\pi\in\mathcal{A}} \Bigg\{ & \gamma A_t(t)h(t,l,i;v)w - h_t(t,l,i;v) + \left(rw + \sum_{k=1}^{n}\pi_k(\mu_k^i - r)\right)\gamma A(t)h(t,l,i;v) \\
& - \frac{1}{2}\sum_{j=1}^{m}\left(\sum_{k=1}^{n}\pi_k\sigma_{kj}^i\right)^2 \gamma^2 A^2(t)h(t,l,i;v) - \sum_{j=1}^{Q} a_{ij}h(t,l,j;v) \\
& - \lambda_i \int_0^\infty \left(h(t,l+y,i;v) - h(t,l,i;v)\right) f_i(y)\mathrm{d}y \Bigg\} = 0.
\end{aligned} \tag{3.7}$$

Differentiating Eq (3.7) with respect to π, we have

$$\boldsymbol{\pi}^*(t) = \frac{(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i^\top)^{-1}\boldsymbol{\alpha}_i}{\gamma A(t)}. \tag{3.8}$$

Substituting Eq (3.8) into Eq (3.7) yields

$$\begin{aligned}
& \gamma A_t(t)h(t,l,i;v)w - h_t(t,l,i;v) + rw\gamma A(t)h(t,l,i;v) + \beta_i h(t,l,i;v) \\
& \quad - \sum_{j=1}^{Q} a_{ij}h(t,l,j;v) - \lambda_i \int_0^\infty \left(h(t,l+y,i;v) - h(t,l,i;v)\right) f_i(y)\mathrm{d}y = 0,
\end{aligned} \tag{3.9}$$

where $\beta_i = \boldsymbol{\alpha}_i^\top(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i^\top)^{-1}\boldsymbol{\alpha}_i - \frac{1}{2}\sum_{j=1}^{m}\left((\boldsymbol{\sigma}_j^i)^\top(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i^\top)^{-1}\boldsymbol{\alpha}_i\right)^2$ with $\boldsymbol{\sigma}_j^i = (\sigma_{1j}^i, \sigma_{2j}^i, \cdots, \sigma_{nj}^i)^\top$. From Eq (3.9), we have

$$\gamma A_t(t)h(t,l,i;v)w + rw\gamma A(t)h(t,l,i;v) = 0, \tag{3.10}$$

and

$$- h_t(t,l,i;v) + \beta_i h(t,l,i;v) - \sum_{j=1}^{Q} a_{ij}h(t,l,j;v)$$

$$-\lambda_i \int_0^\infty \left(h(t, l+y, i; v) - h(t, l, i; v)\right) f_i(y)\mathrm{d}y = 0. \tag{3.11}$$

It follows from Eq (3.10) and the boundary condition in Eq (3.6) that we obtain

$$A(t) = e^{r(T-t)}. \tag{3.12}$$

By the Itô lemma, we have

$$\begin{aligned} & h(T, L(T), X(T); v) - e^{\beta_{X(t)}(T-t)} h(t, L(t), X(t); v) \\ =& \int_t^T \sum_{j=1}^Q a_{X(s)j} e^{\beta_{X(s)}(T-s)} h(s, L(s), X(s); v)\mathrm{d}s + \int_t^T \sum_{j=1}^Q e^{\beta_{X(s)}(T-s)} h(s, L(s), X(s); v)\mathrm{d}M(s, j) \\ &- \int_t^T \beta_{X(s)} e^{\beta_{X(s)}(T-s)} h(s, L(s), X(s); v)\mathrm{d}s + \int_t^T e^{\beta_{X(s)}(T-s)} h_s(s, L(s), X(s); v)\mathrm{d}s \\ &+ \sum_{t<s\leqslant T} e^{\beta_{X(s)}(T-s)} \left(h(s, L(s-) + \Delta L(s), X(s); v) - h(s, L(s-), X(s); v)\right), \end{aligned} \tag{3.13}$$

