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Research article

A flexible bivariate skewed odd-G family of distributions for various Hazard rate profiles in dispersed trends: Theory-driven interdisciplinary joint data analysis

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Abstract: The individual features, temporality, and analysis goals of the random variables determine the distinct challenges that statistical modeling with combined datasets frequently entails. When it comes to areas like stress testing and preventative maintenance, where knowing how reliable a system is and when it will fail is vital, these problems become much more complex. More flexible statistical tools are required in these situations because traditional methods may not work. To deal with the complexities of binary data, especially in cases of imbalance, overdispersion, and changing risk patterns, this study presented two novel bivariate probability models. The Marshall-Olkin shock framework provides the basis for one model, while the Farlie-Gumbel-Morgenstern technique is the basis for the other. Both are mathematically sound and have practical uses since they provide exact analytical formulas for important functions including quantiles, hazard rates, survival functions, and joint probabilities. We focused on two versions that show how these two model families evolved theoretically and how they might be applied in specific cases. Systems where success or failure is represented by binary outcomes across time are practical examples, as are instances where stress levels directly effect the probability of failure. Utilizing comprehensive simulation experiments as support, we employed maximum likelihood estimation to derive the parameters of the models. We then tested the models on three real-world binary datasets, demonstrating their robustness and adaptability in situations where more conventional approaches may fail to do justice to the complexities of the data.

Keywords: statistical model; heavy-tailed distributions; copula techniques; failure analysis; conditional densities; simulation; data analysis
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1. Introduction

Bivariate data modeling is a statistical method employed to examine and comprehend the relationship between two variables. In bivariate data analysis, two sets of data points, commonly designated as X and Y, are examined to ascertain the existence of a significant link or correlation between them. This form of analysis is essential across multiple disciplines, such as economics, social sciences, engineering, biology, and others. A bivariate probability distribution can be employed as a statistical method to analyze such data. A bivariate probability distribution is a probability distribution that addresses two random variables concurrently. It delineates the joint behavior of these two variables and presents the likelihood of various outcomes occurring concurrently for both variables. These distributions are essential in numerous statistical investigations, encompassing regression, hypothesis testing, and decision-making in diverse practical applications. Two often observed bivariate probability distributions are the joint probability mass function (JPMF) and the joint probability density function (JPDF), contingent upon whether the variables are discrete or continuous. The formulation of discrete and continuous bivariate models fundamentally involves the compound "power series class", marginals, combinatorics, copulas, reductions, and conditioning. Recently, there has been a growing interest in the introduction of novel families of composite, weighted, and generalized (G-) bivariate distributions, for instance in Sunoj and Unnikrishnan [1], Sunoj and Sankaran [2], Kundu and Gupta [3], Dimitrakopoulou et al. [4], Bidram [5], Jamalizadeh and Kundu [6], Kundu and Gupta [7], Roozegar and Jafari [8], Arnold et al. [9], Eliwa and El-Morshedy [10,11], El-Morshedy and Eliwa [12], Shahen et al. [13], Adu-Ntim et al. [14], along with references therein.

Given the importance of modeling bivariate data and the complexity of the data that is generated day in and day out, two generators of probabilities are derived based on two different statistical techniques: Marshall-Olkin shock model and Farlie-Gumbel-Morgenstern copulas. The Marshall-Olkin (MO) shock model is a statistical method used to generate bivariate probability distributions, especially to model the dependence between two random variables. This model is frequently used in the insurance industry to analyze and simulate the joint distribution of two insurance claim amounts, but it can also find applications in other fields where dependencies between two variables need to be determined. Here is a high-level overview of the MO shock model and how it can be used to construct bivariate probability distributions:

- Identify the marginal distributions: Start by determining the marginal distributions of the two random variables you want to model. These marginal distributions describe the behavior of each variable independently.
- Select a copula function: The basic concept in the MO shock model is to use the copula function to model the dependency structure between two variables. A copula is a mathematical function that describes the dependence between random variables, while allowing you to model their marginal distributions separately. Common copula families include Gaussian copula, Clayton copula, Gumbel copula, and more. The choice of copula depends on the nature of the dependence you want to capture (for example, linear and tail dependence).
- Define the joint distribution: Using the given copula function, you combine the two marginal distributions to create the joint distribution of the two variables. This involves converting marginal distributions to uniform marginal distributions using the inverse of cumulative distribution functions (CDFs) and then applying the copula function.

The selection of the copula and its characteristics can substantially influence the resultant bivariate distribution. Meticulous selection and calibration of the copula are essential for precisely modeling the relationship between the two variables. The MO shock model is versatile and may be modified to accommodate several forms of dependencies, including both positive and negative tail dependency, as well as symmetric and asymmetric reliance. For additional information regarding the MO shock model methodology, see Marshall and Olkin [15], Babu [16], Sun et al. [17], and Gobbi et al. [18].

Conversely, the Farlie-Gumbel-Morgenstern (FGM) copula is a particular copula function employed in multivariate statistics and probability theory to characterize the dependence structure among random variables. This copula was independently introduced by Farlie, Gumbel, and Morgenstern, therefore its designation. The FGM copula is renowned for its adaptability in modeling diverse dependencies, encompassing both positive and negative tail dependencies, hence proving advantageous for multivariate data with varying connection patterns. The FGM copula is an effective instrument for modeling intricate connections among random variables, especially in the context of severe occurrences and tail behavior. For additional information regarding the FGM methodology, refer to Farlie [19] and Fischer and Klein [20]. The MO and FGM copulas yield two families of bivariate distributions derived from the odd Dagum-G (ODa-G) class, namely the bivariate MO odd Dagum-G (BMOODa-G) and the bivariate FGM odd Dagum-G (BFGMODa-G). The cumulative distribution function (CDF) and probability density function (PDF) of the ODa-G family are correspondingly expressed as

$$F_{\text{ODa-G}}(x;\alpha,\theta,a,\mathbf{\Phi}) = \left(1+\theta\left(\frac{\overline{G}(x;\mathbf{\Phi})}{G(x;\mathbf{\Phi})}\right)^{a}\right)^{-\alpha};\ x,\alpha,\theta,a,\mathbf{\Phi}>0, \tag{1.1}$$

and

$$f_{\text{ODa-G}}(x;\alpha,\theta,a,\mathbf{\Phi}) = \frac{a\alpha\theta g(x;\mathbf{\Phi})\,(G(x,\mathbf{\Phi}))^{a\alpha-1}\left(\overline{G}(x;\mathbf{\Phi})\right)^{a-1}}{\left[G(x,\mathbf{\Phi})^{a}+\theta\overline{G}(x;\mathbf{\Phi})^{a}\right]^{\alpha+1}};\ x,\alpha,\theta,a,\mathbf{\Phi}>0, \tag{1.2}$$

where $G(x,\mathbf{\Phi})$ denotes the baseline cumulative distribution function contingent on the parameter vector Φ, while $\overline{G}(x,\mathbf{\Phi}) = 1 - G(x,\mathbf{\Phi})$ signifies the survival function, and $g(x;\mathbf{\Phi})$ represents the baseline probability density function (see Alghamdi et al. [21]). The Dagum distribution, or the three-parameter Burr type III distribution, is a probability distribution frequently employed in statistics to explain favorably skewed data. This is a continuous probability distribution frequently utilized to characterize income distribution, survival durations, and other occurrences when the data demonstrates a tail-heavy or right-skewed configuration. The Dagum distribution generalizes multiple probability distributions, such as the Weibull, gamma, and Burr distributions. It may converge to these established distributions, contingent upon parameter values. For additional elaborations on Dagum's class, one may see Afify and Alizadeh [22] and Biazatti et al. [23].

It is necessary to investigate and evaluate this model about bivariate random variables using diverse mathematical and statistical techniques due to the distinctive characteristics of this probability distribution and its adaptability in data modeling. Consequently, we introduce and analyze the BMOODa-G and BFGMODa-G families, leading to the subsequent conclusions: The joint cumulative distribution function (JCDF) of the BMOODa-G can be expressed as a mixture of a continuous function and a singular function. It may be possible to come up with clear formulas for the joint probability density function, joint cumulative distribution function, and joint survival function of the two generators. The joint hazard rate function (JHRF) exhibits various shapes contingent upon the parameters of the BMOODa-G and BFGMODa-G. You can use the marginal distributions of these classes to look at distinct patterns of failure rates. The BMOODa-G class is good for modeling situations when maintenance or stress is needed. The two generators have a number of different bivariate distributions that rely on the function that is underneath them. These two groups are good for making asymmetric data. They are good at looking at extreme data, especially the BFGMODa-G family. This model fits better than other distributions that come from the same underlying function.

There are important limits on how the BMOODa-G and BFGMODa-G families can be used. The BMOODa-G family can only describe positive random variables, and it is less flexible when used with margins that are not exponential. Additionally, its capacity to encapsulate dependent structures may be overly simplified in systems affected by several, intricate elements. The BFGMODa-G family is also limited to positive random variables and cannot accurately describe tail dependence or significant asymmetries. This makes it less useful for financial and actuarial datasets with high co-movements. In general, the BMOODa-G family is a good way to simulate shock-based risks and some types of tail dependence. However, it is mostly effective in reliability and insurance situations. The BFGMODa-G family, on the other hand, is easy to work with analytically and works well for mild reliance. However, it cannot handle strong or extreme dependence structures.

The efficacy of both the BMOODa-G and BFGMODa-G families is significantly influenced by the characteristics of the data. The BMOODa-G family operates efficiently with failure-time data, lifetime distributions, and extreme-event risks; yet, its dependence on exponential-type assumptions renders it susceptible to deviations and diminishes its adaptability to intricate data structures. In comparison, the BFGMODa-G family is comparatively simple to fit and comprehend for datasets exhibiting mild reliance; nonetheless, it often underestimates risk in scenarios characterized by stronger or tail dependence.

The structure of the paper is organized as follows: In Section 2, we define the BMOOGE-G family and its corresponding marginal distributions, while also revealing some mathematical and statistical features inside this section. Section 3 examines the BFGMODa-G class and clarifies its attributes. In Section 4, we perform a simulation study that includes both categories. Section 5 demonstrates the practical use and analysis of the BMOODa-G and BFGMODa-G families on a real dataset. Finally, in Section 6, we provide closing observations and delineate prospective directions for further research.

