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Research article

On $IC_{\Phi c}$-subgroups of finite groups

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Abstract: In group theory, the study of how the properties of specific subgroups affect group structure is a research field that remains highly active. For a finite group G and its subgroup S, if $S \cap [S, G] \leq \Phi(S)S_{cG}$ for some CAP-subgroup $S_{cG} \leq S$ of G, we define S as an $IC_{\Phi c}$-subgroup of G. This paper investigates how $IC_{\Phi c}$-subgroups impact finite group structure, yielding novel theorems that generalize prior work and enrich the theory of group structure. Specifically, we establish the following main theorems: (1) Let $P \trianglelefteq G$ be a p-group of order p^n. Suppose there exists an integer k with $1 \leq k < n$ such that: (i) all subgroups of P of order p^k are $IC_{\Phi c}$-subgroups of G; (ii) if $p^k = 2$, then all subgroups of order 4 are also $IC_{\Phi c}$-subgroups of G. Then $P \leq Z_{\mathfrak{U}}(G)$. (2) Given a solvably saturated formation $\mathfrak{F} \supseteq \mathfrak{U}$, let $N \trianglelefteq G$ with $G/N \in \mathfrak{F}$. Suppose for each non-cyclic $P \in Syl_p(F^*(N))$, where p is an arbitrary prime in $\pi(F^*(N))$, the conditions of (1) hold. Then $G \in \mathfrak{F}$.

Keywords: finite group; $IC_{\Phi c}$-subgroup; CAP-subgroup; generalized Fitting subgroup; saturated formation
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1. Introduction

For the entire paper, G consistently represents a finite group. p denotes a prime factor that divides the order of G. $K \cdot \trianglelefteq G$ (resp. $T \lessdot G$) indicates that $K \trianglelefteq G$ and no nontrivial normal subgroup of G is properly contained in K (resp. $T \leq G$ and no proper subgroup of G strictly contains T). Let $\mathcal{S}(G)$, $\mathcal{S}_p(G)$ and $\mathcal{S}_p^k(G)$ respectively denote the set of all subgroups of G, its p-subgroups, and its subgroups of order p^k for a fixed integer k, that is, $\mathcal{S}(G) = \{S \mid S \leq G\}$, $\mathcal{S}_p(G) = \{S \leq G \mid S \text{ is a } p\text{-group}\}$ and $\mathcal{S}_p^k(G) = \{S \leq G \mid |S| = p^k\}$. Other notations are standard and can be referenced in [1, 2].

Investigating the embedding properties of subgroups has been one of the most effective approaches to illustrate the structure of G. Among embedding properties, the cover-avoidance property is an important one. Let $S \in \mathcal{S}(G)$ and M/N be a chief factor of G. We say that S covers M/N if $SM = SN$

and S avoids M/N if $S\cap M = S\cap N$. If S either covers or avoids every chief factor of G, then we say that S has the cover-avoidance property in G, and call S a CAP-subgroup of G. If there exists a chief series Γ_A of G such that S either covers or avoids every chief factor of Γ_A, then S is called a $SCAP$-subgroup of G. The study of the cover-avoidance property has yielded many results (see [3–5]). Furthermore, significant results have also been obtained through the study of its generalizations (see [6,7]). In 2021, IC-property was defined in [8]. This property considers that $S \cap [S,G]$ satisfies certain conditions, where $S \in \mathcal{S}(G)$. With IC-property, many new concepts were introduced, and many results have been obtained using these concepts to characterize the structure of G. For instance, the $IC\overline{s}$-subgroup, which is defined by combining the IC-property with the s-semipermutability, was studied in [9, 10]. In [8], the authors introduced the $IC\Phi$-subgroup. Let $S \in \mathcal{S}(G)$. If $S \cap [S,G] \leq \Phi(S)$, S is defined as an $IC\Phi$-subgroup of G. They obtained novel p-nilpotency and supersolvability criteria for finite groups which later were extended by Kaspczyk in [11]. Combining the IC-property with the cover-avoidance property, in [12], the authors introduced the concepts of an ICC-subgroup and an $ICSC$-subgroup. Let $S \in \mathcal{S}(G)$. If $S \cap [S,G] \leq S_{cG}$ (resp. S_{scG}) for some CAP-subgroup $S_{cG} \leq S$ (resp. $SCAP$-subgroup $S_{scG} \leq S$) of G, S is defined as an ICC-subgroup (resp. $ICSC$-subgroup) of G. Clearly, a CAP-subgroup (resp. $SCAP$-subgroup) of G must be an ICC-subgroup (resp. $ICSC$-subgroup) of G. However, the reverse is generally not true. By using the two concepts above, several novel results on finite group structure have been derived, which extend some previous findings related to the cover-avoidance property. Continuing from the previous research, we present the following concepts.