where $M(s, j)$ is a martingale. Due to Eq (3.11), Eq (3.13) can be rewritten as

$$\begin{aligned} & h(T, L(T), X(T); v) \\ =& e^{\beta_{X(t)}(T-t)} h(t, L(t), X(t); v) + \int_t^T \sum_{j=1}^Q e^{\beta_{X(s)}(T-s)} h(s, L(s), X(s); v)\mathrm{d}M(s, j) \\ &- \int_t^T \int_0^\infty \lambda_{X(s)} e^{\beta_{X(s)}(T-s)} \left(h(s, L(s-) + y, X_s; n) - h(s, L(s-), X(s); v)\right) f_{X(s)}(y)\mathrm{d}y\mathrm{d}s \\ &+ \sum_{t<s\leqslant T} e^{\beta_{X(s)}(T-s)} \left(h(s, L(s-) + \Delta L(s), X(s); v) - h(s, L(s-), X(s); v)\right). \end{aligned} \tag{3.14}$$

Taking the expectation on both sides and then applying Eq (3.6), we obtain

$$\begin{aligned} e^{\beta_i(T-t)} h(t, l, i; v) =& \mathbb{E}\left[h(T, L(T), X(T); v) | L(t) = l, X(t) = i\right] \\ =& \mathbb{E}\left[e^{-\gamma g(L(T); v)} | L(t) = l, X(t) = i\right]. \end{aligned} \tag{3.15}$$

Therefore,

$$h(t, l, i; v) = e^{-\beta_i(T-t)} \mathbb{E}\left[e^{-\gamma g(L(T); v)} | L(t) = l, X(t) = i\right]. \tag{3.16}$$

From Eqs (3.5) and (3.16), we get Eq (3.2). □

Thus, from Eqs (2.12) and (2.13) and Theorem 3.1, the indifference price depends on

$$\varphi(t, l, i; v) \triangleq \mathbb{E}\left[e^{-\gamma v \Pi(L(T))} \mid L(t) = l, X(t) = i\right], \tag{3.17}$$

where

$$\Pi(L(T)) = \int_0^T e^{r(T-s)}\, \mathrm{d}L(s) = \int_0^T \int_0^\infty e^{r(T-s)} x\, N_{X(s)}(\mathrm{d}s \times \mathrm{d}x), \tag{3.18}$$

and $N_{X(t)}(\mathrm{d}t \times \mathrm{d}x)$ is a Poisson random measure with intensity $\lambda_{X(t)} f_{X(t)}(\mathrm{d}x)\,\mathrm{d}t$.

To obtain analytical results for Eq (3.17), we consider a two-state Markov chain for $X(t)$, representing expansion and contraction, with transition matrix

$$\mathbf{A} = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix}. \tag{3.19}$$

To calculate the conditional expectation of Eq (3.17), we introduce the occupation times of the Markov chain $X(t)$. The occupation time of the Markov chain $X(t)$ at state i in interval $[0, t]$ is defined as

$$O_i(t) = \int_0^t \mathrm{I}_{\{X(s)=i\}} \mathrm{d}s, \quad i = 1, 2.$$

Following from [16], the density function of occupation time $O_i(t)$ is

$$\begin{aligned} \zeta_i(t, x) =& e^{-a_i t}\delta_t(x) + e^{-a_i x} e^{-(a-a_i)(t-x)} \Big[a_i I_0\left(2\sqrt{a_i a_j x(t-x)}\right) \\ &+ \sqrt{\frac{a_i a_j x}{t-x}} I_1\left(2\sqrt{a_i a_j x(t-x)}\right) \Big], \quad a = a_1 + a_2,\ i \neq j \in \{1, 2\}, \end{aligned} \tag{3.20}$$

where $I_\rho(z), z > 0$ is the modified Bessel function of the first kind

$$I_\rho(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\rho+1)} \left(\frac{z}{2}\right)^{2k+\rho}.$$

