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*Research article*

# Sharp weak bounds for fractional Hardy-type operators on homogeneous groups

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**Abstract:** In this paper, we establish weak bounds for fractional Hardy operators on central Morrey spaces within the framework of homogeneous groups. We provide precise operator norms and investigate weak bounds for Hardy operators on central Morrey spaces with power weights. These results significantly extend previous findings in the study of Hardy-type operators and contribute to a broader understanding of their behavior in more general settings, particularly within homogeneous groups.

**Keywords:** homogeneous group; fractional Hardy-type operators; sharp weak bounds; central Morrey spaces; weighted estimates
**Mathematics Subject Classification:** 42B35, 47A30

## 1. Introduction

Hardy inequalities have found wide-ranging applications across various fields, including nonlinear and partial differential equations (PDEs) [16, 18], geometry [5, 22], mathematical physics [6, 15]. Enhanced versions of these inequalities, achieved by adding positive terms to the right-hand side, have played a crucial role in understanding critical phenomena in elliptic and polynomial PDEs. These resultant inequalities are of significant importance in understanding critical phenomena in elliptic and polynomial PDEs [25]. The study of Hardy-type operators has therefore garnered significant attention, leading to important developments, including fractional versions that broaden the applicability of these inequalities.

In this paper, we investigate weak-type estimates for fractional Hardy-type operators on homogeneous groups. For foundational concepts related to homogeneous groups, we refer the reader to [11, 21] and the references therein. Recall that a Lie group $\mathbb{G}$, represented by $(\mathbb{R}^n, \circ)$, is called homogeneous if it admits a dilation map $D_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ for each $\gamma > 0$, given by

$$D_\gamma(x) = (\gamma^{\nu_1} x_1, \gamma^{\nu_2} x_2, \ldots, \gamma^{\nu_n} x_n), \qquad \nu_1, \nu_2, \ldots, \nu_n > 0,$$

where $D_\gamma$ is an automorphism of $\mathbb{G}$. The homogeneous dimension of $\mathbb{G}$ is defined as $Q = \nu_1+\nu_2+\cdots+\nu_n$. Homogeneous groups are known to be nilpotent and unimodular, and their Haar measure coincides with the Lebesgue measure on $\mathbb{R}^n$. For a measurable set $\omega \subset \mathbb{G}$ and $\gamma > 0$, the Haar measure satisfies

$$|D_\gamma(\omega)| = \gamma^Q |\omega| \quad \text{and} \quad \int_{\mathbb{G}} f(\gamma x)dx = \gamma^{-Q} \int_{\mathbb{G}} f(x)dx.$$

Moreover, for $g \in L^1(\mathbb{G})$, the following polar decomposition formula holds:

$$\int_{\mathbb{G}} g(x)dx = \int_0^\infty \int_{\mathfrak{S}} g(ry) r^{Q-1} d\sigma(y) dr. \tag{1.1}$$

where $\mathfrak{S} = \{x \in \mathbb{G} : |x| = 1\} \subset \mathbb{G}$ is the unit sphere with respect to a quasi-norm, and $\sigma$ is a unique Radon measure on $\mathfrak{S}$. This representation plays a crucial role in analyzing the distribution of functions on $\mathbb{G}$ in relation to their behavior on the unit sphere. Typical examples of homogeneous groups are Euclidean space $\mathbb{R}^n$ (where $Q = n$), the Heisenberg group, and more general categories such as stratified groups (also known as homogeneous Carnot groups) and graded groups.

For a locally integrable function $f$ on $\mathbb{G}$ and $0 \le \beta < Q$, the fractional Hardy-type operators on homogeneous group $G$ are defined as

$$\mathcal{H}_\beta f(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{Q}}} \int_{B(0,|x|)} f(y)\, dy \quad \text{and} \quad \mathcal{H}_\beta^* f(x) = \int_{\mathbb{G}\setminus B(0,|x|)} \frac{f(y)}{|B(0,|y|)|^{1-\frac{\beta}{Q}}}\, dy. \tag{1.2}$$

When $\beta = 0$, $\mathcal{H}_\beta$ and $\mathcal{H}_\beta^*$ become $\mathcal{H}$ and $\mathcal{H}^*$, respectively.

Comprehending the weak-type estimates of these operators is essential, particularly for managing their behavior on functions exhibiting singularities. While classical $L^p$ estimates provide average control, weak-type inequalities ensure boundedness on more irregular subsets, thus playing an essential role in harmonic analysis and PDE theory. For instance, Zhao et al. [28] established sharp weak $(p, p)$ bounds for Hardy operators, and Lu et al. [17] determined that the norm of the weak $(1, \frac{n}{n-\beta})$ inequality for the fractional Hardy operator is exactly 1. The study of weighted estimates further enriches the theory by introducing a broader, more flexible framework. Gao and Zhao [13] investigated the weighted weak-type boundedness of fractional Hardy operators and their adjoints with power weights, obtaining sharp bounds. Subsequent work by Gao et al. [12] extended some results of [13] to the weighted case, and gave sharp bound for the power weight of $\mathcal{H}_\beta^*$. Recently, Yu and Li [27] extended the results of reference [13] with power weight.

Recently, research has extended beyond Euclidean spaces [8, 12, 13, 27] to homogeneous groups [9, 19, 23], offering deeper insights into the operators' behavior under non-Euclidean geometries. Motivated by these advancements, in this paper, we focus on establishing sharp weak-type bounds for fractional Hardy-type operators on homogeneous groups. By doing so, we contribute to the theoretical understanding of these operators in a broader and more general setting.

Additionally, we explore the framework of central Morrey spaces on homogeneous groups. Recall that the Morrey space $L^{q,\lambda}(\mathbb{R}^n)$, introduced by Morrey [20], characterizes the local behavior of solutions to elliptic PDEs. Alvarez et al. [2] introduced central Morrey spaces with non-power weights, and subsequent works by Fu et al. [10], Wu [26], Zhao et al. [28], Aykol et al. [1], He and Tao [14], and Sarfraz et al. [24] provided sharp bounds for Hardy operators in these spaces. In [12], the authors introduced the weak central Morrey space with non-power weights $WB^{p,\lambda}(\mathbb{R}^n)$, and proved that $\|\mathcal{H}\|_{B^{p,\lambda}(\mathbb{R}^n)\to WB^{p,\lambda}(\mathbb{R}^n)} = 1$, where $1 \le p < \infty$ and $-\frac{1}{p} \le \lambda < 0$.