Definition 1.1. *Let $S \in \mathcal{S}(G)$. If $S \cap [S,G] \leq \Phi(S)S_{cG}$ for some CAP-subgroup $S_{cG} \leq S$ of G, S is defined as an $IC_{\Phi c}$-subgroup of G.*

Definition 1.2. *Let $S \in \mathcal{S}(G)$. If $S \cap [S,G] \leq \Phi(S)S_{scG}$ for some $SCAP$-subgroup $S_{scG} \leq S$ of G, S is defined as an $IC_{\Phi sc}$-subgroup of G.*

For the sake of concise expression, we further define the notations for several sets. Let $\mathcal{ICC}(G)$, $\mathcal{ICSC}(G)$, $\mathcal{IC}_{\Phi c}(G)$, and $\mathcal{IC}_{\Phi sc}(G)$ denote the set of all ICC-subgroups of G, its $ICSC$-subgroups, its $IC_{\Phi c}$-subgroups, and its $IC_{\Phi sc}$-subgroups, respectively.

Obviously, by the definitions, if $S \in \mathcal{IC}_{\Phi c}(G)$, then $S \in \mathcal{IC}_{\Phi sc}(G)$, and if $S \in \mathcal{ICC}(G)$, then $S \in \mathcal{IC}_{\Phi c}(G)$. However, the converses are typically not valid. We can see these from the examples given below.

Example 1.1. *Let $G = A_5 \times S_3$, where A_5 is the alternating group of degree 5 and S_3 is the symmetric group of degree 3. Let a be an element of A_5 of order 3 and b be an element of S_3 of order 3 (clearly, $\langle b\rangle \trianglelefteq S_3$). Take $S = \langle ab\rangle$. It is easy for us to verify that for the chief series $1 \trianglelefteq A_5 \trianglelefteq A_5\langle b\rangle \trianglelefteq G$, S covers $A_5\langle b\rangle/A_5$ and avoids the rest. Hence S is a $SCAP$-subgroup. However, S is not a CAP-subgroup as S neither covers nor avoids the chief factor G/S_3. Obviously, $\Phi(S) = 1$. Since $[ab, xy] = [a, x][b, y]$ for any $xy \in G$ ($x \in A_5$, $y \in S_3$), $[\langle a\rangle, A_5] = A_5$ and $[\langle b\rangle, S_3] = A_3$, we have $[S,G] = A_5\times A_3$. Furthermore, $S \cap [S,G] = S$. Hence $S \in \mathcal{IC}_{\Phi sc}(G)$, but $S \notin \mathcal{IC}_{\Phi c}(G)$.*

Example 1.2. *Let $G = S_5$ and $S = \langle(1234)\rangle$. Clearly, $A_5 \cdot\!\trianglelefteq G$. It is straightforward to verify that S and $\Phi(S) = \langle(13)(24)\rangle$ fail to cover and avoid $A_5/1$. Hence $S_{cG} = 1$ and S is not a CAP-subgroup of G. Note that $[S,G] = A_5$ and $S \cap A_5 = \Phi(S)$. Hence $S \notin \mathcal{ICC}(G)$, but $S \in \mathcal{IC}_{\Phi c}(G)$.*

Naturally, we consider what results will be obtained in the structural theory of groups if the subgroup system is expanded from $\mathcal{ICC}(G)$ to $\mathcal{IC}_{\Phi c}(G)$. The structural properties of G are characterized in this paper under the assumption that the elements in $\mathcal{S}_p^k(G)$ for some fixed integer k belongs to $\mathcal{IC}_{\Phi c}(G)$. Our main results extend the work of [4, 12].

2. Preliminary results

Lemma 2.1. *([13, Lemma 1.2]) Let* $H \leq G$ *and* $1 < \cdots < N < \cdots < M < \cdots < G$ *be a normal series. If H covers (avoids) M/N, then H covers (avoids) any quotient factor between M and N of any refinement of the normal series.*

For future needs, we next present the properties of the two newly defined subgroups.

Lemma 2.2. *Given* $S \in \mathcal{IC}_{\Phi c}(G)$ *and* $K \trianglelefteq G$. *We have the following:*

(1) If $K \leq S$, *then* $S/K \in \mathcal{IC}_{\Phi c}(G/K)$.
(2) If $S \in \mathcal{S}_p(G)$ *and* $(|K|, p) = 1$, *then* $SK/K \in \mathcal{IC}_{\Phi c}(G/K)$.
(3) If $N \trianglelefteq G$ *and* $S \leq N$, *then* $S \in \mathcal{IC}_{\Phi c}(N)$.

