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Research article

A novel multi-granularity variable precision neutrosophic rough set and group decision-making application with three strategies

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Abstract: This paper proposes a novel neutrosophic rough set model to solve group decision-making problems. First, we proposed a novel multi-granularity variable-precision neutrosophic rough set model, which included three basic models: optimistic, pessimistic, and compromise. Second, the properties of the upper and lower approximations of the multi-granularity variable-precision neutrosophic rough set were investigated by means of the residual implications of triangular norms, and their favorable algebraic properties were proved. Finally, the effectiveness, stability, and sensitivity of the three proposed models were verified through a multi-attribute group decision-making example in a single universe, and the experimental results showed that this method could accurately rank the targets. In summary, our method provided multiple strategies and fault tolerance.

Keywords: neutrosophic rough set; multi-granularity; variable precision; group decision-making
Mathematics Subject Classification: 03E72, 90B50, 94D05

Abbreviations:

Single-valued neutrosophic set\number\relation: SVNS\SVNN\SVNR; multi-attribute group decision-making: MAGDM; a family of generalized neutrosophic approximation space: FGNAS; generalized neutrosophic approximation space: GNAS; multi-granularit C-variable precision neutrosophic rough set: MGCVPNRS; optimistic C-variable precision neutrosophic rough set: OCVPNRS; pessimistic C-variable precision neutrosophic rough set: PCVPNRS; compromise C-variable precision neutrosophic rough set: CCVPNRS; information system: IS

1. Introduction

Real-world decision-making is often accompanied by a high degree of uncertainty, imprecision, and conflicting information. Although traditional fuzzy sets, rough sets, and their extended models (such as intuitionistic fuzzy sets, variable precision rough sets, neutrosophic sets, etc.) can handle some problems from different perspectives, there is a lack of models that can effectively integrate their advantages, and they face three major challenges in complex group decision-making. First, it is difficult to simultaneously achieve robustness against neutrosophic data noise, granular fusion of multi-source heterogeneous information, and independent characterization of true, false, and uncertain information within a single framework. Second, most mainstream neutrosophic set models are based on specific triangular norms [1] and lack strict logical support based on more general residual implications and dual operators. Third, single-granularity models are prone to information loss when integrating the preferences of multiple decision-makers, which affects the stability and reliability of results. These limitations restrict the application of existing methods in complex decision-making in key fields such as neutrosophic information, medical diagnosis, and target recognition, and constitute the core motivation of this research.

As a key advancement in the field of rough set theory, multi-granularity technology enables comprehensive analyses of information from different perspectives, thereby more efficiently mining and revealing the structural characteristics of complex data. Currently, its wide applicability has been verified in various fields such as medical diagnosis [2] and target optimization [3]. One study [4] utilizes probabilistic linguistic term sets to represent the preferences and opinions of decision-makers, thereby improving the accuracy and consistency of decision-making. Meanwhile, the feature of variable precision can significantly enhance the anti-interference capability of the model, enabling it to better handle data with uncertainty and noise. For this reason, researchers have proposed a variety of rough set models by integrating the idea of variable precision, such as the models constructed in prior studies [5, 6]. However, the existing related models still have obvious limitations. On the one hand, due to the lack of a dynamic precision adjustment mechanism [7], multi-granularity rough sets either waste effective information due to the absence of fault-tolerant space or affect the final results by forcibly incorporating errors when facing data noise or ambiguous boundaries; on the other hand, although variable precision rough sets can provide a fault-tolerant space for neutrosophic sets through error thresholds and avoid the rigidity problem of hard thresholds, they can only adjust precision under a single granularity [8], which neither enables the separation of multi-dimensional information nor adapts to the differentiated precision requirements in multi-criteria decision-making.

Against the backdrop of the complex contemporary era, the shift from individual decision-making to group decision-making has become increasingly urgent and necessary. On one hand, real-world decision-making issues involve multi-domain knowledge and factors. Constrained by their own knowledge, experience, and cognitive limitations, individual decision-makers struggle to comprehensively and accurately grasp key information, making them prone to misjudgments when dealing with complex problems. For instance, social networks in the big data era require knowledge across multiple domains [9], which is hardly manageable for a single individual. On the other hand, decisions now have a broader impact and are related to the interests of multiple parties. Due to the limited perspective of individual decision-making, it is difficult to fully consider the demands of different groups, resulting in insufficient recognition and feasibility of the decisions made. In contrast, group decision-making can pool the wisdom, experience, and perspectives of multiple individuals; can integrate information more comprehensively; and is conducive to coordinating the interests of multiple parties while enhancing the scientificity and democracy of decisions [10]. This shift is an inevitable outcome of the times and decision-making needs. Liu et al. [11] devised a new variable-weight approach for multi-attribute group decision-making (MAGDM) problems. Su et al. [12] proposed a MAGDM method based on probabilistic linguistic term sets, which aims to evaluate online learning platforms and analyze the situation of self-confidence. Shen et al. [13] constructed a large-scale group decision-making method that integrates dynamic social networks and opinion dynamics. They effectively clustered experts by designing an improved Louvain algorithm, distinguished opinion leaders and followers using structural hole theory and also built a network update mechanism. To improve consensus efficiency, they proposed a hybrid opinion dynamics model by combining the advantages of the DeGroot and Hegselmann–Krause models. Shen et al. [10] developed a multi-objective optimization consensus framework for large-scale group decision-making. They designed a method for determining decision-makers' weights based on structural hole theory and, on this basis, developed a novel clustering method centered on the maximum group consensus level. However, these studies have certain limitations: When processing large-scale data, the models exhibit high computational complexity, which may affect their decision-making efficiency.

On the basis of this analysis, this study conducts an in-depth exploration of the theory and application of multi-granularity variable-precision neutrosophic rough sets (MGVPNRS), with its core contributions reflected in three aspects. First, by means of the residual implication of t-norms and their dual logical operators, variable precision, multi-granularity, and neutrosophic sets are integrated to construct a novel MGVPNRS model. This model has three basic variants: Optimistic, pessimistic, and compromise models. Second, the MGVPNRS model is applied to MAGDM problems, and a reasonable method for formulating decision schemes is designed. Compared with single-granularity models, the multi-granularity framework can more fully accommodate the preference differences among decision-makers and improve the comprehensiveness of decisions. Relying on the fault-tolerant mechanism of variable precision, it effectively reduces the interference of noisy data on the characterization of uncertainty, thereby coping with complex decision-making processes more efficiently. Third, the proposed method is applied to the scenario of medical emergency management and compared with other decision-making methods. Experimental results verify the effectiveness and stability of the method, while also demonstrating a certain degree of sensitivity.

The following outlines the article's structural framework. In Section 2, some fundamental concepts are reviewed. In Section 3, a novel MGVPNRS model is proposed, and its algebraic properties are studied. In Section 4, an approach to MAGDM is constructed on the basis of the proposed model. In Section 5, the effectiveness, stability, and reasonable sensitivity of the proposed model are verified via case applications and comparative evaluations. Lastly, in Section 6, we summarize the work, and an outlook for future work is provided. For improved readability, we list the symbols of some common terms in Table 1.

Table 1. Glossary of symbols.

Object	Symbolic representation
Elements in the universe of discourse $\mathcal{U}$	x, d, e, h, g, q, z
Symbol with a value range of [0,1]	$\alpha, \beta, \gamma, \omega, \zeta, \upsilon, \varrho$
Set	Q, S
Relation	$\aleph, \Upsilon, \Im, \ell$

2. Preliminaries

In the present chapter, we revisit some key notions about fuzzy logic operators (FLOs). Let $I = [0,1]$, and $\Theta : I^2 \longrightarrow I$ denotes a left-continuous t-norm, and $\mapsto: I^2 \longrightarrow I$ is the residuated implication determined by $\forall \upsilon, \zeta \in I, \upsilon \mapsto \zeta = \bigvee\{f \in I : \upsilon\Theta f \leq \zeta\}$. Similarly, $\Xi : I^2 \longrightarrow I$ denotes a right-continuous t-conorm, and $\hookrightarrow: I^2 \longrightarrow I$ is the residuated coimplication determined by $\forall \upsilon, \zeta \in I, \upsilon \hookrightarrow \zeta = \bigwedge\{f \in I : d\Xi f \geq e\}$ [14]. Furthermore, we assume $\neg : I \longrightarrow I$ is an involutive negation. The negative operator $\neg_s$ defined by $\neg_s(\upsilon) = 1 - \upsilon, \upsilon \in I$ is called the standard negative operator, and Θ and Ξ are dual with respect to $\neg$ ($\neg_s$). Table 2 lists some typical examples of Θ, Ξ, and the associated $\mapsto$ and $\hookrightarrow$.

Table 2. Examples of Θ, Ξ, and their associated $\mapsto$ and $\hookrightarrow$.

Θ and Ξ	$\mapsto$ and $\hookrightarrow$
$z\Theta_P q = z \cdot q$	$z \mapsto_P q = \begin{cases} 1, \text{if } z \leq q \\ \frac{q}{z}, \text{otherwise} \end{cases}$
$z\Xi_P q = z + q - z \cdot q$	$z \hookrightarrow_P q = \begin{cases} 0, \text{if } z \geq q \\ \frac{q-z}{1-z}, \text{otherwise} \end{cases}$
$z\Theta_L q = 0 \vee (z + q - 1)$	$z \mapsto_L q = 1 \wedge (1 - z + q)$
$z\Xi_L q = 1 \wedge (z + q)$	$z \hookrightarrow_L q = 0 \vee (q - z)$
$z\Theta_M q = z \wedge q$	$z \mapsto_M q = \begin{cases} 1, \text{if } z \leq q \\ q, \text{otherwise} \end{cases}$
$z\Xi_M q = z \vee q$	$z \hookrightarrow_M q = \begin{cases} 0, \text{if } z \geq q \\ q, \text{otherwise} \end{cases}$
$z\Theta_Y q = max(1 - \sqrt[n]{(1-z)^n + (1-q)^n}), n \in (0, +\infty)$	$z \mapsto_Y q = \begin{cases} 1 - \sqrt[n]{(1-q)^n - (1-z)^n}, \text{if } z > q \\ 1, \text{otherwise} \end{cases}$
$z\Xi_Y q = min(1, \sqrt[n]{z^n + q^n}), n \in (0, +\infty)$	$z \hookrightarrow_Y q = \begin{cases} 0, \text{if } z \geq q \\ \sqrt[n]{q^n - z^n}, \text{otherwise} \end{cases}$
$z\Theta_H q = \frac{zq}{\gamma + (1-\gamma)(z+q-zq)}, \gamma \in (0, +\infty)$	$z \mapsto_H q = \begin{cases} 1, \text{if } z \leq q \\ \frac{\gamma q + (1-\gamma)(z+q-zq)}{z}, \text{otherwise} \end{cases}$
$z\Xi_H q = \frac{z+q-zq-(1-\gamma)zq}{1-(1-\gamma)zq}, \gamma \in (0, +\infty)$	$z \hookrightarrow_H q = \begin{cases} 0, \text{if } z \geq q \\ \frac{q-z+(1-\gamma)zq}{1-z+(1-\gamma)zq}, \text{otherwise} \end{cases}$
$z\Theta_{nM} q = \begin{cases} 0, \text{if } z + q \leq 1 \\ min\{z, q\}, \text{otherwise} \end{cases}$	$z \mapsto_{nM} q = \begin{cases} 1, \text{if } z \leq q \\ max\{1 - z, q\}, \text{otherwise} \end{cases}$
$z\Xi_{nM} q = \begin{cases} 1, \text{if } z + q \geq 1 \\ max\{z, q\}, \text{otherwise} \end{cases}$	$z \hookrightarrow_{nM} q = \begin{cases} 0, \text{if } z > q \\ min\{z, 1 - q\}, \text{otherwise} \end{cases}$

Lemma 1. *For any $\alpha, \beta, \gamma \in I, \gamma_s (s \in S)$, and $\beta_z (z \in Z) \in I$, then*

(1) $\alpha \hookrightarrow (\beta \hookrightarrow \gamma) = \beta \hookrightarrow (\alpha \hookrightarrow \gamma)$ and $\alpha \mapsto (\beta \mapsto \gamma) = \beta \mapsto (\alpha \mapsto \gamma)$;

(2) $\alpha \leq \beta \Leftrightarrow \alpha \mapsto \beta = 1 \Leftrightarrow \beta \hookrightarrow \alpha = 0$;

(3) $(\bigvee_s \gamma_s) \mapsto (\bigwedge_z \beta_z) = \bigwedge_s \bigwedge_z (\gamma_s \mapsto \beta_z)$ and $(\bigwedge_s \gamma_s) \hookrightarrow (\bigvee_z \beta_z) = \bigvee_s \bigvee_z (\gamma_s \hookrightarrow \beta_z)$;

(4) $\alpha\Theta(\beta \mapsto \gamma) \leq \beta \mapsto (\alpha\Theta\gamma)$ and $\alpha\Xi(\beta \hookrightarrow \gamma) \geq \beta \hookrightarrow (\alpha\Xi\gamma)$;

(5) $\bigvee_s (\alpha\Theta\gamma_s) = \alpha\Theta \bigvee_s \gamma_s$ and $\bigwedge_s (\alpha\Xi\gamma_s) = \alpha\Xi \bigwedge_s \gamma_s$.

Definition 1. *[15] A single-valued neutrosophic set (SVNS) L is denoted by* $\forall x \in \mathcal{U}$*, as follows:*

$$L = \{\langle x, T_L(x), I_L(x), F_L(x)\rangle \mid x \in \mathcal{U}\},$$

$T_L(x), I_L(x), F_L(x) \in [0,1]$*, and satisfies* $0 \le T_L(x) + I_L(x) + F_L(x) \le 3$*, the single-valued neutrosophic number (SVNN) in the SVNS is denoted as* $(T_L(x), I_L(x), F_L(x))$*.*

Suppose that $\mathcal{U}$ is a nonempty finite universe, and $I(\mathcal{U})$ stands for a neutrosophic set in $\mathcal{U}$. $\forall \alpha = (\alpha_T, \alpha_I, \alpha_F)$; simultaneously, we utilize $\breve{\alpha}$ to signify the constant neutrosophic set valued α. Set $Q, L \in I(\mathcal{U})$, and $\divideontimes \in \{\cup, \cap, \Theta, \Xi, \mapsto, \hookrightarrow\}$, one define neutrosophic set $Q \divideontimes L$ by $\forall x \in \mathcal{U}$ and $(Q \divideontimes L)(x) = Q(x) \divideontimes L(x)$. For every $Y \subseteq \mathcal{U}$, we define neutrosophic set $\eth_Y$ by $\eth_Y(d) = (1,0,0)$ whenever $d \in Y$ and $\eth_Y(d) = (0,1,1)$ otherwise.

A SVNS $\aleph \in I(\mathcal{U} \times \mathcal{U})$ is called a single-valued neutrosophic relation (SVNR) on $\mathcal{U}$. The pair $(\mathcal{U}, \aleph)$ is termed a generalized neutrosophic approximation space (GNAS). Furthermore, $\forall d, e, h \in \mathcal{U}$ as follows:

(1) If $\aleph(d,d) = (1,0,0)$, then $\aleph$ is reflexive;

(2) If $\aleph(d,e)\Theta\aleph(e,h) \le \aleph(d,h)$, then $\aleph$ is Θ-transitive.

If $\aleph$ satisfies (1) and (2), then $\aleph$ is a Θ-preorder.