To better present our main results, we now extend the definitions of the central Morrey space introduced in [2] and its corresponding weak central Morrey space given in [12] to homogeneous groups equipped with power weights.

For $1 \le p < \infty$, $-\frac{1}{p} \le \lambda < 0$, and $\alpha \in \mathbb{R}$. A function $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ belongs to the central Morrey space $B^{p,\lambda}_\alpha(\mathbb{G})$ if

$$\|f\|_{B^{p,\lambda}_\alpha(\mathbb{G})} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)} |f(x)|^p \cdot |x|^\alpha\, dx\right)^{1/p} < \infty.$$

If $\alpha = 0$, we denote it simply by $B^{p,\lambda}(\mathbb{G})$. Moreover, if $\lambda = -\frac{1}{p}$, $B^{p,\lambda}_\alpha(\mathbb{G})$ coincides with the weighted $L^p$ space $L^p_\alpha(\mathbb{G})$. The corresponding weak central Morrey space $WB^{p,\lambda}_\alpha(\mathbb{G})$ consists of functions $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ such that

$$\|f\|_{WB^{p,\lambda}_\alpha(\mathbb{G})} = \sup_{R>0}|B(0,R)|^{-\lambda-\frac{1}{p}}\sup_{t>0} t\left(\int_{\{x\in B(0,R):\ |f(x)|>t\}} |x|^\alpha\, dx\right)^{1/p} < \infty.$$

When $\alpha = 0$, we denote $WB^{p,\lambda}_\alpha(\mathbb{G})$ by $WB^{p,\lambda}(\mathbb{G})$. Obviously, if $\lambda = -\frac{1}{p}$, then $WB^{p,\lambda}_\alpha(\mathbb{G}) = L^{p,\infty}_\alpha(\mathbb{G})$, which is the weak $L^p_\alpha$ space. It is also clear that $B^{p,\lambda}_\alpha(\mathbb{G}) \subseteq WB^{p,\lambda}_\alpha(\mathbb{G})$ for $1 \le p < \infty$ and $-\frac{1}{p} \le \lambda < 0$.

Our work establishes weak bounds for fractional Hardy operators on central Morrey spaces over homogeneous groups. The employed methodology features a succinct and efficient proof technique, significantly simplifying the demonstration process compared to conventional approaches in operator theory. As an application, we reveal connections between these operators and nonlinear ordinary differential equations of the form (see [3,4]):

$$\lambda\frac{d}{dt}\left([v(t)]^{q/p}[y'(t)]^{q/p'}\right) + \omega(t)[y(t)]^{q/p'} = 0, \tag{1.3}$$

where $v$ and $\omega$ are weight functions on $(0,\infty)$, $v \in C^1(0,\infty)$, $\lambda > 0$, and $y(t)$ is a positive increasing function. Indeed, by using the argument of Theorem 2.2 in [4], we immediately obtain that

$$\left(\int_0^\infty y(x)^q\omega(x)dx\right)^{1/q} \le \lambda^{1/q}\left(\int_0^\infty y'(x)^p\omega(x)dx\right)^{1/p}, \tag{1.4}$$

where $1 < p \le q < \infty$ and $y(x)$ is a solution of Eq (1.3). Note that the inequality (1.4) was connected with the Hardy operator, which can be found in [3].

The key outcomes of our analysis are systematically organized in the following theorems.

**Theorem 1.1.** *Assume that $f \in B^{p,\lambda}_\alpha(\mathbb{G})$ is nonnegative. If $1 < p < \infty$, $-\frac{1}{p} \le \lambda < 0$ and $0 \le \alpha < -\lambda p Q(p-1)$, then the inequality*

$$\|\mathcal{H}f\|_{WB^{p,\lambda}_\alpha(\mathbb{G})} \le \mathcal{A}_{sharp}\|f\|_{B^{p,\lambda}_\alpha(\mathbb{G})}$$

*holds with the sharp constant*

$$\mathcal{A}_{sharp} = \left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\left(\frac{Q}{Q-\alpha/(p-1)}\right)^{\frac{1}{p'}}.$$

**Remark 1.1.** *When $p = 1$, we still attain the sharp weak bound $\mathcal{A}_{sharp} = 1$. The proof closely follows that of Theorem* 1.1*, with nearly identical methods and steps. Indeed, as $p \to 1^+$, we have $\alpha \to 0^+$, and thus*

$$\lim_{(p,\alpha)\to(1^+,0^+)} \left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \left(\frac{Q}{Q-\alpha/(p-1)}\right)^{\frac{1}{p'}} = 1.$$

For the fractional Hardy operators $\mathcal{H}_\beta$, the weak Morrey estimates also hold.

**Theorem 1.2.** *Assume that $f \in B^{p,\lambda_1}(\mathbb{G})$ is nonnegative. If $0 \le \beta < Q$, $1 \le p < q < \infty$, $-\frac{1}{p} \le \lambda_1 < 0$, $-\frac{1}{q} \le \lambda_2 < 0$, $\frac{1}{p} + \lambda_1 = \frac{1}{q} + \lambda_2$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{Q}$, then the inequality*

$$\|\mathcal{H}_\beta f\|_{WB^{q,\lambda_2}(\mathbb{G})} \le \mathcal{B}_{sharp}\|f\|_{B^{p,\lambda_1}(\mathbb{G})}$$

*holds with the best constant satisfactions*

$$\left(\frac{Q}{Q-\beta}\right)\left(\frac{Q-p\beta}{Q}\right)^{\frac{1}{p}} \le \mathcal{B}_{sharp} \le 1. \tag{1.5}$$

**Remark 1.2.** *When $p = 1$, $\mathcal{B}_{sharp} = 1$ in Theorem* 1.2*. And when $\beta = 0$, i.e., $p = q$ and $\lambda_1 = \lambda_2$, Theorem 1.2 reduces to the case $\alpha = 0$ in Theorem 1.1, which can be viewed as a natural extension of Theorem 2.1 in [12] to the setting of homogeneous groups.*

For $\beta \ge 0$, the following results provide the sharp bounds for the norm of the conjugate operator $\mathcal{H}^*_\beta$.