Proof. By hypothesis, $S \cap [S, G] \leq \Phi(S)S_{cG}$.

(1) Since $S/K \cap [S/K, G/K] = (S \cap [S, G])K/K$ and $\Phi(S)S_{cG}K/K = \Phi(S)K/K \cdot S_{cG}K/K \leq \Phi(S/K) \cdot S_{cG}K/K = \Phi(S/K) \cdot (S/K)_{c(G/K)}$, we have $S/K \cap [S/K, S/K] \leq \Phi(S/K) \cdot (S/K)_{c(G/K)}$. Therefore, assertion (1) holds.
(2) Similar to the proof process of (1), we can deduce that (2) holds.
(3) Since $N \trianglelefteq G$, by Lemma 2.1, we have $S_{cG} = S_{cN}$, and $S \cap [S, N] \leq S \cap [S, G] \leq \Phi(S)S_{cG}$. Hence, (3) holds.

□

Lemma 2.3. *Given* $S \in \mathcal{IC}_{\Phi sc}(G)$ *and* $K \trianglelefteq G$. *We have the following:*

(1) If $L \in \mathcal{S}(G)$ *and* $S \leq L$, *then* $S \in \mathcal{IC}_{\Phi sc}(L)$.
(2) If $S \in \mathcal{S}_p(G)$ *and* $(|K|, p) = 1$, *then* $SK/K \in \mathcal{IC}_{\Phi sc}(G/K)$.

Proof. By hypothesis, $S \cap [S, G] \leq \Phi(S)S_{scG}$.

(1) Note that $[S, L] \leq [S, G]$. Then $S \cap [S, L] \leq \Phi(S)S_{scG}$. Note that $SCAP$-subgroups possess the property: if $H \leq K \leq G$ and H is a $SCAP$-subgroup of G, then H is a $SCAP$-subgroup of K. This property implies $S_{scG} = S_{scL}$. Therefore, assertion (1) holds.
(2) Through routine calculations following the same approach as in Lemma 2.2(1)'s proof, we can easily prove (2).

□

Lemma 2.4. *([2, IV, Theorem 6.10]) For local formations* $\mathfrak{F}$, $G^{\mathfrak{F}}$ *centralizes* $Z_{\mathfrak{F}}(G)$.

Lemma 2.5. *([14, Chapter X, Corollary 13.7]) For a quasinilpotent group G, the Fitting subgroup $F(G)$ coincides with the hypercenter $Z_\infty(G)$.*

Lemma 2.6. *([2, IX, Remark 2.7]) Given the conditions* $F(G) = 1$ *and* $G = F^*(G)$*, it follows that G must equal* $\mathrm{Soc}(G)$.

In order to state it simply, we use the following notations. Let $P \in \mathcal{S}_p(G)$ and $|P| = p^n$. P is said to satisfy $\triangle_1$ (resp. $\triangle_2$, $\triangle_3$) if

$\triangle_1$: for any $P_1 \lessdot P$, $P_1 \in \mathcal{IC}_{\Phi c}(G)$.

$\triangle_2$: all subgroups of order p of P, and if $P \in \mathcal{S}_2(G)$ is non-abelian, cyclic subgroups of order 4 as well, belong to $\mathcal{IC}_{\Phi sc}(G)$.

$\triangle_3$: an integer k with $1 \leq k < n$ exists for which $\mathcal{S}_p^k(P) \subseteq \mathcal{IC}_{\Phi c}(G)$, and if $p^k = 2$, assume moreover that $\mathcal{S}_2^2(P) \subseteq \mathcal{IC}_{\Phi c}(G)$.

3. Main results

To prove Theorem 3.1, one of the important conclusions of this paper, first, we establish two lemmas.

Lemma 3.1. *Let* $P \in \mathcal{S}_p(G)$ *and* $P \trianglelefteq G$*. If* $\triangle_1$ *or* $\triangle_2$ *is satisfied by P, then* $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$.

Proof. (1) If P satisfies $\triangle_1$.

Assume $\Phi(P) \neq 1$. By Lemma 2.2(1), the quotient group $P/\Phi(P)$ satisfies $\triangle_1$. Applying induction on $|P|$, we deduce that $P/\Phi(P)$ is an element of $\mathcal{S}(Z_{\mathfrak{U}}(G/\Phi(P)))$. Hence $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ by [12, Lemma 2.9].