2. The BMOODa-G family: Structure and characteristic functions

Let $U_1 \sim$ ODa-G$(\alpha_1, \theta, a, \mathbf{\Phi})$, $U_2 \sim$ ODa-G$(\alpha_2, \theta, a, \mathbf{\Phi})$, and $U_3 \sim$ ODa-G$(\alpha_3, \theta, a, \mathbf{\Phi})$ be independent random variables. Consider X_1 to be defined as the maximum of U_1 and U_3, and assume X_2 as the maximum of U_2 and U_3. The bivariate vector $\mathbf{X} = (X_1, X_2)$ follows the distribution BMOODa-G$(\alpha_1, \alpha_2, \alpha_3, \theta, a, \mathbf{\Phi})$. The JCDF of $\mathbf{X}$ for $x_1, x_2 > 0$ and for $z = \min\{x_1, x_2\}$ can be articulated as

$$\begin{aligned} F_{\text{BMOODa-G}}(x_1, x_2) &= \left(1 + \theta\left(\frac{\overline{G}(x_1; \mathbf{\Phi})}{G(x_1; \mathbf{\Phi})}\right)^{a}\right)^{-\alpha_1} \left(1 + \theta\left(\frac{\overline{G}(x_2; \mathbf{\Phi})}{G(x_2; \mathbf{\Phi})}\right)^{a}\right)^{-\alpha_2} \left(1 + \theta\left(\frac{\overline{G}(z; \mathbf{\Phi})}{G(z; \mathbf{\Phi})}\right)^{a}\right)^{-\alpha_3} \\ &= F_{\text{ODa-G}}(x_1; \alpha_1, \theta, a, \mathbf{\Phi})\, F_{\text{ODa-G}}(x_2; \alpha_2, \theta, a, \mathbf{\Phi})\, F_{\text{ODa-G}}(z; \alpha_3, \theta, a, \mathbf{\Phi}) \\ &= \begin{cases} F_{\text{ODa-G}}(x_1; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi})\, F_{\text{ODa-G}}(x_2; \alpha_2, \theta, a, \mathbf{\Phi}) & \text{if } x_1 < x_2, \\ F_{\text{ODa-G}}(x_1; \alpha_1, \theta, a, \mathbf{\Phi})\, F_{\text{ODa-G}}(x_2; \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}) & \text{if } x_2 < x_1, \\ F_{\text{ODa-G}}(x; \alpha_1 + \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}) & \text{if } x_1 = x_2 = x. \end{cases} \end{aligned} \tag{2.1}$$

The corresponding JPDF of the BMOODa-G family can be formulated as

$$f_{\text{BMOODa-G}}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } x_1 < x_2, \\ f_2(x_1, x_2) & \text{if } x_2 < x_1, \\ f_3(x) & \text{if } x_1 = x_2 = x, \end{cases} \tag{2.2}$$

where

$$f_1(x_1, x_2) = f_{\text{ODa-G}}(x_1; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi}) f_{\text{ODa-G}}(x_2; \alpha_2, \theta, a, \mathbf{\Phi}),$$
$$f_2(x_1, x_2) = f_{\text{ODa-G}}(x_1; \alpha_1, \theta, a, \mathbf{\Phi}) f_{\text{ODa-G}}(x_2; \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}),$$

and

$$f_3(x) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{\text{ODa-G}}(x; \alpha_1 + \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}).$$

The expressions $f_i(x_1, x_2)$; $i = 1, 2$ can be easily obtained by differentiating Eq (2.1) with respect to x_i; $i = 1, 2$. Let us first assume that $x_1 < x_2$. Then,

$$F_{\text{BMOODa-G}}(x_1, x_2) = F_{\text{ODa-G}}(x_1; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi}) F_{\text{ODa-G}}(x_2; \alpha_2, \theta, a, \mathbf{\Phi}).$$

Hence, differentiating this function with respect to x_1 and x_2, the expression of $f_1(x_1, x_2)$ can be derived as shown in Eq (2.2). Similarly, $f_2(x_1, x_2)$ is given when $x_2 < x_1$. However, $f_3(x)$ cannot be derived in a similar way. For this reason, the following identity is employed to derive $f_3(x)$, where

$$\int_0^\infty f_3(x)\, dx + \int_0^\infty \int_0^{x_2} f_1(x_1, x_2)\, dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1, x_2)\, dx_2 dx_1 = 1.$$

Let,

$$I_1 = \int_0^\infty \int_0^{x_2} f_1(x_1, x_2)\, dx_1 dx_2 \text{ and } I_2 = \int_0^\infty \int_0^{x_1} f_2(x_1, x_2)\, dx_2 dx_1,$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^{x_2} f_{\text{ODa-G}}(x_1; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi}) f_{\text{ODa-G}}(x_2; \alpha_2, \theta, a, \mathbf{\Phi})\, dx_1 dx_2 \\ &= \int_0^\infty \frac{a\alpha_2 \theta g(x_2; \mathbf{\Phi}) (G(x_2, \mathbf{\Phi}))^{a(\alpha_1+\alpha_2+\alpha_3)-1} \left(\overline{G}(x_2; \mathbf{\Phi})\right)^{a-1}}{\left[G(x_2, \mathbf{\Phi})^a + \theta \overline{G}(x_2; \mathbf{\Phi})^a\right]^{\alpha_1+\alpha_2+\alpha_3+1}} dx_2. \end{aligned} \tag{2.3}$$

Similarly,

$$I_2 = \int_0^\infty \frac{a\alpha_1 \theta g(x_1; \mathbf{\Phi}) (G(x_1, \mathbf{\Phi}))^{a(\alpha_1+\alpha_2+\alpha_3)-1} \left(\overline{G}(x_1; \mathbf{\Phi})\right)^{a-1}}{\left[G(x_1, \mathbf{\Phi})^a + \theta \overline{G}(x_1; \mathbf{\Phi})^a\right]^{\alpha_1+\alpha_2+\alpha_3+1}} dx_1. \tag{2.4}$$

From Eqs (2.3) and (2.4), we get

$$\int_0^\infty f_3(x)\, dx = \int_0^\infty \frac{a(\alpha_1 + \alpha_2 + \alpha_3) \theta g(x; \mathbf{\Phi}) (G(x, \mathbf{\Phi}))^{a(\alpha_1+\alpha_2+\alpha_3)-1} \left(\overline{G}(x; \mathbf{\Phi})\right)^{a-1}}{\left[G(x, \mathbf{\Phi})^a + \theta \overline{G}(x; \mathbf{\Phi})^a\right]^{\alpha_1+\alpha_2+\alpha_3+1}} dx_1$$

$$-\int_0^{\infty}\frac{a\alpha_2\theta g(x;\mathbf{\Phi})(G(x,\mathbf{\Phi}))^{a(\alpha_1+\alpha_2+\alpha_3)-1}\left(\overline{G}(x;\mathbf{\Phi})\right)^{a-1}}{\left[G(x,\mathbf{\Phi})^a+\theta\overline{G}(x;\mathbf{\Phi})^a\right]^{\alpha_1+\alpha_2+\alpha_3+1}}dx_2$$

$$-\int_0^{\infty}\frac{a\alpha_1\theta g(x;\mathbf{\Phi})(G(x,\mathbf{\Phi}))^{a(\alpha_1+\alpha_2+\alpha_3)-1}\left(\overline{G}(x;\mathbf{\Phi})\right)^{a-1}}{\left[G(x,\mathbf{\Phi})^a+\theta\overline{G}(x;\mathbf{\Phi})^a\right]^{\alpha_1+\alpha_2+\alpha_3+1}}dx_1$$

$$=\frac{a\alpha_3\theta g(x;\mathbf{\Phi})(G(x,\mathbf{\Phi}))^{a(\alpha_1+\alpha_2+\alpha_3)-1}\left(\overline{G}(x;\mathbf{\Phi})\right)^{a-1}}{\left[G(x,\mathbf{\Phi})^a+\theta\overline{G}(x;\mathbf{\Phi})^a\right]^{\alpha_1+\alpha_2+\alpha_3+1}}.$$

Then,

$$f_3(x)=\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}f_{\text{ODa-G}}(x;\alpha_1+\alpha_2+\alpha_3,\theta,a,\mathbf{\Phi}).$$

Further, the marginal CDFs for the BMOODa-G family can be represented as follows

$$F_{X_i}(x_i)=P[\max(U_i,U_3)\leq x_i]=F_{\text{ODa-G}}(x_i;\alpha_i+\alpha_3,\theta,a,\mathbf{\Phi});\ i=1,\ 2. \tag{2.5}$$

The corresponding marginal PDFs can be expressed as follows

$$f_{X_i}(x_i)=f_{\text{ODa-G}}(x_i;\alpha_i+\alpha_3,\theta,a,\mathbf{\Phi});\ x_i>0. \tag{2.6}$$

The CDF and PDF marginals can be represented as a linear representation as follows

$$F_{X_i}(x_i)=\sum_{k,m\geq 0}v_{k,m}^{(i)}Q_{-m}^{(i)}(x_i;\mathbf{\Phi});\ i=1,2, \tag{2.7}$$

and

$$f_{X_i}(x_i)=\sum_{k,m\geq 0}v_{k,m}^{(i)}q_{-m}^{(i)}(x_i;\mathbf{\Phi});\ i=1,2, \tag{2.8}$$

respectively, where $Q_{-m}^{(i)}(x_i;\mathbf{\Phi}) = G^{-m}(x_i;\mathbf{\Phi})$ represents the CDF of the exponentiated-G (exp-G) family of distributions, with a power parameter $(-m)$,

$$v_{k,m}^{(i)}=(-1)^{ak-m}\binom{-\alpha_i-\alpha_3}{k}\binom{ak}{m}\theta^k,$$

and

$$q_{-m}^{(i)}(x_i;\mathbf{\Phi})=-mg(x_i;\mathbf{\Phi})G^{-m-1}(x_i;\mathbf{\Phi}),$$

represents the PDF of the exp-G family of distributions with a power parameter of $(-m)$. The joint survival function (JSF) of the BMOODa-G family can be proposed as

$$S_{\text{BMOODa-G}}(x_1,x_2)=\begin{cases}S_1(x_1,x_2) & \text{if } x_1<x_2,\\ S_2(x_1,x_2) & \text{if } x_2<x_1,\\ S_3(x,x) & \text{if } x_1=x_2=x,\end{cases} \tag{2.9}$$

where

$$S_1(x_1,x_2)=1-\left(1+\theta\left(\frac{\overline{G}(x_2;\mathbf{\Phi})}{G(x_2;\mathbf{\Phi})}\right)^a\right)^{-(\alpha_2+\alpha_3)}-\left(1+\theta\left(\frac{\overline{G}(x_1;\mathbf{\Phi})}{G(x_1;\mathbf{\Phi})}\right)^a\right)^{-(\alpha_1+\alpha_3)}$$

$$\times\left[1-\left(1+\theta\left(\frac{\overline{G}(x_2;\mathbf{\Phi})}{G(x_2;\mathbf{\Phi})}\right)^{a}\right)^{-\alpha_2}\right],$$

$$S_2(x_1,x_2)=1-\left(1+\theta\left(\frac{\overline{G}(x_1;\mathbf{\Phi})}{G(x_1;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_1+\alpha_3)}-\left(1+\theta\left(\frac{\overline{G}(x_2;\mathbf{\Phi})}{G(x_2;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_2+\alpha_3)}$$
$$\times\left[1-\left(1+\theta\left(\frac{\overline{G}(x_1;\mathbf{\Phi})}{G(x_1;\mathbf{\Phi})}\right)^{a}\right)^{-\alpha_1}\right],$$

and

$$S_3(x)=1-\left(1+\theta\left(\frac{\overline{G}(x;\mathbf{\Phi})}{G(x;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_1+\alpha_3)}-\left(1+\theta\left(\frac{\overline{G}(x;\mathbf{\Phi})}{G(x;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_2+\alpha_3)}$$
$$+\left(1+\theta\left(\frac{\overline{G}(x;\mathbf{\Phi})}{G(x;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_1+\alpha_2+\alpha_3)}.$$

If the random vector X has the BMOODa-G family, then the joint hazard rate function (JHRF) can be expressed by using $h(x_1,x_2)=f(x_1,x_2)/S(x_1,x_2)$, whereas the joint reversed hazard rate function (JRHRF) can be proposed as $v(x_1,x_2)=f(x_1,x_2)/F(x_1,x_2)$. The distribution of $W=\max\{X_1,X_2\}$ is given by

$$F_W(w)=P(\max\{X_1,X_2\}\le w)=\left(1+\theta\left(\frac{\overline{G}(w;\mathbf{\Phi})}{G(w;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_1+\alpha_2+\alpha_3)}. \tag{2.10}$$