**Theorem 1.3.** *Assume that $f \in L^p_\alpha(\mathbb{G})$ is nonnegative. Let $0 < \beta < Q$, $1 < p < \frac{Q+\alpha}{\beta}$, and assume that $\frac{Q+\gamma}{q} = \frac{Q+\alpha}{p} - \beta$. If $f \in L^p_\alpha(\mathbb{G})$, then the inequality*

$$\|\mathcal{H}^*_\beta f\|_{L^{q,\infty}_\gamma(\mathbb{G})} \le \mathcal{B}^*_{sharp}\|f\|_{L^p_\alpha(\mathbb{G})}$$

*holds with the sharp constant*

$$\mathcal{B}^*_{sharp} = \left(\frac{q}{p'}\right)^{\frac{1}{p'}} \left(\frac{Q}{Q+\gamma}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{\beta}{Q}+\frac{1}{q}-\frac{1}{p}}.$$

**Remark 1.3.** *When $\beta = 0$, the condition $1 < p < \frac{Q+\alpha}{\beta}$ should be interpreted as $1 < p < \infty$, and Theorem 1.3 remains valid in this case. In particular, when the homogeneous group $\mathbb{G}$ reduces to the Euclidean space $\mathbb{R}^n$, so that $Q = n$, Theorem 1.3 specializes to the result in [12, Theorem 2.4].*

**Remark 1.4.** *When $p = 1$, we also obtain the sharp bound $\mathcal{B}^*_{sharp} = \left(\frac{Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{\beta}{Q}+\frac{1}{q}-1}$. The method is similar to the proof of Theorem* 1.3.

Throughout this paper, $|\mathfrak{S}|$ denotes the volume of the unit sphere. Let $1 < p, p' < \infty$, where $p$ and $p'$ are conjugate indices, i.e., $1/p + 1/p' = 1$. Formally, we will also define $p = 1$ as the conjugate to $p' = \infty$, and vice versa.

## 2. Proofs of main results

In this section, we will prove our main results.

*Proof of Theorem 1.1.* Since $-\frac{1}{p} \leq \lambda < 0$ and $0 \leq \alpha < -\lambda p Q(p-1)$, we deduce that $\frac{\alpha}{p-1} < Q$. Applying Hölder's inequality yields

$$\begin{aligned} |\mathcal{H}f(x)| &\leq \frac{1}{V_Q|x|^Q} \left( \int_{B(0,|x|)} |y|^{-\frac{\alpha p'}{p}} \, dy \right)^{\frac{1}{p'}} \left( \int_{B(0,|x|)} |f(y)|^p \cdot |y|^\alpha \, dy \right)^{\frac{1}{p}} \\ &\leq V_Q^\lambda \left( \frac{Q}{Q - \frac{\alpha}{p-1}} \right)^{\frac{1}{p'}} |x|^{Q\lambda - \frac{\alpha}{p}} \cdot \|f\|_{B_\alpha^{p,\lambda}(\mathbb{G})}, \end{aligned} \tag{2.1}$$

where $V_Q$ is the volume measure of the unit sphere in the homogeneous group $\mathbb{G}$ and $V_Q = \frac{|\mathfrak{S}|}{Q}$.

Inequalities (2.1) and $Q\lambda - \frac{\alpha}{p} < 0$ imply that

$$\|\mathcal{H}f\|_{WB_\alpha^{p,\lambda}(\mathbb{G})} \leq \sup_{R>0} |B(0,R)|^{-\lambda-\frac{1}{p}} \sup_{t>0} t \left( \int_{\mathbb{G}} \chi_{\{x\in B(0,R):\, |x|<\left(\frac{t}{A}\right)^{\frac{1}{(Q\lambda-\alpha/p)}}\}}(x) \cdot |x|^\alpha \, dx \right)^{1/p},$$

where $A = V_Q^\lambda \left(\frac{Q}{Q-\alpha/(p-1)}\right)^{1/p'} \|f\|_{B_\alpha^{p,\lambda}(\mathbb{G})}$. Let $K := \left(\frac{t}{A}\right)^{1/(Q\lambda-\frac{\alpha}{p})}$, we will now consider two cases: $0 < R \leq K$ and $R > K$.

**Case I**: $0 < R \leq K$. Since $-Q\lambda + \frac{\alpha}{p} > 0$, it follows that

$$\begin{aligned} \|\mathcal{H}f\|_{WB_\alpha^{p,\lambda}(\mathbb{G})} &\leq \sup_{0<R\leq K} |B(0,R)|^{-\lambda-\frac{1}{p}} \sup_{t>0} t \left( |\mathfrak{S}| \int_0^R r^{\alpha+Q-1} \, dr \right)^{\frac{1}{p}} \\ &= V_Q^{-\lambda} \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} \sup_{t>0} \sup_{0<R\leq K} t \cdot R^{-Q\lambda+\frac{\alpha}{p}} \\ &= V_Q^{-\lambda} \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} A = \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} \left( \frac{Q}{Q-\alpha/(p-1)} \right)^{\frac{1}{p'}} \|f\|_{B_\alpha^{p,\lambda}(\mathbb{G})}. \end{aligned}$$