Assume $\Phi(P) = 1$. Then for any $P_1 \lessdot P$, $P_1 \in \mathcal{ICC}(G)$. Hence, by [12, Theorem 3.1], $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$.

(2) If P satisfies $\triangle_2$.

For any $N \trianglelefteq G$ with $N < P$, it is evident that $\triangle_2$ is satisfied by N. Then, by induction, $N \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Let $T \trianglelefteq G$ such that $P/T \cdot\!\trianglelefteq G/T$. Then $T \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. If T is not unique, then we can obtain $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Therefore, we may as well assume that T is unique below. When $\Omega(C) < P$, we have $\Omega(C) \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. [12, Lemma 2.9] then gives $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. If $\Omega(C) = P$, then the order of elements in P is p or 4. Let y be an element of P but not an element of T. Then $Y = \langle y \rangle \in \mathcal{IC}_{\Phi sc}(G)$. Note that T is unique and $Y^G = P$.

If $[Y, G] = P$, then $Y = Y \cap P \leq \Phi(Y)Y_{scG}$. It follows that $Y = Y_{scG}$. Note that Y cannot avoid P/T. Hence Y must cover P/T, that is, $P = TY$. Then P/T is cyclic and so $P/T \in \mathcal{S}(Z_{\mathfrak{U}}(G/T))$. Hence $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ since $T \in \mathcal{S}(Z_{\mathfrak{U}}(G))$.

If $[Y, G] < P$, then $[Y, G] \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Since $Y[Y, G] = P$, we have $P/[Y, G]$ is cyclic and so $P/[Y, G] \in \mathcal{S}(Z_{\mathfrak{U}}(G/[Y, G]))$. Hence $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. □

Since $\mathcal{IC}_{\Phi c}(G) \subseteq \mathcal{IC}_{\Phi sc}(G)$, replacing $\mathcal{IC}_{\Phi sc}(G)$ by $\mathcal{IC}_{\Phi c}(G)$ in the statement of $\triangle_2$ preserves the validity of Lemma 3.1's conclusion.

Lemma 3.2. *Let* $P \in \mathcal{S}_2(G)$*. Suppose* $K \cdot\!\trianglelefteq G$ *exists with* $|K| = 2$ *and* $K \leq P$*, and for any* $S \in \mathcal{S}_2^2(P)$*,* $S \in \mathcal{IC}_{\Phi c}(G)$*. Then for any* $A \leq P$ *with* $|A| = 2$*,* $A \in \mathcal{IC}_{\Phi c}(G)$.

Proof. Clearly, $K \in \mathcal{IC}_{\Phi c}(G)$. Thus, we can assume that $A \neq K$. Then $B = AK$ is elementary abelian of order 4. Hence $B \cap [B, G] \leq \Phi(B)B_{cG}$. If $A \cap [A, G] = 1$, obviously, $A \in \mathcal{IC}_{\Phi c}(G)$. If $A \cap [A, G] = A$, then $A \leq [A, G]$, and so $A \leq B \cap [B, G] \leq \Phi(B)B_{cG} = B_{cG} \leq B$. Note that $|A| = 2$ and $|B| = 4$. Hence, we have either $A = B_{cG}$ or $B = B_{cG}$. If $A = B_{cG}$, then $A \in \mathcal{IC}_{\Phi c}(G)$.

Now assume that $B = B_{cG}$. Let both V and F be normal in G with $V \trianglelefteq F$ and $F/V \cdot\!\trianglelefteq G/V$. Then B either covers or avoids F/V. If B avoids F/V, then $B \cap F = B \cap V$. It follows that $A \cap F = A \cap V$. Hence A avoids F/V.

If B covers F/V, then $BF = BV$. First, we prove that $|F/V| = 2$. Suppose that is not the case. Clearly, $F/V \leq BV/V$. Note that $|B| = 4$ and $|F/V| > 2$, so $F/V = BV/V$. If $K \leq V$, then $|F/V| = |AKV/V| = |AV/V| \leq 2$, a contradiction. Hence $K \cap V = 1$, $KV/V \trianglelefteq G/V$ with $|KV/V| = 2$ and $KV/V < F/V$, this contradicts $F/V \cdot\!\trianglelefteq G/V$. Thus, $|F/V| = 2$. If $A \leq V$, then A avoids F/V. If $A \nleq V$, then either $A \cap F = 1$ or $AV = F$, that is, A either avoids or covers F/V. Hence $A \in \mathcal{IC}_{\Phi c}(G)$; this completes the proof. □