This distribution of the maximum of two random variables is often encountered in reliability analysis, where W represents the lifetime of a system, and X_1 and X_2 represent the lifetimes of its individual components. The distribution of $Y=\min\{X_1,X_2\}$ can be proposed as

$$\begin{aligned}F_Y(y)&=P(\min\{X_1,X_2\}\le y)\\&=\left(1+\theta\left(\frac{\overline{G}(y;\mathbf{\Phi})}{G(y;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_1+\alpha_3)}+\left(1+\theta\left(\frac{\overline{G}(y;\mathbf{\Phi})}{G(y;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_2+\alpha_3)}\\&\quad-\left(1+\theta\left(\frac{\overline{G}(y;\mathbf{\Phi})}{G(y;\mathbf{\Phi})}\right)^{a}\right)^{-(\alpha_1+\alpha_2+\alpha_3)}.\end{aligned} \tag{2.11}$$

This distribution of the minimum of two random variables is often encountered in reliability analysis and order statistics, where Y represents the smallest value among a set of random observations. The conditional PDF of X_i given $X_j=x_j$, $(i,j=1,2;\ i\neq j)$ can be presented as

$$f_{X_i|X_j=x_j}\left(x_i|x_j\right)=\begin{cases}f_1\left(x_i|x_j\right) & \text{if } x_i<x_j,\\ f_2\left(x_i|x_j\right) & \text{if } x_j<x_i,\\ f_3\left(x_i|x_j\right) & \text{if } x_i=x_j=x,\end{cases}$$

where

$$f_1\left(x_i|x_j\right)=\frac{f_1\left(x_i,x_j\right)}{f_{X_j}\left(x_j\right)}=\frac{f_{\text{ODa-G}}(x_i;\alpha_i+\alpha_3,\theta,a,\mathbf{\Phi})\,f_{\text{ODa-G}}\left(x_j;\alpha_j,\theta,a,\mathbf{\Phi}\right)}{f_{\text{ODa-G}}\left(x_j;\alpha_j+\alpha_3,\theta,a,\mathbf{\Phi}\right)},$$

$$f_2\left(x_i|x_j\right) = \frac{f_2\left(x_i, x_j\right)}{f_{X_j}\left(x_j\right)} = f_{\text{ODa-G}}\left(x_i; \alpha_i, \theta, a, \mathbf{\Phi}\right),$$

and

$$f_3\left(x_i|x_j\right) = \frac{f_3\left(x_i, x_j\right)}{f_{X_j}\left(x_j\right)} = \frac{\alpha_3}{\alpha_i + \alpha_3} F_{\text{ODa-G}}\left(x; \alpha_i, \theta, a, \mathbf{\Phi}\right).$$

Conditional PDF is a fundamental concept in probability and statistics, and is used in various statistical analyzes and modeling tasks to understand how a random variable depends on or is affected by another random variable.

2.1. Some statistical characteristics of the BMOODa-G family

2.1.1. Median correlation

Domma [24] derived the median correlation coefficient for the bivariate distributions as a form

$$D_{X_1,X_2} = 4F_{X_1,X_2}\left(D_{X_1}, D_{X_2}\right) - 1, \tag{2.12}$$

where D_{X_1} and D_{X_2} denote the median of X_1 and X_2, respectively. If $X_1 \sim$ ODa-G$(\alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi})$ and $X_2 \sim$ ODa-G$(\alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi})$, then

$$D_{X_i} = G^{-1}\left[\left(\left(\frac{1}{\theta}\left\{U^{-\frac{1}{\alpha_i+\alpha_3}} - 1\right\}\right)^{\frac{1}{a}} + 1\right)^{-1}\right];\ i = 1, 2, \tag{2.13}$$

where U has a uniform $U(0, 1)$ distribution and $G^{-1}(.)$ is the baseline quantile function. Therefore, the coefficient of median correlation between X_1 and X_2 at $U = 0.5$ is given by

$$D_{X_1,X_2} = \begin{cases} 4F_{\text{ODa-G}}\left(D_{X_1}; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi}\right) F_{\text{ODa-G}}\left(D_{X_2}; \alpha_2, \theta, a, \mathbf{\Phi}\right) - 1 & \text{if } x_1 < x_2, \\ 4F_{\text{ODa-G}}\left(D_{X_1}; \alpha_1, \theta, a, \mathbf{\Phi}\right) F_{\text{ODa-G}}\left(D_{X_2}; \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}\right) - 1 & \text{if } x_2 < x_1, \\ 4F_{\text{ODa-G}}\left(D_{X_1}; \alpha_1 + \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}\right) - 1 & \text{if } x_1 = x_2 = x. \end{cases} \tag{2.14}$$

The median correlation coefficient has the advantage of being less sensitive to outliers than some other measures of correlation, like the mean correlation coefficient. It is a robust summary statistic that can help you understand the typical relationship between two variables in your dataset.

2.1.2. Product moments and associated descriptive statistical metrics

Moments play a crucial role in probability theory, as they help characterize probability distributions and provide insights into their properties. For example, moments are used to derive measures like skewness and kurtosis, which describe the shape of a distribution, and they are used in various statistical techniques, including moment-based estimation and hypothesis testing. The rth moment of X_i, say $\mu_i^{(r)}$, can be defined as

$$\mu_i^{(r)} = \mathbf{E}\left(X_i^r\right) = \int_0^\infty x_i^r f_{X_i}\left(x_i\right) dx_i. \tag{2.15}$$

Hence, by using Eq (2.8), we get

$$\mu_i^{(r)} = \sum_{k,m=0}^{\infty} v_{k,m}^{(i)} \int_0^{\infty} x_i^r q_{-m}^{(i)} (x_i; \mathbf{\Phi}_i)\, dx_i = \sum_{k,m=0}^{\infty} v_{k,m}^{(i)} \mathbf{E}\left(W_{i,-m}^r\right), \tag{2.16}$$

where $W_{i,-m}^r; i = 1, 2$ is a random variable having the exp-G CDF with power parameter $(-m)$. Setting $r = 1$ in Eq (2.16), we get the mean of $X_i; i = 1, 2$. Thus, the nth central moment of X_i, say $L_i^{(n)}$, is given by

$$L_i^{(n)} = \sum_{r=0}^{n} \sum_{k,m=0}^{\infty} \left(-\mu_i^{(1)}\right)^{n-r} \binom{n}{r} v_{k,m}^{(i)} \mathbf{E}\left(W_{i,-m}^r\right); i = 1, 2. \tag{2.17}$$

The product moment of the random variable $X_1^r X_2^r$ can be expressed as

$$\begin{aligned}
\mathbf{E}\left(X_1^r X_2^r\right) = & \int_0^{\infty}\int_0^{x_2} x_1^r x_2^r f_{\text{ODa-G}}\left(x_1; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi}\right) f_{\text{ODa-G}}\left(x_2; \alpha_2, \theta, a, \mathbf{\Phi}\right) dx_1 dx_2 \\
& + \int_0^{\infty}\int_0^{x_1} x_1^r x_2^r f_{\text{ODa-G}}\left(x_1; \alpha_1, \theta, a, \mathbf{\Phi}\right) f_{\text{ODa-G}}\left(x_2; \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}\right) dx_2 dx_1 \\
& + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \int_0^{\infty} x^{2r} f_{\text{ODa-G}}\left(x; \alpha_1 + \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi}\right) dx, \\
& = \sum_{k,m\geq 0} v_{k,m}^{(1)} v_{k,m}^{(2)} \int_0^{\infty}\int_0^{x_2} x_1^r x_2^r q_{-m}^{(1)}\left(x_1; \mathbf{\Phi}\right) q_{-m}^{(2)}\left(x_2; \mathbf{\Phi}\right) dx_1 dx_2 \\
& + \sum_{k,m\geq 0} v_{k,m}^{(1)} v_{k,m}^{(2)} \int_0^{\infty}\int_0^{x_1} x_1^r x_2^r q_{-m}^{(1)}\left(x_1; \mathbf{\Phi}\right) q_{-m}^{(2)}\left(x_2; \mathbf{\Phi}\right) dx_2 dx_1 \\
& + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \sum_{k,m\geq 0} v_{k,m} \int_0^{\infty} x^{2r} q_{-m}\left(x; \mathbf{\Phi}\right) dx.
\end{aligned} \tag{2.18}$$

Using Eqs (2.16) and (2.18) when $r = 1$, we get the covariance of the bivariate distribution as follows

$$\mathbf{cov}\left(X_1, X_2\right) = \mathbf{E}\left(X_1 X_2\right) - \left(\sum_{k,m=0}^{\infty} u_{k,m}^{(1)} \mathbf{E}\left(W_{1,-m}^1\right)\right)\left(\sum_{k,m=0}^{\infty} u_{k,m}^{(2)} \mathbf{E}\left(W_{2,-m}^2\right)\right). \tag{2.19}$$

where $cov\left(X_1, X_2\right) = E\left(X_1 X_2\right) - E\left(X_1\right) E\left(X_2\right)$. Moreover, the correlation of X_1 and X_2 is defined by $\rho = \frac{\mathbf{cov}(X_1,X_2)}{\sqrt{Var(X_1)Var(X_2)}}$, where $-1 \leq \rho \leq 1$ and $Var\left(X_i\right) = \mu_i^{(2)} - \left(\mu_i^{(1)}\right)^2; i = 1, 2$. The bivariate skewness and kurtosis can be formulated as

$$\begin{aligned}
\text{Skewness} = & \frac{1}{\left(1 - \rho^2\right)^3} \left[Z_{30}^2 + Z_{03}^2 + 3\left(1 + 2\rho^2\right)\left(Z_{12}^2 + Z_{21}^2\right) - 2\rho^3 Z_{30} Z_{03}\right. \\
& \left. + 6\rho\left\{Z_{30}\left(\rho Z_{12} - Z_{21}\right) + Z_{03}\left(\rho Z_{21} - Z_{12}\right) - \left(2 + \rho^2\right) Z_{21} Z_{12}\right\}\right]
\end{aligned} \tag{2.20}$$

and

$$\text{kurtosis} = \frac{Z_{40} + Z_{04} + 2Z_{22} + 4\rho\left(\rho Z_{22} - Z_{13} - Z_{31}\right)}{\left(1-\rho^2\right)^2}, \tag{2.21}$$

where $Z_{sl} = E\left(X_1^s X_2^l\right) / \left[\sqrt{Var\left(X_1\right)}\right]^s \left[\sqrt{Var\left(X_2\right)}\right]^l$. Bivariate skewness and kurtosis are extensions of the concepts of skewness and kurtosis from univariate statistics to the bivariate case. These measures provide information about the shape and asymmetry of the joint distribution of two random variables. Bivariate skewness quantifies the degree and direction of asymmetry in the joint distribution, while bivariate kurtosis measures the tailedness or thickness of the joint distribution's tails. They help analysts better understand the shape and behavior of multivariate data, which can be important for making informed decisions and modeling dependencies between variables.