**Case II**: $R > K$. It is easy to compute that

$$\begin{aligned} \|\mathcal{H}f\|_{WB_\alpha^{p,\lambda}(\mathbb{G})} &\leq \sup_{R>K} |B(0,R)|^{-\lambda-\frac{1}{p}} \sup_{t>0} t \left( |\mathfrak{S}| \int_0^K r^{\alpha+Q-1} \, dr \right)^{\frac{1}{p}} \\ &= V_Q^{-\lambda} \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} \sup_{t>0} \sup_{R>K} t \cdot R^{-Q(\lambda+\frac{1}{p})} K^{\frac{\alpha+Q}{p}} \\ &= V_Q^{-\lambda} \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} \sup_{t>0} t \cdot K^{-Q\lambda+\frac{\alpha}{p}} = \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} \left( \frac{Q}{Q-\alpha/(p-1)} \right)^{\frac{1}{p'}} \|f\|_{B_\alpha^{p,\lambda}(\mathbb{G})}. \end{aligned}$$

Combining Cases I and II, we obtain

$$\|\mathcal{H}f\|_{WB_\alpha^{p,\lambda}(\mathbb{G})} \leq \left( \frac{Q}{Q+\alpha} \right)^{\frac{1}{p}} \left( \frac{Q}{Q-\alpha/(p-1)} \right)^{\frac{1}{p'}} \|f\|_{B_\alpha^{p,\lambda}(\mathbb{G})}. \tag{2.2}$$

Next, we will demonstrate that the constant $\mathcal{A}_{\mathrm{sharp}} = \left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\left(\frac{Q}{Q-\alpha/(p-1)}\right)^{\frac{1}{p'}}$ is optimal, that is, show the existence of a function $f_0 \in B_\alpha^{p,\lambda}(\mathbb{G})$ that satisfies the following equation:

$$\|\mathcal{H}f_0\|_{WB_\alpha^{p,\lambda}(\mathbb{G})} = \mathcal{A}_{\mathrm{sharp}}\|f_0\|_{B_\alpha^{p,\lambda}(\mathbb{G})}. \tag{2.3}$$

Suppose that there exists a function in the simple form $f_0(x) = |x|^m\chi_{\{x:\,|x|\le 1\}}(x)$, where the real number $m \in (\lambda Q - \alpha/p, 0)$. In this case, it suffices to prove that Eq (2.3) has at least one solution for $m$. When $0 < R \le 1$, it follows from $m > \lambda Q - \frac{\alpha}{p}$ and $\lambda \ge -\frac{1}{p}$ that $m > -\frac{Q+\alpha}{p}$. Thus, $\mathcal{F}$ can be computed as

$$\mathcal{F} := \int_{B(0,R)} |f_0(x)|^p \cdot |x|^\alpha\, dx = |\mathfrak{S}| \int_0^R r^{Q+pm+\alpha-1}\, dr = \left(\frac{|\mathfrak{S}|}{Q+pm+\alpha}\right) R^{Q+pm+\alpha}.$$

When $R > 1$, it is also easy to calculate that

$$\mathcal{F} = |\mathfrak{S}| \int_0^1 r^{Q+pm+\alpha-1}\, dr = \frac{|\mathfrak{S}|}{Q+pm+\alpha}.$$

Therefore, combining the above two cases and the definition of $B_\alpha^{p,\lambda}(\mathbb{G})$, we have

$$\begin{aligned}\|f_0\|_{B_\alpha^{p,\lambda}(\mathbb{G})} &= \max\left\{\sup_{0<R\le 1}\left(\frac{\mathcal{F}}{|B(0,R)|^{1+\lambda p}}\right)^{1/p},\ \sup_{R>1}\left(\frac{\mathcal{F}}{|B(0,R)|^{1+\lambda p}}\right)^{1/p}\right\}\\ &= V_Q^{-\lambda}\left(\frac{Q}{Q+pm+\alpha}\right)^{\frac{1}{p}} \max\left\{\sup_{0<R\le 1} R^{m+\frac{\alpha}{p}-Q\lambda},\ \sup_{R>1} R^{-Q\lambda-\frac{Q}{p}}\right\} = V_Q^{-\lambda}\left(\frac{Q}{Q+pm+\alpha}\right)^{\frac{1}{p}}. \end{aligned} \tag{2.4}$$

Substituting the function $f_0(x)$ into the Hardy-type operator $\mathcal{H}f$, i.e., Eq (1.2) with $\beta = 0$, we compute $\mathcal{H}f_0(x)$ as follows:

$$\mathcal{H}f_0(x) = \begin{cases} \dfrac{Q}{m+Q}\cdot|x|^m, & 0 \le |x| \le 1;\\ \dfrac{Q}{m+Q}\cdot|x|^{-Q}, & |x| > 1. \end{cases}$$

Let $\mathcal{D} := \frac{Q}{m+Q}$, then we have

$$\mathcal{E} := \{x \in B(0,R) \colon |\mathcal{H}f_0(x)| > t\} = \left\{x \in B(0,1) \colon |x| < \left(\frac{\mathcal{D}}{t}\right)^{-\frac{1}{m}}\right\} \cup \left\{1 \le |x| < R \colon |x| < \left(\frac{\mathcal{D}}{t}\right)^{\frac{1}{Q}}\right\}.$$

To obtain our results, we divide $\mathcal{D}$ into two parts: $\mathcal{D} > t$ and $\mathcal{D} \le t$.

**Case I**: $\mathcal{D} > t$. In this case, we have $\left(\frac{\mathcal{D}}{t}\right)^{-\frac{1}{m}} > 1$ and $\left(\frac{\mathcal{D}}{t}\right)^{\frac{1}{Q}} > 1$.