Definition 2. *[15] If Q and S be two SVNSs on* $\mathcal{U}$*, we define:*

(1) $Q \subseteq S$ *if and only if(iff)* $\forall x \in \mathcal{U}$, $T_Q(x) \le T_S(x), I_Q(x) \ge I_S(x)$*, and* $F_Q(x) \ge F_S(x)$*;*

(2) $\neg Q = \{(x, F_Q(x), 1 - I_Q(x), T_Q(x)) | x \in \mathcal{U}\}$*;*

(3) $Q \cap S = \{(x, T_Q(x) \wedge T_S(x), I_Q(x) \vee I_S(x), F_Q(x) \vee F_S(x)) | x \in \mathcal{U}\}$*;*

(4) $Q \cup S = \{(x, T_Q(x) \vee T_S(x), I_Q(x) \wedge I_S(x), F_Q(x) \wedge F_S(x)) | x \in \mathcal{U}\}$*.*

Definition 3. *[16] Let* $\alpha_1 = (T_{\alpha_1}, I_{\alpha_1}, F_{\alpha_1})$ *and* $\alpha_2 = (T_{\alpha_2}, I_{\alpha_2}, F_{\alpha_2})$ *represent two SVNNs, the operation rules are as follows:*

(1) $\lambda\alpha_1 = (1 - (1 - T_{\alpha_1})^\lambda, (I_{\alpha_1})^\lambda, (F_{\alpha_1})^\lambda)$*;*

(2) $\alpha_1^\lambda = ((T_{\alpha_1})^\lambda, 1 - (1 - I_{\alpha_1})^\lambda, 1 - (1 - F_{\alpha_1})^\lambda)$*;*

(3) $\alpha_1 \oplus \alpha_2 = (T_{\alpha_1} + T_{\alpha_2} - T_{\alpha_1} \cdot T_{\alpha_2}, I_{\alpha_1} \cdot I_{\alpha_2}, F_{\alpha_1} \cdot F_{\alpha_2})$*;*

(4) $\alpha_1 \odot \alpha_2 = (T_{\alpha_1} \cdot T_{\alpha_2}, I_{\alpha_1} + I_{\alpha_2} - I_{\alpha_1} \cdot I_{\alpha_2}, F_{\alpha_1} + F_{\alpha_2} - F_{\alpha_1} \cdot F_{\alpha_2})$*.*

Definition 4. *[17] Let Q and S be two SVNSs in* $\mathcal{U}$*, where "*Ξ*" and "*Θ*" denote t-conorm and t-norm.*

(1) The union of Q and S is a SVNS G, denoted $G = Q\Xi S$*, where* $\forall x \in \mathcal{U}$, $T_{Q\Xi S}(x) = T_Q(x)\Xi T_S(x)$*,*$I_{Q\Xi S}(x) = I_Q(x)\Theta I_S(x)$*, and* $F_{Q\Xi S}(x) = F_Q(x)\Theta F_S(x)$*.*

(2) The intersection of Q and S is a SVNS K, denoted $K = Q\Theta S$*, where* $\forall x \in \mathcal{U}$, $T_{Q\Theta S}(x) = T_Q(x)\Theta T_S(x)$*,* $I_{Q\Theta S}(x) = I_Q(x)\Xi I_S(x)$*, and* $F_{Q\Theta S}(x) = F_Q(x)\Xi F_S(x)$*.*

3. Novel models of variable-precision neutrosophic rough sets

Novel variable-precision neutrosophic rough sets (VPNRSs) (i.e.,single-granularity VPNRS and multi-granularity VPNRS) models will be introduced in the following subsections, with an emphasis on the relevant properties of the MGVPNRS model, since single-granularity is a special case of multi-granularity.

3.1. Definition and basic properties

Next, we propose a single-granularity VPNRS model.

For any $\mathcal{Z} \subseteq \mathcal{U}$, let $|\mathcal{Z}|$ represents its cardinality. For $\omega \in I$, we write

$$\mathcal{U}_\omega = \{\mathcal{Z} \subseteq \mathcal{U} : |\mathcal{Z}| \geq \omega|\mathcal{U}|\}.$$

Definition 5. *Assume that $(\mathcal{U}, \aleph)$ is a GNAS, $\omega \in I$, and $Q \in I(\mathcal{U})$. One defines $\underline{\aleph}^\omega(Q), \overline{\aleph}^\omega(Q) \in I(\mathcal{U})$: $\forall g \in \mathcal{U}$,*

$$T_{\underline{\aleph}^\omega(Q)}(g) = \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Z}} \Big(T_\aleph(g, x) \mapsto T_Q(x)\Big),$$

$$I_{\underline{\aleph}^\omega(Q)}(g) = \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Z}} \Big(I_\aleph(g, x) \hookrightarrow I_Q(x)\Big),$$

$$F_{\underline{\aleph}^\omega(Q)}(g) = \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Z}} \Big(F_\aleph(g, x) \hookrightarrow F_Q(x)\Big),$$

$$T_{\overline{\aleph}^\omega(Q)}(g) = \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Z}} \Big(F_\aleph(g, x) \hookrightarrow T_Q(x)\Big),$$

$$I_{\overline{\aleph}^\omega(Q)}(g) = \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Z}} \Big((1 - I_\aleph(g, x)) \mapsto I_Q(x)\Big),$$

$$F_{\overline{\aleph}^\omega(Q)}(g) = \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Z}} \Big(T_\aleph(g, x) \mapsto F_Q(x)\Big).$$

Next, we will introduce the MGCVPNRS model and its properties.

Definition 6. *(Optimistic model) Let $\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ be a family of GNAS (referred to as FGNAS), and $Q \in I(\mathcal{U})$. Then the pair $(\underline{O\Upsilon}^\omega(Q), \overline{O\Upsilon}^\omega(Q))$ is called an optimistic C-variable precision neutrosophic rough set (OCVPNRS) of Q, where $\underline{O\Upsilon}^\omega(Q)$ and $\overline{O\Upsilon}^\omega(Q)$ are determined via the following:*

$$T_{\underline{O\Upsilon}^\omega(Q)}(g) = \bigvee_i \bigvee_{g \in \mathcal{Y} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Y}} \Big(T_{\aleph_i}(g, x) \mapsto T_Q(x)\Big),$$

$$I_{\underline{O\Upsilon}^\omega(Q)}(g) = \bigwedge_i \bigwedge_{g \in \mathcal{Y} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Y}} \Big(I_{\aleph_i}(g, x) \hookrightarrow I_Q(x)\Big),$$

$$F_{\underline{O\Upsilon}^\omega(Q)}(g) = \bigwedge_i \bigwedge_{g \in \mathcal{Y} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Y}} \Big(F_{\aleph_i}(g, x) \hookrightarrow F_Q(x)\Big),$$

$$T_{\overline{O\Upsilon}^\omega(Q)}(g) = \bigwedge_i \bigwedge_{g \in \mathcal{Y} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Y}} \Big(F_{\aleph_i}(g, x) \hookrightarrow T_Q(x)\Big),$$

$$I_{\overline{O\Upsilon}^\omega(Q)}(g) = \bigvee_i \bigvee_{g \in \mathcal{Y} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Y}} \Big((1 - I_{\aleph_i}(g, x)) \mapsto I_Q(x)\Big),$$

$$F_{\overline{O\Upsilon}^\omega(Q)}(g) = \bigvee_i \bigvee_{g \in \mathcal{Y} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Y}} \Big(T_{\aleph_i}(g, x) \mapsto F_Q(x)\Big).$$

Next, we present several theorems on OCVPNRSs and provide their proofs.

Theorem 1. *Let $\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ is a FGNAS, $\breve{\alpha} \in I$, Q, and $S \in I(\mathcal{U})$.*
(1) $Q \subseteq S \Rightarrow \underline{O\Upsilon}^{\omega}(Q) \subseteq \underline{O\Upsilon}^{\omega}(S), \overline{O\Upsilon}^{\omega}(Q) \subseteq \overline{O\Upsilon}^{\omega}(S)$;
(2) $\underline{O\Upsilon}^{\omega}(Q \cap S) \subseteq \underline{O\Upsilon}^{\omega}(Q) \cap \underline{O\Upsilon}^{\omega}(S), \overline{O\Upsilon}^{\omega}(Q \cup S) \supseteq \overline{O\Upsilon}^{\omega}(Q) \cup \overline{O\Upsilon}^{\omega}(S)$;
(3) $\underline{O\Upsilon}^{\omega}(\breve{\alpha} \mapsto Q) \subseteq \breve{\alpha} \mapsto \underline{O\Upsilon}^{\omega}(Q), \overline{O\Upsilon}^{\omega}(\breve{\alpha} \hookrightarrow Q) \supseteq \breve{\alpha} \hookrightarrow \overline{O\Upsilon}^{\omega}(Q)$;
(4) $\underline{O\Upsilon}^{\omega}(\breve{\alpha}) \supseteq \breve{\alpha}, \overline{O\Upsilon}^{\omega}(\breve{\alpha}) \subseteq \breve{\alpha}$. Particularly, $\underline{O\Upsilon}^{\omega}(\mathcal{U}) = \mathcal{U}, \overline{O\Upsilon}^{\omega}(\emptyset) = \emptyset$;
(5) $\underline{O\Upsilon}^{\omega}(\breve{\alpha}\Theta Q) \supseteq \breve{\alpha}\Theta\underline{O\Upsilon}^{\omega}(Q)$ and $\breve{\alpha}\Xi\overline{O\Upsilon}^{\omega}(Q) \supseteq \overline{O\Upsilon}^{\omega}(\breve{\alpha}\Xi Q)$.

Proof. (1) This is proved direct inference from Definition 6 and Lemma 1 (2).
(2) This follows directly from (1).
(3) Let $g \in \mathcal{U}$, then by Definition 6, Lemma 1 (3) and (6), we have

$$\begin{aligned}
T_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\mapsto Q)}(g) &= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto T_{(\breve{\alpha}\mapsto Q)}(x)\Big) \\
&= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(\alpha_1 \mapsto (T_{\aleph_i}(g,x) \mapsto T_Q(x))\Big) \\
&\leq \alpha_1 \mapsto \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} (T_{\aleph_i}(g,x) \mapsto T_Q(x)) \\
&= \Big(T_{\breve{\alpha}} \mapsto T_{\underline{O\Upsilon}^{\omega}(Q)}\Big)(g),
\end{aligned}$$

$$\begin{aligned}
T_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\mapsto Q)}(g) &= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto T_{(\breve{\alpha}\mapsto Q)}(x)\Big) \\
&= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(\alpha_1 \mapsto (T_{\aleph_i}(g,x) \mapsto T_Q(x))\Big) \\
&\leq \alpha_1 \mapsto \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} (T_{\aleph_i}(g,x) \mapsto T_Q(x)) \\
&= \Big(T_{\breve{\alpha}} \mapsto T_{\underline{O\Upsilon}^{\omega}(Q)}\Big)(g),
\end{aligned}$$

$$\begin{aligned}
I_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\mapsto Q)}(g) &= \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow I_{(\breve{\alpha}\mapsto Q)}(x)\Big) \\
&= \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(\alpha_2 \hookrightarrow (I_{\aleph_i}(g,x) \hookrightarrow I_Q(x))\Big) \\
&\geq \alpha_2 \hookrightarrow \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (I_{\aleph_i}(g,x) \hookrightarrow I_Q(x)) \\
&= \Big(I_{\breve{\alpha}} \hookrightarrow I_{\underline{O\Upsilon}^{\omega}(Q)}\Big)(g),
\end{aligned}$$

$$F_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\mapsto Q)}(g) = \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow F_{(\breve{\alpha}\mapsto Q)}(x)\Big)$$

$$
\begin{aligned}
&= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(\alpha_3 \hookrightarrow (F_{\aleph_i}(g,x) \hookrightarrow F_Q(x))\Big)\\
&\geq \alpha_3 \hookrightarrow \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (F_{\aleph_i}(g,x) \hookrightarrow F_Q(x))\\
&= \Big(F_{\breve{\alpha}} \hookrightarrow F_{\underline{O\Upsilon}^{\omega}(Q)}\Big)(g).
\end{aligned}
$$

Thus, $\underline{O\Upsilon}^{\omega}(\breve{\alpha} \mapsto Q) \leq \breve{\alpha} \mapsto \underline{O\Upsilon}^{\omega}(Q)$. Likewise, we can get $\overline{O\Upsilon}^{\omega}(\breve{\alpha} \hookrightarrow Q) \geq \breve{\alpha} \hookrightarrow \overline{O\Upsilon}^{\omega}(Q)$.

(4) By Lemma 1 (1) and (4), we have

$$
T_{\underline{O\Upsilon}^{\omega}(\breve{\alpha})}(g) = \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto T_{\breve{\alpha}}(x)\Big) \geq \alpha_1,
$$

$$
I_{\underline{O\Upsilon}^{\omega}(\breve{\alpha})}(g) = \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow I_{\breve{\alpha}}(x)\Big) \leq \alpha_2,
$$

$$
F_{\underline{O\Upsilon}^{\omega}(\breve{\alpha})}(g) = \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow F_{\breve{\alpha}}(x)\Big) \leq \alpha_3.
$$

Therefore, $\underline{O\Upsilon}^{\omega}(\breve{\alpha}) \geq \breve{\alpha}$. Likewise, $\overline{O\Upsilon}^{\omega}(\breve{\alpha}) \leq \breve{\alpha}$.

(5) For any $g \in \mathcal{U}$, we have

$$
\begin{aligned}
T_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\Theta Q)}(g) &= \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto (T_{\breve{\alpha}\Theta Q})(x)\Big)\\
&\geq \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(\alpha_1\Theta(T_{\aleph_i}(g,x) \mapsto T_Q(x))\Big)\\
&= \alpha_1\Theta \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} (T_{\aleph_i}(g,x) \mapsto T_Q(x))\\
&= \Big(T_{\breve{\alpha}}\Theta T_{\underline{O\Upsilon}^{\omega}(Q)}\Big)(g),
\end{aligned}
$$

$$
\begin{aligned}
I_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\Theta Q)}(g) &= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow (I_{\breve{\alpha}\Theta Q})(x)\Big)\\
&\leq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(\alpha_2\Xi(I_{\aleph_i}(g,x) \hookrightarrow I_Q(x))\Big)\\
&= \alpha_2\Xi \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (I_{\aleph_i}(g,x) \hookrightarrow I_Q(x))\\
&= \Big(I_{\breve{\alpha}}\Xi I_{\underline{O\Upsilon}^{\omega}(Q)}\Big)(g),
\end{aligned}
$$

$$
F_{\underline{O\Upsilon}^{\omega}(\breve{\alpha}\Theta Q)}(g) = \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow (F_{\breve{\alpha}\Theta Q})(x)\Big)
$$

$$
\begin{aligned}
&\le \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(\alpha_3\Xi(F_{\aleph_i}(g,x) \hookrightarrow F_Q(x))\Big) \\
&= \alpha_3\Xi \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (F_{\aleph_i}(g,x) \hookrightarrow F_Q(x)) \\
&= \Big(F_{\breve{\alpha}}\Xi F_{\underline{O\Upsilon^\omega(Q)}}\Big)(g).
\end{aligned}
$$

Thus, $\underline{O\Upsilon}^\omega(\breve{\alpha}\Theta Q) \ge \breve{\alpha}\Theta\underline{O\Upsilon}^\omega(Q)$. Likewise, we can get $\breve{\alpha}\Xi\overline{O\Upsilon}^\omega(Q) \ge \overline{O\Upsilon}^\omega(\breve{\alpha}\Xi Q)$. □

It can be seen that the optimistic model possesses the basic properties of rough sets. It not only satisfies the monotonicity of set inclusion relations but also preserves set inclusion relations under union and intersection operations. Additionally, it satisfies set inclusion relations under the operations of continuous triangular norms and their induced residual implications.

