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Research article

Lie symmetry analysis for the fractal Black-Scholes equation of option pricing

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Abstract: In recent years, numerous advanced technologies have been applied to financial research, yet few scholars have utilized Lie symmetry theory. Focusing on the fractal characteristics of financial markets, this paper constructed a fractal Black-Scholes (B-S) equation of option pricing incorporating He's fractal derivative, as well as studied its solution and dynamic properties through the Lie symmetry analysis method. First, the fractal equation was transformed into an equivalent continuous equation using the improved fractal two-scale transform. Then, the geometric vector fields, symmetry reductions, and exact solutions of the equation were derived. Finally, combined with the data from Linnan Dongsheng Film and Television Co., Ltd., the dynamic impacts of parameters such as fractal dimension, volatility, and risk-free interest rate on option prices were simulated and analyzed. The study found that the fractal dimension has a significant regulatory effect on the sensitivity of option prices, and volatility and risk-free interest rate showed phased influence characteristics by affecting the price fluctuation space and time value. This research provided a new theoretical method and analytical tool for option pricing in fractal market environments.

Keywords: fractal B-S equation of option pricing; Lie symmetry analysis method; improved fractal two-scale transform; exact solution; dynamic behavior
Mathematics Subject Classification: 35Q91, 91G20

1. Introduction

In the financial market, options are among the most important and popular financial derivatives. Therefore, pricing options is an important issue in both theory and practice. The Black-Scholes (B-S) model was proposed by Black, Scholes [1], and Merton [2] in 1973, which provides an approximate description of the behavior of the underlying asset. Most financial systems rely on the efficient market hypothesis (EMH), which includes the B-S formula for option pricing. However, this system has flaws because the EMH is based on the assumption that economic processes follow a normal distribution (Gaussian random walk). This leads to biases in predicting the future volatility of market prices. Therefore, a new financial risk assessment model is needed to address this deficiency. The most reliable mathematical tool currently is based on Lévy statistics and a nonstationary fractional diffusion equation characterized by the Lévy index. The fractional aspects of the B-S model have been discussed extensively in the literature from various perspectives. For instance, there are the time-fractional B-S equations with Caputo derivatives [3] and Liouville-Caputo fractional derivatives [4], as well as the pricing fractional B-S equations with Weyl fractional derivatives [5,6] or those combining both Caputo and Weyl derivatives [7]. Araneda [8] presented a multi-fractal option pricing formula, and Zhang et al. [9] studied fractional B-S equations and their solution techniques. Recently, the numerical pricing of American and European options under the time-fractional B-S equation has been extensively studied [10–12]. Li et al. [13] conducted a fractional study on bank data using fractal-fractional Caputo derivatives, while Li et al. [14] developed a mathematical model for financial bubbles under fractal-fractional Caputo derivatives. Garcin [15] proposed a method to extract the realized Hurst exponent from logarithmic prices. Bianchi et al. [16,17] also investigated the relevance of multi-fractal processes with random exponents and fractal stokes models in financial research. Regarding the Hurst exponent and volatility, other recent studies include the work of Di Persio and Turatta [18] as well as Zournatzidou and Floros [19]. Despite the numerous advantages of these methods, as well as those using fractional derivative operators, there is a lack of sufficiently detailed analysis of the B-S equation in fractal dimensions.

Some studies have attempted to apply fractal analysis in financial research and the estimation of financial volatility. Research shows that high-risk assets have higher fractal dimensions, while stable assets possess lower fractal dimension. The fractal dimension has been verified as a practical tool, and since stock market prices are fractal objects, the fractal dimension can be used to measure the degree of complexity [20, 21]. Fractal dimensional analysis has also been applied to study the Indian financial market [22] and to examine the statistical characteristics of volatility in the stock markets of New York, Tokyo, Taiwan, South Korea, Singapore, and Hong Kong [23]. In addition, [24] studied the properties of spiral waves in the $\lambda - \omega$ system under fractal dimension. He et al. discussed the applications from two-scale thermodynamics to fractal variational principles as well as future challenges[25], and demonstrated the application by combining it with the fractal Zhiber-Shabat oscillator [26]. [27] integrated jump processes into the sub-mixed fractional Brownian motion, constructed a pricing model for fractal barrier options, and derived the corresponding pricing formula. These fractal derivatives are regarded as extensions of the Leibniz derivative in discontinuous fractal media. In particular, He's fractal derivative can effectively simulate the impact of discontinuous patterns on the properties of solutions. Through the improved fractal two-scale transform, fractal spaces can be converted into continuous problems, making it possible to effectively solve fractal differential equations and further conduct a detailed analysis of the dynamic behavior of the system under different fractal dimensions. The Lie symmetry analysis method plays a crucial role in the study of partial differential equations (PDEs). Based on this, Abbas et al. investigated the Lie group analysis, solitary wave solutions, and conservation laws of the Schamel-Burger's equation [28]. This method has been extended to the study of multi-dimensional PDEs. Regarding (2+1)-dimensional equations, examples include the Ito equation [29], the Zakharov-Kuznetsov equation [30], the Date-Jimbo-Kashiwara-Miwa equation [31], the Schwarzian Korteweg-de Vries equation [32], and the (3+1)-dimensional Gardner-KP equation[33]. The Lie symmetry analysis method has also been applied to option pricing problems in the field of financial mathematics. For example, Liu et al.[34] studied the B-S model for option pricing with dividend yields in financial markets via Lie symmetry analysis. [35] carried out Lie symmetry analysis on (1+2)-dimensional non-autonomous evolution equations in financial mathematics. Tseng et al. explored the option pricing problem of trading accounts using this method [36]. Considering that the price changes of the underlying asset can be regarded as a fractal transmission system, we introduce the He's fractal derivative to construct a fractal B-S option pricing equation. First, it is transformed into an equivalent equation using the improved fractal two-scale transform, and then the equation is analyzed using Lie symmetry methods. The fractal B-S option pricing equation is as follows