When $0 < R \le (\frac{\mathcal{D}}{t})^{\frac{1}{Q}}$, the set $\mathcal{E} = \{x \colon x \in B(0,R)\}$. Due to $\frac{\alpha}{Q} < p - 1$, we can obtain

$$1 + \lambda - \frac{\alpha}{pQ} > 1 + \lambda - \frac{p-1}{p} > \lambda + \frac{1}{p} > 0. \tag{2.5}$$

Thus, (2.5) allows us to conclude that

$$\begin{aligned}
\|\mathcal{H}f_0\|_{WB^{p,\lambda}_{\alpha}(\mathbb{G})} &= \sup_{0<R<(\frac{\mathcal{D}}{t})^{1/Q}} \left(V_Q R^Q\right)^{-\lambda-\frac{1}{p}} \sup_{t<\mathcal{D}} t\left(|\mathfrak{S}| \int_0^R r^{\alpha+Q-1}\,dr\right)^{\frac{1}{p}} \\
&= V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \sup_{0<R<(\frac{\mathcal{D}}{t})^{1/Q}} \sup_{t<\mathcal{D}} t\cdot R^{-Q\lambda+\frac{\alpha}{p}} \\
&= V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \sup_{t<\mathcal{D}} t\cdot\left(\frac{\mathcal{D}}{t}\right)^{-\lambda+\frac{\alpha}{pQ}} = V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\mathcal{D}.
\end{aligned}$$

When $R > (\frac{\mathcal{D}}{t})^{\frac{1}{Q}}$, the set $\mathcal{E}$ can be calculated as $\left\{x\colon 0<|x|<\left(\frac{\mathcal{D}}{t}\right)^{1/Q}\right\}$. Thus, (2.5) also gives that

$$\begin{aligned}
\|\mathcal{H}f_0\|_{WB^{p,\lambda}_{\alpha}(\mathbb{G})} &= \sup_{R>(\frac{\mathcal{D}}{t})^{1/Q}} \left(V_Q R^Q\right)^{-\lambda-\frac{1}{p}} \sup_{t<\mathcal{D}} t\left(|\mathfrak{S}| \int_0^{(\frac{\mathcal{D}}{t})^{1/Q}} r^{\alpha+Q-1}\,dr\right)^{\frac{1}{p}} \\
&= V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \sup_{R>(\frac{\mathcal{D}}{t})^{1/Q}} \sup_{t<\mathcal{D}} t\cdot\left(\frac{\mathcal{D}}{t}\right)^{\frac{Q+\alpha}{Qp}}\cdot R^{-Q(\lambda+\frac{1}{p})} = V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\mathcal{D}.
\end{aligned}$$

**Case II**: $\mathcal{D} < t$. In this case, we can easily obtain $\left(\frac{\mathcal{D}}{t}\right)^{-\frac{1}{m}} < 1$, $\left(\frac{\mathcal{D}}{t}\right)^{\frac{1}{Q}} < 1$ and $\left\{1\le|x|<R\colon |x|<\left(\frac{\mathcal{D}}{t}\right)^{\frac{1}{Q}}\right\} = \emptyset$. When $0 < R < \left(\frac{\mathcal{D}}{t}\right)^{-\frac{1}{m}}$, the set $\mathcal{E}$ is equal to $\{x\colon x\in B(0,R)\}$. The inequality $m > \lambda Q - \frac{\alpha}{p}$ allows us to deduce that

$$\begin{aligned}
\|\mathcal{H}f_0\|_{WB^{p,\lambda}_{\alpha}(\mathbb{G})} &= \sup_{0<R<(\frac{\mathcal{D}}{t})^{-1/m}} \left(V_Q R^Q\right)^{-\lambda-\frac{1}{p}} \sup_{t>\mathcal{D}} t\left(|\mathfrak{S}| \int_0^R r^{\alpha+Q-1}\,dr\right)^{\frac{1}{p}} \\
&= V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \sup_{0<R<(\frac{\mathcal{D}}{t})^{-1/m}} \sup_{t>\mathcal{D}} t\cdot R^{-Q\lambda+\frac{\alpha}{p}} \\
&= V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \sup_{t>\mathcal{D}} t\cdot\left(\frac{\mathcal{D}}{t}\right)^{\frac{Q\lambda}{m}-\frac{\alpha}{mp}} = V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\mathcal{D}.
\end{aligned}$$

When $R \ge \left(\frac{\mathcal{D}}{t}\right)^{-\frac{1}{m}}$, the set $\mathcal{E}$ can be computed as $\left\{x\colon 0<|x|<\left(\frac{\mathcal{D}}{t}\right)^{-1/m}\right\}$. The inequality $m > \lambda Q - \frac{\alpha}{p}$ also gives that

$$\begin{aligned}
\|\mathcal{H}f_0\|_{WB^{p,\lambda}_{\alpha}(\mathbb{G})} &= \sup_{R\ge(\frac{\mathcal{D}}{t})^{-1/m}} \left(V_Q R^Q\right)^{-\lambda-\frac{1}{p}} \sup_{t>\mathcal{D}} t\left(|\mathfrak{S}| \int_0^{(\frac{\mathcal{D}}{t})^{-1/m}} r^{\alpha+Q-1}\,dr\right)^{\frac{1}{p}} \\
&= V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \sup_{R\ge(\frac{\mathcal{D}}{t})^{-1/m}} \sup_{t>\mathcal{D}} t\cdot\left(\frac{\mathcal{D}}{t}\right)^{\frac{Q+\alpha}{-mp}}\cdot R^{-Q(\lambda+\frac{1}{p})} = V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\mathcal{D}.
\end{aligned}$$

Combining Cases I and II, we obtain

$$\|\mathcal{H} f_0\|_{WB^{p,\lambda}_{\alpha}(\mathbb{G})} = V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}} \mathcal{D} = V_Q^{-\lambda}\left(\frac{Q}{Q+\alpha}\right)^{\frac{1}{p}}\left(\frac{Q}{m+Q}\right). \tag{2.6}$$

Therefore, using (2.4) and (2.6), Eq (2.3) can be rewritten as

$$\left(Q-\tfrac{\alpha}{p-1}\right)^{\frac{1}{p'}}(Q+pm+\alpha)^{\frac{1}{p}} = m+Q.$$

Define the function

$$T(m) = \left(Q-\tfrac{\alpha}{p-1}\right)^{\frac{1}{p'}}(Q+pm+\alpha)^{\frac{1}{p}} - m - Q,$$

where $m \in (Q\lambda - \alpha/p, 0)$. Moreover, it is straightforward to obtain its derivative

$$T'(m) = \left(Q-\tfrac{\alpha}{p-1}\right)^{\frac{1}{p'}}(Q+pm+\alpha)^{\frac{1}{p}-1} - 1.$$

Let $m_0 := -\frac{\alpha}{p-1}$. Thus, it is easy to check that $m_0 \in (Q\lambda - \alpha/p, 0)$. And it is not difficult to conclude that $T(m)$ is increasing on $(Q\lambda - \alpha/p, m_0)$ and decreasing on $(m_0, 0)$. Hence, we have

$$T(m) \le T(m_0) = 0.$$

Therefore, the function $T(m)$ has a unique solution $m_0$ in the interval $(Q\lambda - \alpha/p, 0)$. We conclude that there exists a function $f_0 \in B^{p,\lambda}_{\alpha}(\mathbb{G})$ such that Eq (2.3) holds. This completes the proof of Theorem 1.1.