2.2. The BMOODa-exponential distribution

The exponential (Ex) distribution is a fundamental probability distribution used to model events that occur randomly over time, with a constant rate of occurrence. It is characterized by its rate parameter (δ) and is particularly useful for understanding processes with memoryless properties. Suppose the CDF and PDF of the baseline distribution are given by $G(x;\delta) = 1 - e^{-\delta x}$ and $g(x;\delta) = \delta e^{-\delta x}$, for $\delta > 0$ and $x > 0$. Then, the JCDF and JSF of the BMOODa-exponential (BODa-Ex) distribution can be formulated as

$$F_{\text{BMOODa-Ex}}\left(x_1, x_2\right) = \left(1 + \theta\left(\frac{e^{-\delta x_1}}{1 - e^{-\delta x_1}}\right)^a\right)^{-\alpha_1} \left(1 + \theta\left(\frac{e^{-\delta x_2}}{1 - e^{-\delta x_2}}\right)^a\right)^{-\alpha_2} \left(1 + \theta\left(\frac{e^{-\delta z}}{1 - e^{-\delta z}}\right)^a\right)^{-\alpha_3}. \tag{2.22}$$

$$\begin{aligned} S_{\text{BMOODa-Ex}}\left(x_1, x_2\right) = 1 &- \left(1 + \theta\left(\frac{e^{-\delta x_1}}{1 - e^{-\delta x_1}}\right)^a\right)^{-(\alpha_1+\alpha_3)} - \left(1 + \theta\left(\frac{e^{-\delta x_2}}{1 - e^{-\delta x_2}}\right)^a\right)^{-(\alpha_2+\alpha_3)} \\ &+ F_{\text{BMOODa-Ex}}\left(x_1, x_2\right). \end{aligned} \tag{2.23}$$

The JPDF of the BMOODa-Ex distribution can be proposed as

$$f_{\text{BMOODa-Ex}}\left(x_1, x_2\right) = \begin{cases} f_1\left(x_1, x_2\right) & \text{if } x_1 < x_2, \\ f_2\left(x_1, x_2\right) & \text{if } x_2 < x_1, \\ f_3(x) \quad \text{if } x_1 = x_2 = x, \end{cases} \tag{2.24}$$

where

$$f_1\left(x_1, x_2\right) = \frac{a^2\alpha_2\left(\alpha_1+\alpha_3\right)\theta^2\delta^2 e^{-\delta x_1}\left(1-e^{-\delta x_1}\right)^{a(\alpha_1+\alpha_3)-1}\left(e^{-\delta x_1}\right)^{a-1}}{\left[\left(1-e^{-\delta x_1}\right)^a + \theta e^{-\delta a x_1}\right]^{\alpha_1+\alpha_3+1}} \frac{e^{-\delta x_2}\left(1-e^{-\delta x_2}\right)^{a\alpha_2-1}\left(e^{-\delta x_2}\right)^{a-1}}{\left[\left(1-e^{-\delta x_2}\right)^a + \theta e^{-\delta a x_2}\right]^{\alpha_2+1}},$$

$$f_2\left(x_1, x_2\right) = \frac{a^2\alpha_1\left(\alpha_2+\alpha_3\right)\theta^2\delta^2 e^{-\delta x_1}\left(1-e^{-\delta x_1}\right)^{a\alpha_1-1}\left(e^{-\delta x_1}\right)^{a-1}}{\left[\left(1-e^{-\delta x_1}\right)^a + \theta e^{-\delta a x_1}\right]^{\alpha_1+1}} \frac{e^{-\delta x_2}\left(1-e^{-\delta x_2}\right)^{a(\alpha_2+\alpha_3)-1}\left(e^{-\delta x_2}\right)^{a-1}}{\left[\left(1-e^{-\delta x_2}\right)^a + \theta e^{-\delta a x_2}\right]^{(\alpha_2+\alpha_3)+1}}$$

and

$$f_3(x, x) = \frac{a\theta\alpha_3\delta e^{-\delta x}\left(1-e^{-\delta x}\right)^{a(\alpha_1+\alpha_2+\alpha_3)-1}\left(e^{-\delta x}\right)^{a-1}}{\left[\left(1-e^{-\delta x}\right)^a + \theta e^{-\delta a x}\right]^{\alpha_1+\alpha_2+\alpha_3+1}}.$$

Figures 1–3 show different diagrams of the JPDF distribution of BMOODa-Ex under different specific schemes as follows: (a) $\alpha_1 = 0.1$, $\alpha_2 = 0.03$, $\alpha_3 = 0.09$, $\theta = 0.7$, $a = 6.5$, $\delta = 0.1$; (b) $\alpha_1 = 0.8$, $\alpha_2 = 0.3$, $\alpha_3 = 0.5$, $\theta = 0.2$, $a = 6.5$, $\delta = 0.09$; and (c) $\alpha_1 = 1.9$, $\alpha_2 = 0.03$, $\alpha_3 = 0.9$, $\theta = 0.7$, $a = 6.5$, $\delta = 0.5$.

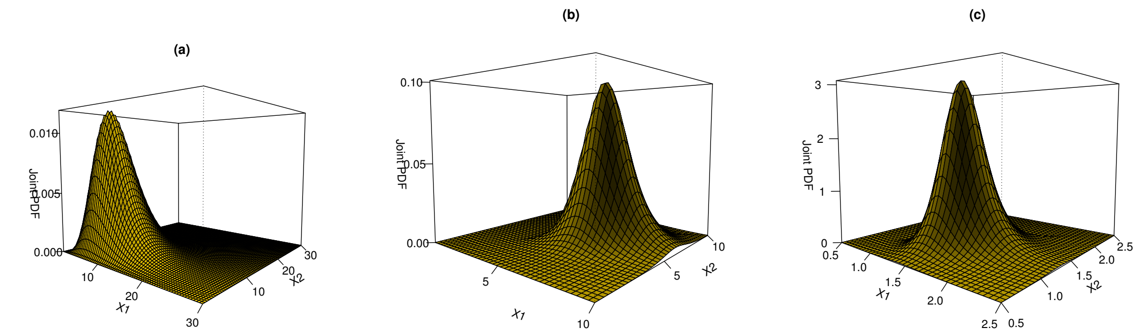

Figure 1. The JPDF of the BMOODa-Ex model.

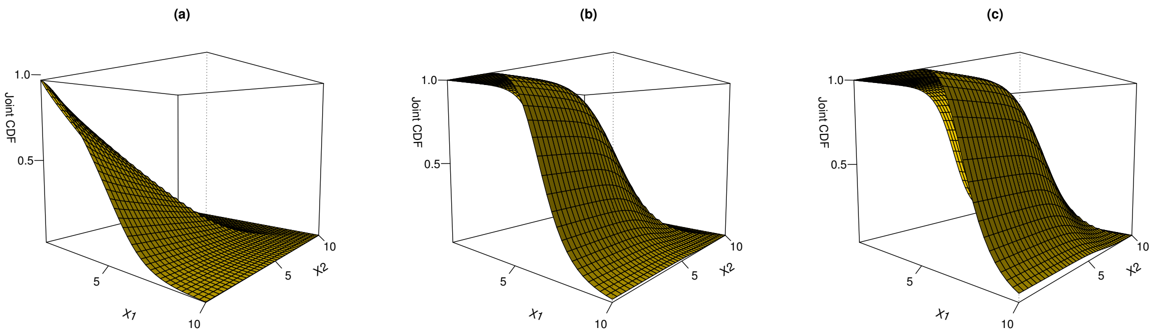

Figure 2. The JSF of the BMoODa-Ex model.

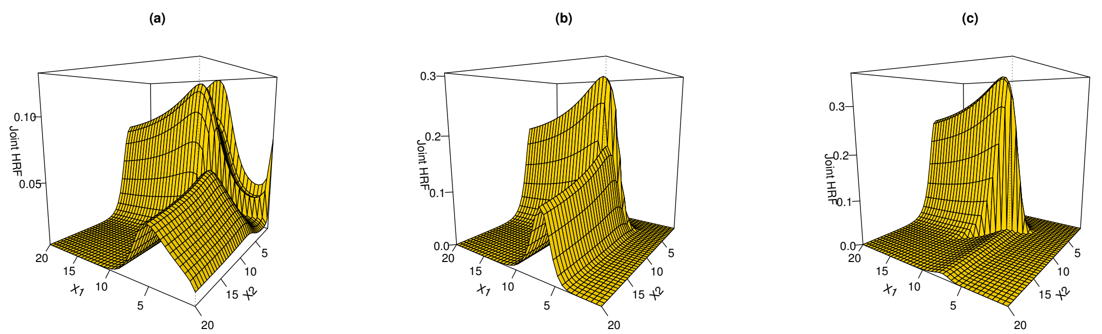

Figure 3. The JHRF of the BMOODa-Ex model.

Figures 1–3 collectively depict the primary structural characteristics of the proposed bivariate model. The JPDFs validate unimodality while emphasizing asymmetry and heavy-tailed characteristics, which account for the potential of extreme joint outcomes. The JHRFs exhibit both bathtub and inverted-bathtub forms, indicating non-monotonic risk dynamics that are more realistic than strictly growing or decreasing dangers. Furthermore, the JCDFs demonstrate sensitivity to parameter selections, indicating areas that exhibit tight increase under specific conditions and highlighting the influence of dependence structures on distributional behavior. The graphical results collectively confirm the model's adaptability in depicting intricate dependencies, asymmetric hazards, and varied danger forms. Figure 4(a) lists the plots of the conditional PDF of X_1 given that $x_2 = 5, 15, 25$, whereas the attitude of the conditional PDF of $X_{2,}$ given that $x_1 = 5, 12, 20$ can be reported in Figure 4(b) for $\alpha_1 = 9$, $\alpha_2 = 5$, $\alpha_3 = 9$, $\theta = 20.9$, $a = 5.5$, and $\delta = 0.05$.

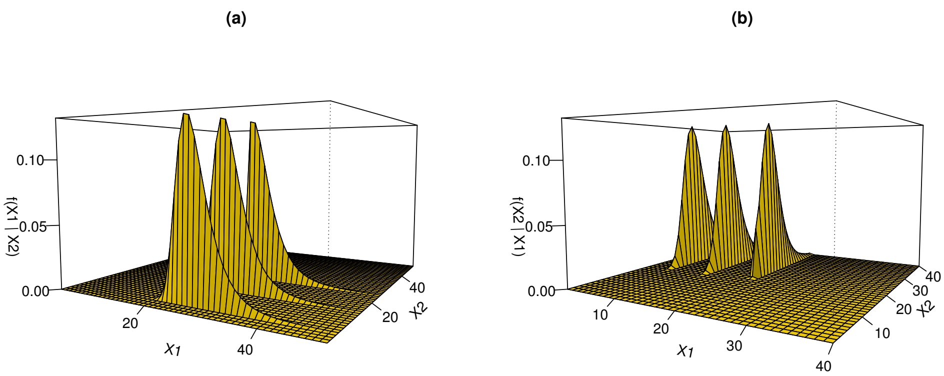

Figure 4. The conditional PDF of the BMOODa-Ex model.

2.3. Maximum likelihood estimation of BMOODa-G class parameters

Maximum likelihood estimation (MLE) is a statistical method used for estimating the parameters of a probability distribution or a statistical model. The fundamental idea behind MLE is to find the set of parameter values that maximizes the likelihood function, which measures the likelihood of observing the given data under a specific set of parameter values. The values of the parameters ϖ "as an example" that maximize the likelihood function are called the maximum likelihood estimators. These estimators provide the parameter values that are most likely to have generated the observed data. MLEs are often used in hypothesis testing and for constructing confidence intervals. For example, you can use the likelihood ratio test or the Wald test to assess the significance of estimated parameters. Further, MLE can be used for model selection by comparing the likelihood of different models and selecting the model with the highest likelihood. In this section, the maximum likelihood method is used to estimate the unknown parameters of the BMOODa-G family. Suppose $((x_{11}, x_{21}), ..., (x_{1n}, x_{2n}))$ is a random sample from BMOODa-G family. Consider the following notation: $I_1 = \{i;\ x_{1i} > x_{2i}\}$, $I_2 = \{i;\ x_{1i} < x_{2i}\}$, $I_3 = \{i;\ x_{1i} = x_{2i} = x_i\}$, $I = I_1 \cup I_2 \cup I_3$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_3| = n_3$, and $n_1 + n_2 + n_3 = n$.