$$\frac{\partial V}{\partial t^{\alpha}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

where V represents the price of the option, S is a nonnegative value, representing the price of the underlying asset, r is the risk-free interest rate, t is the time value less than or equal to the maturity date T, σ is the volatility, and α is the fractal dimension.

The main contributions of this paper are reflected in three aspects: First, it introduces He's fractal derivative to construct the fractal B-S option pricing equation, which is more in line with the fractal characteristics of financial markets. Second, it combines the improved fractal two-scale transform with Lie symmetry analysis to systematically derive the geometric vector fields, symmetric reductions, and some exact solutions of the fractal B-S equation, providing a new analytical tool for option pricing in fractal markets. Third, it reveals the regulatory effect of fractal dimension on the sensitivity of option prices and the laws of phased influence of volatility and risk-free interest rate through numerical simulations, making up for the deficiency of existing studies in the analysis of the dynamic characteristics of the B-S equation under fractal dimension. The paper is organized as follows. To start, a basic theory of fractal calculus is presented in Section 2. Section 3 gives all of the geometric vector fields of the equations on the basis of the arbitrary parameters by use of the Lie symmetry analysis method and improved fractal two-scale transform. In Section 4, the symmetry reductions of the equations are investigated and the exponentiated solutions and the similarity solutions to the equations are also worked out. In Section 5, we discuss the dynamic behavior of the fractal B-S equation of option pricing under the influence of different parameters. The final Section 6 gives the conclusion.

2. Preliminaries

To make our presentations self-contained and easy to understand, we present briefly some basic notations and properties of the fractal derivative.

Definition 1. The fractal derivative with respect to t is defined as:

$$\frac{\partial u}{\partial t^{\alpha}}(t_0, x) = \Gamma(1+\alpha) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{u(t,x) - u(t_0,x)}{(t-t_0)^{\alpha}}, \tag{2.1}$$

where Δt is the smallest time scale, and α is the fractal dimension.

The following rules and formulae are very useful for practical applications:

(1) The chain rules:

$$\frac{\partial}{\partial t^{\alpha}}\Big(\frac{\partial u}{\partial t^{\beta}}\Big) = \frac{\partial}{\partial t^{\beta}}\Big(\frac{\partial u}{\partial t^{\alpha}}\Big), \tag{2.2}$$

$$\frac{\partial}{\partial t^{\alpha}}[\phi(u)] = \frac{\partial \phi}{\partial u}\Big(\frac{\partial u}{\partial t^{\alpha}}\Big). \tag{2.3}$$

(2) The differential and integration formulae:

$$\frac{\partial t^{m}}{\partial t^{\alpha}} = \frac{m}{\alpha} t^{m-\alpha}, \tag{2.4}$$

$$\int_{t_0^{\alpha}}^{t_1^{\alpha}} t^{m} dt^{\alpha} = \frac{\alpha}{m+\alpha}\Big[t_1^{\alpha(m+\alpha)} - t_0^{\alpha(m+\alpha)}\Big]. \tag{2.5}$$

3. Lie symmetry analysis for the fractal B-S equation of option pricing (1.1)

To begin, in order to facilitate solving Eq (1.1) and considering its financial significance, we introduce the improved fractal two-scale transform

$$\iota = -(T-t)^{\alpha}, \tag{3.1}$$

then the fractal B-S equation of option pricing (1.1) is transformed into

$$\frac{\partial V}{\partial \iota} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3.2}$$