**Remark 2.1.** *It can be shown that* (2.2) *still holds when* $\alpha \in (\lambda p Q, 0)$. *However, no function of the simple form* $f_0(x) = |x|^m \chi_{\{x:\,|x|\le 1\}}(x)$ *satisfies Eq* (2.3), *not even for* $m < 0$.

*Proof of Theorem 1.2.* By Hölder's inequality and $\lambda_2 - \lambda_1 = \frac{\beta}{Q}$, we obtain

$$|\mathcal{H}_\beta f(x)| \le |B(0,|x|)|^{\lambda_2}\left(\frac{1}{|B(0,|x|)|^{1+\lambda_1 p}}\int_{B(0,|x|)}|f(y)|^p\,dy\right)^{1/p} \le V_Q^{\lambda_2}\|f\|_{B^{p,\lambda_1}(\mathbb{G})}\cdot|x|^{Q\lambda_2} := \mathcal{T}|x|^{Q\lambda_2}.$$

Therefore, we have

$$\|\mathcal{H}_\beta f\|_{WB^{q,\lambda_2}(\mathbb{G})} \le \sup_{R>0}|B(0,R)|^{-\lambda_2-\frac{1}{q}}\sup_{t>0} t\left(\int_{\mathbb{G}}\chi_{\left\{x\in B(0,R):\,|x|<\left(\frac{t}{\mathcal{T}}\right)^{\frac{1}{Q\lambda_2}}\right\}}dx\right)^{1/q}.$$

When $0 < R \le \left(\frac{t}{\mathcal{T}}\right)^{\frac{1}{Q\lambda_2}}$, it is easy to see that $\left\{x \in B(0,R)\colon |x| < \left(\frac{t}{\mathcal{T}}\right)^{\frac{1}{Q\lambda_2}}\right\} = \{x \in B(0,R)\}$. Based on this case, we obtain

$$\|\mathcal{H}_\beta f\|_{WB^{q,\lambda_2}(\mathbb{G})} \le \sup_{0<R\le\left(\frac{t}{\mathcal{T}}\right)^{1/Q\lambda_2}}\sup_{t>0} V_Q^{-\lambda_2}\cdot tR^{-Q\lambda_2} = V_Q^{-\lambda_2}\sup_{t>0} t\left(\tfrac{t}{\mathcal{T}}\right)^{-1} = \|f\|_{B^{p,\lambda_1}(\mathbb{G})}. \tag{2.7}$$

When $R > \left(\frac{t}{\mathcal{T}}\right)^{\frac{1}{Q\lambda_2}}$, we can also obtain $\left\{x \in B(0,R)\colon |x| < \left(\frac{t}{\mathcal{T}}\right)^{\frac{1}{Q\lambda_2}}\right\} = \left\{x\colon |x| < \left(\frac{t}{\mathcal{T}}\right)^{\frac{1}{Q\lambda_2}}\right\}$. This case gives that

$$\|\mathcal{H}_\beta f\|_{WB^{q,\lambda_2}(\mathbb{G})} \le \sup_{R>\left(\frac{t}{\mathcal{T}}\right)^{1/Q\lambda_2}}\sup_{t>0} V_Q^{-\lambda_2} tR^{-Q(\lambda_2+\frac{1}{q})}\cdot\left(\frac{t}{\mathcal{T}}\right)^{1/q\lambda_2} = V_Q^{-\lambda_2}\sup_{t>0} t\left(\tfrac{t}{\mathcal{T}}\right)^{-1} = \|f\|_{B^{p,\lambda_1}(\mathbb{G})}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\|\mathcal{H}_\beta f\|_{WB^{q,\lambda_2}(\mathbb{G})} \leq 1 \cdot \|f\|_{B^{p,\lambda_1}(\mathbb{G})}.$$

Thus, we have derived the right-hand side of inequality (1.5), namely $\mathcal{B}_{\text{sharp}} \leq 1$. Next, we turn to the proof of the lower bound of inequality (1.5). Taking $f_0(x) = |x|^{-\beta}\chi_{\{x:\,|x|\leq 1\}}(x)$. When $0 < R \leq 1$, we can compute that

$$I := \int_{B(0,R)} |f_0(x)|^p \, dx = |\mathfrak{S}| \int_0^R r^{-p\beta+Q-1} \, dr = \frac{|\mathfrak{S}|}{Q - p\beta} R^{Q-p\beta}.$$

When $R > 1$, we also have

$$I = |\mathfrak{S}| \int_0^1 r^{-p\beta+Q-1} \, dr = \frac{|\mathfrak{S}|}{Q - p\beta}.$$

Combining the above two cases, the $B^{p,\lambda}(\mathbb{G})$-norm of $f_0$ can be obtained directly, noting that for $-\beta - Q\lambda_1 = -Q\lambda_2 > 0$,

$$\begin{aligned} \|f_0\|_{B^{p,\lambda}(\mathbb{G})} &= \max\left\{ \sup_{0<R\leq 1} \left( \frac{\mathcal{I}}{|B(0,R)|^{1+\lambda_1 p}} \right)^{1/p}, \sup_{R>1} \left( \frac{\mathcal{I}}{|B(0,R)|^{1+\lambda_1 p}} \right)^{1/p} \right\} \\ &= V_Q^{-\lambda_1} \left( \frac{Q}{Q - p\beta} \right)^{\frac{1}{p}} \max\left\{ \sup_{0\leq R\leq 1} R^{-\beta-Q\lambda_1}, \sup_{R>1} R^{-Q(\lambda_1+\frac{1}{p})} \right\} \\ &= V_Q^{-\lambda_1} \left( \frac{Q}{Q - p\beta} \right)^{\frac{1}{p}}. \end{aligned} \tag{2.9}$$