Theorem 3.2. *For the two-state Markov chain, the conditional expectation in Eq* (3.17) *is given by*

$$\varphi(t, l, i; v) = e^{-\gamma v l e^{r(T-t)}} \psi(t, l, i; v), \tag{3.21}$$

where

$$\begin{aligned} \psi(t, l, i; v) = \int_0^{T-t} \exp\Big\{ &\int_t^{t+t_1} \int_0^\infty [e^{-\gamma v e^{r(T-s)} x} - 1] \lambda_i f_i(x) \mathrm{d}x \mathrm{d}s \\ &+ \int_{t+t_1}^T \int_0^\infty [e^{-\gamma v e^{r(T-s)} x} - 1] \lambda_j f_j(x) \mathrm{d}x \mathrm{d}s \Big\} \zeta_i(T-t, t_1) \mathrm{d}t_1. \end{aligned} \tag{3.22}$$

Proof. By the tower property of conditional expectation, we first write

$$\begin{aligned} \varphi(t, l, i; v) &= \mathbb{E}\left[\exp\left\{ -\gamma v \left(l e^{r(T-t)} + \int_t^T \int_0^\infty e^{r(T-s)} x N_{X(s)}(\mathrm{d}s \times \mathrm{d}x) \right) \right\} \middle| L(t) = l, X(t) = i \right] \\ &= e^{-\gamma v l e^{r(T-t)}} \mathbb{E}\left[\mathbb{E}\left[\exp\left\{ -\gamma v \int_t^T \int_0^\infty e^{r(T-s)} x N_{X(s)}(\mathrm{d}s \times \mathrm{d}x) \right\} \middle| \mathcal{F}_T^X \right] \middle| X(t) = i \right]. \end{aligned} \tag{3.23}$$

Following Proposition 3.6 in [17], we obtain

$$\mathbb{E}\left[\exp\left\{ -\gamma v \int_t^T \int_0^\infty e^{r(T-s)} x N_{X(s)}(\mathrm{d}s \times \mathrm{d}x) \right\} \middle| \mathcal{F}_T^X \right]$$

$$= \exp\left\{\int_t^T \int_0^\infty \left(e^{-\gamma v e^{r(T-s)}x} - 1\right) \lambda_{X(s)} f_{X(s)}(x) \mathrm{d}x \mathrm{d}s\right\}. \tag{3.24}$$

Next, the distribution of the time spent in each regime is characterized by the occupation time density $\zeta_i(\cdot,\cdot)$ in Eq (3.20). Thus, by integrating over all possible occupation times t_1 in state i during $[t, T]$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\int_t^T \int_0^\infty \left(e^{-\gamma v e^{r(T-s)}x} - 1\right) \lambda_{X(s)} f_{X(s)}(x) \mathrm{d}x \mathrm{d}s\right\} \middle| X(t) = i\right] \\
&= \int_0^{T-t} \exp\left\{ \int_t^{t+t_1} \int_0^\infty \left(e^{-\gamma v e^{r(T-s)}x} - 1\right) \lambda_i f_i(x) \mathrm{d}x \mathrm{d}s \right. \\
&\qquad \left. + \int_{t+t_1}^T \int_0^\infty \left(e^{-\gamma v e^{r(T-s)}x} - 1\right) \lambda_j f_j(x) \mathrm{d}x \mathrm{d}s\right\} \zeta_i(T-t, t_1) \mathrm{d}t_1.
\end{aligned} \tag{3.25}$$

Combining Eqs (3.23)–(3.25) yields Eq (3.21). □

Remark 3.1. *Note that before Theorem 3.2, without loss of generality, we assume that the Markov chain only has two states. When the Markov chain has more than two states, we cannot obtain the closed-form expressions of the density function of occupation time in the literature, which makes it very difficult to derive the analytic expressions of $\varphi(t, l, i; v)$.*

Theorem 3.3. *For the two-state Markov chain, the indifference price of the NPL-backed securities is*

$$p^u(t) = (1 - v)l - \frac{1}{\gamma} e^{-r(T-t)} \ln\left(\frac{\psi(t, l, i; 1)}{\psi(t, l, i; v)}\right), \tag{3.26}$$

where $\psi(t, l, i; v)$ is given by Eq (3.22).