Definition 7. *(Pessimistic model) Let $\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ be a FGNAS, and $Q \in I(\mathcal{U})$. The couple $(\underline{P\Upsilon}^\omega(Q), \overline{P\Upsilon}^\omega(Q)$ is termed the pessimistic C-variable precision neutrosophic rough set (PCVPNRS) of Q, where $\underline{P\Upsilon}^\omega(Q)$ and $\overline{P\Upsilon}^\omega(Q))$ are determined via the following:*

$$T_{\underline{P\Upsilon}^\omega(Q)} = \bigwedge_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto T_Q(x)\Big),$$

$$I_{\underline{P\Upsilon}^\omega(Q)} = \bigvee_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow I_Q(x)\Big),$$

$$F_{\underline{P\Upsilon}^\omega(Q)} = \bigvee_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow F_Q(x)\Big),$$

$$T_{\overline{P\Upsilon}^\omega(Q)} = \bigvee_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow T_Q(x)\Big),$$

$$I_{\overline{P\Upsilon}^\omega(Q)} = \bigwedge_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big((1 - I_{\aleph_i}(g,x)) \mapsto I_Q(x)\Big),$$

$$F_{\overline{P\Upsilon}^\omega(Q)} = \bigwedge_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto F_Q(x)\Big).$$

The following theorem describes the algebraic properties of the pessimistic model. Next, we enunciate several theorems of the PCVPNRS.

Theorem 2. *Let $\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ be a FGNAS. $\breve{\alpha} \in I$, $Q, S \in I(\mathcal{U})$.*
(1) $Q \subseteq S \Rightarrow \underline{P\Upsilon}^\omega(Q) \subseteq \underline{P\Upsilon}^\omega(S)$ and $\overline{P\Upsilon}^\omega(Q) \subseteq \overline{P\Upsilon}^\omega(S)$;
(2) $\underline{P\Upsilon}^\omega(Q \cap S) \subseteq \underline{P\Upsilon}^\omega(Q) \cap \underline{P\Upsilon}^\omega(S)$ and $\overline{P\Upsilon}^\omega(Q \cup S) \supseteq \overline{P\Upsilon}^\omega(Q) \cup \overline{P\Upsilon}^\omega(S)$;
(3) $\underline{P\Upsilon}^\omega(\breve{\alpha} \mapsto Q) \subseteq \breve{\alpha} \mapsto \underline{P\Upsilon}^\omega(Q)$ and $\overline{P\Upsilon}^\omega(\breve{\alpha} \hookrightarrow Q) \supseteq \breve{\alpha} \hookrightarrow \overline{P\Upsilon}^\omega(Q)$;
(4) $\underline{P\Upsilon}^\omega(\breve{\alpha}) \supseteq \breve{\alpha}, \overline{P\Upsilon}^\omega(\breve{\alpha}) \subseteq \breve{\alpha}$. Particularly, $\underline{P\Upsilon}^\omega(\mathcal{U}) = \mathcal{U}$, $\overline{P\Upsilon}^\omega(\emptyset) = \emptyset$;
(5) $\underline{P\Upsilon}^\omega(\breve{\alpha}\Theta Q) \supseteq \breve{\alpha}\Theta\underline{P\Upsilon}^\omega(Q)$ and $\breve{\alpha}\Xi\overline{P\Upsilon}^\omega(Q) \supseteq \overline{P\Upsilon}^\omega(\breve{\alpha}\Xi Q)$.

Proof. This bears a resemblance to Theorem 1 and has the same properties as the optimistic model. □

The optimistic and pessimistic models represent two extreme cases, thus leading to the introduction of a compromise model.

Definition 8. *(Compromise model) Let* $\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ *be a FGNAS, and* $Q \in I(\mathcal{U})$*, the couple* $(\underline{C\Upsilon}^{\omega}(Q), \overline{C\Upsilon}^{\omega}(Q)$ *termed the compromise C-variable precision neutrosophic rough set (CCVPNRS) of* Q*, where* $\underline{C\Upsilon}^{\omega}(Q)$ *and* $\overline{C\Upsilon}^{\omega}(Q))$ *are determined via THE following:*

$$T_{\underline{C\Upsilon}^{\omega}(Q)}(g) = \rho T_{\underline{O\Upsilon}^{\omega}(Q)}(g) + (1-\rho)T_{\underline{P\Upsilon}^{\omega}(Q)}(g),$$

$$I_{\underline{C\Upsilon}^{\omega}(Q)}(g) = \rho I_{\underline{O\Upsilon}^{\omega}(Q)}(g) + (1-\rho)I_{\underline{P\Upsilon}^{\omega}(Q)}(g),$$
$$F_{\underline{C\Upsilon}^{\omega}(Q)}(g) = \rho F_{\underline{O\Upsilon}^{\omega}(Q)}(g) + (1-\rho)F_{\underline{P\Upsilon}^{\omega}(Q)}(g),$$
$$T_{\overline{C\Upsilon}^{\omega}(Q)}(g) = \rho T_{\overline{O\Upsilon}^{\omega}(Q)}(g) + (1-\rho)T_{\overline{P\Upsilon}^{\omega}(Q)}(g),$$
$$I_{\overline{C\Upsilon}^{\omega}(Q)}(g) = \rho I_{\overline{O\Upsilon}^{\omega}(Q)}(g) + (1-\rho)I_{\overline{P\Upsilon}^{\omega}(Q)}(g),$$
$$F_{\overline{C\Upsilon}^{\omega}(Q)}(g) = \rho F_{\overline{O\Upsilon}^{\omega}(Q)}(g) + (1-\rho)F_{\overline{P\Upsilon}^{\omega}(Q)}(g).$$

Theorem 3. *Let* $(\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\})$ *be a FGNAS.* $Q, S \in I(\mathcal{U})$.

(1) $Q \subseteq S \Rightarrow \underline{C\Upsilon}^{\omega}(Q) \subseteq \underline{C\Upsilon}^{\omega}(S), \overline{C\Upsilon}^{\omega}(Q) \subseteq \overline{C\Upsilon}^{\omega}(S)$;

(2) $\underline{C\Upsilon}^{\omega}(Q \cap S) \subseteq \underline{C\Upsilon}^{\omega}(Q) \cap \underline{C\Upsilon}^{\omega}(S), \overline{C\Upsilon}^{\omega}(Q \cup S) \supseteq \overline{C\Upsilon}^{\omega}(Q) \cup \overline{C\Upsilon}^{\omega}(S)$;

(3) $\underline{C\Upsilon}^{\omega}(\breve{\alpha}) \supseteq \breve{\alpha}, \overline{C\Upsilon}^{\omega}(\breve{\alpha}) \subseteq \breve{\alpha}$*. Particularly,* $\underline{C\Upsilon}^{\omega}(\mathcal{U}) = \mathcal{U}, \overline{C\Upsilon}^{\omega}(\emptyset) = \emptyset$.

Proof. (1)–(3) have similar proofs to Theorems 1 and 2. □

Definition 9. *(Distributive property)* Θ *and* Ξ *are said to satisfy the distributive property provided that for any* $e, g, r \in I$*, the following holds:*

$$e\Theta(g+r) = e\Theta g + e\Theta r,\ e\Xi(g+r) = e\Xi g + e\Xi r.$$

Remark 1. *The distributive property holds when* $\Theta = \Theta_P$.

Theorem 4. *Let* $\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ *is a FGNAS,* $\Theta = \Theta_P$*, then*

$$\underline{C\Upsilon}^{\omega}(\breve{\alpha}\Theta_P Q) \supseteq \breve{\alpha}\Theta_P \underline{C\Upsilon}^{\omega}(Q).$$

Proof. By Theorem 1 (5) and Theorem 2 (5), and Definition 9.

$$\begin{aligned}
T_{\underline{C\Upsilon}^{\omega}(\breve{\alpha}\Theta_P Q)} &= \rho \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \big(T_{\aleph_i}(g,x) \mapsto (T_{\breve{\alpha}\Theta_p Q})(x)\big) + (1-\rho)\bigwedge_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \big(T_{\aleph_i}(g,x) \mapsto T_{(\breve{\alpha}\Theta_p Q)}(x)\big)\\
&= \rho \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \big(T_{\aleph_i}(g,x) \mapsto (\alpha_1\Theta_p T_Q(x))\big) + (1-\rho)\bigwedge_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \big(T_{\aleph_i}(g,x) \mapsto (\alpha_1\Theta_P T_Q(x))\big)\\
&\geq \alpha_1\Theta_P\rho \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} (T_{\aleph_i}(g,x) \mapsto T_Q(x)) + \alpha_1\Theta_p(1-\rho)\bigwedge_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} (T_{\aleph_i}(g,x) \mapsto T_Q(x))\\
&= T_{\underline{C\Upsilon}(\alpha)}\Theta_P T_{\underline{C\Upsilon}^{\omega}(Q)},\\
I_{\underline{C\Upsilon}^{\omega}(\breve{\alpha}\Theta_P Q)} &= \rho \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \big(I_{\aleph_i}(g,x) \hookrightarrow I_{(\breve{\alpha}\Theta_p Q)}(x)\big) + (1-\rho)\bigvee_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \big(I_{\aleph_i}(g,x) \hookrightarrow I_{(\breve{\alpha}\Theta_p Q)}(x)\big)\\
&= \rho \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \big(I_{\aleph_i}(g,x) \hookrightarrow (\alpha_2\Xi_p I_Q(x))\big) + (1-\rho)\bigvee_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \big(I_{\aleph_i}(g,x) \hookrightarrow (\alpha_2\Xi_P I_Q(x))\big)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_2 \Xi_P \rho \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (I_{\aleph_i}(g,x) \hookrightarrow I_Q(x)) + \alpha_2 \Xi_p (1-\rho) \bigvee_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (I_{\aleph_i}(g,x) \hookrightarrow I_Q(x)) \\
&= I_{\underline{C\Upsilon}(\breve{\alpha})} \Xi_P I_{\underline{C\Upsilon}^\omega(Q)}, \\
F_{\underline{C\Upsilon}^\omega(\breve{\alpha}\Theta_P Q)} &= \rho \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \left(F_{\aleph_i}(g,x) \hookrightarrow F_{(\breve{\alpha}\Theta_p Q)}(x)\right) + (1-\rho) \bigvee_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \left(F_{\aleph_i}(g,x) \hookrightarrow F_{(\breve{\alpha}\Theta_p Q)}(x)\right) \\
&= \rho \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \left(F_{\aleph_i}(g,x) \hookrightarrow (\alpha_2 \Xi_p F_Q(x))\right) + (1-\rho) \bigvee_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \left(F_{\aleph_i}(g,x) \hookrightarrow (\alpha_2 \Xi_P F_Q(x))\right) \\
&\leq \alpha_2 \Xi_P \rho \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (F_{\aleph_i}(g,x) \hookrightarrow F_Q(x)) + \alpha_2 \Xi_p (1-\rho) \bigvee_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} (F_{\aleph_i}(g,x) \hookrightarrow F_Q(x)) \\
&= F_{\underline{C\Upsilon}(\breve{\alpha})} \Xi_P F_{\underline{C\Upsilon}^\omega(Q)}.
\end{aligned}$$

Therefore,

$$\underline{C\Upsilon}^\omega(\breve{\alpha}\Theta_P Q) \supseteq \breve{\alpha}\Theta_P \underline{C\Upsilon}^\omega(Q).$$

□

It is not difficult to find that the compromise model only satisfies the monotonicity of the set inclusion relations and the preserves set inclusion relations under union and intersection operations.

Next, we illustrate that the inequality in Theorem 4 does not satisfy other logical operators.

Demonstration of Theorem 4. Let $\mathcal{U} = \{w, m, n\}$, $\Theta = \Theta_L$, $\Xi = \Xi_L$, and

$$\aleph_1 = \begin{Bmatrix} (0.4, 0.5, 0.4) & (0.5, 0.7, 0.1) & (1, 0.8, 0.8) \\ (0.5, 0.6, 1.0) & (0.2, 0.6, 0.4) & (0.9, 0.2, 0.4) \\ (0.9, 0.2, 0.4) & (0.8, 0.9, 1) & (0.6, 1.0, 0.0) \end{Bmatrix},$$

$$\aleph_2 = \begin{Bmatrix} (0.9, 0.2, 0.4) & (0.3, 0.9, 0.1) & (0.1, 0.7, 0.0) \\ (0.4, 0.5, 0.1) & (0.0, 0.1, 0.7) & (1.0, 0.8, 0.8) \\ (1.0, 0.5, 0.0) & (0.4, 0.4, 0.2) & (0.1, 0.5, 0.4) \end{Bmatrix},$$

$$\aleph_3 = \begin{Bmatrix} (0.7, 0.7, 0.0) & (0.4, 0.8, 0.9) & (1.0, 0.4, 0.5) \\ (0.8, 0.2, 0.1) & (1.0, 0.1, 0.8) & (0.1, 0.3, 0.5) \\ (0.0, 0.8, 1.0) & (1.0, 0.0, 1.0) & (1.0, 1.0, 0.0) \end{Bmatrix},$$

take

$$Q = \frac{(0.2, 0.6, 0.4)}{w} + \frac{(0.5, 0.4, 1.0)}{m} + \frac{(0.7, 0.1, 0.5)}{n}$$

and $\breve{\alpha} = (0.2, 0.6, 0.1)$, $\rho = 0.5$, and $\omega = 0.7$. This proves that

$$\underline{C\Upsilon}^\omega(\breve{\alpha}\Theta_L Q) = \frac{(0.2517, 0.2, 0.6)}{w} + \frac{(0.1515, 0.3162, 0.4243)}{m} + \frac{(0.3072, 0.3464, 0.6325)}{n},$$

$$\breve{\alpha}\Theta_L \underline{C\Upsilon}^\omega(Q) = \frac{(0.2, 0.6, 0.6)}{w} + \frac{(0.2, 0.6, 0.4243)}{m} + \frac{(0.2, 0.6, 0.6325)}{n}.$$

Therefore, $\underline{C\Upsilon}^\omega(\breve{\alpha}\Theta Q) \supseteq \breve{\alpha}\Theta\underline{C\Upsilon}^\omega(Q)$ cannot be compared in terms of magnitude. Similarly, we find that $\breve{\alpha}\Xi\overline{C\Upsilon}^\omega(Q) \supseteq \overline{C\Upsilon}^\omega(\breve{\alpha}\Xi Q)$ does not hold.

Remark 2. *Observe that when $\rho = 0$ (respectively, $\rho = 1$), the CCVPNRS transforms into the PCVPNRS (respectively, the OCVPNRS). Thus, as the parameter ρ approaches 0 (respectively, ρ approaches 1), it implies that the compromise model gravitates towards the PCVPNRS (respectively, the OCVPNRS).*

Observe that when $\rho = 0$ (respectively, $\rho = 1$), the CCVPNRS reduces to the PCVPNRS (respectively, the OCVPNRS). Consequently, as ρ tends to 0 (respectively, ρ approaches 1), the CCVPNRS demonstrates a tendency to lean toward the PCVPNRS (correspondingly, the OCVPNRS).