Let $P \trianglelefteq G$ and $|P| = p^n$. Lemma 3.1 above shows that when $k = n - 1$ and $\mathcal{S}_p^k(P) \subseteq \mathcal{IC}_{\Phi c}(G)$ or when $k = 1$ and $\mathcal{S}_p^k(P) \subseteq \mathcal{IC}_{\Phi sc}(G)$ (if $p = 2$, and moreover the cyclic subgroups in $\mathcal{S}_2^2(P)$ belong to $\mathcal{IC}_{\Phi sc}(G)$), we have $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. This lemma only considers the cases of maximal and minimal subgroups of P. Next, we generalize this conclusion to a more general case.

Theorem 3.1. *Let $P \in \mathcal{S}_p(G)$ and $P \trianglelefteq G$. Suppose that P satisfies $\triangle_3$. Then $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$.*

Proof. Take (G, P) as a minimal counterexample. Let $K \cdot\!\trianglelefteq G$ and $K \leq P$. Then

(1) $1 < k < n - 1$.

It follows directly from Lemma 3.1.

(2) Let $L < P$ and $L \trianglelefteq G$. If $|L| > p^k$, then $L \in \mathcal{S}(Z_{\mathfrak{U}}(G))$.

Clearly, (G, L) satisfies the hypothesis of the theorem. Hence, the conclusion is obvious.

(3) $K < P$.

Suppose that is not the case, that is, K is equal to P. Let $S < P$ and $S \in \mathcal{S}_p^k(G)$. By hypothesis, $S \in \mathcal{IC}_{\Phi c}(G)$. Note that $K \cdot\!\trianglelefteq G$ and $1 < S < P = K$, clearly, $[S, G] \neq 1$. Since $S[S, G] = S^G \leq P = K$, we have $[S, G] = P$. Hence $S = S \cap [S, G] \leq \Phi(S)S_{cG}$, implying $S = \Phi(S)S_{cG} = S_{cG}$. As K is a minimal normal subgroup in G, necessarily $S = 1$, which contradicts $S \in \mathcal{S}_p^k(G)$.

(4) $|K| < p^k$.

If $|K| > p^k$, then $K \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ by (2) and (3). From this, it can be deduced that $p^k < |K| = p$, which contradicts (1).

If $|K| = p^k$, then, by (1), both $|K|$ and $|P : K|$ are greater than p. From (2), we immediately conclude that $P/K \cdot\!\trianglelefteq G/K$. Let $A/K < P/K$ and $|A/K| = p$. Obviously, $A < P$ and $A \ntrianglelefteq G$. Because K is non-cyclic, there must exist $S \lessdot A$ such that $A = KS$. Then, both K and S belong to $\mathcal{S}_p^k(G)$. By hypothesis, $S \cap [S, G] \leq \Phi(S)S_{cG}$. If $K \cap S = 1$, then $|A| = |KS| = |K||S|$". It follows that $p^k = |S| = |A|/|K| = p$, that is, $k = 1$, which contradicts to (1). Hence, we have $K \cap S \neq 1$. If $[K \cap S, G] = 1$, then $K \cap S \trianglelefteq G$. Since $K \cdot\!\trianglelefteq G$ and $K \cap S \neq 1$, we have $K \leq S$ and so $A = KS = S$, which is impossible since $S \lessdot A$. So, we have $[K \cap S, G] \neq 1$. Thus, $1 < [K \cap S, G] \leq [K, G] \leq K$. From the above inclusion relations, since $K \cdot\!\trianglelefteq G$, it follows that $K = [K, G] = [K \cap S, G] \leq [S, G] \leq P$. Therefore, by $P/K \cdot\!\trianglelefteq G/K$, we have $[S, G] = K$ or $[S, G] = P$. Assume that $[S, G] = K$. Then $A = KS = [S, G]S = S^G \trianglelefteq G$, a contradiction. Hence $[S, G] = P$, then $S = S \cap [S, G] \leq \Phi(S)S_{cG}$, and so $S = \Phi(S)S_{cG} = S_{cG}$. Thus, S avoids or covers $K/1$. Neither will happen, a contradiction.

(5) $P/K \in \mathcal{S}(Z_{\mathfrak{U}}(G/K))$.