Proof. Applying Theorem 3.2 and plugging the related expectations into the price formula, we have

$$p^u(t) = -\frac{1}{\gamma} e^{-r(T-t)} \ln\left(\frac{\varphi(t, l, i; 1)}{\varphi(t, l, i; v)}\right).$$

Direct substitution and simplification yield Eq (3.26). □

Remark 3.2. *The parameter u can be interpreted as the proportion of cash flow sold in the senior tranche. When considering the risk and return characteristics of different tranches, certain results can be derived from the perspective of fair premium pricing. However, under the utility-based framework of indifference pricing, some technical challenges remain unresolved.*

4. Numerical results

In this section, we assume that the repayment amount $Y(t)$ follows an exponential distribution, with density

$$f_i(y) = \beta_i e^{-\beta_i y}$$

for $y \geq 0$.

To demonstrate the theoretical findings, we perform numerical simulations using a representative parameter configuration, as summarized in Table 1. For simplicity, the non-performing loans in the pool are treated as a single aggregate entity. The parameter values presented in Table 1 indicate that $\lambda_1 < \lambda_2$, $\beta_1 > \beta_2$, and $a_1 > a_2$. Here, State 1 corresponds to a contraction regime, whereas State 2 represents an expansion regime.

Table 1. The economic meaning of the model parameters.

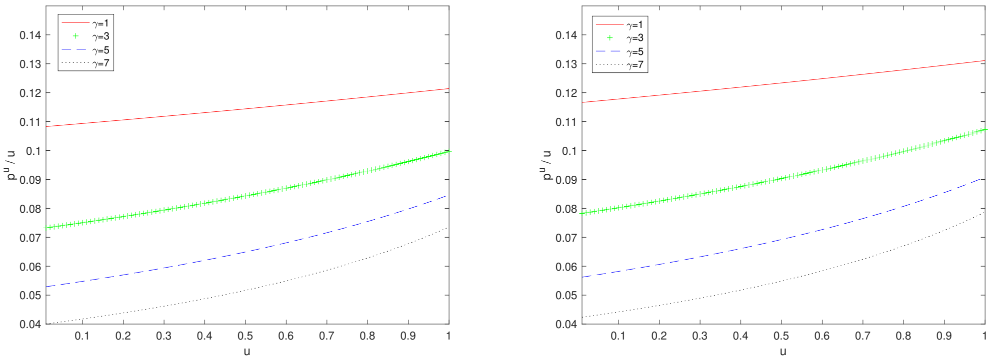
Parameters	Value	Economic meaning
r	0.02	the risk-free rate
T	3	investment horizon
γ	3	the absolute aversion coefficient of the bank
λ_1	0.3	$0.3\Delta(t)$ repayment events occurred during the time interval $(t, t + \Delta(t))$ in regime 1
λ_2	0.5	$0.5\Delta(t)$ repayment events occurred during the time interval $(t, t + \Delta(t))$ in regime 2
β_1	10	the reciprocal of the expected repayment amounts in regime 1
β_2	8	the reciprocal of the expected repayment amounts in regime 2
a_1	0.6	the reciprocal of the average sojourn time in regime 1
a_2	0.3	the reciprocal of the average sojourn time in regime 2

Remark 4.1. *If sufficient data on non-performing loan recoveries are available, specifically the amounts recovered at the end of each year, the parameter estimates can be obtained using maximum likelihood estimation. The detailed parameter estimation methodology is presented in the appendix.*

Figures 1–3 illustrate the relationships between the securitization proportion u and the unit indifference price p^u/u under different model parameters and economic regimes. Across all scenarios, the indifference price increases with u, indicating that a higher securitization proportion leads to a higher price. Moreover, for a given u, the indifference price is consistently higher in the expansion regime than in the contraction regime, suggesting that banks can achieve better prices for NPL-backed securities during periods of economic growth.