The likelihood function is given as

$$l(\alpha_1, \alpha_2, \alpha_3, \theta, a, \mathbf{\Phi}) = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_i, x_i). \tag{2.25}$$

The log-likelihood function can be expressed as

$$\begin{aligned} L(\alpha_1, \alpha_2, \alpha_3, \theta, a, \mathbf{\Phi}) = & \sum_{i=1}^{n_1} \ln\left[f_{\text{ODa-G}}(x_{1i}; \alpha_1 + \alpha_3, \theta, a, \mathbf{\Phi})\right] + \sum_{i=1}^{n_1} \ln\left[f_{\text{ODa-G}}(x_{2i}; \alpha_2, \theta, a, \mathbf{\Phi})\right] \\ & + \sum_{i=1}^{n_2} \ln\left[f_{\text{ODa-G}}(x_{1i}; \alpha_1, \theta, a, \mathbf{\Phi})\right] + \sum_{i=1}^{n_2} \ln\left[f_{\text{ODa-G}}(x_{2i}; \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi})\right] \\ & + n_3 \ln\left(\frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\right) \sum_{i=1}^{n_3} \ln\left[f_{\text{ODa-G}}(x_i; \alpha_1 + \alpha_2 + \alpha_3, \theta, a, \mathbf{\Phi})\right]. \end{aligned} \tag{2.26}$$

Differentiating the log-likelihood with respect to α_1, α_2, α_3, θ, a, and Φ, respectively, and setting the results equal to zero, we have

$$\begin{aligned} \frac{\partial L}{\partial \alpha_1} = & \frac{n_1}{\alpha_1 + \alpha_3} + a \sum_{i=1}^{n_1} \ln(G(x_{1i}; \mathbf{\Phi})) - \sum_{i=1}^{n_1} \ln\left(G(x_{1i}; \mathbf{\Phi})^a + \theta \overline{G}(x_{1i}; \mathbf{\Phi})^a\right) \\ & + \frac{n_2}{\alpha_1} + a \sum_{i=1}^{n_2} \ln(G(x_{1i}; \mathbf{\Phi})) - \sum_{i=1}^{n_2} \ln\left(G(x_{1i}; \mathbf{\Phi})^a + \theta \overline{G}(x_{1i}; \mathbf{\Phi})^a\right) \\ & + a \sum_{i=1}^{n_3} \ln(G(x_i; \mathbf{\Phi})) - \sum_{i=1}^{n_3} \ln\left(G(x_i; \mathbf{\Phi})^a + \theta \overline{G}(x_i; \mathbf{\Phi})^a\right), \end{aligned} \tag{2.27}$$

$$\begin{aligned} \frac{\partial L}{\partial \alpha_2} = & \frac{n_1}{\alpha_2} + a \sum_{i=1}^{n_1} \ln(G(x_{2i}; \mathbf{\Phi})) - \sum_{i=1}^{n_1} \ln\left(G(x_{2i}; \mathbf{\Phi})^a + \theta \overline{G}(x_{2i}; \mathbf{\Phi})^a\right) \\ & + \frac{n_2}{\alpha_2 + \alpha_3} + a \sum_{i=1}^{n_2} \ln(G(x_{2i}; \mathbf{\Phi})) - \sum_{i=1}^{n_2} \ln\left(G(x_{2i}; \mathbf{\Phi})^a + \theta \overline{G}(x_{2i}; \mathbf{\Phi})^a\right) \\ & + a \sum_{i=1}^{n_3} \ln(G(x_i; \mathbf{\Phi})) - \sum_{i=1}^{n_3} \ln\left(G(x_i; \mathbf{\Phi})^a + \theta \overline{G}(x_i; \mathbf{\Phi})^a\right), \end{aligned} \tag{2.28}$$

$$\begin{aligned} \frac{\partial L}{\partial \alpha_3} = & \frac{n_1}{\alpha_1 + \alpha_3} + a \sum_{i=1}^{n_1} \ln(G(x_{1i}; \mathbf{\Phi})) - \sum_{i=1}^{n_1} \ln\left(G(x_{1i}; \mathbf{\Phi})^a + \theta \overline{G}(x_{1i}; \mathbf{\Phi})^a\right) \\ & + \frac{n_2}{\alpha_2 + \alpha_3} + a \sum_{i=1}^{n_2} \ln(G(x_{2i}; \mathbf{\Phi})) - \sum_{i=1}^{n_2} \ln\left(G(x_{2i}; \mathbf{\Phi})^a + \theta \overline{G}(x_{2i}; \mathbf{\Phi})^a\right) \\ & + \frac{n_3}{\alpha_3} + a \sum_{i=1}^{n_3} \ln(G(x_i; \mathbf{\Phi})) - \sum_{i=1}^{n_3} \ln\left(G(x_i; \mathbf{\Phi})^a + \theta \overline{G}(x_i; \mathbf{\Phi})^a\right), \end{aligned} \tag{2.29}$$

$$\frac{\partial L}{\partial \theta}=\frac{2n_1}{\theta}-(\alpha_1+\alpha_3+1)\sum_{i=1}^{n_1}\frac{\overline{G}(x_{1i};\mathbf{\Phi})^a}{G(x_{1i};\mathbf{\Phi})^a+\theta\overline{G}(x_{1i};\mathbf{\Phi})^a}-(\alpha_2+1)\sum_{i=1}^{n_1}\frac{\overline{G}(x_{2i};\mathbf{\Phi})^a}{G(x_{2i};\mathbf{\Phi})^a+\theta\overline{G}(x_{2i};\mathbf{\Phi})^a}$$
$$+\frac{2n_2}{\theta}-(\alpha_1+1)\sum_{i=1}^{n_2}\frac{\overline{G}(x_{1i};\mathbf{\Phi})^a}{G(x_{1i};\mathbf{\Phi})^a+\theta\overline{G}(x_{1i};\mathbf{\Phi})^a}-(\alpha_2+\alpha_3+1)\sum_{i=1}^{n_2}\frac{\overline{G}(x_{2i};\mathbf{\Phi})^a}{G(x_{2i};\mathbf{\Phi})^a+\theta\overline{G}(x_{2i};\mathbf{\Phi})^a}$$
$$+\frac{n_3}{\theta}-(\alpha_1+\alpha_2+\alpha_3+1)\sum_{i=1}^{n_3}\frac{\overline{G}(x_i;\mathbf{\Phi})^a}{G(x_i;\mathbf{\Phi})^a+\theta\overline{G}(x_i;\mathbf{\Phi})^a}, \tag{2.30}$$

$$\frac{\partial L}{\partial a}=\frac{2(n_1+n_2)+n_3}{a}+(\alpha_1+\alpha_3)\sum_{i=1}^{n_1}\ln\left(G(x_{1i};\mathbf{\Phi})\right)+\sum_{i=1}^{n_1}\ln\left(\overline{G}(x_{1i};\mathbf{\Phi})\right)$$
$$-(\alpha_1+\alpha_3+1)\sum_{i=1}^{n_1}\frac{\Psi(x_{1i};a,\theta,\mathbf{\Phi})}{G(x_{1i};\mathbf{\Phi})^a+\theta\overline{G}(x_{1i};\mathbf{\Phi})^a}+\alpha_2\sum_{i=1}^{n_1}\ln\left(G(x_{2i};\mathbf{\Phi})\right)+\sum_{i=1}^{n_1}\ln\left(\overline{G}(x_{2i};\mathbf{\Phi})\right)$$
$$-(\alpha_2+1)\sum_{i=1}^{n_1}\frac{\Psi(x_{2i};a,\theta,\mathbf{\Phi})}{G(x_{2i};\mathbf{\Phi})^a+\theta\overline{G}(x_{2i};\mathbf{\Phi})^a}+\alpha_1\sum_{i=1}^{n_2}\ln\left(G(x_{1i};\mathbf{\Phi})\right)+\sum_{i=1}^{n_2}\ln\left(\overline{G}(x_{1i};\mathbf{\Phi})\right)$$
$$-(\alpha_1+1)\sum_{i=1}^{n_2}\frac{\Psi(x_{1i};a,\theta,\mathbf{\Phi})}{G(x_{1i};\mathbf{\Phi})^a+\theta\overline{G}(x_{1i};\mathbf{\Phi})^a}+(\alpha_2+\alpha_3)\sum_{i=1}^{n_2}\ln\left(G(x_{2i};\mathbf{\Phi})\right)+\sum_{i=1}^{n_2}\ln\left(\overline{G}(x_{2i};\mathbf{\Phi})\right)$$
$$-(\alpha_2+\alpha_3+1)\sum_{i=1}^{n_2}\frac{\Psi(x_{2i};a,\theta,\mathbf{\Phi})}{G(x_{2i};\mathbf{\Phi})^a+\theta\overline{G}(x_{2i};\mathbf{\Phi})^a}+(\alpha_1+\alpha_2+\alpha_3)\sum_{i=1}^{n_3}\ln\left(G(x_i;\mathbf{\Phi})\right)$$
$$+\sum_{i=1}^{n_3}\ln\left(\overline{G}(x_i;\mathbf{\Phi})\right)-(\alpha_1+\alpha_2+\alpha_3+1)\sum_{i=1}^{n_3}\frac{\Psi(x_i;a,\theta,\mathbf{\Phi})}{G(x_i;\mathbf{\Phi})^a+\theta\overline{G}(x_i;\mathbf{\Phi})^a}, \tag{2.31}$$

and

$$\frac{\partial L}{\partial \mathbf{\Phi}}=\sum_{i=1}^{n_1}\frac{\Psi_1(x_{1i};\mathbf{\Phi})}{g(x_{1i};\mathbf{\Phi})}+(a(\alpha_1+\alpha_3)-1)\sum_{i=1}^{n_1}\frac{\Psi_2(x_{1i};\mathbf{\Phi})}{G(x_{1i};\mathbf{\Phi})}+(a-1)\sum_{i=1}^{n_1}\frac{-\Psi_2(x_{1i};\mathbf{\Phi})}{\overline{G}(x_{1i};\mathbf{\Phi})}$$
$$-(\alpha_1+\alpha_3+1)\sum_{i=1}^{n_1}\frac{a\Psi_2(x_{1i};\mathbf{\Phi})\left(G(x_{1i};\mathbf{\Phi})^{a-1}-\theta\overline{G}(x_{1i};\mathbf{\Phi})^{a-1}\right)}{G(x_{1i};\mathbf{\Phi})^a+\theta\overline{G}(x_{1i};\mathbf{\Phi})^a}+\sum_{i=1}^{n_1}\frac{\Psi_1(x_{2i};\mathbf{\Phi})}{g(x_{2i};\mathbf{\Phi})}$$
$$+(a\alpha_2-1)\sum_{i=1}^{n_1}\frac{\Psi_2(x_{2i};\mathbf{\Phi})}{G(x_{2i};\mathbf{\Phi})}+(a-1)\sum_{i=1}^{n_1}\frac{-\Psi_2(x_{2i};\mathbf{\Phi})}{\overline{G}(x_{2i};\mathbf{\Phi})}+\sum_{i=1}^{n_2}\frac{\Psi_1(x_{1i};\mathbf{\Phi})}{g(x_{1i};\mathbf{\Phi})}$$
$$-(\alpha_2+1)\sum_{i=1}^{n_1}\frac{a\Psi_2(x_{2i};\mathbf{\Phi})\left(G(x_{2i};\mathbf{\Phi})^{a-1}-\theta\overline{G}(x_{2i};\mathbf{\Phi})^{a-1}\right)}{G(x_{2i};\mathbf{\Phi})^a+\theta\overline{G}(x_{2i};\mathbf{\Phi})^a}+(a\alpha_1-1)\sum_{i=1}^{n_2}\frac{\Psi_2(x_{1i};\mathbf{\Phi})}{G(x_{1i};\mathbf{\Phi})}$$
$$+(a-1)\sum_{i=1}^{n_2}\frac{-\Psi_2(x_{1i};\mathbf{\Phi})}{\overline{G}(x_{1i};\mathbf{\Phi})}-(\alpha_1+1)\sum_{i=1}^{n_2}\frac{a\Psi_2(x_{1i};\mathbf{\Phi})\left(G(x_{1i};\mathbf{\Phi})^{a-1}-\theta\overline{G}(x_{1i};\mathbf{\Phi})^{a-1}\right)}{G(x_{1i};\mathbf{\Phi})^a+\theta\overline{G}(x_{1i};\mathbf{\Phi})^a}$$
$$+\sum_{i=1}^{n_2}\frac{\Psi_1(x_{2i};\mathbf{\Phi})}{g(x_{2i};\mathbf{\Phi})}+(a(\alpha_2+\alpha_3)-1)\sum_{i=1}^{n_2}\frac{\Psi_2(x_{2i};\mathbf{\Phi})}{G(x_{2i};\mathbf{\Phi})}+(a-1)\sum_{i=1}^{n_2}\frac{-\Psi_2(x_{2i};\mathbf{\Phi})}{\overline{G}(x_{2i};\mathbf{\Phi})}$$
$$-(\alpha_2+\alpha_3+1)\sum_{i=1}^{n_2}\frac{a\Psi_2(x_{2i};\mathbf{\Phi})\left(G(x_{2i};\mathbf{\Phi})^{a-1}-\theta\overline{G}(x_{2i};\mathbf{\Phi})^{a-1}\right)}{G(x_{2i};\mathbf{\Phi})^a+\theta\overline{G}(x_{2i};\mathbf{\Phi})^a}+\sum_{i=1}^{n_3}\frac{\Psi_1(x_i;\mathbf{\Phi})}{g(x_i;\mathbf{\Phi})}$$