In what follows, we use Lie symmetry analysis method to find all Lie symmetry algebras for the Eq (3.2). We know that the geometric vector fields of Eq (3.2) can be written in the following form

$$W = \zeta(S,\iota,V)\partial_S + \eta(S,\iota,V)\partial_\iota + \psi(S,\iota,V)\partial_V, \tag{3.3}$$

where $\zeta(S,\iota,V), \eta(S,\iota,V), \psi(S,\iota,V)$ are the coefficient functions to be determined. The symmetry group of Eq (3.2) will be derived from the vector field (3.3). Applying the second prolongation $pr^{(2)}W = pr^{(1)}W + \psi^{SS}\frac{\partial}{\partial V_{SS}} + \psi^{S\iota}\frac{\partial}{\partial V_{S\iota}} + \psi^{\iota\iota}\frac{\partial}{\partial V_{\iota\iota}}$ to Eq (3.2) and solving the determining equations, we find that the coefficient functions ζ, η and ψ must satisfy the following condition

$$\begin{aligned}
&\zeta = \frac{1}{2}S\eta_\iota \log S + S\rho, \qquad \psi = \mu(S,\iota)V + \nu(S,\iota),\\
&\mu = \frac{1}{4\sigma^2}\eta_{\iota\iota}\log^2 S + \Big(\frac{1}{4} - \frac{r}{2\sigma^2}\Big)\eta_\iota \log S + \frac{1}{\sigma^2}\rho_\iota \log S + \kappa,\\
&\frac{1}{4\sigma^2}\eta_{\iota\iota\iota}\log^2 S + \frac{1}{\sigma^2}\rho_{\iota\iota}\log S + \frac{1}{4}\eta_{\iota\iota} - \Big(\frac{1}{8}\sigma^2 + \frac{1}{2}r + \frac{r^2}{2\sigma^2}\Big)\eta_\iota - \Big(\frac{1}{2} - \frac{r}{\sigma^2}\Big)\rho_\iota + \kappa_\iota = 0,
\end{aligned} \tag{3.4}$$

where $\zeta, \eta, \psi, \rho, \kappa$ are the coefficient functions to be determined. By solving the equations, we get the vector field of Eq (3.2) as follows

$$\begin{aligned}
&W_1 = \partial_\iota, \quad W_2 = S\partial_S, \quad W_3 = V\partial_V, \quad W_4 = \sigma^2 S\iota\partial_S + \Big[\log S + (\frac{1}{2}\sigma^2 - r)\iota\Big]V\partial_V,\\
&W_5 = \sigma^2 S\iota\log S\,\partial_S + 2\sigma^2\iota\partial_\iota + \Big[(\frac{1}{2}\sigma^2 - r)\log S + ((\frac{1}{2}\sigma^2 - r)^2 + 2\sigma^2 r)\iota\Big]V\partial_V,\\
&W_6 = 2\sigma^2 S\iota\log S\,\partial_S + 2\sigma^2\iota^2\partial_\iota\\
&\qquad + \Big[(\sigma^2 - 2r)\iota\log S + \log^2 S - \sigma^2\iota + ((\frac{1}{2}\sigma^2 - r)^2 + 2\sigma^2 r)\iota^2\Big]V\partial_V, \quad W_\nu = \nu\partial_V,
\end{aligned} \tag{3.5}$$

where the function $\nu = \nu(S, \iota)$ satisfies Eq (3.2). It is easy to prove that a set of bases of Lie algebra of Eq (3.2) is $\{W_1, \ldots, W_6, W_\nu\}$. Furthermore, the one-parameter groups G_i generated by $W_i (i = 1, \ldots, 6, \nu)$ are presented as

$$\begin{aligned}
&G_1 : (S, \iota, V) \to (S, \iota + \epsilon, V), \quad G_2 : (S, \iota, V) \to (e^{\epsilon} S, \iota, V),\\
&G_3 : (S, \iota, V) \to (S, \iota, e^{\epsilon} V), \quad G_4 : (S, \iota, V) \to (S e^{\sigma^2 \epsilon \iota}, \iota, S^{\epsilon} e^{(\frac{1}{2}\sigma^2 - r)\epsilon\iota + \frac{1}{2}\sigma^2 \epsilon^2 \iota} V),\\
&G_5 : (S, \iota, V) \to \Big(S^{\delta}, \delta^2 \iota, \mathtt{exp}\Big[(\frac{1}{2} - \frac{r}{\sigma^2})(\delta - 1)\mathtt{log} S + (\frac{1}{8}\sigma^2 + \frac{1}{2} r + \frac{r^2}{2\sigma^2})(\delta^2 - 1)\iota\Big] V\Big),\\
&G_6 : (S, \iota, V) \to \Big(S^{\frac{1}{1 - 2\sigma^2 \epsilon\iota}}, \frac{\iota}{1 - 2\sigma^2 \epsilon \iota},\\
&\qquad \sqrt{1 - 2\sigma^2 \epsilon \iota}\,\mathtt{exp}\Big\{\Big[(\frac{1}{2} - \frac{r}{\sigma^2})\mathtt{log} S + \frac{1}{2\sigma^2 \iota}\mathtt{log}^2 S + (\frac{1}{8}\sigma^2 + \frac{1}{2} r + \frac{r^2}{2\sigma^2})\iota\Big]\frac{2\sigma^2 \epsilon \iota}{1 - 2\sigma^2 \epsilon \iota}\Big\} V\Big),\\
&G_\nu : (S, \iota, V) \to (S, \iota, V + \epsilon \nu),
\end{aligned} \tag{3.6}$$