By (4), $|K| < p^k$, that is, $|K| \leq p^{k-1}$. Suppose $p \neq 2$ or $p = 2$ with $|K| < 2^{k-1}$. Since $\mathcal{S}_p^k(P) \subseteq \mathcal{IC}_{\Phi c}(G)$, by Lemma 2.2(1), for any $S/K \in \mathcal{S}(P/K)$ with $|S/K| = \frac{p^k}{|K|} := p^l$ (when $p \neq 2$, $p \leq p^l <$

$|P/K|$; when $p = 2$, $p < p^l < |P/K|$), we have $S/K \in \mathcal{IC}_{\Phi c}(G/K)$, that is, $\mathcal{S}_p^l(P/K) \subseteq \mathcal{IC}_{\Phi c}(G/K)$. Consequently, $(G/K, P/K)$ fulfills the assumed conditions. Hence $P/K \in \mathcal{S}(Z_{\mathfrak{U}}(G/K))$. Now assume $p = 2$ and $|K| = 2^{k-1}$. In this case, since $\frac{p^k}{|K|} = 2$, we have $\mathcal{S}_2^1(P/K) \subseteq \mathcal{IC}_{\Phi c}(G/K)$. Let $U/K \in \mathcal{S}_2^2(P/K)$ be an arbitrary cyclic subgroup of order 4. We show $U/K \in \mathcal{IC}_{\Phi c}(G/K)$.

Assume for contradiction $|K| = 2$. Then $2^k = 2^2$. By hypothesis, $\mathcal{S}_2^2(P) \subseteq \mathcal{IC}_{\Phi c}(G)$. By Lemmas 3.1 and 3.2, $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$, a contradiction. Thus, $|K| > 2$.

Assume $K \leq \Phi(U)$. Then U must be cyclic, and so K must be cyclic, leading to a contradiction.

Hence $K \nleq \Phi(U)$. Then, there exists $U_1 \lessdot U$ such that $U = KU_1$. Obviously, $|U_1| = 2^k$. By hypothesis, $U_1 \cap [U_1, G] \leq \Phi(U_1)(U_1)_{cG}$. From $|K| > 2$ and $K \cdot\!\trianglelefteq G$, it is straightforward to infer that neither $K \cap U_1$ nor $[K \cap U_1, G]$ is equal to 1, then $1 < [K \cap U_1, G] \leq [K, G] \leq K$. By $K \cdot\!\trianglelefteq G$ again, it can be concluded that $K = [K, G] = [K \cap U_1, G] \leq [U_1, G]$. So $U \cap [U, G] = KU_1 \cap [KU_1, G] = KU_1 \cap [U_1, G]^K[K, G] = KU_1 \cap [U_1, G] = K(U_1 \cap [U_1, G]) \leq K\Phi(U_1)(U_1)_{cG} \leq \Phi(U)K(U_1)_{cG}$, and $K(U_1)_{cG} = U_{cG}$. Hence $U \in \mathcal{IC}_{\Phi c}(G)$, and so $U/K \in \mathcal{IC}_{\Phi c}(G/K)$. This implies that the cyclic subgroups in $\mathcal{S}_2^2(P)$ belong to $\mathcal{IC}_{\Phi c}(G)$. By Lemma 3.1, $P/K \in \mathcal{S}(Z_{\mathfrak{U}}(G/K))$.

(6) The final contradiction.

By (4) and (5), there exists $W \trianglelefteq G$ such that $K < W < P$ and $|W| > p^k$. Then $W \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ by (2). Thus $|K| = p$, and thus $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ since $P/K \in \mathcal{S}(Z_{\mathfrak{U}}(G/K))$, the final contradiction. □

The following two theorems investigate the influence of elements in $\mathcal{IC}_{\Phi sc}(G)$ and $\mathcal{IC}_{\Phi c}(G)$ on the p-supersolvability of G and the solvability of $F^*(G)$.

Theorem 3.2. *Let $E \trianglelefteq G$ and $P \in Syl_p(E)$. Suppose that P satisfies $\triangle_2$. Then $E \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$.*

Proof. Take (G, E) as a minimal counterexample.

(1) $O_{p'}(E) = 1$.

By Lemma 2.3(2), this is obvious.

(2) $E = G$ and $O_{p'}(G) = 1$.

If $E < G$, then (E, E) satisfies the hypothesis by Lemma 2.3(1). Hence E is p-supersolvable. By (1) and [15, Theorem 2.1.6], $P \trianglelefteq E$ and so $P \trianglelefteq G$. Then, by Lemma 3.1, $P \in \mathcal{S}(Z_{\mathfrak{U}}(G))$, and so $E \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$, a contradiction. Hence $E = G$ and $O_{p'}(G) = 1$.