We collectively refer to the three models of OCVPNRS, PCVPNRS, and CCVPNRS as MGCVPNRSs.

The subsequent proposition outlines certain distinctive properties of MGCVPNRSs that set them apart from other neutrosophic rough sets.

Proposition 1. *For a FGNAS $(\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\})$, $\omega \in I$, and then*

(1) If $H \subseteq \mathcal{U}$, $\omega = \frac{|H|}{|\mathcal{U}|}$, then $H \subseteq \underline{O\Upsilon}^{\omega}(H)$, $\overline{O\Upsilon}^{\omega}(\mathcal{U} - H) \subseteq \mathcal{U} - H$;

(2) Suppose $Q, S \in I(\mathcal{U})$, and $\omega > 0.5$,

$$\underline{O\Upsilon}^{\omega}(Q) \cup \underline{O\Upsilon}^{\omega}(S) \subseteq \underline{O\Upsilon}^{2\omega-1}(Q \cup S),$$

$$\overline{O\Upsilon}^{2\omega-1}(Q \cap S) \subseteq \overline{O\Upsilon}^{\omega}(Q) \cap \overline{O\Upsilon}^{\omega}(S).$$

Proof. (1) Let $\omega = \frac{|H|}{|\mathcal{U}|}$, then $H \in \mathcal{U}_\omega$ and $\forall g \in H$, have

$$T_{\underline{O\Upsilon}^{\omega}(H)}(g) = \bigvee_i \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Z}} \Big(T_{\aleph_i}(g, x) \mapsto T_H(x)\Big) \geq \bigvee_i \bigwedge_{x \in H} \Big(T_{\aleph_i}(g, x) \mapsto T_H(x)\Big) = 1,$$

$$I_{\underline{O\Upsilon}^{\omega}(H)}(g) = \bigwedge_i \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Z}} \Big(I_{\aleph_i}(g, x) \hookrightarrow I_H(x)\Big) \leq \bigwedge_i \bigvee_{x \in H} \Big(I_{\aleph_i}(g, x) \hookrightarrow I_H(x)\Big) = 0,$$

$$F_{\underline{O\Upsilon}^{\omega}(H)}(g) = \bigwedge_i \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigvee_{x \in \mathcal{Z}} \Big(F_{\aleph_i}(g, x) \hookrightarrow F_H(x)\Big) \leq \bigwedge_i \bigvee_{x \in H} \Big(F_{\aleph_i}(g, x) \hookrightarrow F_H(x)\Big) = 0.$$

Thus $H \subseteq \underline{O\Upsilon}^{\omega}(H)$. The same line of reasoning applies to the proof of the other formulae.

(2) Let $\omega > 0.5$, so $\omega \geq 2\omega - 1$, and then $\mathcal{U}_\omega \subseteq \mathcal{U}_{2\omega-1}$. Hence, $\forall Q, S \in I(\mathcal{U}), \forall g \in \mathcal{U}$, and

$$\begin{aligned} T_{\left(\underline{O\Upsilon}^{\omega}(Q) \cup \underline{O\Upsilon}^{\omega}(S)\right)}(g) &= T_{\underline{O\Upsilon}^{\omega}(Q)}(g) \vee T_{\underline{O\Upsilon}^{\omega}(S)}(g) \\ &\leq T_{\underline{O\Upsilon}^{\omega}(Q \cup S)}(g) && \text{by Theorem 1 (1)} \\ &= \bigvee_i \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_\omega} \bigwedge_{x \in \mathcal{Z}} \Big(T_{\aleph_i}(g, x) \mapsto T_{(Q \cup S)}(g)\Big) \\ &\leq \bigvee_i \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{2\omega-1}} \bigwedge_{x \in \mathcal{Z}} \Big(T_{\aleph_i}(g, x) \mapsto T_{(Q \cup S)}(g)\Big) && \text{by } \mathcal{U}_\omega \subseteq \mathcal{U}_{2\omega-1} \\ &= T_{\underline{O\Upsilon}^{2\omega-1}(Q \cup S)}(g), \end{aligned}$$

$$I_{\left(\underline{O\Upsilon}^{\omega}(Q) \cup \underline{O\Upsilon}^{\omega}(S)\right)}(g) = I_{\underline{O\Upsilon}^{\omega}(Q)}(g) \vee I_{\underline{O\Upsilon}^{\omega}(S)}(g)$$

$$
\begin{aligned}
&\geq I_{\underline{O\Upsilon}^{\omega}(Q\cup S)}(g) && \text{by Theorem 1 (1)}\\
&= \bigwedge_{i}\bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}}\bigvee_{x\in\mathcal{Z}}\left(I_{\aleph_i}(g,x)\hookrightarrow I_{(Q\cup S)}(g)\right)\\
&\geq \bigwedge_{i}\bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{2\omega-1}}\bigvee_{x\in\mathcal{Z}}\left(I_{\aleph_i}(g,x)\hookrightarrow I_{(Q\cup S)}(g)\right) && \text{by } \mathcal{U}_{\omega}\subseteq\mathcal{U}_{2\omega-1}\\
&= I_{\underline{O\Upsilon}^{2\omega-1}(Q\cup S)}(g),
\end{aligned}
$$

$$
\begin{aligned}
F_{\left(\underline{O\Upsilon}^{\omega}(Q)\cup\underline{O\Upsilon}^{\omega}(S)\right)}(g) &= F_{\underline{O\Upsilon}^{\omega}(Q)}(g)\vee F_{\underline{O\Upsilon}^{\omega}(S)}(g)\\
&\geq F_{\underline{O\Upsilon}^{\omega}(Q\cup S)}(g) && \text{by Theorem 1 (1)}\\
&= \bigwedge_{i}\bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}}\bigvee_{x\in\mathcal{Z}}\left(F_{\aleph_i}(g,x)\hookrightarrow F_{(Q\cup S)}(g)\right)\\
&\geq \bigwedge_{i}\bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{2\omega-1}}\bigvee_{x\in\mathcal{Z}}\left(F_{\aleph_i}(g,x)\hookrightarrow F_{(Q\cup S)}(g)\right) && \text{by } \mathcal{U}_{\omega}\subseteq\mathcal{U}_{2\omega-1}\\
&= F_{\underline{O\Upsilon}^{2\omega-1}(Q\cup S)}(g).
\end{aligned}
$$

This implies that $\underline{O\Upsilon}^{\omega}(Q)\cup\underline{O\Upsilon}^{\omega}(S)\subseteq\underline{O\Upsilon}^{2\omega-1}(Q\cup S)$. Another formulation is also proved in a similar way. □

Proposition 2. *For a FGNAS ($\mathcal{U},\Upsilon=\{\aleph_1,\aleph_2,\cdots,\aleph_s\}$) and $\omega\in I$, we have*
(1) If $H\subseteq\mathcal{U}$, $\omega=\frac{|H|}{|\mathcal{U}|}$, then $H\subseteq\underline{P\Upsilon}^{\omega}(H)$, $\overline{P\Upsilon}^{\omega}(\mathcal{U}-H)\subseteq\mathcal{U}-H$;
(2) Let $Q,S\in I(\mathcal{U})$, $\omega>0.5$,

$$
\underline{P\Upsilon}^{\omega}(Q)\cup\underline{P\Upsilon}^{\omega}(S)\subseteq\underline{P\Upsilon}^{2\omega-1}(Q\cup S),
$$

$$
\overline{P\Upsilon}^{2\omega-1}(Q\cap S)\subseteq\overline{P\Upsilon}^{\omega}(Q)\cap\overline{P\Upsilon}^{\omega}(S).
$$

Proof. Similar to Proposition 1. □

Form Propositions 1 and 2, it follows that the following proposition holds:

Proposition 3. *For a FGNAS ($\mathcal{U},\Upsilon=\{\aleph_1,\aleph_2,\cdots,\aleph_s\}$) and $\omega\in I$, we have*
(1) If $H\subseteq\mathcal{U}$, $\omega=\frac{|H|}{|\mathcal{U}|}$, then $H\subseteq\underline{Q\Upsilon}^{\omega}(H)$, $\overline{Q\Upsilon}^{\omega}(\mathcal{U}-H)\subseteq\mathcal{U}-H$;
(2) Let $Q,S\in I(\mathcal{U})$, $\omega>0.5$,

$$
\underline{Q\Upsilon}^{\omega}(Q)\cup\underline{Q\Upsilon}^{\omega}(S)\subseteq\underline{Q\Upsilon}^{2\omega-1}(Q\cup S),
$$

$$
\overline{Q\Upsilon}^{2\omega-1}(Q\cap S)\subseteq\overline{Q\Upsilon}^{\omega}(Q)\cap\overline{Q\Upsilon}^{\omega}(S).
$$

Next, we prove that the MGCVPNRS satisfies duality.

Theorem 5. *Assume $\mathcal{U},\Upsilon=\{\aleph_1,\aleph_2,\cdots,\aleph_s\}$ be a FGNAS, $\forall Q\in I(\mathcal{U})$, and $\omega\in I$. If Θ,Ξ are dual, and $\neg$ is the standard negation, then*
(1) $\neg(\underline{O\Upsilon}^{\omega}(Q))=\overline{O\Upsilon}^{\omega}(\neg Q)$ and $\neg(\overline{O\Upsilon}^{\omega}(Q))=\underline{O\Upsilon}^{\omega}(\neg Q)$;
(2) $\neg(\underline{P\Upsilon}^{\omega}(Q))=\overline{P\Upsilon}^{\omega}(\neg Q)$ and $\neg(\overline{P\Upsilon}^{\omega}(Q))=\underline{P\Upsilon}^{\omega}(\neg Q)$;
(3) $\neg(\underline{C\Upsilon}^{\omega}(Q))=\overline{C\Upsilon}^{\omega}(\neg Q)$ and $\neg(\overline{C\Upsilon}^{\omega}(Q))=\underline{C\Upsilon}^{\omega}(\neg Q)$.

Proof. (1) Take $\forall Q \in I(\mathcal{U})$, $\forall g \in \mathcal{U}$, and Θ, Ξ are dual, and $\neg$ is the standard negation

$$\begin{aligned} T_{\underline{O\Upsilon^{\omega}}(\neg Q)}(g) &= \bigvee_{i} \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigwedge_{x \in \mathcal{Z}} \left(T_{\aleph_i}(g, x) \mapsto T_{(\neg Q)}(x)\right) \\ &= \bigvee_{i} \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigwedge_{x \in \mathcal{Z}} \left(T_{\aleph_i}(g, x) \mapsto F_{(Q)}(x)\right) \\ &= F_{\overline{O\Upsilon}^{\omega}(Q)}(g) \\ &= T_{\neg(\overline{O\Upsilon}^{\omega}(Q))}(g), \end{aligned}$$

$$\begin{aligned} I_{\underline{O\Upsilon^{\omega}}(\neg Q)}(g) &= \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigvee_{x \in \mathcal{Z}} \left(I_{\aleph_i}(g, x) \hookrightarrow I_{(\neg Q)}(x)\right) \\ &= \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigvee_{x \in \mathcal{Z}} \left(I_{\aleph_i}(g, x) \hookrightarrow (1 - I_{Q}(x))\right) \\ &= 1 - \bigvee_{i} \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigwedge_{x \in \mathcal{Z}} \left((1 - I_{\aleph_i}(g, x)) \mapsto I_{Q}(x)\right) \\ &= 1 - I_{\overline{O\Upsilon}^{\omega}(Q)}(g) \\ &= I_{\neg(\overline{O\Upsilon}^{\omega}(Q))}(g), \end{aligned}$$

$$\begin{aligned} F_{\underline{O\Upsilon^{\omega}}(\neg Q)}(g) &= \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigvee_{x \in \mathcal{Z}} \left(F_{\aleph_i}(g, x) \hookrightarrow F_{(\neg Q)}(x)\right) \\ &= \bigvee_{i} \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigwedge_{x \in \mathcal{Z}} \left(F_{\aleph_i}(g, x) \hookrightarrow T_{(Q)}(x)\right) \\ &= T_{\overline{O\Upsilon}^{\omega}(Q)}(g) \\ &= F_{\neg(\overline{O\Upsilon}^{\omega}(Q))}(g). \end{aligned}$$

Therefore, $\neg(\overline{O\Upsilon}^{\omega}(Q)) = \underline{O\Upsilon}^{\omega}(\neg Q)$. In a similar way, we can get $\neg(\underline{O\Upsilon}^{\omega}(Q)) = \overline{O\Upsilon}^{\omega}(\neg Q)$.

Parts (2) and (3) follow the proof of (1). □

This means that the optimistic model, the pessimistic model, and the compromise model all satisfy the duality property.

Finally, we investigate the relationship between MGCVPNRS-defined constructs for two different neutrosophic relation families.

Proposition 4. *Assume that* $\Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$, $\Im = \{\ell_1, \ell_2, \cdots, \ell_t\}$ *are families of SVNR on* $\mathcal{U}$, *for* $\forall Q \in I(\mathcal{U})$, *and* $\forall \omega \in I$.

(1) When $s = t$, $\aleph_i \geq \ell_i, i = 1, 2, \cdots, s$,

$$\underline{O\Upsilon}^{\omega}(Q) \subseteq \underline{O\Im}^{\omega}(Q),\ \overline{O\Upsilon}^{\omega}(Q) \supseteq \overline{O\Im}^{\omega}(Q);$$

(2) When $\Im \subseteq \Upsilon$,

$$\underline{O\Upsilon}^{\omega}(Q) \supseteq \underline{O\Im}^{\omega}(Q),\ \overline{O\Upsilon}^{\omega}(Q) \subseteq \overline{O\Im}^{\omega}(Q).$$

Proof. (1) Take $g \in \mathcal{U}$.

$$\begin{aligned} T_{\underline{O\Upsilon}^{\omega}(Q)}(g) &= \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigwedge_{x\in\mathcal{Z}} \left(T_{\aleph_i}(g,x) \mapsto T_Q(x)\right) \\ &\leq \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigwedge_{x\in\mathcal{Z}} \left(T_{\ell_i}(g,x) \mapsto T_Q(x)\right) \\ &= T_{\underline{O\mathfrak{I}}^{\omega}(Q)}(g), \end{aligned}$$

$$\begin{aligned} I_{\underline{O\Upsilon}^{\omega}(Q)}(g) &= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(I_{\aleph_i}(g,x) \hookrightarrow I_Q(x)\right) \\ &\geq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(I_{\ell_i}(g,x) \hookrightarrow I_Q(x)\right) \\ &= I_{\underline{O\mathfrak{I}}^{\omega}(Q)}(g), \end{aligned}$$

$$\begin{aligned} F_{\underline{O\Upsilon}^{\omega}(Q)}(g) &= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(F_{\aleph_i}(g,x) \hookrightarrow F_Q(x)\right) \\ &\geq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(F_{\ell_i}(g,x) \hookrightarrow F_Q(x)\right) \\ &= F_{\underline{O\mathfrak{I}}^{\omega}(Q)}(g). \end{aligned}$$

Therefore, $\underline{O\Upsilon}^{\omega}(Q) \subseteq \underline{O\mathfrak{I}}^{\omega}(Q)$. In the same way, we can find that $\overline{O\Upsilon}^{\omega}(Q) \supseteq \overline{O\mathfrak{I}}^{\omega}(Q)$.
(2) Take $g \in \mathcal{U}$.

$$\begin{aligned} T_{\underline{O\Upsilon}^{\omega}(Q)}(g) &= \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigwedge_{x\in\mathcal{Z}} \left(T_{\aleph_i}(g,x) \mapsto T_Q(x)\right) \\ &\geq \bigvee_{i} \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigwedge_{x\in\mathcal{Z}} \left(T_{\ell_i}(g,x) \mapsto T_Q(x)\right) \\ &= T_{\underline{O\mathfrak{I}}^{\omega}(Q)}(g), \end{aligned}$$

$$\begin{aligned} I_{\underline{O\Upsilon}^{\omega}(Q)}(g) &= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(I_{\aleph_i}(g,x) \hookrightarrow I_Q(x)\right) \\ &\leq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(I_{\ell_i}(g,x) \hookrightarrow I_Q(x)\right) \\ &= I_{\underline{O\mathfrak{I}}^{\omega}(Q)}(g), \end{aligned}$$

$$\begin{aligned} F_{\underline{O\Upsilon}^{\omega}(Q)}(g) &= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(F_{\aleph_i}(g,x) \hookrightarrow F_Q(x)\right) \\ &\leq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_{\omega}} \bigvee_{x\in\mathcal{Z}} \left(F_{\ell_i}(g,x) \hookrightarrow F_Q(x)\right) \end{aligned}$$

$$= \quad F_{\underline{O\mathfrak{I}^{\omega}}(Q)}(g).$$

Therefore, $\underline{O\Upsilon^{\omega}}(Q) \supseteq \underline{O\mathfrak{I}^{\omega}}(Q)$. Likewise, we can see that $\overline{O\Upsilon}^{\omega}(Q) \subseteq \overline{O\mathfrak{I}}^{\omega}(Q)$. □

Proposition 5. *Assume that* $\Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$, $\mathfrak{I} = \{\ell_1, \ell_2, \cdots, \ell_t\}$ *are families of SVNR on* $\mathcal{U}$*, for* $\forall Q \in I(\mathcal{U})$ *and* $\forall \omega \in I$.