$$+ \left(a\left(\alpha_1+\alpha_2+\alpha_3\right)-1\right)\sum_{i=1}^{n_3}\frac{\Psi_2\left(x_i;\mathbf{\Phi}\right)}{G\left(x_i;\mathbf{\Phi}\right)}+(a-1)\sum_{i=1}^{n_3}\frac{-\Psi_2\left(x_i;\mathbf{\Phi}\right)}{\overline{G}\left(x_i;\mathbf{\Phi}\right)}$$
$$-\left(\alpha_1+\alpha_2+\alpha_3+1\right)\sum_{i=1}^{n_3}\frac{a\Psi_2\left(x_i;\mathbf{\Phi}\right)\left(G\left(x_i;\mathbf{\Phi}\right)^{a-1}-\theta\overline{G}\left(x_i;\mathbf{\Phi}\right)^{a-1}\right)}{G\left(x_i;\mathbf{\Phi}\right)^a+\theta\overline{G}\left(x_i;\mathbf{\Phi}\right)^a}, \tag{2.32}$$

where $\Psi\left(x;a,\theta,\mathbf{\Phi}\right) = G\left(x;\mathbf{\Phi}\right)^a\ln G\left(x;\mathbf{\Phi}\right)+\theta\overline{G}\left(x;\mathbf{\Phi}\right)^a\ln\overline{G}\left(x;\mathbf{\Phi}\right)$, $\Psi_1\left(x;\mathbf{\Phi}\right) = \frac{\partial}{\partial\mathbf{\Phi}}g\left(x;\mathbf{\Phi}\right)$, and $\Psi_2\left(x;\mathbf{\Phi}\right) = \frac{\partial}{\partial\mathbf{\Phi}}G\left(x;\mathbf{\Phi}\right)$. The maximum likelihood estimators can be obtained by optimizing the *FindArgMax* function in Mathematica or the *optim* function in R. However, Mathematica was employed for parameter estimation, while R was used for visualization in our study.

3. The BFGMODa-G family: Structure and characteristic functions

According to the FGM copula function, the JCDF of the BFGMODa-G can be defined as

$$\begin{aligned} F_{\text{BFGMODa-G}}\left(x_1,x_2\right) &= \left(1+\theta_1\left(\frac{\overline{G}\left(x_1;\mathbf{\Phi}_1\right)}{G\left(x_1;\mathbf{\Phi}_1\right)}\right)^{a_1}\right)^{-\alpha_1}\left(1+\theta_2\left(\frac{\overline{G}\left(x_2;\mathbf{\Phi}_2\right)}{G\left(x_2;\mathbf{\Phi}_2\right)}\right)^{a_2}\right)^{-\alpha_2} \\ &\times\left[1+\gamma\left(1-\left(1+\theta_1\left(\frac{\overline{G}\left(x_1;\mathbf{\Phi}_1\right)}{G\left(x_1;\mathbf{\Phi}_1\right)}\right)^{a_1}\right)^{-\alpha_1}\right)\left(1-\left(1+\theta_2\left(\frac{\overline{G}\left(x_2;\mathbf{\Phi}_2\right)}{G\left(x_2;\mathbf{\Phi}_2\right)}\right)^{a_2}\right)^{-\alpha_2}\right)\right] \\ &= F_{\text{ODa-G}}\left(x_1;\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1\right)F_{\text{ODa-G}}\left(x_2;\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2\right)\left[1+\gamma\right. \\ &\left.\times\left(1-F_{\text{ODa-G}}\left(x_1;\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1\right)\right)\left(1-F_{\text{ODa-G}}\left(x_2;\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2\right)\right)\right]. \end{aligned} \tag{3.1}$$

The corresponding JPDF of the BFGMODa-G family can be expressed as

$$\begin{aligned} f_{\text{BFGMODa-G}}\left(x_1,x_2\right) &= f_{\text{ODa-G}}\left(x_1;\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1\right)f_{\text{ODa-G}}\left(x_2;\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2\right) \\ &\times\left[1+\gamma\left(1-2F_{\text{ODa-G}}\left(x_1;\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1\right)\right)\left(1-2F_{\text{ODa-G}}\left(x_2;\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2\right)\right)\right]. \end{aligned} \tag{3.2}$$

Further, the marginal CDFs and its corresponding PDFs for the BFGMODa-G family can be represented as

$$F^*_{X_i}\left(x_i\right) = F_{\text{ODa-G}}\left(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i\right); x_i>0,\ i=1,\ 2, \tag{3.3}$$

and

$$f^*_{X_i}\left(x_i\right) = f_{\text{ODa-G}}\left(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i\right);\ x_i>0. \tag{3.4}$$

respectively. These marginals can be represented as a linear representation as follows

$$F^*_{X_i}\left(x_i\right) = \sum_{k,m\geq 0}u^{(i)}_{k,m}H^{(i)}_{-m}\left(x_i;\mathbf{\Phi}_i\right);\ x_i>0,\ i=1,2, \tag{3.5}$$

and

$$f^*_{X_i}\left(x_i\right) = \sum_{k,m\geq 0}u^{(i)}_{k,m}h^{(i)}_{-m}\left(x_i;\mathbf{\Phi}_i\right);\ x_i>0,\ i=1,2, \tag{3.6}$$

where $H^{(i)}_{-m}\left(x_i;\mathbf{\Phi}_i\right) = G^{-m}\left(x_i;\mathbf{\Phi}_i\right)$ represents the CDF of the exponentiated-G (exp-G) family of distributions, with a power parameter $(-m)$, $u^{(i)}_{k,m} = (-1)^{a_ik-m}\binom{-\alpha_i}{k}\binom{a_ik}{m}\theta_i^k$ and $h^{(i)}_{-m}\left(x_i;\mathbf{\Phi}_i\right) =$

$-mg(x_i;\mathbf{\Phi}_i)\,G^{-m-1}(x_i;\mathbf{\Phi}_i)$ represents the PDF of the exp-G family of distributions with a power parameter of $(-m)$. The JSF of the BFGMODa-G family is proposed as

$$S_{\text{BFGMODa-G}}(x_1,x_2)=1-\left(1+\theta_1\left(\frac{\overline{G}(x_1;\mathbf{\Phi}_1)}{G(x_1;\mathbf{\Phi}_1)}\right)^{a_1}\right)^{-\alpha_1}-\left(1+\theta_2\left(\frac{\overline{G}(x_2;\mathbf{\Phi}_2)}{G(x_2;\mathbf{\Phi}_2)}\right)^{a_2}\right)^{-\alpha_2}+F_{\text{BFGMODa-G}}(x_1,x_2). \tag{3.7}$$

If the random vector X has the BFGMODa-G family, then the conditional PDF, CDF, and SF of X_i given $X_j=x_j$ $(i,j=1,2;\ i\neq j)$ can be formulated as

$$\begin{aligned}f^*_{X_i|X_j=x_j}\left(x_i|x_j\right)&=f_{\text{ODa-G}}(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i)\left[1+\gamma\left(1-2F_{\text{ODa-G}}(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i)\right)\right.\\&\quad\left.\times\left(1-2F_{\text{ODa-G}}\left(x_j;\alpha_j,\theta_j,a_j,\mathbf{\Phi}_j\right)\right)\right],\end{aligned} \tag{3.8}$$

$$\begin{aligned}F^*_{X_i|X_j=x_j}\left(x_i|x_j\right)&=P\left(X_i\leq x_i|X_j=x_j\right)\\&=F_{\text{ODa-G}}(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i)\left[1+\gamma\left(1-F_{\text{ODa-G}}(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i)\right)\right.\\&\quad\left.\times\left(1-2F_{\text{ODa-G}}\left(x_j;\alpha_j,\theta_j,a_j,\mathbf{\Phi}_j\right)\right)\right],\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}S^*_{X_i|X_j=x_j}\left(x_i|x_j\right)&=P\left(X_i>x_i|X_j=x_j\right)\\&=\left(1-F_{\text{ODa-G}}(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i)\right)\left[1-\gamma F_{\text{ODa-G}}(x_i;\alpha_i,\theta_i,a_i,\mathbf{\Phi}_i)\right.\\&\quad\left.\times\left(1-2F_{\text{ODa-G}}\left(x_j;\alpha_j,\theta_j,a_j,\mathbf{\Phi}_j\right)\right)\right].\end{aligned} \tag{3.10}$$

The conditional survival function is often used in survival analysis, which is a branch of statistics that deals with analyzing the time until an event of interest occurs, such as death, failure, or an event related to a specific outcome. Conditional survival functions can be valuable in various fields, including medical research, where they can be used to estimate the probability of a patient surviving beyond a certain time, given that the patient has already survived for a specific duration. This information can be useful for making informed decisions about treatment and follow-up care.