(3) Let $N \trianglelefteq G$ and $N < G$. Then $N \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$. In other words, $Z_{p\mathfrak{U}}(G)$ is the unique maximal normal subgroup of G.

Since $N \cap P \in Syl_p(N)$ and $N \cap P$ satisfies $\triangle_2$, we have $N \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$.

(4) $O_p(G) \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ and $O_p(G) \in Syl_p(Z_{p\mathfrak{U}}(G))$.

Obviously, $O_p(G)$ satisfies $\triangle_2$, by Lemma 3.1, $O_p(G) \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Note that $\mathcal{S}(Z_{\mathfrak{U}}(G)) \subseteq \mathcal{S}(Z_{p\mathfrak{U}}(G))$. Then $O_p(G) \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$. Since $O_{p'}(Z_{p\mathfrak{U}}(G)) \leq O_{p'}(G) = 1$ and $Z_{p\mathfrak{U}}(G)$ is p-supersolvable, we have $O_p(G) \in Syl_p(Z_{p\mathfrak{U}}(G))$ by [15, Theorem 2.1.6].

(5) $O_p(G)$, $Z(G)$, and $Z_{\mathfrak{U}}(G)$ are equal to each other.

Assume that $G^{\mathfrak{U}} < G$. Then, by (3), we have $G^{\mathfrak{U}} \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$. This implies that G is p-supersolvable, leading to a contradiction. Hence $G^{\mathfrak{U}} = G$. Then, by Lemma 2.4, $Z(G)$ contains $Z_{\mathfrak{U}}(G)$. Since $O_{p'}(Z(G)) \leq O_{p'}(G)$ is equal to 1, we have $Z(G)$ contained in $O_p(G)$. Note that $O_p(G) \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Hence $O_p(G)$, $Z(G)$, and $Z_{\mathfrak{U}}(G)$ are equal to each other.

(6) The final contradiction.

By [1, IV, Theorem 5.5] and (3), it follows that there is an element $y \in P$ with order p or 4, and $y \notin Z(G)$. Clearly, $Y = \langle y \rangle \in \mathcal{IC}_{\Phi sc}(G)$. Assume that $Y_{scG} = Y$. Then, by (3), Y covers or avoids G/T, where $T := Z_{p\mathfrak{U}}(G)$. If Y covers G/T, then $G = YT$. Hence G/T is cyclic. From this, we may deduce that $G \in \mathcal{S}(Z_{p\mathfrak{U}}(G))$, a contradiction. If Y avoids G/T, then $Y \leq T$. Hence $Y \leq O_p(G) = Z(G)$ by (4) and (5), this contradicts $y \notin Z(G)$. Thus, we have $Y_{scG} \leq \Phi(Y)$. Furthermore, $Y \cap [Y, G] \leq \Phi(Y)$. From this inclusion relation, it is immediate that $[Y, G] < G$. By (3), we have $[Y, G] \leq T$. Then $Y^G T = Y[Y, G]T = YT = G$. Similarly, a contradiction is derived again, completing the proof. □

When we set $E = G$ in the above theorem, the following conclusion can immediately be obtained.

Corollary 3.1. *Let $P \in Syl_p(G)$. Then G is p-supersolvable whenever P satisfies $\triangle_2$.*

Theorem 3.3. *Let $p = \min \pi(F^*(G))$ and $P \in Syl_p(F^*(G))$. Then $F^*(G)$ is solvable whenever P satisfies $\triangle_3$.*

Proof. Take G as a minimal counterexample. Let $K \cdot\!\trianglelefteq G$ and $K \leq F(G)$. Obviously, $p = 2$.

(1) $F^*(G) = G$.

By $F^*(F^*(G)) = F^*(G)$ and Lemma 2.2(3), this is obvious.

(2) G/K is not solvable. $K \in \mathcal{S}_2(G)$ is elementary abelian, and $|K| = 2^k$.

From $K \leq F(G)$, we know K is solvable. Then G/K is not solvable, as otherwise G would be solvable.

Since $K \cdot\!\trianglelefteq G$ and $K \leq F(G)$, a prime number r exists for which K belongs to $\mathcal{S}_r(G)$ and is elementary abelian. By (1), $F^*(G/K) = G/K$. If $r > 2$, then $PK/K \in Syl_p(G/K)$. Note that $p = 2$; we consequently find PK/K satisfies $\triangle_3$ by Lemma 2.2(2). Hence G/K is solvable, which contradicts the above. Hence $r = 2$. Let us prove that $|K| = 2^k$.

If $|K| > 2^k$, then, by Theorem 3.1, $K \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Hence, by the minimality of K, $|K| = 2$ and $k = 0$, a contradiction.