(1) When $s = t$, $\aleph_i \geq \ell_i, i = 1, 2, \cdots, s$,

$$\underline{P\Upsilon^{\omega}}(Q) \subseteq \underline{P\mathfrak{I}^{\omega}}(Q), \ \overline{P\Upsilon}^{\omega}(Q) \supseteq \overline{P\mathfrak{I}}^{\omega}(Q).$$

(2) When $\mathfrak{I} \subseteq \Upsilon$,

$$\underline{P\Upsilon^{\omega}}(Q) \subseteq \underline{P\mathfrak{I}^{\omega}}(Q), \ \overline{P\Upsilon}^{\omega}(Q) \supseteq \overline{P\mathfrak{I}}^{\omega}(Q).$$

Proof. Similar to Proposition 4. □

Proposition 6. *Assume that* $\Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$, $\mathfrak{I} = \{\ell_1, \ell_2, \cdots, \ell_t\}$ *are families of SVNR on* $\mathcal{U}$ *and* $\aleph_i \geq \ell_i$*, for* $\forall Q \in I(\mathcal{U})$*,and* $\forall \omega \in I$.

$$\underline{C\Upsilon^{\omega}}(Q) \subseteq \underline{C\mathfrak{I}^{\omega}}(Q), \ \ \overline{C\Upsilon}^{\omega}(Q) \supseteq \overline{C\mathfrak{I}}^{\omega}(Q).$$

Proof. It can be proved by Propositions 4 and 5. □

3.2. Comparable property and idempotent property

In the current part, it will be shown that the MGCVPNRS fulfills the comparability property whenever Υ has reflexivity. Furthermore, it will be demonstrated that the idempotent property is satisfied by the OCVPNRS when Υ is a preorder.

Definition 10. *Let* $\Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ *be a neutrosophic relation family on* $\mathcal{U}$.

(1) We call Υ *reflexive if each* $\aleph_i$ *is reflexive.*

(2) We call Υ *partially reflexive if there is an* $\aleph_i$ *that is reflexive.*

Theorem 6. *Let* $(\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\})$ *is a reflexive FGNAS and* $\omega \in I$*. We then have* $\forall Q \in I(\mathcal{U})$, $\underline{O\Upsilon^{\omega}}(Q) \subseteq Q \subseteq \overline{O\Upsilon}^{\omega}(Q)$, $\underline{P\Upsilon^{\omega}}(Q) \subseteq Q \subseteq \overline{P\Upsilon}^{\omega}(Q)$*, and* $\underline{C\Upsilon^{\omega}}(Q) \subseteq Q \subseteq \overline{C\Upsilon}^{\omega}(Q)$.

Proof. Let $g \in \mathcal{U}$, in which case

$$T_{\underline{O\Upsilon^{\omega}}(Q)}(g) = \bigvee_{i} \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigwedge_{x \in \mathcal{Z}} \Big(T_{\aleph_i}(g, x) \mapsto T_Q(x)\Big) \leq \bigvee_{i} \bigvee_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \Big(T_{\aleph_i}(g, g) \mapsto T_Q(g)\Big) = T_Q(g),$$

$$I_{\underline{O\Upsilon^{\omega}}(Q)}(g) = \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigvee_{x \in \mathcal{Z}} \Big(I_{\aleph_i}(g, x) \hookrightarrow I_Q(x)\Big) \geq \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \Big(I_{\aleph_i}(g, g) \hookrightarrow I_Q(g)\Big) = I_Q(g),$$

$$F_{\underline{O\Upsilon^{\omega}}(Q)}(g) = \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigvee_{x \in \mathcal{Z}} \Big(F_{\aleph_i}(g, x) \hookrightarrow F_Q(x)\Big) \geq \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \Big(F_{\aleph_i}(g, g) \hookrightarrow F_Q(g)\Big) = F_Q(g),$$

and

$$T_{\overline{O\Upsilon}^{\omega}(Q)}(g) = \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \bigvee_{x \in \mathcal{Z}} \Big(F_{\aleph_i}(g, x) \hookrightarrow T_Q(x)\Big) \geq \bigwedge_{i} \bigwedge_{g \in \mathcal{Z} \in \mathcal{U}_{\omega}} \Big(F_{\aleph_i}(g, g) \hookrightarrow T_Q(g)\Big) = T_Q(g),$$

$$I_{\overline{O\Upsilon}^{\omega}(Q)}(g) = \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big((1 - I_{\aleph_i}(g,x)) \mapsto I_Q(x)\Big) \leq \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \Big((1 - I_{\aleph_i}(g,g)) \hookrightarrow I_Q(g)\Big) = I_Q(g),$$

$$F_{\overline{O\Upsilon}^{\omega}(Q)}(g) = \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto F_Q(x)\Big) \leq \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \Big(T_{\aleph_i}(g,g) \hookrightarrow F_Q(g)\Big) = F_Q(g).$$

Thus, $\underline{O\Upsilon}^{\omega}(Q) \subseteq Q \subseteq \overline{O\Upsilon}^{\omega}(Q)$. In the same way, it is inferred that $\underline{P\Upsilon}^{\omega}(Q) \subseteq Q \subseteq \overline{P\Upsilon}^{\omega}(Q)$, and $\underline{C\Upsilon}^{\omega}(Q) \subseteq Q \subseteq \overline{C\Upsilon}^{\omega}(Q)$. □

This indicates that the optimistic model, the pessimistic model, and the compromise model all satisfy the comparability property.

Form Theorems 1 (4), 2 (4), and 3 (3); Propositions 1 (1), 2 (1), and 3 (1); and Theorem 6, the following corollary is drawn.

Now, the idempotent property is examined as follows.

Definition 11. *Assume that $\Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\}$ is a SVNR family on $\mathcal{U}$.*
(1) We call Υ Θ-transitive if every $\aleph_i$ is Θ-transitive.
(2) We call Υ Ξ-transitive if every $\aleph_i$ is Ξ-transitive.

Proposition 7. *Suppose $(\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\})$ is a FGNAS, $\omega \in I$, and $Q \in I(\mathcal{U})$. Then*
(1) $\underline{O\Upsilon}^{\omega}(Q) \subseteq \underline{O\Upsilon}^{\omega}(\underline{O\Upsilon}^{\omega}(Q))$, while Υ is Θ-transitive.
(2) $\overline{O\Upsilon}^{\omega}(\overline{O\Upsilon}^{\omega}(Q)) \subseteq \overline{O\Upsilon}^{\omega}(Q)$, while Υ is Ξ-transitive.

Proof. (1) Suppose that $Q \in I(\mathcal{U})$, with respect to $g \in \mathcal{U}$.

$$\begin{aligned}
T_{\underline{O\Upsilon}^{\omega}(\underline{O\Upsilon}^{\omega}(Q))}(g) &= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto T_{\underline{O\Upsilon}^{\omega}(Q)}(x)\Big) \\
&= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto \bigvee_{j=1} \bigvee_{x\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{q\in\mathcal{Z}} \Big(T_{\aleph_j}(x,q) \mapsto T_Q(q)\Big)\Big) \\
&\geq \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \bigwedge_{q\in\mathcal{Z}} \Big(T_{\aleph_i}(g,x) \mapsto \Big(T_{\aleph_i}(x,q) \mapsto T_Q(q)\Big)\Big) \\
&= \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{x\in\mathcal{Z}} \bigwedge_{q\in\mathcal{Z}} \Big(\Big(T_{\aleph_i}(g,x) \Theta T_{\aleph_i}(x,q)\Big) \mapsto T_Q(q)\Big) \\
&\geq \bigvee_i \bigvee_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigwedge_{q\in\mathcal{Z}} \Big(T_{\aleph_i}(g,q) \mapsto T_Q(q)\Big) \qquad \text{by the } \Theta-\text{transitive property} \\
&= T_{\underline{O\Upsilon}^{\omega}(Q)}(g),
\end{aligned}$$

$$\begin{aligned}
I_{\underline{O\Upsilon}^{\omega}(\underline{O\Upsilon}^{\omega}(Q))}(g) &= \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow I_{\underline{O\Upsilon}^{\omega}(Q)}(x)\Big) \\
&= \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow \bigwedge_{j=1} \bigwedge_{x\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{q\in\mathcal{Z}} \Big(I_{\aleph_j}(x,q) \hookrightarrow I_Q(q)\Big)\Big) \\
&\leq \bigwedge_i \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \bigvee_{q\in\mathcal{Z}} \Big(I_{\aleph_i}(g,x) \hookrightarrow \Big(I_{\aleph_i}(x,q) \hookrightarrow I_Q(q)\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \bigvee_{q\in\mathcal{Z}} \Big(\big(I_{\aleph_i}(g,x)\Xi I_{\aleph_i}(x,q)\big) \hookrightarrow I_Q(q)\Big) \\
&\leq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{q\in\mathcal{Z}} \big(I_{\aleph_i}(g,q) \hookrightarrow I_Q(q)\big) \qquad \text{by the } \Xi\text{−transitive property} \\
&= I_{\underline{O\Upsilon}^\omega(Q)}(g),
\end{aligned}
$$

$$
\begin{aligned}
F_{\underline{O\Upsilon}^\omega(\underline{O\Upsilon}^\omega(Q))}(g) &= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \big(F_{\aleph_i}(g,x) \hookrightarrow F_{\underline{O\Upsilon}^\omega(Q)}(x)\big) \\
&= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow \bigwedge_{j=1} \bigwedge_{x\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{q\in\mathcal{Z}} \big(F_{\aleph_j}(x,q) \hookrightarrow F_Q(q)\big)\Big) \\
&\leq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \bigvee_{q\in\mathcal{Z}} \Big(F_{\aleph_i}(g,x) \hookrightarrow \big(F_{\aleph_i}(x,q) \hookrightarrow F_Q(q)\big)\Big) \\
&= \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{x\in\mathcal{Z}} \bigvee_{q\in\mathcal{Z}} \Big(\big(F_{\aleph_i}(g,x)\Xi F_{\aleph_i}(x,q)\big) \hookrightarrow F_Q(q)\Big) \\
&\leq \bigwedge_{i} \bigwedge_{g\in\mathcal{Z}\in\mathcal{U}_\omega} \bigvee_{q\in\mathcal{Z}} \big(F_{\aleph_i}(g,q) \hookrightarrow F_Q(q)\big) \qquad \text{by the } \Xi\text{−transitive property} \\
&= F_{\underline{O\Upsilon}^\omega(Q)}(g).
\end{aligned}
$$

Therefore, $\underline{O\Upsilon}^\omega(Q) \subseteq \underline{O\Upsilon}^\omega(\underline{O\Upsilon}^\omega(Q))$.

(2) Similar to (1). □

According to Theorem 6 and Proposition 7, we derive the idempotent property for the OCVPNRS.

Theorem 7. *Presume $(\mathcal{U}, \Upsilon = \{\aleph_1, \aleph_2, \cdots, \aleph_s\})$ is a reflexive FGNAS, $\omega \in I$, and $Q \in I(\mathcal{U})$. Then*
(1) $\underline{O\Upsilon}^\omega(Q) = \underline{O\Upsilon}^\omega(\underline{O\Upsilon}^\omega(Q))$, while Υ is Θ-transitive.
(2) $\overline{O\Upsilon}^\omega(\overline{O\Upsilon}^\omega(Q)) = \overline{O\Upsilon}^\omega(Q)$, while Υ is Ξ-transitive.

For the PCVPNRS and CCVPNRS, it cannot be verified that they satisfy Proposition 7. So, it seems that it may be impossible to talk about idempotent properties for the PCVPNRS and CCVPNRS.

4. The construction of the MAGDM model

The MAGDM problem is increasingly pervasive in our daily lives. MAGDM involves selecting or ranking all feasible alternatives on the basis of diverse criteria. A multitude of approaches exist for solving MAGDM problems. In this study, an algorithm grounded in the MGVPNRS model is proposed to solve the MAGDM problem. Assume that $\mathcal{V} = \{x_1, x_2, ..., x_n\}$ is the object set and that $AT = \{t_1, t_2, ..., t_n\}$ is the attribute set, $\omega = \{\omega_1, \omega_2, ..., \omega_n\}$ is the attribute weight, and $T = \{T_1, T_2, ..., T_n\}$ is the decision column. Let $\aleph \in I(\mathcal{U} \times \mathcal{U})$ be a SVNR from $\mathcal{U}$ to $\mathcal{U}$. With the Algorithm 1, we obtain the optimal solution.

Algorithm 1. The decision-making process.

Input Multi-granulation neutrosophic decision IS, and two parameters ω, ρ.
Output Select the optimal medical emergency rescue system.
Step 1: Normalized SVNN decision-making matrix.
Step 2: According to Eqs (4.1)–(4.3), obtain the attribute weights and the decision column.
Step 3: By Definition 6, we compute $\overline{O\Upsilon}^{\omega}(Q)$ and $\underline{O\Upsilon}^{\omega}(Q)$ of Q.
Subsequently, we proceed to aggregate the experts.
Step 4: According to the Definition 16, we calculated the score functions for objects x_1 to x_5.
Step 5: Using the score function as the criterion, sort each alternative scheme.

Definition 12. *In an neutrosophic information system (IS), the SVNR $\aleph$ defined below, is called a distance-based SVNR with $\forall h_s, h_k \in \mathcal{U}$*

$$\aleph(h_s, h_k) = \left(1 - \sqrt{\frac{\sum_{l=1}^{t}(T_{sl} - T_{kl})^2}{t}}, \sqrt{\frac{\sum_{l=1}^{t}(I_{sl} - I_{kl})^2}{t}}, \sqrt{\frac{\sum_{l=1}^{t}(F_{sl} - F_{kl})^2}{t}}\right).$$

Obviously, $\aleph$ is a tolerance.