3.1. Some statistical properties of the BFGMODa-G family

3.1.1. Median correlation

If $X_1\sim$ ODa-G$(\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1)$ and $X_2\sim$ ODa-G$(\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2)$, then the median of the marginals can be formulated as

$$M_{X_i}=G^{-1}\left[\left(\left(\frac{1}{\theta_i}\left\{U^{-\frac{1}{\alpha_i}}-1\right\}\right)^{\frac{1}{a_i}}+1\right)^{-1}\right];\ i=1,2, \tag{3.11}$$

where U has a uniform $U(0,1)$ distribution and $G^{-1}(.)$ is the baseline quantile function. Therefore, the coefficient of median correlation between X_1 and X_2 at $U=0.5$ is given by

$$\begin{aligned}M_{X_1,X_2}&=4F^*_{\text{ODa-G}}(M_{X_1};\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1)\,F^*_{\text{ODa-G}}(M_{X_2};\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2)\\&\quad\times\left[1+\gamma\left(1-F^*_{\text{ODa-G}}(M_{X_1};\alpha_1,\theta_1,a_1,\mathbf{\Phi}_1)\right)\left(1-F^*_{\text{ODa-G}}(M_{X_2};\alpha_2,\theta_2,a_2,\mathbf{\Phi}_2)\right)\right]-1.\end{aligned} \tag{3.12}$$

3.1.2. Product moments and covariance

According to Eq (3.6), the rth moment of X_i is given by

$$\mu_i^{(r)} = \sum_{k,m=0}^{\infty} u_{k,m}^{(i)} \int_0^{\infty} x_i^r h_{-m}^{(i)} (x_i; \boldsymbol{\Phi}_i)\, dx_i = \sum_{k,m=0}^{\infty} u_{k,m}^{(i)} \mathbf{E}\left(Y_{i,-m}^r\right), \tag{3.13}$$

where $Y_{i,-m}^r; i = 1,2$ is a random variable having the exp-G CDF with power parameter $(-m)$. Setting $r = 1$ in Eq (3.13), we get the mean of $X_i; i = 1,2$. Thus, the nth central moment of X_i, say $L_i^{(n)}$, is given by

$$L_i^{(n)} = \sum_{r=0}^{n} \sum_{k,m=0}^{\infty} \left(-\mu_i^{(1)}\right)^{n-r} \binom{n}{r} u_{k,m}^{(i)} \mathbf{E}\left(Y_{i,-m}^r\right); i = 1,2. \tag{3.14}$$

The product moment of the random variable $X_1^r X_2^r$ can be expressed as

$$\mathbf{E}\left(X_1^r X_2^r\right) = \sum_{k,m\geq 0} u_{k,m}^{(1)} u_{k,m}^{(2)} \int_0^{\infty}\int_0^{\infty} x_1^r x_2^r h_{-m}^{(1)} (x_1; \boldsymbol{\Phi}_1)\, h_{-m}^{(2)} (x_2; \boldsymbol{\Phi}_2) \left[1 + \gamma \prod_{i=1}^{2} \left(1 - 2\sum_{k,m\geq 0} u_{k,m}^{(i)} H_{-m}^{(i)} (x_i; \boldsymbol{\Phi}_i)\right)\right] dx_1 dx_2. \tag{3.15}$$

Utilizing Eqs (3.13) and (3.15) when $r = 1$, we get the covariance of the the BFGMODa-G family as the following equation

$$\begin{aligned}\mathbf{cov}(X_1, X_2) =& \sum_{k,m\geq 0} u_{k,m}^{(1)} u_{k,m}^{(2)} \int_0^{\infty}\int_0^{\infty} x_1 x_2 h_{-m}^{(1)} (x_1; \boldsymbol{\Phi}_1)\, h_{-m}^{(2)} (x_2; \boldsymbol{\Phi}_2) \left[1 + \gamma \prod_{i=1}^{2} \left(1 - 2\sum_{k,m\geq 0} u_{k,m}^{(i)} H_{-m}^{(i)} (x_i; \boldsymbol{\Phi}_i)\right)\right] dx_1 dx_2 \\ &- \left(\sum_{k,m=0}^{\infty} u_{k,m}^{(1)} \mathbf{E}\left(Y_{1,-m}^1\right)\right)\left(\sum_{k,m=0}^{\infty} u_{k,m}^{(2)} \mathbf{E}\left(Y_{2,-m}^2\right)\right).\end{aligned} \tag{3.16}$$

Covariance is a way to assess how much two random variables vary at the same time. It shows if the variables are related in a straight line and if they tend to go up or down together. It basically measures how much two random variables change together. So, it is commonly utilized in finance, economics, and data analysis, among other fields. It is also used to figure out several crucial statistical measures, such as the correlation coefficient, which makes the variance a number between -1 and 1 so that it is easier to understand and compare how variables are related.

3.2. The BFGMODa-exponential distribution

If the random vector X has the BFGMODa-exponential (BFGMODaEx) distribution, then the JPDF can be presented as

$$\begin{aligned} f_{\text{BFGMODaEx}} (x_1, x_2) =& \frac{a_1 \alpha_1 \theta_1 \delta_1 e^{-\delta_1 x_1} \left(1 - e^{-\delta_1 x_1}\right)^{a_1 \alpha_1 - 1} \left(e^{-\delta_1 x_1}\right)^{a_1 - 1}}{\left[(1 - e^{-\delta_1 x_1})^{a_1} + \theta_1 e^{-a_1 \delta_1 x_1}\right]^{\alpha_1 + 1}} \\ &\times \frac{a_2 \alpha_2 \theta_2 \delta_2 e^{-\delta_2 x_2} \left(1 - e^{-\delta_2 x_2}\right)^{a_2 \alpha_2 - 1} \left(e^{-\delta_2 x_2}\right)^{a_2 - 1}}{\left[(1 - e^{-\delta_2 x_2})^{a_2} + \theta_2 e^{-a_2 \delta_2 x_2}\right]^{\alpha_2 + 1}} \end{aligned}$$

$$\times\left[1+\gamma\left(1-2\left(1+\theta_1\left(\frac{e^{-\delta_1 x_1}}{1-e^{-\delta_1 x_1}}\right)^{a_1}\right)^{-\alpha_1}\right)\left(1-2\left(1+\theta_2\left(\frac{e^{-\delta_2 x_2}}{1-e^{-\delta_2 x_2}}\right)^{a_2}\right)^{-\alpha_2}\right)\right]. \tag{3.17}$$

Further, the JCDF and its corresponding JSF of the BFGMODaEx distribution can be proposed as

$$F_{\text{BFGMODaEx}}(x_1,x_2)=\left(1+\theta_1\left(\frac{e^{-\delta_1 x_1}}{1-e^{-\delta_1 x_1}}\right)^{a_1}\right)^{-\alpha_1}\left(1+\theta_2\left(\frac{e^{-\delta_2 x_2}}{1-e^{-\delta_2 x_2}}\right)^{a_2}\right)^{-\alpha_2}$$
$$\times\left[1+\gamma\left(1-\left(1+\theta_1\left(\frac{e^{-\delta_1 x_1}}{1-e^{-\delta_1 x_1}}\right)^{a_1}\right)^{-\alpha_1}\right)\left(1-\left(1+\theta_2\left(\frac{e^{-\delta_2 x_2}}{1-e^{-\delta_2 x_2}}\right)^{a_2}\right)^{-\alpha_2}\right)\right], \tag{3.18}$$

and

$$S_{\text{BFGMODaEx}}(x_1,x_2)=1-\left(1+\theta_1\left(\frac{e^{-\delta_1 x_1}}{1-e^{-\delta_1 x_1}}\right)^{a_1}\right)^{-\alpha_1}-\left(1+\theta_2\left(\frac{e^{-\delta_2 x_2}}{1-e^{-\delta_2 x_2}}\right)^{a_2}\right)^{-\alpha_2}+F_{\text{BFGMODa-Ex}}(x_1,x_2). \tag{3.19}$$

Figures 5–7 display various diagrams of the JPDF, JSF, and JHRF of the BFGMODaEx under different specific schemes: (a) $\alpha_1=7$, $\alpha_2=6$, $\theta_1=0.1$, $\theta_2=0.6$, $a_1=0.7$, $a_2=0.3$, $\delta_1=0.1$, $\delta_2=0.3$, $\gamma=-0.2$; (b) $\alpha_1=7$, $\alpha_2=6$, $\theta_1=0.1$, $\theta_2=0.3$, $a_1=0.7$, $a_2=0.3$, $\delta_1=0.1$, $\delta_2=0.3$, $\gamma=-0.2$; and (c) $\alpha_1=7$, $\alpha_2=6$, $\theta_1=0.1$, $\theta_2=0.3$, $a_1=7$, $a_2=3$, $\delta_1=0.1$, $\delta_2=0.3$, $\gamma=-0.9$.

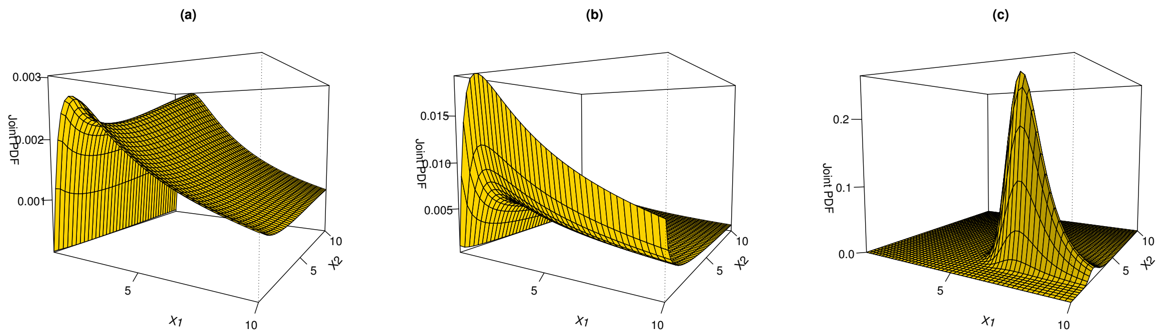

Figure 5. The JPDF of the BFGMODaEx models.

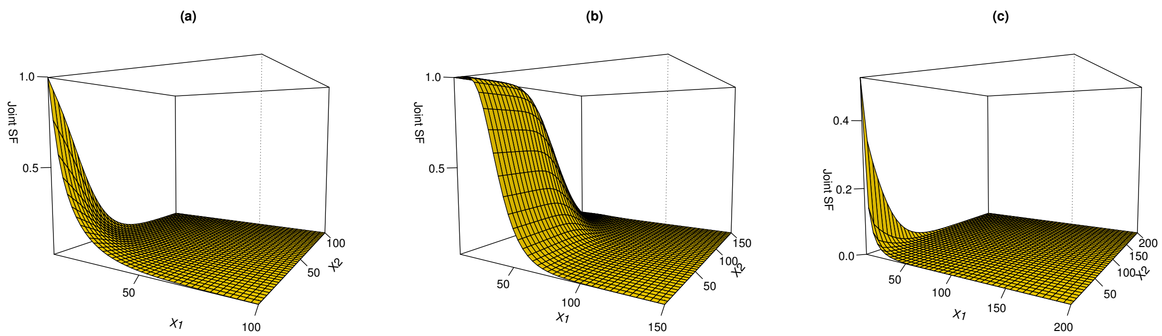

Figure 6. The JSF of the BFGMODaEx models.

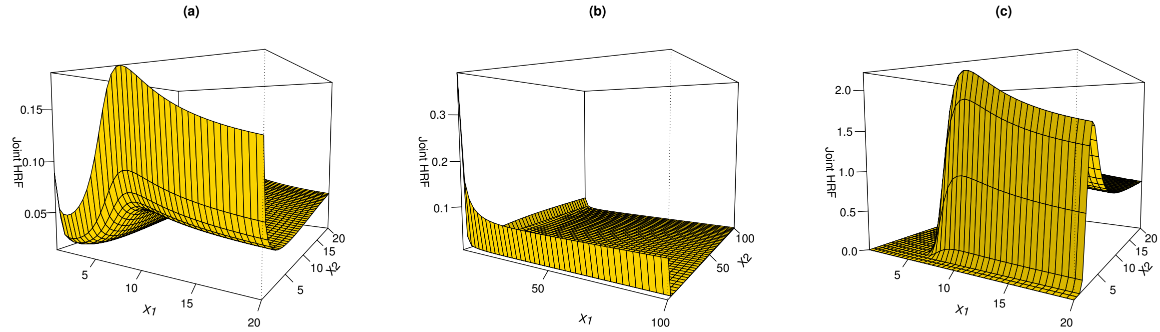

Figure 7. The JHRF of the BFGMODaEx models.