Figures 1–3 show that the unit indifference price p^u/u is an increasing function of the securitization ratio u. This indicates that as the proportion of assets sold increases, the bank demands a higher unit indifference price. The bank is willing to discount its riskiest assets but insists on maintaining the original price for the portion of assets that already carries low risk. Initially, when selling the first unit, the bank's primary objective is to rapidly reduce risk, making price a secondary consideration. The bank is inclined to accept a price as long as it covers the expected value and provides slight compensation. However, by the time the last units are sold, risk mitigation is no longer the primary concern. The bank's main goal shifts to profit maximization and it is no longer willing to accept a discount for risk reduction. Instead, it requires the buyer to pay a price close to the full expected value of the asset. This phenomenon occurs because securitization represents a transition from a high-risk to a low-risk state. As more units are sold, the risk profile of the remaining assets improves. Consequently, the bank's valuation of these remaining assets approaches their full expected value, which naturally leads to a higher unit price.

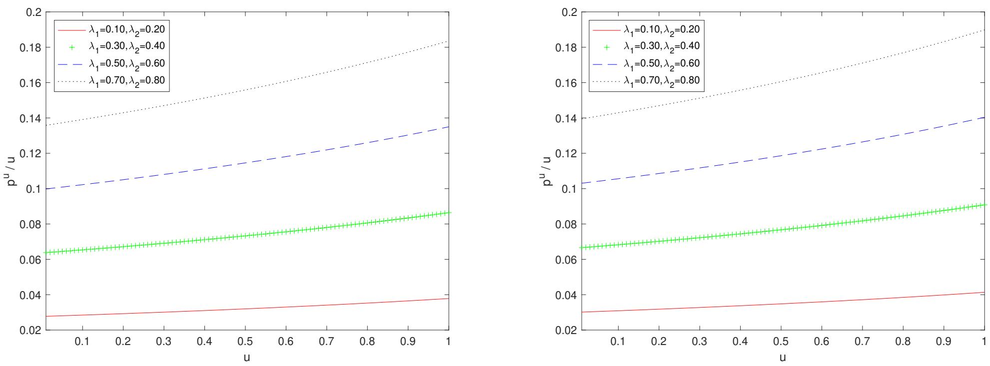
Figure 1 shows that, holding u fixed, the indifference price decreases as the absolute risk aversion coefficient γ increases. This reflects the fact that more risk-averse banks are willing to sell at lower prices. This is because risk-averse banks prioritize the safety of assets over potential returns. Thus, according to expected utility theory, a bank seeking to rapidly reduce risk exposure is willing to accept a lower price in exchange for certainty.



(a) $X(0) = 1$. **(b)** $X(0) = 2$.

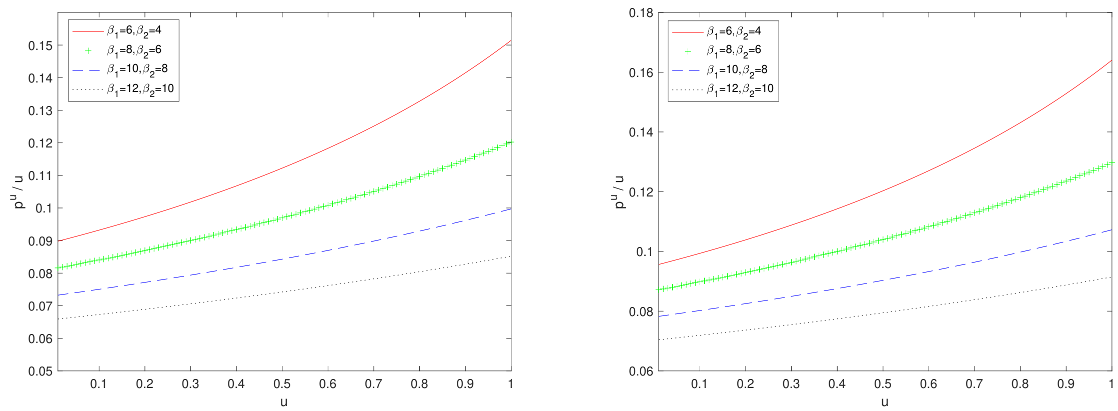
Figure 1. The relationships between proportion u of securitization and the unit indifference price p^u/u for different absolute risk aversion coefficients γ.