If $|K| < 2^k$, then, by Lemma 2.2(1), when $|K| < 2^{k-1}$, P/K satisfies $\triangle_3$, so G/K is solvable, a contradiction. When $|K| = 2^{k-1}$. Following the argument for Theorem 3.1(5), we deduce that G/K satisfies Corollary 3.1's hypothesis and is solvable, a contradiction.

Hence $|K| = 2^k$.

(3) $F(G) = 1$.

Otherwise, according to (2), we obtain $|K| = 2^k$. Further, Lemma 2.5 implies that $F(G) \in \mathcal{S}(Z_{\mathfrak{U}}(G))$, so $|K| = 2$. This yields $k = 1$. Using hypothesis and Corollary 3.1, we can confirm that G is 2-supersolvable. If $O_{p'}(G) \neq 1$, then, from the proof in the second paragraph of (2), $G/O_{p'}(G)$ is solvable. Note that $p = 2$, it follows that G is solvable, a contradiction. Hence $O_{p'}(G) = 1$. Then, by [15, Theorem 2.1.6], we establish that $P \trianglelefteq G$. As a consequence, G is proven to be solvable, leading to a contradiction.

(4) Deriving the ultimate contradiction.

By (1), (3), and Lemma 2.6, $G = Soc(G)$. Then, in G, all chief factors are non-abelian and simple. Let $S \leq P$ with order 2^k. Then $S \in \mathcal{IC}_{\Phi c}(G)$. Since a 2-subgroup of G cannot cover a chief factor of G that is non-abelian and simple, we have $S_{cG} = 1$ and S satisfies $S \cap [S, G] \leq \Phi(S)$. Hence, using [11, Theorem 1.3], G is 2-nilpotent. The solvability of G follows, thereby yielding the last contradiction. □

Formation, which is involved in many articles, studies the structure of groups from a broader perspective (see [12, Theorem 3.8] and [4, Main Theorem]). With regard to $IC_{\Phi c}$-subgroups, from the perspective of formations, we have the following conclusion.

Theorem 3.4. *Given a solvably saturated formation* $\mathfrak{F} \supseteq \mathfrak{U}$*, let* $N \trianglelefteq G$ *with* $G/N \in \mathfrak{F}$*. Then* $G \in \mathfrak{F}$ *whenever each non-cyclic* $P \in Syl_p(F^*(N))$ *satisfies* $\triangle_3$*, where* p *is an arbitrary prime number belonging to* $\pi(F^*(N))$*.*

Proof. Let $r = \min \pi(F^*(N))$ and $R \in Syl_r(F^*(N))$. First, consider when R is cyclic; then $F^*(N)$ is r-nilpotent, from which we get that $F^*(N)$ is solvable. Alternatively, if R is not cyclic, as a consequence of Lemma 2.2(3) and Theorem 3.3, we conclude that $F^*(N)$ is solvable. Therefore, $F^*(N) = F(N)$.

Let $Q \in Syl_q(F(N))$. Then $Q \trianglelefteq G$. If Q is not cyclic, then $Q \in \mathcal{S}(Z_{\mathfrak{U}}(G))$ by Theorem 3.1. If Q is cyclic, obviously, $Q \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Hence $F(N) \in \mathcal{S}(Z_{\mathfrak{U}}(G))$. Note that $G/N \in \mathfrak{F}$. By combining [12, Lemma 2.6] with [12, Lemma 2.7], we obtain $G \in \mathfrak{F}$. □

In particular, when $\mathfrak{F}$ is taken as the class of nilpotent groups or the class of supersolvable groups, the conclusion stated in the aforementioned theorem is obviously valid.

4. Conclusions

Note that CAP-subgroups and ICC-subgroups of G belong to $\mathcal{IC}_{\Phi c}(G)$, and if $\mathfrak{F}$ represents a saturated formation, it necessarily follows that it is solvably saturated. Therefore, by Theorems 3.1 and 3.4, we can obtain [4, Main Theorem] and [12, Theorems 3.5 and 3.8]. Hence, our conclusions expand the existing results and contribute new findings to the theory of group structure. In future work, certain elements of $\mathcal{IC}_{\Phi c}(G)$ may be employed to investigate the local versions of Theorem 3.4 (for example, p-nilpotency), a strategy that potentially offers new insights into the group structure by connecting local properties to global behavior.

Author contributions

Huajie Zheng: Writing-original draft; Yong Xu: Conceptualization, supervision, writing-review & editing; Songtao Guo: Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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