where $\delta = e^{\sigma^2 \epsilon}$, and $\nu = \nu(S, \iota)$ satisfies Eq (3.2).

4. Symmetry reductions and exact solutions to the fractal B-S equation of option pricing (1.1)

In Section 3, we considered the symmetry and symmetry groups of Eq (3.2). Now, let's investigate its symmetric reductions and exact solutions.

Case 1. Solution for the generator W_1.

For W_1, we arrive at the following reduced ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 \zeta^2 f'' + r\zeta f' - rf = 0, \tag{4.1}$$

where$f' = \frac{df}{d\zeta}$. Considering the above Euler equation, we have the corresponding characteristic equation $\frac{1}{2}\sigma^2 \lambda^2 - (\frac{1}{2}\sigma^2 - r)\lambda - r = 0$, and solving it yields
$\lambda_{1,2} = \Big(\frac{1}{2}\sigma^2 - r \pm \sqrt{\Delta}\Big)/\sigma^2, \Delta = \frac{1}{4}\sigma^4 + r^2 + \sigma^2 r > 0$; hence, (4.1) admits the general solution $f = c_1 \zeta^{\lambda_1} + c_2 \zeta^{\lambda_2}$. Finally, we derive the exact solution of Eq (1.1) as follows:

$$V(S, t) = c_1 S^{\lambda_1} + c_2 S^{\lambda_2}, \tag{4.2}$$

where c_1 and c_2 are arbitrary constants, and $\lambda_{1,2} = \Big(\frac{1}{2}\sigma^2 - r \pm \sqrt{\Delta}\Big)/\sigma^2$ are two real roots to the characteristic equation, respectively.

Case 2. Solution for the generator W_2.

For W_2, we have the following reduced ODE:

$$f' - rf = 0. \tag{4.3}$$

Using the method of separation of variables, we can obtain that $f = e^{r\zeta}$. Then, the exact solution to Eq (1.1) is obtained as follows

$$V(S, t) = c\,\mathtt{exp}\Big[-r(T - t)^{\alpha}\Big], \tag{4.4}$$

where c is an arbitrary constant.

Case 3. Solution for the generator W_4.

We consider the following similarity transformation in terms of W_4

$$\zeta = \iota, \quad \theta = \log V - \frac{1}{2\sigma^2\iota}\Big[\log S + (\frac{1}{2}\sigma^2 - r)\iota\Big]^2, \tag{4.5}$$

and the similarity solution is $\theta = f(\zeta)$, then $V = \exp\Big[f(\iota) + \frac{1}{2\sigma^2\iota}(\log S + (\frac{1}{2}\sigma^2 - r)\iota)^2\Big]$. Thus, we have the reduced ODE:

$$2\zeta f^{'} - 2r\zeta + 1 = 0. \tag{4.6}$$

Solving Eq (4.6), we have $f(\zeta) = -\frac{1}{2}\log\zeta + r\zeta + c_1$. Then, we have the exact solution to Eq (1.1)

$$V(S,t) = c\exp\Big\{\frac{-1}{2\sigma^2(T-t)^\alpha}\Big[\log S - (\frac{1}{2}\sigma^2 - r)(T-t)^\alpha\Big]^2 - \frac{1}{2}\log V - r(T-t)^\alpha\Big\}. \tag{4.7}$$

Case 4. Solution for the generator W_6.