Figure 2 demonstrates that the indifference price increases with the repayment intensity λ; as borrowers repay more frequently, the expected cash flows rise, leading to higher security prices. In contrast, Figure 3 reveals that the indifference price declines as β increases, since a higher β corresponds to smaller expected repayment amounts, thereby reducing the value of the securitized NPLs. An improvement in repayment ability increases the amount and certainty of expected cash flows, thereby driving the indifference price of the security higher. The price of a non-performing loan-backed security is fundamentally determined by the quality of its underlying non-performing loans. An enhancement in borrowers' repayment ability directly leads to an improvement in the overall credit quality of the asset pool; what was once an asset "likely to incur losses" transforms into one with the potential to recover a significant portion. As asset quality improves, the security becomes more attractive to investors. This increased demand further pushes its indifference price upward.



(a) $X(0) = 1$. **(b)** $X(0) = 2$.

Figure 2. The relationships between proportion u of securitization and the unit indifference price p^u/u for different repayment intensities λ_1 and λ_2.



(a) $X(0) = 1$.   **(b)** $X(0) = 2$.

Figure 3. The relationships between proportion u of securitization and the unit indifference price p^u/u for different parameters β_1 and β_2.

5. Conclusions

This paper presents a unified valuation framework for NPL-backed securities in the presence of regime-switching macroeconomic conditions. By modeling the random payment times of NPLs with a Cox process featuring regime-dependent intensities, and adopting an exponential utility-based indifference pricing approach, we are able to accommodate both market incompleteness and investor risk preferences. Closed-form solutions for the indifference price are derived under a two-regime Markov chain representing expansion and contraction states. Numerical experiments reveal several key insights: the indifference price increases with the securitization proportion and repayment intensity, but decreases with higher risk aversion and larger repayment scale parameters. Moreover, prices are consistently higher during economic expansion than contraction, highlighting the importance of macroeconomic dynamics in NPL valuation. These results provide a practical reference for both banks and investors in structuring and pricing NPL transactions under uncertainty.

Author contributions

Wanrong Mu: Investigation, Methodology, Writing—original draft, Validation, Software; Congjin Zhou: Conceptualization, Investigation, Supervision, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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Supplementary

Parameters estimation methodology

Let $\boldsymbol{\theta}$ be the vector of parameters. Without loss of generality, following [18], we assume that the initial state $X(0)$ of the Markov chain has a stationary distribution

$$\mathrm{P}\left(X(0)=1\right)=\frac{a_2}{a_1+a_2} \text{ and } \mathrm{P}(X(0)=2)=\frac{a_1}{a_1+a_2}. \tag{S.1}$$

Proposition S.1. *Assume that $X(s)=i$ for $0\leqslant s\leqslant t$, the density function of $L(t)$ can be represented as*

$$f(x;i,t,\boldsymbol{\theta})=\sum_{l=1}^{\infty}\frac{\lambda_i^l x^{l-1}e^{-\lambda_i x}}{\Gamma(l)}e^{-\lambda_i t}\frac{(\lambda_i t)^l}{l!} \tag{S.2}$$

where $i=1,2$.

Proof. For sufficiently small h such that $x\leqslant L(t)\leqslant x+h$, we have

$$\mathrm{P}_i\left(x\leqslant L(t)\leqslant x+h;\boldsymbol{\theta}\right)$$

$$= \sum_{l=1}^{\infty} \mathrm{P}_i \left(x \leqslant \sum_{m=0}^{l} Y_m \leqslant x + h, N(t) = l; \boldsymbol{\theta} \right)$$
$$= \sum_{l=1}^{\infty} \mathrm{P}_i \left(x \leqslant \sum_{m=0}^{l} Y_m \leqslant x + h; \boldsymbol{\theta} \middle| N(t) = l \right) \mathrm{P}_i \left(N(t) = l; \boldsymbol{\theta} \right)$$
$$= \sum_{l=1}^{\infty} \int_x^{x+h} \frac{\lambda_i^l s^{l-1} e^{-\lambda_i s}}{\Gamma(l)} \, \mathrm{d}s \, e^{-\lambda_i t} \frac{(\lambda_i t)^l}{l!}. \tag{S.3}$$

Dividing by h and taking $h \to 0$, we obtain the density function of $L(t)$ in Eq (S.2) □

Denote by

$$\mathrm{d}L_i(t) = Y_i(t) \mathrm{d}N_i(t), \tag{S.4}$$

where $N_i(t)$ is a Poisson process with parameter λ_i, and the density function of Y_i is $f_i(y)$.