Definition 13. *[18] Let $H = (h_1, h_2, ...h_n)(k = 1, 2, 3, ..., n)$. S is the SVNS, and its neutrosophic entropy is defined as*

$$E(S) = 1 - \frac{1}{n}\sum_{h_k \in H}(T_S(h_k) + F_S(h_k)) \cdot |I_S(h_k) - I_{\neg S}(h_k)|. \tag{4.1}$$

The neutrosophic entropy measures the indeterminacy within the attribute's data. Higher entropy implies a higher level of indeterminacy. Below is the formula for calculating the weight ω_j of the j-th attribute:

$$\omega_j = \frac{1 - E(h_j)}{\sum_j^n(1 - E(h_j))}. \tag{4.2}$$

The greater the weight, the higher the certainty and the lower the uncertainty of the data for that attribute. Consequently, such data provide more information when distinguishing between different evaluation objects and hold greater significance. Conversely, an attribute with a smaller weight indicates that its data are generally uncertain or ambiguous, and thus contributes less to distinguishing between evaluation objects.

Remark 3. *Let $x_j = \langle T_j, I_j, F_j\rangle (j = 1, 2, ..., n)$ is a set of SVNNs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the attribute weight, satisfying $\omega \in [0, 1]$ and $\sum_{j=1}^{n} \omega = 1$.*

Definition 14. *[19] Let $a_k = \langle T_k, I_k, F_k\rangle (k = 1, 2, 3, ...s)$ be a set of SVNNs, for which we have defined a new SVNN:*

$$SNNWA(a_1, a_2, a_3, ..., a_s) = \langle 1 - \prod_{k=1}^{s}(1 - T_k)^{\omega_k}, \prod_{k=1}^{s}(I_k)^{\omega_k}, \prod_{k=1}^{s}(F_k)^{\omega_k}\rangle. \tag{4.3}$$

Definition 15. *[20] We define the sum of two SVNSs as follows:*

$$Q \boxplus T = \left\{\langle \kappa, Q(\kappa) \oplus T(\kappa) | \kappa \in \mathcal{U}\rangle\right\}. \tag{4.4}$$

Definition 16. *If we assume that $Y = (T_Y, I_Y, F_Y)$ is a SVNN, the score of Y can be defined as follows:*

$$S(Y) = T_Y - I_Y - F_Y. \tag{4.5}$$

5. The application in MAGDM

In this section, building upon the proposed MGVPNRS model and incorporating Zhang's dataset [21], this study systematically develops a methodological decision-making framework.

Earthquakes possess formidable destructive capabilities, and accurately predicting them remains an extremely arduous task. Moreover, the aftermath of an earthquake can severely disrupt social order. In light of these factors, it is crucial for us to fully recognize the significance of medical assistance in disaster relief operations. In particular, when their lives are in peril, individuals strongly desire to obtain prompt and effective emergency aid. Hence, in the new century, minimizing the impact of disasters and enhancing the efficiency of medical rescue efforts carry substantial practical implications.

Currently, five regions' medical emergency rescue systems are awaiting evaluation. In order to select the optimal solution from these systems, decision-makers need to comprehensively consider various factors. This is a typical multi-attribute group decision-making problem.

On the basis of the analysis above, we will implement our model for evaluating medical emergency rescue systems. We will validate the effectiveness and reliability of our method, test its sensitivity and stability through parameter analysis, and also conduct a parameter analysis and discuss the advantages of our method in comparison with other methods.

Example 1. *Let $\mathcal{V} = \{h_1, h_2, h_3, h_4, h_5\}$ denote five medical emergency rescue systems, and let $AT = \{k_1, k_2, k_3, k_4, k_5, k_6\}$ represent six conditional attributes (the diagnostic assessment proficiency, the perception of risk-related information, the capacity to handle information from diverse sources, the resilience against interference when analyzing information, the ability of accurate positioning, the coordination competence of a heterogeneous team), and they are all beneficial attributes. Through our decision-making method of ranking, we can figure out which system is the optimal choice. The decision-making expert group consists of five experts. The experts evaluate these five systems using the six indicators k_1, k_2, k_3, k_4, k_5, and k_6, and the index evaluation value takes the form of a SVNN. Tables 3–7 presents the decision matrix.*

Let $\omega = 0.9$, $\rho = 0.5$, $\Theta = \Theta_p$, and $\Xi = \Xi_p$. Using the optimistic approach as an example, we detail the computation steps, then present the rankings for all three methods (optimistic, pessimistic, and compromise).

Next, on the basis of the Algorithm 1, we present the specific steps.

(1) Introduce the IS

Tables 3–7 present the evaluation information of five regions' medical emergency rescue systems by five experts, in which the attribute values are represented by g_{rj} ($1 \le r \le 5, 1 \le j \le 6$), $g_{rj} = (T_{rj}(x),\ I_{rj}(x),\ F_{rj}(x))_{5\times 6}$. Since all six indicators are benefit-type indicators, we do not need to normalize this.

(2) Deriving the attribute weights and determining decision attributes.

Using Eqs (4.1) and (4.2), we obtained the attribute weight (keeping the calculation results to four decimal places, the same applies below).

$$\omega_1 = \{0.1974, 0.1470, 0.2223, 0.1534, 0.1200, 0.1598\},$$

$$\omega_2 = \{0.1668, 0.1707, 0.2356, 0.1648, 0.1257, 0.1363\},$$

$$\omega_3 = \{0.2032, 0.1840, 0.1372, 0.1471, 0.1121, 0.2164\},$$

$$\omega_4 = \{0.2619, 0.1580, 0.1509, 0.1673, 0.1706, 0.0913\},$$

$$\omega_5 = \{0.2041, 0.2083, 0.1867, 0.2203, 0.1194, 0.0612\}.$$

The decision column is obtained by Eq (4.3):

$$\begin{aligned} T_1 = \{&(h_1, 0.5529, 0.4231, 0.4912), (h_2, 0.6576, 0.3424, 0.3368),\\ &(h_3, 0.6375, 0.3879, 0.4020), (h_4, 0.4756, 0.5550, 0.6009),\\ &(h_5, 0.5598, 0.4932, 0.4536)\},\end{aligned}$$

$$\begin{aligned} T_2 = \{&(h_1, 0.5350, 0.4381, 0.5102), (h_2, 0.7002, 0.2564, 0.2769),\\ &(h_3, 0.6065, 0.4080, 0.4047), (h_4, 0.5179, 0.4920, 0.4955),\\ &(h_5, 0.5616, 0.4441, 0.5096)\},\end{aligned}$$

$$\begin{aligned} T_3 = \{&(h_1, 0.5309, 0.4736, 0.5126), (h_2, 0.6552, 0.3482, 0.3923),\\ &(h_3, 0.6599, 0.3383, 0.3579), (h_4, 0.4353, 0.5838, 0.6197),\\ &(h_5, 0.0.4427, 0.5510, 0.5282)\},\end{aligned}$$

$$\begin{aligned} T_4 = \{&(h_1, 0.5411, 0.4457, 0.5308), (h_2, 0.7261, 0.2103, 0.2381),\\ &(h_3, 0.6767, 0.2834, 0.3451), (h_4, 0.6007, 0.3809, 0.4641),\\ &(h_5, 0.5323, 0.4548, 0.5329)\},\end{aligned}$$

$$\begin{aligned} T_5 = \{&(h_1, 0.5523, 0.4576, 0.4570), (h_2, 0.7001, 0.2593, 0.2777),\\ &(h_3, 0.6325, 0.3656, 0.3956), (h_4, 0.5124, 0.4744, 0.5716),\\ &(h_5, 0.5557, 0.3855, 0.4542)\}.\end{aligned}$$

Table 3. Information about the five medical emergency systems of expert Group 1.

$\mathcal{V}/AT$	k_1	k_2	k_3	k_4	k_5	k_6
h_1	(0.70, 0.25, 0.30)	(0.15, 0.65, 0.85)	(0.25, 0.65, 0.70)	(0.65, 0.45, 0.50)	(0.65, 0.50, 0.45)	(0.70, 0.25, 0.35)
h_2	(0.65, 0.30, 0.35)	(0.55, 0.45, 0.50)	(0.75, 0.20, 0.15)	(0.65, 0.40, 0.50)	(0.60, 0.55, 0.50)	(0.65, 0.40, 0.35)
h_3	(0.60, 0.45, 0.40)	(0.60, 0.40, 0.45)	(0.70, 0.25, 0.30)	(0.60, 0.50, 0.55)	(0.70, 0.30, 0.35)	(0.60, 0.55, 0.45)
h_4	(0.45, 0.40, 0.55)	(0.35, 0.65, 0.70)	(0.55, 0.60, 0.65)	(0.45, 0.65, 0.60)	(0.65, 0.50, 0.45)	(0.35, 0.60, 0.65)
h_5	(0.55, 0.60, 0.40)	(0.40, 0.55, 0.65)	(0.60, 0.50, 0.45)	(0.70, 0.30, 0.35)	(0.45, 0.65, 0.60)	(0.55, 0.45, 0.40)

Table 4. Information about the five investment companies of expert Group 2.

$\mathcal{V}/AT$	k_1	k_2	k_3	k_4	k_5	k_6
h_1	(0.65, 0.35, 0.40)	(0.35, 0.65, 0.70)	(0.35, 0.55, 0.65)	(0.60, 0.45, 0.50)	(0.55, 0.40, 0.45)	(0.70, 0.25, 0.35)
h_2	(0.70, 0.25, 0.20)	(0.65, 0.40, 0.35)	(0.80, 0.10, 0.15)	(0.60, 0.45, 0.50)	(0.70, 0.25, 0.35)	(0.65, 0.40, 0.35)
h_3	(0.65, 0.30, 0.35)	(0.50, 0.55, 0.45)	(0.70, 0.25, 0.30)	(0.45, 0.65, 0.60)	(0.65, 0.50, 0.45)	(0.60, 0.45, 0.40)
h_4	(0.50, 0.45, 0.45)	(0.40, 0.65, 0.60)	(0.60, 0.35, 0.40)	(0.50, 0.55, 0.60)	(0.60, 0.55, 0.50)	(0.45, 0.55, 0.50)
h_5	(0.60, 0.50, 0.45)	(0.35, 0.70, 0.60)	(0.55, 0.45, 0.65)	(0.75, 0.20, 0.25)	(0.50, 0.45, 0.65)	(0.50, 0.55, 0.60)

Table 5. Information about the five medical emergency systems of expert Group 3.

$\mathcal{V}/AT$	k_1	k_2	k_3	k_4	k_5	k_6
h_1	(0.55, 0.60, 0.40)	(0.20, 0.60, 0.80)	(0.30, 0.65, 0.75)	(0.50, 0.60, 0.65)	(0.60, 0.55, 0.50)	(0.75, 0.20, 0.30)
h_2	(0.65, 0.30, 0.35)	(0.65, 0.40, 0.35)	(0.60, 0.45, 0.50)	(0.65, 0.35, 0.45)	(0.65, 0.50, 0.45)	(0.70, 0.25, 0.35)
h_3	(0.75, 0.20, 0.20)	(0.60, 0.35, 0.40)	(0.65, 0.35, 0.45)	(0.65, 0.35, 0.45)	(0.60, 0.55, 0.50)	(0.65, 0.40, 0.55)
h_4	(0.35, 0.65, 0.60)	(0.35, 0.65, 0.70)	(0.55, 0.50, 0.65)	(0.55, 0.45, 0.65)	(0.45, 0.65, 0.60)	(0.40, 0.60, 0.55)
h_5	(0.50, 0.45, 0.45)	(0.30, 0.70, 0.65)	(0.45, 0.65, 0.50)	(0.40, 0.55, 0.65)	(0.35, 0.65, 0.70)	(0.55, 0.45, 0.40)

Table 6. Information about the five medical emergency systems of expert Group 4.

$\mathcal{V}/AT$	k_1	k_2	k_3	k_4	k_5	k_6
h_1	(0.65, 0.35, 0.40)	(0.25, 0.60, 0.75)	(0.40, 0.55, 0.65)	(0.60, 0.45, 0.55)	(0.50, 0.55, 0.60)	(0.70, 0.25, 0.35)
h_2	(0.80, 0.10, 0.15)	(0.70, 0.20, 0.25)	(0.75, 0.20, 0.15)	(0.50, 0.65, 0.60)	(0.80, 0.15, 0.20)	(0.60, 0.50, 0.45)
h_3	(0.65, 0.25, 0.35)	(0.75, 0.20, 0.20)	(0.65, 0.35, 0.40)	(0.65, 0.35, 0.45)	(0.70, 0.25, 0.35)	(0.65, 0.45, 0.40)
h_4	(0.70, 0.20, 0.30)	(0.45, 0.50, 0.55)	(0.45, 0.55, 0.60)	(0.65, 0.40, 0.50)	(0.65, 0.45, 0.50)	(0.50, 0.55, 0.60)
h_5	(0.60, 0.40, 0.50)	(0.40, 0.55, 0.65)	(0.35, 0.65, 0.70)	(0.70, 0.25, 0.35)	(0.50, 0.55, 0.60)	(0.45, 0.55, 0.50)

Table 7. Information about the five medical emergency systems of expert Group 5.

$\mathcal{V}/AT$	k_1	k_2	k_3	k_4	k_5	k_6
h_1	(0.55, 0.35, 0.50)	(0.35, 0.70, 0.60)	(0.45, 0.65, 0.50)	(0.65, 0.35, 0.45)	(0.70, 0.35, 0.25)	(0.65, 0.40, 0.35)
h_2	(0.75, 0.15, 0.20)	(0.70, 0.20, 0.25)	(0.70, 0.25, 0.30)	(0.65, 0.45, 0.35)	(0.75, 0.30, 0.25)	(0.55, 0.45, 0.50)
h_3	(0.60, 0.40, 0.50)	(0.60, 0.35, 0.40)	(0.65, 0.45, 0.35)	(0.70, 0.25, 0.35)	(0.65, 0.40, 0.35)	(0.45, 0.55, 0.50)
h_4	(0.65, 0.25, 0.35)	(0.25, 0.60, 0.75)	(0.50, 0.70, 0.65)	(0.55, 0.45, 0.65)	(0.50, 0.55, 0.60)	(0.60, 0.50, 0.45)
h_5	(0.55, 0.50, 0.60)	(0.20, 0.65, 0.75)	(0.40, 0.60, 0.65)	(0.80, 0.10, 0.15)	(0.60, 0.50, 0.45)	(0.50, 0.55, 0.60)

(3) Calculate the neutrosophic relations
From Definition 12, we obtain SVNRs. The results are shown in Tables 8–12.