We take into account the following similarity transformation

$$\zeta = \iota^{-1}\log S, \quad \theta = \log V + \frac{1}{2}\log\iota - (\frac{1}{2} - \frac{r}{\sigma^2})\log S - (\frac{1}{8}\sigma^2 + \frac{1}{2}r + \frac{r^2}{2\sigma^2})\iota - \frac{1}{2\sigma^2}\iota^{-1}\log^2 S, \tag{4.8}$$

then we get the reduced ODE

$$f^{''} + f^{'2} = 0. \tag{4.9}$$

Solving (4.9), we have $f(\zeta) = \log|\zeta + c_1| + c_3$. Then, the exact solution to Eq (1.1) is given as

$$\begin{aligned} V(S,t) =& c_2\Big[\frac{-1}{(T-t)^\alpha}\log S + c_1\Big]\exp\Big[(\frac{1}{2} - \frac{r}{\sigma^2})\log S - (\frac{1}{8}\sigma^2 + \frac{1}{2}r + \frac{r^2}{2\sigma^2})(T-t)^\alpha \\ & - \frac{1}{2\sigma^2(T-t)^\alpha}\log^2 S + \frac{1}{2}\log(T-t)^\alpha\Big]. \end{aligned} \tag{4.10}$$

Case 5. Solution for the generator $W = W_2 + \rho W_3$($\rho \neq 0$ is an arbitrary constant).