By the definition of $\mathcal{H}_\beta$, we have

$$\mathcal{H}_\beta f_0(x) = \begin{cases} V_Q^{\frac{\beta}{Q}} \dfrac{Q}{Q-\beta}, & 0 \leq |x| \leq 1; \\ V_Q^{\frac{\beta}{Q}} \dfrac{Q}{Q-\beta} \cdot |x|^{\beta - Q}, & |x| > 1. \end{cases}$$

Let $\mathcal{M} := \frac{Q}{Q-\beta} V_Q^{\frac{\beta}{Q}}$, it is evident that $|\mathcal{H}_\beta f_0(x)| \leq \mathcal{M}$. Thus, we can choose $t < \mathcal{M}$, such that

$$\begin{aligned} \left\{ x \in B(0,R) \colon |\mathcal{H}_\beta f_0(x)| > t \right\} &= \{ x \in B(0,1) \colon \mathcal{M} > t \} \cup \left\{ 1 \leq |x| \leq R \colon |x| < \left( \tfrac{t}{\mathcal{M}} \right)^{1/(\beta - Q)} \right\} \\ &= \left\{ x \in B(0,R) \colon |x| < \left( \tfrac{t}{\mathcal{M}} \right)^{1/(\beta-Q)} \right\}. \end{aligned}$$

Similar to the estimates of inequalities (2.7) and (2.8), we have $\|\mathcal{H}_\beta f_0\|_{WB^{q,\lambda_2}(\mathbb{G})} = V_Q^{-\lambda_2} \frac{Q}{Q-\beta}$. Therefore,

$$\mathcal{B}_{\text{sharp}} = \|\mathcal{H}_\beta\|_{B^{p,\lambda_1}(\mathbb{G}) \to WB^{q,\lambda_2}(\mathbb{G})} \geq \frac{\|\mathcal{H}_\beta f_0\|_{WB^{q,\lambda_2}(\mathbb{G})}}{\|f_0\|_{B^{p,\lambda_1}(\mathbb{G})}} = \left( \frac{Q}{Q-\beta} \right) \left( \frac{Q - p\beta}{Q} \right)^{\frac{1}{p}}.$$

Thus, Theorem 1.2 is proved.

*Proof of Theorem 1.3.* By Hölder's inequality, and since $p < \frac{Q+\alpha}{\beta}$, $\frac{Q+\gamma}{q} = \frac{Q+\alpha}{p} - \beta$, we have

$$\begin{aligned}|\mathcal{H}^*_\beta f(x)| &\leq \left(\int_{\mathbb{G}\setminus B(0,|x|)} |y|^{-\frac{p'\alpha}{p}} \cdot |B(0,|y|)|^{\frac{p'\beta}{Q}-p'}\, dy\right)^{1/p'} \left(\int_{\mathbb{G}\setminus B(0,|x|)} |f(y)|^p |y|^\alpha\, dy\right)^{1/p} \\ &\leq V_Q^{\frac{\beta}{Q}-\frac{1}{p}} \left(\frac{q}{p'}\right)^{\frac{1}{p'}} \left(\frac{Q}{Q+\gamma}\right)^{\frac{1}{p'}} \|f\|_{L^p_\alpha(\mathbb{G})} \cdot |x|^{-\frac{Q+\gamma}{q}} := \mathcal{L} \cdot |x|^{-\frac{Q+\gamma}{q}}.\end{aligned}$$

Therefore, we have

$$\|\mathcal{H}^*_\beta f\|_{L^{q,\infty}_\gamma(\mathbb{G})} \leq \sup_{t>0} t \left(|\mathfrak{S}| \int_0^{\left(\frac{\mathcal{L}}{t}\right)^{\frac{q}{Q+\gamma}}} r^{Q+\gamma-1}\, dr\right)^{1/q} = \left(\frac{q}{p'}\right)^{\frac{1}{p'}} \left(\frac{Q}{Q+\gamma}\right)^{\frac{1}{p'}+\frac{1}{q}} V_Q^{\frac{\beta}{Q}+\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p_\alpha(\mathbb{G})}.$$

Next, we will demonstrate that the constant $\mathcal{B}^*_{\text{sharp}}$ is optimal, that is, show the existence of a function $f_0 \in L^p_\alpha(\mathbb{G})$ that satisfies the following equation:

$$\|\mathcal{H}^*_\beta f_0\|_{L^{q,\infty}_\gamma(\mathbb{G})} = \mathcal{B}^*_{\text{sharp}} \|f_0\|_{L^p_\alpha(\mathbb{G})}. \tag{2.10}$$

Suppose there exists a function of the simple form $f_0 = |x|^m \chi_{\{x:\,|x|\geq 1\}}(x)$, where the real number $m < -\frac{Q+\alpha}{p}$. In this case, it suffices to show that Eq (2.10) admits at least one solution for $m$.