Table 8. The SVNR of Decision-maker 1.

	h_1	h_2	h_3	h_4	h_5
h_1	(1, 0, 0)	(0.7362, 0.2131, 0.2677)	(0.7323, 0.2566, 0.2424)	(0.7568, 0.1768, 0.1768)	(0.7869, 0.2000, 0.1633)
h_2	(0.7362, 0.2131, 0.2677)	(1, 0, 0)	(0.9388, 0.1429, 0.1021)	(0.7949, 0.2291, 0.2685)	(0.8775, 0.1882, 0.1581)
h_3	(0.7323, 0.2566, 0.2424)	(0.9388, 0.1429, 0.1021)	(1, 0, 0)	(0.8197, 0.2051, 0.2082)	(0.8542, 0.2160, 0.1671)
h_4	(0.7568, 0.1768, 0.1768)	(0.7949, 0.2291, 0.2685)	(0.8197, 0.2051, 0.2082)	(1, 0, 0)	(0.8380, 0.2160, 0.1671)
h_5	(0.7869, 0.2000, 0.1633)	(0.8775, 0.1882, 0.1581)	(0.8542, 0.2160, 0.1671)	(0.8380, 0.1947, 0.1882)	(1, 0, 0)

Table 9. The SVNR of Decision-maker 2.

	h_1	h_2	h_3	h_4	h_5
h_1	(1, 0, 0)	(0.7691, 0.2309, 0.2654)	(0.8232, 0.1791, 0.1826)	(0.8354, 0.1696, 0.1354)	(0.8661, 0.1780, 0.1720)
h_2	(0.7691, 0.2309, 0.2654)	(1, 0, 0)	(0.8979, 0.1594, 0.1137)	(0.8163, 0.2189, 0.2010)	(0.7969, 0.2582, 0.3136)
h_3	(0.8232, 0.1791, 0.1826)	(0.8979, 0.1594, 0.1137)	(1, 0, 0)	(0.8920, 0.1041, 0.0957)	(0.7969, 0.2582, 0.3136)
h_4	(0.8354, 0.1696, 0.1354)	(0.8163, 0.2189, 0.2010)	(0.8920, 0.1041, 0.0957)	(1, 0, 0)	(0.8775, 0.1568, 0.1904)
h_5	(0.8661, 0.1780, 0.1720)	(0.7969, 0.2582, 0.3136)	(0.8317, 0.2300, 0.2441)	(0.8775, 0.1568, 0.19074)	(1, 0, 0)

Table 10. The SVNR of Decision-maker 3.

	h_1	h_2	h_3	h_4	h_5
h_1	$(1, 0, 0)$	$(0.7655, 0.1990, 0.2282)$	$(0.7568, 0.2630, 0.2354)$	$(0.7869, 0.1915, 0.1486)$	$(0.8432, 0.1339, 0.1514)$
h_2	$(0.7655, 0.1990, 0.2282)$	$(1, 0, 0)$	$(0.9423, 0.0890, 0.0707)$	$(0.7682, 0.2389, 0.2273)$	$(0.7611, 0.2062, 0.1848)$
h_3	$(0.7568, 0.2630, 0.2354)$	$(0.9423, 0.0890, 0.0707)$	$(1, 0, 0)$	$(0.7664, 0.2500, 0.2517)$	$(0.7664, 0.2336, 0.1871)$
h_4	$(0.7869, 0.1915, 0.1486)$	$(0.7682, 0.2389, 0.2273)$	$(0.7664, 0.2500, 0.2517)$	$(1, 0, 0)$	$(0.8775, 0.1275, 0.1155)$
h_5	$(0.8432, 0.1339, 0.1514)$	$(0.7611, 0.2062, 0.1848)$	$(0.7664, 0.2336, 0.1871)$	$(0.8775, 0.1275, 0.1155)$	$(1, 0, 0)$

Table 11. The SVNR of Decision-maker 4.

	h_1	h_2	h_3	h_4	h_5
h_1	$(1, 0, 0)$	$(0.7239, 0.3182, 0.3500)$	$(0.7559, 0.2415, 0.2716)$	$(0.8646, 0.1500, 0.1458)$	$(0.8709, 0.1555, 0.1190)$
h_2	$(0.7239, 0.3182, 0.3500)$	$(1, 0, 0)$	$(0.8920, 0.1568, 0.1594)$	$(0.8096, 0.2508, 0.2700)$	$(0.7284, 0.3506, 0.3674)$
h_3	$(0.7559, 0.2415, 0.2716)$	$(0.8920, 0.1568, 0.1594)$	$(1, 0, 0)$	$(0.8380, 0.1756, 0.1958)$	$(0.7773, 0.2398, 0.2574)$
h_4	$(0.8646, 0.1500, 0.1458)$	$(0.8096, 0.2508, 0.2700)$	$(0.8380, 0.1756, 0.1958)$	$(1, 0, 0)$	$(0.9087, 0.1190, 0.1307)$
h_5	$(0.8709, 0.1555, 0.1190)$	$(0.7284, 0.3506, 0.3674)$	$(0.7773, 0.2398, 0.2574)$	$(0.9087, 0.1190, 0.1307)$	$(1, 0, 0)$

Table 12. The SVNR of Decision-maker 5.

	h_1	h_2	h_3	h_4	h_5
h_1	$(1, 0, 0)$	$(0.8010, 0.2784, 0.2179)$	$(0.8419, 0.1826, 0.1323)$	$(0.8882, 0.1173, 0.2000)$	$(0.8845, 0.1500, 0.2031)$
h_2	$(0.8010, 0.2784, 0.2179)$	$(1, 0, 0)$	$(0.9021, 0.1756, 0.1443)$	$(0.7664, 0.2700, 0.3189)$	$(0.7331, 0.3215, 0.3221)$
h_3	$(0.8419, 0.1826, 0.1323)$	$(0.9021, 0.1756, 0.1443)$	$(1, 0, 0)$	$(0.8107, 0.1882, 0.2550)$	$(0.8000, 0.1607, 0.2170)$
h_4	$(0.8882, 0.1173, 0.2000)$	$(0.7664, 0.2700, 0.3189)$	$(0.8107, 0.1882, 0.2550)$	$(1, 0, 0)$	$(0.8677, 0.1837, 0.2441)$
h_5	$(0.8845, 0.1500, 0.2031)$	$(0.7331, 0.3215, 0.3221)$	$(0.8000, 0.1607, 0.2170)$	$(0.8677, 0.1837, 0.2441)$	$(1, 0, 0)$

(4) Calculate the $\overline{O\Upsilon}^{\omega}(Q)$ and $\underline{O\Upsilon}^{\omega}(Q)$ of Q

By the formula in Definition 6, the optimistic lower and upper bounds $\overline{O\Upsilon}^{\omega}(Q)$ and $\underline{O\Upsilon}^{\omega}(Q)$ of Q are derived. The results are shown in Table 13.

Table 13. The optimistic lower and upper approximations.

	$\underline{O\Upsilon}^{\omega}$	$\overline{O\Upsilon}^{\omega}$
h_1	$(0.5529, 0.4381, 0.4644)$	$(0.5529, 0.4347, 0.4729)$
h_2	$(0.7261, 0.2103, 0.2781)$	$(0.6552, 0.3482, 0.3798)$
h_3	$(0.6767, 0.2834, 0.3710)$	$(0.6375, 0.3879, 0.3588)$
h_4	$(0.5858, 0.3811, 0.4641)$	$(0.5422, 0.4510, 0.4670)$
h_5	$(0.5616, 0.3855, 0.4542)$	$(0.5575, 0.4387, 0.4670)$

(5) Calculate the score function by Definition 16 as shown in Table 14.

Table 14. Score function.

$\mathcal{V}/AT$	$\underline{O\Upsilon}^{\omega} \boxplus \overline{O\Upsilon}^{\omega}$	$\underline{P\Upsilon}^{\omega} \boxplus \overline{P\Upsilon}^{\omega}$	$\underline{C\Upsilon}^{\omega} \boxplus \overline{C\Upsilon}^{\omega}$	S_{OR}	S_{PR}	S_{CR}
h_1	$(0.8001, 0.1904, 0.2196)$	$(0.8179, 0.1497, 0.1820)$	$(0.8092, 0.1688, 0.1999)$	0.3900	0.4862	0.4404
h_2	$(0.9056, 0.0732, 0.1056)$	$(0.8813, 0.0953, 0.1209)$	$(0.8941, 0.0835, 0.1130)$	0.7267	0.6651	0.6976
h_3	$(0.8828, 0.1099, 0.1331)$	$(0.8603, 0.1110, 0.1324)$	$(0.8720, 0.1105, 0.1327)$	0.6398	0.6169	0.6288
h_4	$(0.8104, 0.1719, 0.2167)$	$(0.7881, 0.1639, 0.1823)$	$(0.7996, 0.1678, 0.1987)$	0.4218	0.4420	0.4330
h_5	$(0.8060, 0.1691, 0.2121)$	$(0.7733, 0.1784, 0.1863)$	$(0.7903, 0.1737, 0.1988)$	0.4248	0.4085	0.4178

(6) Table 15 demonstrates that various strategies yield distinct rankings, yet the optimal selection consistently remain h_2.

Table 15. Ranking of different strategies.

ρ	Strategies	Ranking
0	*Pessmistic*	$h_2 > h_3 > h_1 > h_4 > h_5$
0.5	*Compromise*	$h_2 > h_3 > h_1 > h_4 > h_5$
1	*Optimistic*	$h_2 > h_3 > h_5 > h_4 > h_1$

5.1. Parameter analysis

The introduced model includes two parameters, ω and ρ, alongside multiple logic operators. This section evaluates how ω affects the rankings in the compromise model, while ρ influences the ranking of the other three strategies.

5.1.1. Discussion of ρ

Fixing $\omega = 0.9$, $\mapsto = \mapsto_P$, $\hookrightarrow = \hookrightarrow_P$, let $\rho \in (0, 1)$ be a step length of 0.1. Table 16 presents the results.

As shown in Table 16, the rankings of the three strategies change with the variation of ρ. This shows that our model exhibits sensitivity to parameters. When the value of ρ approaches 0, this implies that decision-makers with a neutral stance tend to make optimistic judgments. Similarly, when the value of ρ approaches 1, this shows that decision-makers with a neutral stance tend to make pessimistic judgments.

As shown in Figure 1, as ρ increases, the scores of h_2 and h_3 also increase continuously; in contrast, the score of h_1 shows a downward trend, which is also a key factor contributing to the change in the rankings. Nevertheless, for each strategy, the optimal choice remains h_2, indicating that our model maintains relative stability.

Table 16. Ranking of different values of ρ.

ρ	$S(h_1)$	$S(h_2)$	$S(h_3)$	$S(h_4)$	$S(h_5)$	Ranking
0.1	0.4774	0.6719	0.6194	0.4404	0.4106	$h_2 > h_3 > h_1 > h_4 > h_5$
0.2	0.4684	0.6785	0.6218	0.4387	0.4125	$h_2 > h_3 > h_1 > h_4 > h_5$
0.3	0.4593	0.6850	0.6242	0.4369	0.4144	$h_2 > h_3 > h_1 > h_4 > h_5$
0.4	0.4500	0.6914	0.6265	0.4350	0.4161	$h_2 > h_3 > h_1 > h_4 > h_5$
0.6	0.4308	0.7037	0.6311	0.4309	0.4194	$h_2 > h_3 > h_4 > h_5 > h_1$
0.7	0.4209	0.7096	0.6333	0.4288	0.4208	$h_2 > h_3 > h_4 > h_1 > h_5$
0.8	0.4108	0.7154	0.6355	0.4265	0.4222	$h_2 > h_3 > h_4 > h_5 > h_1$
0.9	0.4005	0.7211	0.6376	0.4242	0.4236	$h_2 > h_3 > h_4 > h_5 > h_1$

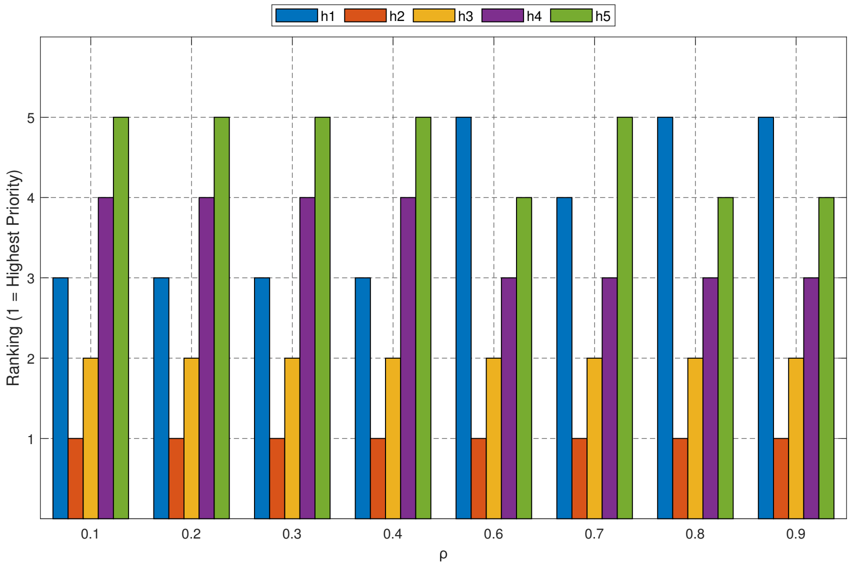

Figure 1. Comparison of ranking results under different ρ values.

5.1.2. Discussion of ω

Subsequently, we consider how variations in the parameter ω affect the ranking outcomes. It should be noted that 1–ω represents the proportion of incorrect or missing data. Generally, the proportion remains within a low range, so 1–ω is usually close to 0. In other words, ω is roughly 1. Therefore, in Example 1, we set $\rho = 0.5$, $\mapsto = \mapsto_P, \hookrightarrow = \hookrightarrow_P$ and let ω range from 0.6 to 1 with an increment of 0.1. The outcomes can be seen in Table 17.

The data in Table 17 show that for different strategies, the sorting results change as ω varies, which reflects the sensitivity of our method.

When ω takes the values of 0.7 and 0.8, the ranking of h_1 drops sharply to the last place, while the rankings of h_4 and h_5 rise. This indicates that the impact of the optimistic strategy on the scores of h_1, h_4, and h_5 is significantly different from that of other strategies. When ω takes the values of 0.9 and 1, the ranking of h_5 rises further and surpasses h_4, while h_1 remains in the last place. This shows that when ω is 0.9 or above, the performance of h_5 under the optimistic strategy continues to improve.

As shown in Figure 2, regardless of changes in the rankings, the optimal choice remains consistent, always being h_2. This reflects the stability of the proposed method. Meanwhile, the rankings fluctuate with changes in ω, which indicates that the proposed method has a certain degree of sensitivity.

Table 17. Ranking of different values of ω.