We have the following similarity transformation

$$\zeta = \iota, \quad \theta = S^{-\rho}V, \tag{4.11}$$

and the similarity solution $\theta = f(\zeta)$, that is,

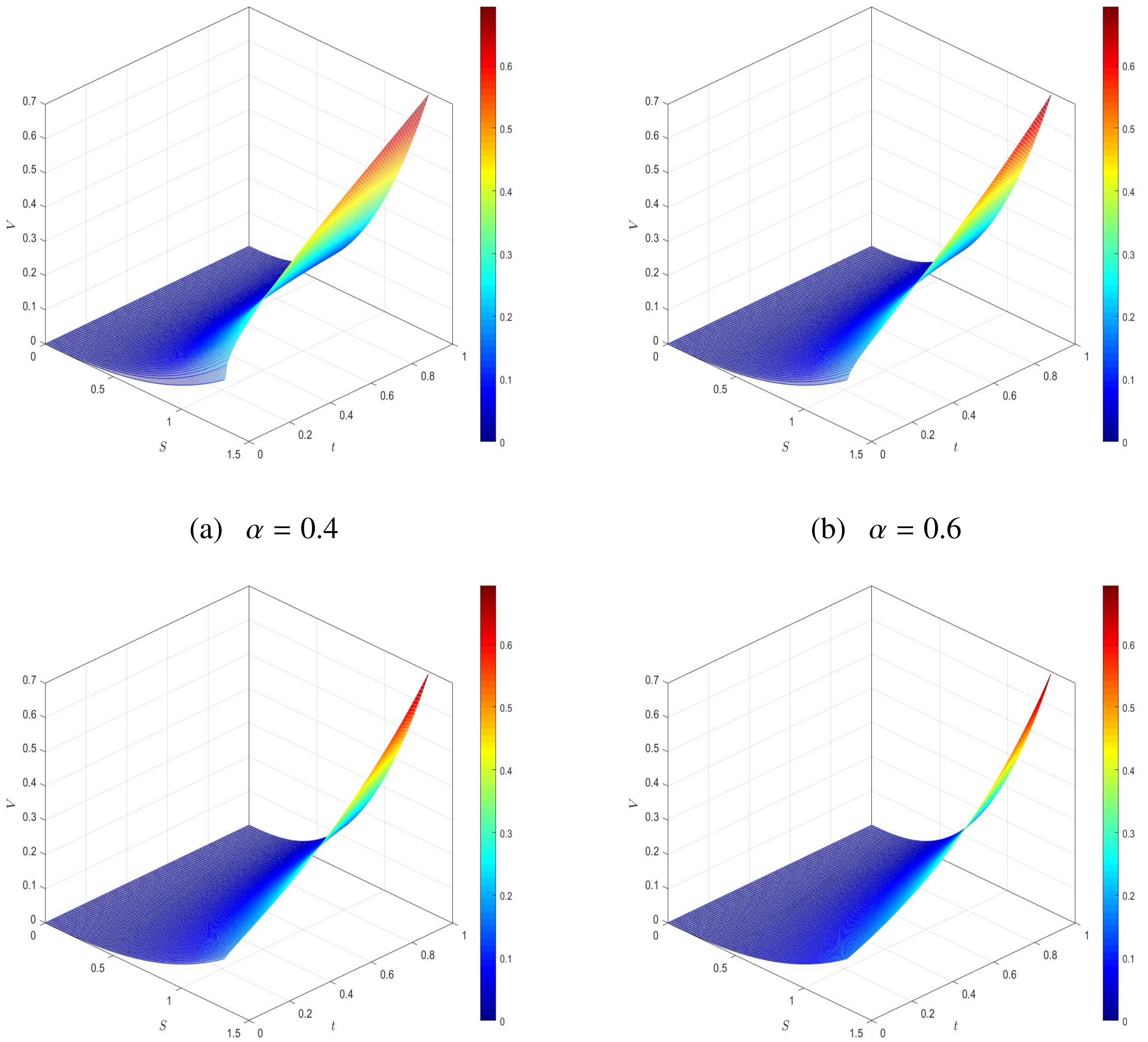
$$V = S^\rho f(\zeta). \tag{4.12}$$

Inserting (4.12) into (3.2), we get the reduced ODE

$$f^{'} + \frac{1}{2}\rho\sigma^2(\rho - 1)f + \rho r f - r f = 0. \tag{4.13}$$

By applying the method of separation of variables to solve Eq (4.13), we obtain $f = c\texttt{exp}\Big[-(\frac{1}{2}\rho^2\sigma^2 - \frac{1}{2}\sigma^2\rho + r\rho - r)\zeta\Big]$. Then, we have the exact solution to Eq (1.1)

$$V(S,t) = cS^{\rho}\texttt{exp}\Big[(\frac{1}{2}\rho^2\sigma^2 - \frac{1}{2}\sigma^2\rho + r\rho - r)(T-t)^{\alpha}\Big]. \quad (4.14)$$



(a) $\alpha = 0.4$ (b) $\alpha = 0.6$

(c) $\alpha = 0.8$ (d) $\alpha = 1$

Figure 1. Effect of different fractal dimensions α on the option price for $c = 0.07, \rho = 2.6662, r = 0.0358, \sigma = 0.82$, and $T = 1$.

5. Discussion on dynamic characteristics of the fractal B-S equation of option pricing (1.1)

In the following, we focus on analyzing the dynamic characteristics in terms of (4.14), where some of the parameters are derived from the data of Linnan Dongsheng Film and Television Co., Ltd. All figures are depicted by using Matlab. Figure 1 shows the impact of different fractal dimensions α on the option price. Under the given parameters ($c = 0.07, \rho = 2.6662, r = 0.0358, \sigma = 0.82$ and $T = 1$), the fractal dimension α has a significant impact on the changes in option price V with asset price S and time t. Overall, regardless of whether α takes the value of 0.4, 0.6, 0.8, or 1, the option price V shows a common trend of rising with the increase in asset price S and growing as time t progresses. This conforms to the basic logic that the higher the asset price and the longer the remaining term, the greater the option value. The difference lies in the regulatory role of the fractal dimension: the smaller the α, the more sensitive the option price is to the fluctuations of S and t, and the steeper the corresponding three-dimensional surface. This is because a lower α corresponds to a strong fractal market, where asset price fluctuations have strong long-term correlation and pronounced clustering. The larger the α, the more gradual the change in V. This is because a higher α weakens the fractal characteristics of the market, making it closer to the random walk assumption of the classic pricing model, highlighting the key impact of the fractal structure on the dynamic law of option pricing.

Figures 2 and 3 illustrate the dynamic behavior of the option price V as it changes with the asset price S and time t under different fractal dimension α values. Figure 2 shows that, at a fixed point in time, as the asset price S increases, the option price V exhibits nonlinear growth. Different α values affect the growth rate; the smaller the α value, the faster the option price grows. This indicates that a lower fractal dimension corresponds to more sensitive changes in the option price, potentially reflecting stronger fractal characteristics of the market, that is, price changes are more volatile and complex. Figure 3, on the other hand, illustrates the decay pattern of the option price V over time t. At a constant asset price, the option price gradually declines as time t progresses. Similarly, a smaller α value results in a faster decline rate of the option price. This suggests that in a fractal market, a lower α value may cause the option price to be more responsive to changes in time t. Overall, a decrease in the fractal dimension α enhances the sensitivity of the option price to both the asset price S and time t, reflecting stronger fractal properties of the market.

Figures 4 and 5 demonstrate the impacts of volatility σ and risk-free interest rate r on the option price, respectively, under the condition that the asset price S is fixed. In Figure 4, the curves corresponding to different volatility levels σ show that the option price V generally decreases over time t, and higher volatility leads to higher option prices. This occurs because increased volatility amplifies fluctuations in the asset price, raising both the probability of future profitability and the potential payoff space for the option, thereby directly elevating its value. As expiration approaches, the differences in option prices corresponding to different volatility curves narrow. At this point, the option price is more determined by the relationship between the actual asset price and the strike price, and the impact of volatility diminishes. Figure 5 demonstrates the mechanism of the risk-free interest rate r on option prices V. Regardless of the interest rate level, option prices exhibit a decreasing trend as time advances, reflecting the gradual decay of time value as expiration approaches. Comparing curves with different interest rates, higher risk-free rates correspond to higher initial option prices. As time increases, curves with different r values gradually converge. This is because as the expiration approaches, the proportion of the option's time value decreases, and the intrinsic value begins to dominate the pricing. Consequently, the impact of the risk-free interest rate is lessened, revealing that the effect of the risk-free interest rate on the option price has dynamic and stage-specific characteristics.