Lemma S.1. *Given $O_i(t) = s$ for $0 < s < t$, the process $L(t)$ can be written as*

$$L(t) = L_i(s) + L_j(t - s), \tag{S.5}$$

where $i, j = 1, 2$ and $i \neq j$.

Proof. Given $O_i(t) = s$ for $0 < s < t$, from the additivity, the stationary increments property, and the independent increments property of the Poisson processes, $L(t)$ can be written as Eq (S.5) by summing up the jumps in each state. □

Proposition S.2. *Given $O_i(t) = s$ for $0 < s < t$, the density function of $L(t)$ can be represented as*

$$f(x; i, s, t, \boldsymbol{\theta}) = \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \int_0^x \frac{\lambda_i^{l_1} y^{l_1-1} e^{-\lambda_i y}}{\Gamma(l_1)} \frac{\lambda_j^{l_2} (x-y)^{l_2-1} e^{-\lambda_j (x-y)}}{\Gamma(l_2)} \, \mathrm{d}y$$
$$\times e^{-\lambda_i s} \frac{(\lambda_i s)^{l_1}}{l_1!} e^{-\lambda_j (t-s)} \frac{(\lambda_j (t-s))^{l_2}}{l_2!}, \tag{S.6}$$

where $i, j = 1, 2$ and $i \neq j$.

Proof. Taking sufficiently small h such that $x \leqslant L(t) \leqslant x + h$, from Lemma S.1, we have

$$\mathrm{P}_i \left(x \leqslant L(t) \leqslant x + h; \boldsymbol{\theta} | O_i(t) = s \right)$$
$$= \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \mathrm{P}_i \left(x \leqslant L_i(s) + L_j(t-s) \leqslant x + h; \boldsymbol{\theta} | O_i(t) = s, N_i(s) = l_1, N_j(t-s) = l_2 \right)$$
$$\times \mathrm{P}_i \left(N_i(s) = l_1, N_j(t-s) = l_2; \boldsymbol{\theta} \middle| O_i(t) = s \right)$$
$$= \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \int_0^{x+h} \int_{x-y}^{x+h-y} \frac{\lambda_i^{l_1} y^{l_1-1} e^{-\lambda_i y}}{\Gamma(l_1)} \frac{\lambda_j^{l_2} z^{l_2-1} e^{-\lambda_j z}}{\Gamma(l_2)} \, \mathrm{d}z \, \mathrm{d}y$$
$$\times e^{-\lambda_i s} \frac{(\lambda_i s)^{l_1}}{l_1!} e^{-\lambda_j (t-s)} \frac{(\lambda_j (t-s))^{l_2}}{l_2!}. \tag{S.7}$$

Dividing by h and taking $h \to 0$, we obtain the density function of $L(t)$ in Eq (S.6) □

We denote by $M(t)$ the number of state transitions up to time t. From [19, p.283], $M(t)$ is a Poisson process with parameter $a = a_1 + a_2$. Thus,

$$\mathrm{P}_i\left(X(t) = j, M(t) = n\right) = \frac{a - a_j}{a}\frac{(at)^n}{n!}e^{-at}. \tag{S.8}$$