Strategies	ω	$S(h_1)$	$S(h_2)$	$S(h_3)$	$S(h_4)$	$S(h_5)$	Ranking
Pessmistic	0.6	0.3473	0.7389	0.6126	0.2770	0.3097	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.7	0.3790	0.6938	0.6013	0.3186	0.3222	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.8	0.3790	0.6938	0.6013	0.3186	0.3222	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.9	0.4862	0.6651	0.6169	0.4420	0.4085	$h_2 > h_3 > h_1 > h_4 > h_5$
	1	0.4862	0.6651	0.6169	0.4420	0.4085	$h_2 > h_3 > h_1 > h_4 > h_5$
Compromise	0.6	0.3473	0.7389	0.6104	0.2645	0.3015	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.7	0.3790	0.6938	0.6013	0.3186	0.3222	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.8	0.3790	0.6938	0.6013	0.3186	0.3222	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.9	0.4404	0.6976	0.6288	0.4330	0.4178	$h_2 > h_3 > h_1 > h_4 > h_5$
	1	0.4404	0.6976	0.6288	0.4330	0.4178	$h_2 > h_3 > h_1 > h_4 > h_5$
Optimistic	0.6	0.3473	0.7297	0.6175	0.3048	0.3279	$h_2 > h_3 > h_1 > h_5 > h_4$
	0.7	0.3637	0.7281	0.6132	0.3824	0.3813	$h_2 > h_3 > h_4 > h_5 > h_1$
	0.8	0.3637	0.7281	0.6132	0.3824	0.3813	$h_2 > h_3 > h_4 > h_5 > h_1$
	0.9	0.3900	0.7267	0.6398	0.4218	0.4248	$h_2 > h_3 > h_5 > h_4 > h_1$
	1	0.3900	0.7267	0.6398	0.4218	0.4248	$h_2 > h_3 > h_5 > h_4 > h_1$

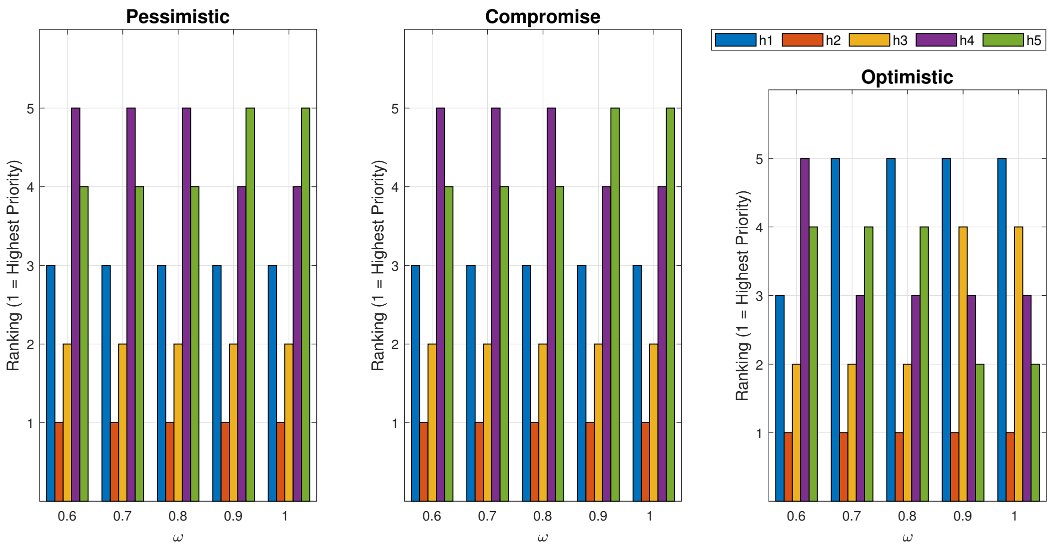

Figure 2. Comparison of ranking results of different strategies with parameter variations.

5.1.3. Analysis of fuzzy logic operators

Example 1 extends the analysis by first applying the $(\mapsto_P, \hookrightarrow_P)$ operator and then augmenting the experiment with four fuzzy logic operators: $(\mapsto_L, \hookrightarrow_L)$, $(\mapsto_H, \hookrightarrow_H)$, and $(\mapsto_Y, \hookrightarrow_Y)$. This allows us to examine how different aggregation strategies influence the final ranking outcomes. We set $\omega = 0.9$ and $\rho = 0.5$. Table 18 displays the results.

Table 18. Ranking of different fuzzy logic operators.

Strategies	FLO	$S(h_1)$	$S(h_2)$	$S(h_3)$	$S(h_4)$	$S(h_5)$	Ranking
Pessmistic	$(\mapsto_L, \hookrightarrow_L)$	0.3292	0.7389	0.6175	0.2976	0.2693	$h_2 > h_3 > h_1 > h_5 > h_4$
	$(\mapsto_{nM}, \hookrightarrow_{nM})$	0.4475	0.4816	0.4687	0.4780	0.4728	$h_2 > h_4 > h_5 > h_3 > h_1$
	$(\mapsto_P, \hookrightarrow_P)$	0.4862	0.6651	0.6169	0.4420	0.4085	$h_2 > h_3 > h_1 > h_4 > h_5$
	$(\mapsto_H, \hookrightarrow_H)$	−0.0296	0.0570	0.0411	−0.0997	−0.0526	$h_2 > h_3 > h_1 > h_5 > h_4$
	$(\mapsto_Y, \hookrightarrow_Y)$	0.4937	0.6097	0.5916	0.5068	0.4564	$h_2 > h_3 > h_4 > h_1 > h_5$
Compromise	$(\mapsto_L, \hookrightarrow_L)$	0.3365	0.7389	0.6195	0.2797	0.2831	$h_2 > h_3 > h_1 > h_5 > h_4$
	$(\mapsto_{nM}, \hookrightarrow_{nM})$	0.4713	0.4803	0.4836	0.4873	0.4816	$h_4 > h_3 > h_5 > h_2 > h_1$
	$(\mapsto_P, \hookrightarrow_P)$	0.4005	0.7211	0.6376	0.4242	0.4236	$h_2 > h_3 > h_4 > h_5 > h_1$
	$(\mapsto_H, \hookrightarrow_H)$	0.0436	0.2635	0.1608	0.0254	0.0480	$h_2 > h_3 > h_5 > h_1 > h_4$
	$(\mapsto_Y, \hookrightarrow_Y)$	0.4681	0.6139	0.5721	0.4790	0.4553	$h_2 > h_3 > h_4 > h_1 > h_5$
Optimistic	$(\mapsto_L, \hookrightarrow_L)$	0.3373	0.7389	0.6198	0.2776	0.2846	$h_2 > h_3 > h_1 > h_5 > h_4$
	$(\mapsto_{nM}, \hookrightarrow_{nM})$	0.4937	0.4789	0.4945	0.4961	0.4903	$h_4 > h_3 > h_1 > h_5 > h_2$
	$(\mapsto_P, \hookrightarrow_P)$	0.3900	0.7267	0.6398	0.4218	0.4248	$h_2 > h_3 > h_5 > h_4 > h_1$
	$(\mapsto_H, \hookrightarrow_H)$	0.1106	0.4159	0.2643	0.1353	0.1382	$h_2 > h_3 > h_5 > h_4 > h_1$
	$(\mapsto_Y, \hookrightarrow_Y)$	0.4406	0.6177	0.5505	0.4486	0.4521	$h_2 > h_3 > h_5 > h_4 > h_1$

As demonstrated in Table 18, the orderings for distinct logical operators differ. For $(\mapsto_L,\hookrightarrow_L)$, $(\mapsto_H,\hookrightarrow_H)$, and $(\mapsto_Y,\hookrightarrow_Y)$, the best selections are always the same, that is h_2. The ranking varies due to different logical operators, but the optimal choice remains unchanged, which demonstrates the stability of our method. For $(\mapsto_{nM},\hookrightarrow_{nM})$ and for different strategies, the optimal choices are different. This result indicates that the ordering outcomes are influenced by FLOs and exhibit certain sensitivity.

5.2. Comparisons with other methods

In this section, we systematically analyze and compare nine decision-making methods proposed by previous researchers, covering both their theoretical and experimental aspects.

(1) In terms of theory

Decision-making methods come in different varieties. As shown in Table 19, we compared the basic theories and frameworks of 10 decision-making models (including our own). Our analysis leads to the following key conclusions.

- In terms of model selection, references [22–24] focus on interval-valued neutrosophic sets, references [21,25,26] conduct research on neutrosophic sets, reference [27] involves intuitionistic fuzzy sets, and references [28, 29] target interval-valued intuitionistic fuzzy sets. Compared with these models, the proposed model in this paper enhances fault tolerance and reduces the interference caused by noisy data through its variable precision characteristic.

- For multiple strategies: We make full use of the advantages of multi-granularity, as it offers three strategies for our method: optimistic, pessimistic, and compromise. This allows decision-makers to choose the most appropriate solution depending on their personal preferences (shaped by factors such as personality, cognition, and thinking styles). Furthermore, by incorporating the uncertainty characteristics of the neutrosophic set, we put forward the multi-granularity neutrosophic rough set. This approach strengthens the model's capability to handle multi-feature and multi-source information.

- For the type of decision-making: Compared with individual decision-making, group decision-making can effectively curb individual cognitive biases, avoid risks caused by subjective tendencies in individual decision-making, and more comprehensively balance multiple interests.

Table 19. The rankings for different methods.

Methods	Fault tolerance	Multi-strategy	Decision-making types
Our	Variable precision	Multi-granularity	Group
Nancy and Garg [25]	None	None	Individual
Selvachandran et al. [26]	None	None	Individual
Zhang et al. [21]	None	None	Group
Stanujkić et al. [22]	None	None	Group
Liu et al. [23]	None	None	Group
Liu et al. [24]	None	None	Group
Garg et al. [27]	None	None	Group
Zhou et al. [28]	None	None	Group
Tu et al. [29]	None	None	Individual

(2) In terms of experiment

There are significant differences in the ranking results of different methods, and the optimal choices also vary. As shown in Table 20, among the 12 evaluated methods, 10 methods (including the three strategies we proposed) consider h_2 to rank first, and 2 methods consider h_2 to rank second. One method considers h_3 to rank first, and 9 methods consider h_3 to rank second. Therefore, the probability that the medical diagnosis system h_2 ranks in the top two is 100%. Thus, the alternative h_2 is the optimal choice. The possibility that the medical diagnosis system h_3 enters the top two is 83.3%. Therefore, h_3 can be used as the sub-optimal choice for the medical emergency management system.

To further investigate the effectiveness of ranking the results among different methods, Spearman's correlation coefficient (SCC) is applied for analysis. The mathematical formula of SCC is given below:

$$SCC = 1 - \frac{6\sum_i^n r_i^2}{n(n^2-1)},$$

where n represents the number of objects and r_i is the rank difference of the same object between two sets of data. We computed the SCCs among 12 methods, and the results are presented in Figure 3.

In the Figure 3, a color that is closer to yellow indicates a stronger positive correlation, while a color that is closer to blue implies a stronger negative correlation. The majority of the methods fall within the yellow region, which suggests that a strong positive correlation exists among most of these methods, e.g., our method and the methods in references [21–29]. Therefore, the method we proposed is highly consistent with these methods. Furthermore, Liu et al.'s [23] method exhibits relatively low correlation coefficients with other methods, which suggests that methodological differences significantly influence the correlation coefficients. However, the optimal and worst choices remain consistent across all methods. Hence, it is credible and efficient.

Table 20. The rankings of different methods.

Methods	Rankings	Best selection
Ours (optimistic)	$h_2 > h_3 > h_5 > h_4 > h_1$	h_2
Ours (compromise)	$h_2 > h_3 > h_1 > h_4 > h_5$	h_2
Ours (pessimistic)	$h_2 > h_3 > h_1 > h_4 > h_5$	h_2
Nancy and Garg [25]	$h_2 > h_3 > h_1 > h_5 > h_4$	h_2
Selvachandran et al. [26]	$h_2 > h_3 > h_1 > h_4 > h_5$	h_2
Zhang et al. [21]	$h_2 > h_3 > h_1 > h_5 > h_4$	h_2
Stanujkić et al. [22]	$h_1 > h_2 > h_3 > h_5 > h_4$	h_1
Liu et al. [23]	$h_3 > h_2 > h_4 > h_5 > h_1$	h_3
Liu et al. [24]	$h_2 > h_5 > h_4 > h_3 > h_1$	h_2
Garg et al. [27]	$h_2 > h_3 > h_1 > h_5 > h_4$	h_2
Zhou et al. [28]	$h_2 > h_3 > h_1 > h_5 > h_4$	h_2
Tu et al. [29]	$h_2 > h_3 > h_1 > h_5 > h_4$	h_2

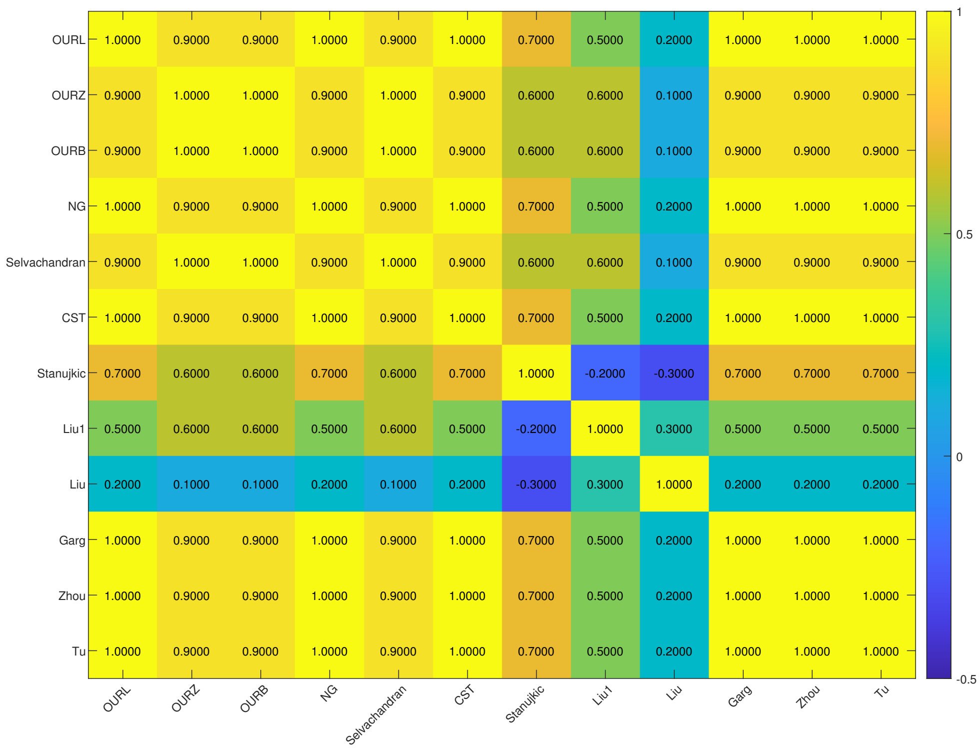

Figure 3. SCCs of different methods.

6. Conclusions

We proposed a novel MGVPNRS model and investigated its theoretical foundations and applications. The key summaries of this article are presented below.

(1) We establish a MGVPNRS based on the residual implication of t-norms and the residual co-implication of t-conorms, and investigate their fundamental properties, thereby enriching the theoretical framework of neutrosophic rough sets.

(2) By integrating variable precision and multi-granularity, we construct a MGVPNRS model. This model includes three submodels: optimistic, pessimistic, and compromise. It can fully take decision-makers' subjective preferences into account, features high fault tolerance, and can effectively reduce the probability of decision-making errors. Furthermore, it is applied to solve MAGDM problems.

(3) Finally, through a series of parametric analyses and contrastive experiments, we further validate the superiority of our model.

In the long run, our research will focus on the following areas.

(1) We will engage in in-depth research on various fuzzy logic operators to further broaden the model architecture of neutrosophic rough set theory.

(2) Neutrosophic set theory can be applied to other fields, such as the case of feature selection [30] and pattern recognition [31]. Meanwhile, we will study a novel aggregation operator, similarity measure, and distance measure, aiming to provide effective tools for uncertain information fusion in complex systems.

(3) Future research will focus on exploring efficient processing strategies suitable for large-scale data [32, 33] and extending the relevant research to other rough set models [34, 35].

Author contributions

Hongyuan Zheng: Conceptualization, investigation, methodology, resources, software, writing – original draft, writing – review and editing. **Chunxin Bo**: Conceptualization, methodology, writing – original draft, supervisor, writing – review and editing. **Lingqiang Li**: Investigation, resources. **Lu Wang**: Investigation, writing – review and editing. **Wenjie Jiang**: Writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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