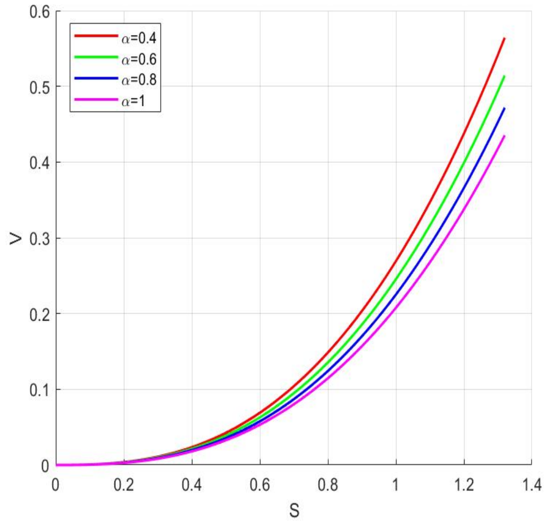


Figure 2. $c = 0.07, \rho = 2.6662, r = 0.0358, \sigma = 0.82$, and $t = 0.6$.

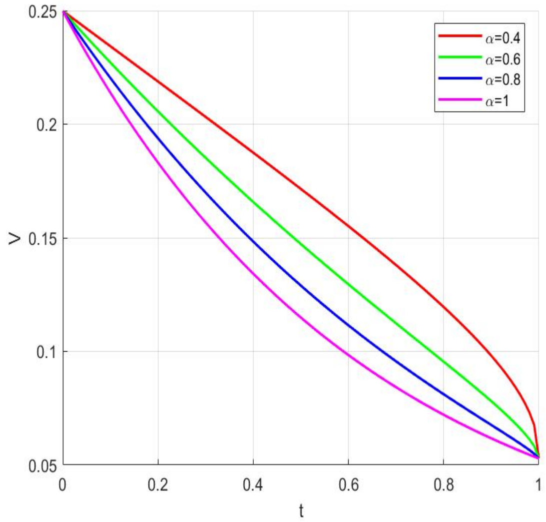


Figure 3. $c = 0.07, \rho = 2.6662, r = 0.0358, \sigma = 0.82$, and $S = 0.9$.

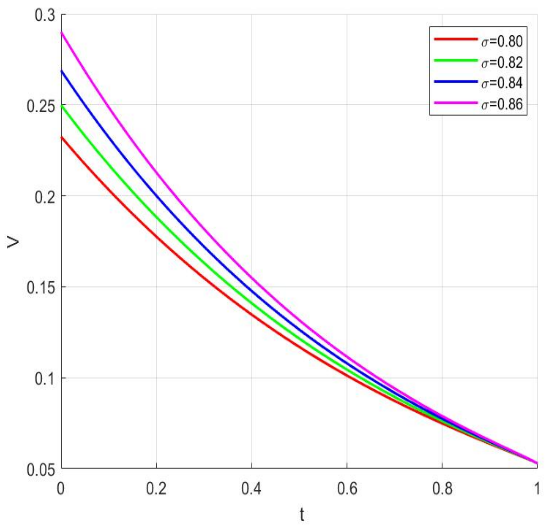


Figure 4. $c = 0.07, \rho = 2.6662, r = 0.0358, \alpha = 0.9$, and $S = 0.9$.

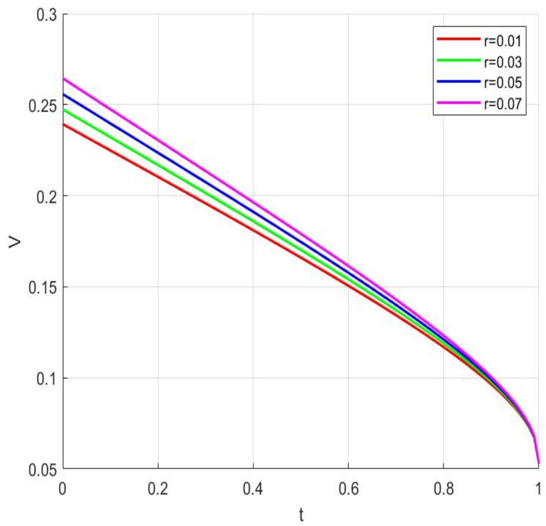


Figure 5. $c = 0.07, \rho = 2.6662, \alpha = 0.4, \sigma = 0.82$, and $S = 0.9$.