From [19, pp.285–286], given $X(t) = i$ and $M(t) = n$, the conditional distribution of occupation time can be represented as

$$\begin{aligned}\mathrm{P}_i(n, s, t, i) &= \mathrm{P}(O_i(t) \le s | X(t) = i, M(t) = n, X(0) = i)\\ &= \sum_{k=2}^{n}\binom{n-1}{k-2}\left(\frac{a_j}{a}\right)^{k-2}\left(\frac{a_i}{a}\right)^{n-k+1}C(k, n, s, t),\end{aligned} \tag{S.9}$$

and given $X(t) = j$ and $M(t) = n$, where $j \neq i$, the conditional distribution of occupation time can be represented as

$$\begin{aligned}\mathrm{P}_i(n, s, t, j) &= \mathrm{P}(O_i(t) \le s | X(t) = j, M(t) = n, X(0) = i)\\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}\left(\frac{a_j}{a}\right)^{k-1}\left(\frac{a_i}{a}\right)^{n-k}C(k, n, s, t)\end{aligned} \tag{S.10}$$

where

$$C(k, n, s, t) = \sum_{j=k}^{n}\binom{n}{j}\left(\frac{s}{t}\right)^{j}\left(1 - \frac{s}{t}\right)^{n-j},\ 0 < s < t$$

for $i, j = 1, 2$. It is easy to see that

$$\frac{\partial \mathrm{P}_i(n, s, t, i)}{\partial s} = \sum_{k=2}^{n}\binom{n-1}{k-2}\left(\frac{a_j}{a}\right)^{k-2}\left(\frac{a_i}{a}\right)^{n-k+1}C_s(k, n, s, t),$$

and

$$\frac{\partial \mathrm{P}_i(n, s, t, j)}{\partial s} = \sum_{k=1}^{n}\binom{n-1}{k-1}\left(\frac{a_j}{a}\right)^{k-1}\left(\frac{a_i}{a}\right)^{n-k}C_s(k, n, s, t),$$

where

$$C_s(k, n, s, t) = \frac{\partial C(k, n, s, t)}{\partial s} = \sum_{j=k}^{n}\binom{n}{j}\left(\frac{s}{t}\right)^{j-1}\left(1 - \frac{s}{t}\right)^{n-j-1}\frac{jt - ns}{t^2},\ 0 < s < t.$$

To simplify the expression, for $0 < s < t$, we define

$$\Theta_i(s, t, j) = \sum_{n=2}^{\infty}\mathrm{P}\left(X(t) = j, M(t) = n | X(0) = i\right)\frac{\partial}{\partial s}\mathrm{P}_i(n, s, t, j), \tag{S.11}$$

where the right hand side has two cases, one for $i \neq j$ and the other for $i = j$. Suppose there are observed values $L_1, L_2, \cdots, L_H$. From [20, 21], the likelihood function of the model parameters can be written as

$$L(\boldsymbol{\theta}) = \prod_{k=0}^{H} \sum_{i=1}^{2} \{(f(L_k, X((k+1)\Delta t) = i; \boldsymbol{\theta} | X(k\Delta t) = i) \\ + f(L_k, X((k+1)\Delta t) = j; \boldsymbol{\theta} | X(k\Delta t) = i))\, P(X(k\Delta t) = i)\},$$

where

$$\begin{aligned} & f(L_k, X((k+1)\Delta t) = i; \boldsymbol{\theta} | X(k\Delta t) = i) \\ = & e^{-a_i \Delta t} f(L_k; i, \Delta t, \boldsymbol{\theta}) + \int_0^{\Delta t} f(L_k; i, s, \Delta t, \boldsymbol{\theta}) \Theta_i(s, \Delta t, i) ds \end{aligned} \tag{S.12}$$

and

$$f(L_k, X((k+1)\Delta t) = j; \boldsymbol{\theta} | X(k\Delta t) = i) = \int_0^{\Delta t} f(L_k; i, s, \Delta t, \boldsymbol{\theta}) \Theta_i(s, \Delta t, j) ds, \tag{S.13}$$

where $i, j = 1, 2$ and $i \neq j$. The maximum likelihood estimation is to find the estimated values $\boldsymbol{\theta}^*$ of model parameters that maximize the likelihood of occurrence of the observed data, which is

$$\boldsymbol{\theta}^* = \text{argmax} L(\boldsymbol{\theta}). \tag{S.14}$$