Since $pm + \alpha + Q < 0$, it follows that

$$\begin{aligned}\|f_0\|_{L^p_\alpha(\mathbb{G})} &= \left(\int_{\mathbb{G}} |x|^{pm} \chi_{\{x:\,|x|\geq 1\}}(x) \cdot |x|^\alpha\, dx\right)^{1/p} = \left(|\mathfrak{S}| \int_1^{+\infty} r^{pm+\alpha+Q-1}\, dr\right)^{1/p} \\ &= \left(\frac{Q}{-pm-\alpha-Q}\right)^{\frac{1}{p}} V_Q^{\frac{1}{p}}.\end{aligned} \tag{2.11}$$

Observe that $m + \beta < m + \frac{Q+\alpha}{p} < 0$; thus, we can conclude that

$$\mathcal{H}^*_\beta f_0(x) = \begin{cases} \dfrac{Q}{-m-\beta} V_Q^{\frac{\beta}{Q}}, & 0 \leq |x| \leq 1; \\ \dfrac{Q}{-m-\beta} V_Q^{\frac{\beta}{Q}} \cdot |x|^{m+\beta}, & |x| > 1. \end{cases}$$

Let $\mathcal{K} := \frac{Q}{-m-\beta} V_Q^{\frac{\beta}{Q}}$, and it is evident that $|\mathcal{H}^*_\beta f_0(x)| \leq \mathcal{K}$. When $t \geq \mathcal{K}$, it implies that

$$\left\{x \in \mathbb{G} \colon |\mathcal{H}^*_\beta f_0(x)| > t\right\} = \emptyset.$$

Thus, we only need to consider the case $t < \mathcal{K}$, and the following identity holds:

$$\begin{aligned}\left\{x \in \mathbb{G} \colon |\mathcal{H}^*_\beta f_0(x)| > t\right\} &= \{0 \leq |x| \leq 1 \colon \mathcal{K} > t\} \cup \{|x| > 1 \colon \mathcal{K}|x|^{m+\beta} > t\} \\ &= \left\{x \in \mathbb{G} \colon |x| < \left(\tfrac{t}{\mathcal{K}}\right)^{1/(m+\beta)}\right\}.\end{aligned}$$

Furthermore, note that

$$1 + \frac{Q+\gamma}{q(m+\beta)} = 1 + \frac{(Q+\alpha)/p - \beta}{m+\beta} = \frac{m + (Q+\alpha)/p}{m+\beta} > 0.$$

Therefore, we have

$$\|\mathcal{H}_\beta^* f_0\|_{L_\gamma^{q,\infty}(\mathbb{G})} = \sup_{0<t<\mathcal{K}} t\left(|\mathfrak{S}|\int_0^{\left(\frac{t}{\mathcal{K}}\right)^{1/(m+\beta)}} r^{Q+\gamma-1}\,dr\right)^{1/q}$$
$$= \sup_{0<t<\mathcal{K}}\left(\frac{|\mathfrak{S}|}{Q+\gamma}\right)^{\frac{1}{q}}\mathcal{K}^{-\frac{Q+\gamma}{q(m+\beta)}}\cdot t^{1+\frac{Q+\gamma}{q(m+\beta)}} = \left(\frac{Q}{-m-\beta}\right)\left(\frac{Q}{Q+\gamma}\right)^{\frac{1}{q}} V_Q^{\frac{\beta}{Q}+\frac{1}{q}}. \tag{2.12}$$

Hence, by combining (2.11) and (2.12), Eq (2.10) simplifies to the following form:

$$\frac{Q}{-m-\beta} = \left(\frac{Q}{-pm-\alpha-Q}\right)^{\frac{1}{p}}\left(\frac{qQ}{p'(Q+\gamma)}\right)^{\frac{1}{p'}}.$$

Furthermore, using the relation $\frac{Q+\gamma}{q} = \frac{Q+\alpha}{p} - \beta$, the equation can be further simplified as follows:

$$\frac{1}{-m-\beta} = \left(\frac{1}{-pm-\alpha-Q}\right)^{\frac{1}{p}}\left(\frac{p-1}{Q+\alpha-p\beta}\right)^{\frac{1}{p'}}. \tag{2.13}$$

Taking the logarithm on both sides of Eq (2.13), we obtain

$$-ln(-m-\beta) = -\frac{1}{p}ln(-mp-\alpha-Q) + \frac{1}{p'}ln\left(\frac{p-1}{Q+\alpha-p\beta}\right).$$

Define the function

$$\varphi(m) = ln(-m-\beta) - \frac{1}{p}ln(-mp-\alpha-Q) + \frac{1}{p'}ln\left(\frac{p-1}{Q+\alpha-p\beta}\right),$$

where $m \in (-\infty, -\frac{Q+\alpha}{p})$. Then, we can calculate its derivative

$$\varphi'(m) = \frac{(p-1)m+Q+\alpha-\beta}{(m+\beta)(mp+Q+\alpha)}.$$

Let $m_0 := \frac{\beta-Q-\alpha}{p-1}$, then we obtain $m_0 < -\frac{Q+\alpha}{p}$. It is not difficult to conclude that $\varphi(m)$ is decreasing on $(-\infty, m_0)$ and increasing on $(m_0, -\frac{Q+\alpha}{p})$. Hence, we have

$$\varphi(m) \geq \varphi(m_0) = 0.$$

Therefore, the function $\varphi(m)$ has a unique solution $m_0$ in the interval $(-\infty, -\frac{Q+\alpha}{p})$. Thus, we complete the proof of Theorem 1.3.

## 3. Conclusions

In this paper, we have established sharp weak-type estimates for both classical and fractional Hardy operators and their adjoints on homogeneous groups, within the frameworks of weighted Lebesgue spaces and central Morrey spaces. Specifically, in weighted central Morrey spaces, we obtain the sharp weak bound $\mathcal{A}_{\text{sharp}}$ for the classical Hardy operator $\mathcal{H}$, and accurately characterized the influence of the homogeneous dimension $Q$ and the weight parameter $\alpha$ on the value of the constant. For the fractional case, we establish the sharp weak-type estimates of the fractional Hardy operator $\mathcal{H}_\beta$ under the conditions $\frac{1}{p} + \lambda_1 = \frac{1}{q} + \lambda_2$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{Q}$. In weighted Lebesgue spaces, we derive the sharp weak bound $\mathcal{B}^*_{\text{sharp}}$ for the adjoint operator $\mathcal{H}^*_\beta$. Furthermore, we reveal a connection between these Hardy-type operators and a class of nonlinear ordinary differential equations, highlighting the interplay between harmonic analysis and differential equations.

## Author contributions

Hongbin Liu, Juan Zhang and Qianjun He: Conceptualization, Investigation, Methodology, Software, Writing–original draft, Writing–review and editing; Juan Zhang and Qianjun He: Validation, Supervision; Juan Zhang: Funding acquisition, Project administration. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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