Figures 6 and 7 respectively examine the combined effects of σ, r, and α. It can be seen from Figure 6 that as the volatility σ increases, the option price V shows an obvious upward trend. This is because volatility reflects the degree of fluctuation in the price of the underlying asset; the higher the volatility, the greater the possibility that the price of the underlying asset will change in a direction favorable to the option holder, and the more potential profit opportunities the option contains, so the option price will rise accordingly. As the fractal dimension α increases, the dependence of the option price on this fractal characteristic decreases, which will cause the option price to decline to some extent. Figure 7 shows that as the risk-free interest rate r increases, the option price V exhibits an upward trend. This is because as the risk-free interest rate rises, the cost that option holders incur to obtain the option right will increase correspondingly, which in turn drives up the option price. In contrast, as α increases, the option price V shows a downward trend, which reflects the reverse regulatory effect of the fractal dimension α on the option price.

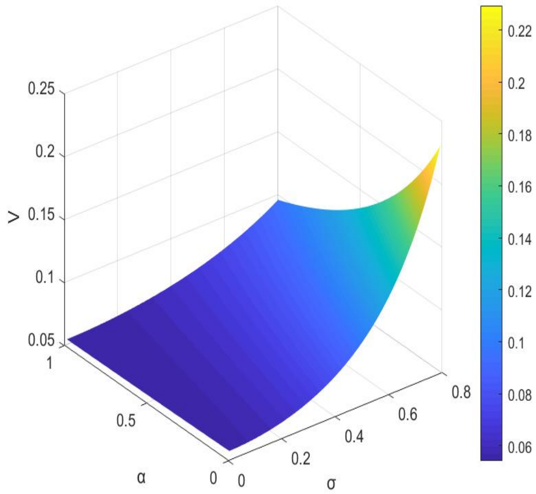


Figure 6. $c = 0.07, \rho = 2.6662, r = 0.0358, S = 0.9$, and $t = 0.6$.

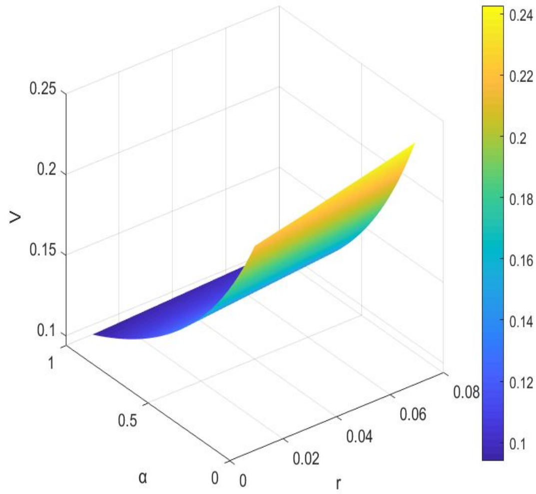


Figure 7. $c = 0.07, \rho = 2.6662, \sigma = 0.80, S = 0.9$, and $t = 0.6$.

6. Conclusions

This paper introduces He's fractal derivative to construct a fractal B-S option pricing equation. Through the improved fractal two-scale transform, the problem in fractal space can be converted into a continuous problem, facilitating analysis of the dynamic behavior of the system under different fractal dimensions.

In the research process, first, the fractal equation is transformed into an equivalent continuous equation using an improved fractal two-scale transform. Then, using the Lie symmetry analysis method, the geometric vector fields, symmetry reductions, and exact solutions of the equation are derived. The paper also analyzes the dynamic impact of parameters such as fractal dimension, volatility, and risk-free interest rate on option prices, supported by example data. The results show that the fractal dimension has a significant regulatory effect on the sensitivity of option prices. This is because a lower fractal dimension corresponds to stronger fractal characteristics of the market, where asset price fluctuations have stronger long-term correlation and pronounced clustering. The impact of volatility and risk-free interest rate exhibits phased characteristics. Volatility increases option value by amplifying the price fluctuation space, but its impact weakens as the expiration date approaches. The risk-free interest rate initially raises the option price through the time value premium, but as the expiration date approaches, the proportion of time value decreases, and the intrinsic value dominates the pricing, so its impact weakens and the curves corresponding to different interest rate levels gradually converge.

The innovation of this research lies in applying the Lie symmetry analysis method to the study of the fractal B-S option pricing equation, providing a new theoretical framework and analytical tool for option pricing in a fractal market environment. By analyzing its dynamic characteristics, it deepens the understanding of the relationship between financial market complexity and derivative pricing, which is significant for improving the option pricing model, enhancing pricing accuracy, and providing strong theoretical support for financial risk assessment and investment decision-making.

Author contributions

Yuan Yue: Conceptualization, formal analysis, methodology, software, resources, writing-original draft; Chao Yue: Funding acquisition, investigation, supervision, validation, visualization, editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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