Figures 5–7 depict the principal structural characteristics of the suggested bivariate model. The JPDFs validate unimodality while emphasizing asymmetry and heavy-tailed characteristics, reflecting the potential for extreme joint outcomes. The JHRFs exhibit both bathtub and inverted-bathtub patterns, indicating non-monotonic risk dynamics that are more realistic than strictly monotonic behaviors. The JSFs exhibit sensitivity to parameter selections, indicating areas of steep decline under specific conditions and highlighting the influence of dependence structures on distributional characteristics. The graphical findings illustrate the model's adaptability in depicting intricate connections, asymmetric hazards, and varied hazard patterns. Figure 8 shows some plots of the conditional PDF of $X_{1,}$ given that $x_2 = 5, 15, 25$ under scheme (a): $\alpha_1 = 7$, $\alpha_2 = 6$, $\theta_1 = 2$, $\theta_2 = 0.6$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\gamma = -0.2$. The attitude of the conditional PDF of $X_{2,}$ given that $x_1 = 5, 12, 20$ under scheme (b): $\alpha_1 = 7, \alpha_2 = 5, \theta_1 = 0.2, \theta_2 = 6, a_1 = 0.7, a_2 = 0.3, \delta_1 = 0.1, \delta_2 = 0.9, \gamma = -0.2$ is shown in Figure 8.

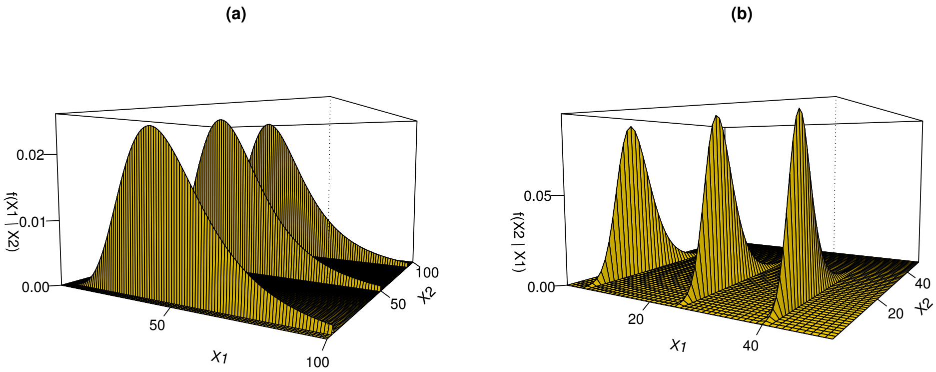

Figure 8. The conditional PDF of the BFGMODaEx model.

Conditional PDFs are used in various statistical and scientific applications to model and analyze the behavior of random variables under specific conditions or events. They provide a way to focus on the probability distribution relevant to a particular context or situation, allowing for more precise and context-specific probability modeling.

3.3. Maximum likelihood estimation of BFGMODa-G class parameters

In this section, the maximum likelihood method is applied to estimate the unknown parameters of the BFGMODa-G family. Assume $((x_{11}, x_{21}), \ ..., \ (x_{1n}, x_{2n}))$ is a random sample from the BFGMODa-G class. Then, the likelihood function of the sample of size n is given by

$$l(\boldsymbol{\Upsilon}) = \prod_{i=1}^{n} f_{\text{BFGMODa-G}}(x_{1i}, x_{2i}). \tag{3.20}$$

where $\boldsymbol{\Upsilon} = (\alpha_1, \alpha_2, \theta_1, \theta_2, a_1, a_2, \delta_1, \delta_2, \gamma)$. The log-likelihood function can be expressed as

$$\begin{aligned} L(\boldsymbol{\Upsilon}) = &\sum_{i=1}^{n} \ln\left[f_{\text{ODa-G}}(x_{1i}; \alpha_1, \theta_1, a_1, \boldsymbol{\Phi}_1)\right] + \sum_{i=1}^{n} \ln\left[f_{\text{ODa-G}}(x_{2i}; \alpha_2, \theta_2, a_2, \boldsymbol{\Phi}_2)\right] \\ &+ \sum_{i=1}^{n} \ln\left[1 + \gamma\left(1 - 2F_{\text{ODa-G}}(x_{1i}; \alpha_1, \theta_1, a_1, \boldsymbol{\Phi}_1)\right)\left(1 - 2F_{\text{ODa-G}}(x_{2i}; \alpha_2, \theta_2, a_2, \boldsymbol{\Phi}_2)\right)\right]. \end{aligned} \tag{3.21}$$

Differentiating the log-likelihood with respect to α_1, α_2, θ_1, θ_2, a_1, a_2, γ, and Φ, respectively, and setting the results equal to zero, we get

$$\begin{aligned} \frac{\partial L}{\partial \alpha_j} = &\frac{n}{\alpha_j} + a_j \sum_{i=1}^{n} \ln\left(G\left(x_{ji}; \boldsymbol{\Phi}_j\right)\right) - \sum_{i=1}^{n} \ln\left(G\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j} + \theta_j \overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j}\right) \\ &+ \sum_{i=1}^{n} \frac{2\gamma\left(1 - 2\left(1 + \theta_k\left(\frac{\overline{G}(x_{ki}; \boldsymbol{\Phi}_k)}{G(x_{ki}, \boldsymbol{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)\left(1 + \theta_j\left(\frac{\overline{G}(x_{ji}; \boldsymbol{\Phi}_j)}{G(x_{ji}, \boldsymbol{\Phi}_j)}\right)^{a_j}\right)^{-\alpha_j} \ln\left(1 + \theta_j\left(\frac{\overline{G}(x_{ji}; \boldsymbol{\Phi}_j)}{G(x_{ji}, \boldsymbol{\Phi}_j)}\right)^{a_j}\right)}{1 + \gamma\left(1 - 2\left(1 + \theta_j\left(\frac{\overline{G}(x_{1i}; \boldsymbol{\Phi}_j)}{G(x_{1i}, \boldsymbol{\Phi}_j)}\right)^{a_j}\right)^{-\alpha_j}\right)\left(1 - 2\left(1 + \theta_k\left(\frac{\overline{G}(x_{2i}; \boldsymbol{\Phi}_k)}{G(x_{2i}, \boldsymbol{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \frac{\partial L}{\partial \theta_j} = &\frac{n}{\theta_j} - \left(\alpha_j + 1\right) \sum_{i=1}^{n} \frac{\overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j}}{G\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j} + \theta_j \overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j}} \\ &+ \sum_{i=1}^{n} \frac{2\gamma\alpha_j\left(1 - 2\left(1 + \theta_k\left(\frac{\overline{G}(x_{ki}; \boldsymbol{\Phi}_k)}{G(x_{ki}, \boldsymbol{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)\left(1 + \theta_j\left(\frac{\overline{G}(x_{ji}; \boldsymbol{\Phi}_j)}{G(x_{ji}, \boldsymbol{\Phi}_j)}\right)^{a_j}\right)^{-\alpha_j - 1} \left(\frac{\overline{G}(x_{ji}; \boldsymbol{\Phi}_j)}{G(x_{ji}, \boldsymbol{\Phi}_j)}\right)^{a_j}}{1 + \gamma\left(1 - 2\left(1 + \theta_j\left(\frac{\overline{G}(x_{1i}; \boldsymbol{\Phi}_j)}{G(x_{1i}, \boldsymbol{\Phi}_j)}\right)^{a_j}\right)^{-\alpha_j}\right)\left(1 - 2\left(1 + \theta_k\left(\frac{\overline{G}(x_{2i}; \boldsymbol{\Phi}_k)}{G(x_{2i}, \boldsymbol{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} \frac{\partial L}{\partial a_j} = &\frac{n}{a_j} + \alpha_j \sum_{i=1}^{n} \ln\left(G\left(x_{ji}; \boldsymbol{\Phi}_j\right)\right) + \sum_{i=1}^{n} \ln\left(\overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)\right) \\ &- \left(\alpha_j + 1\right) \sum_{i=1}^{n} \frac{G\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j} \ln G\left(x_{ji}; \boldsymbol{\Phi}_j\right) + \theta_j \overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j} \ln \overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)}{G\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j} + \theta_j \overline{G}\left(x_{ji}; \boldsymbol{\Phi}_j\right)^{a_j}} \end{aligned}$$

$$+\sum_{i=1}^{n}\frac{2\gamma\alpha_j\theta_j\left(1-2\left(1+\theta_k\left(\frac{\overline{G}(x_{ki};\mathbf{\Phi}_k)}{G(x_{ki};\mathbf{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)\left(1+\theta_j\left(\frac{\overline{G}\left(x_{ji};\mathbf{\Phi}_j\right)}{G\left(x_{ji},\mathbf{\Phi}_j\right)}\right)^{a_j}\right)^{-\alpha_j-1}\left(\frac{\overline{G}\left(x_{ji};\mathbf{\Phi}_j\right)}{G\left(x_{ji},\mathbf{\Phi}_j\right)}\right)^{a_j}\ln\left(\frac{\overline{G}\left(x_{ji};\mathbf{\Phi}_j\right)}{G\left(x_{ji},\mathbf{\Phi}_j\right)}\right)}{+\gamma\left(1-2\left(1+\theta_j\left(\frac{\overline{G}(x_{1i};\mathbf{\Phi}_j)}{G(x_{1i},\mathbf{\Phi}_j)}\right)^{a_j}\right)^{-\alpha_j}\right)\left(1-2\left(1+\theta_k\left(\frac{\overline{G}(x_{2i};\mathbf{\Phi}_k)}{G(x_{2i},\mathbf{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)}, \tag{3.24}$$

$$\frac{\partial L}{\partial\gamma}=\sum_{i=1}^{n}\frac{\left(1-2\left(1+\theta_1\left(\frac{\overline{G}(x_{1i};\mathbf{\Phi}_1)}{G(x_{1i},\mathbf{\Phi}_1)}\right)^{a_1}\right)^{-\alpha_1}\right)\left(1-2\left(1+\theta_2\left(\frac{\overline{G}(x_{2i};\mathbf{\Phi}_2)}{G(x_{2i},\mathbf{\Phi}_2)}\right)^{a_2}\right)^{-\alpha_2}\right)}{1+\gamma\left(1-2\left(1+\theta_1\left(\frac{\overline{G}(x_{1i};\mathbf{\Phi}_1)}{G(x_{1i},\mathbf{\Phi}_1)}\right)^{a_1}\right)^{-\alpha_1}\right)\left(1-2\left(1+\theta_2\left(\frac{\overline{G}(x_{2i};\mathbf{\Phi}_2)}{G(x_{2i},\mathbf{\Phi}_2)}\right)^{a_2}\right)^{-\alpha_2}\right)}, \tag{3.25}$$

and

$$\begin{aligned}\frac{\partial L}{\partial\mathbf{\Phi}_j}&=\sum_{i=1}^{n}\frac{\Psi_1\left(x_{ji};\mathbf{\Phi}_j\right)}{g\left(x_{ji};\mathbf{\Phi}_j\right)}+\left(a_j\alpha_j-1\right)\sum_{i=1}^{n}\frac{\Psi_2\left(x_{ji};\mathbf{\Phi}_j\right)}{G\left(x_{ji};\mathbf{\Phi}_j\right)}+\left(a_j-1\right)\sum_{i=1}^{n}\frac{-\Psi_2\left(x_{ji};\mathbf{\Phi}_j\right)}{\overline{G}\left(x_{ji};\mathbf{\Phi}_j\right)}\\&-(\alpha_1+1)\sum_{i=1}^{n}\frac{a_j\Psi_2\left(x_{ji};\mathbf{\Phi}_j\right)\left(G\left(x_{ji};\mathbf{\Phi}\right)^{a_j-1}-\theta_j\overline{G}\left(x_{ji};\mathbf{\Phi}\right)^{a_j-1}\right)}{G\left(x_{ji};\mathbf{\Phi}_j\right)^{a_j}+\theta_j\overline{G}\left(x_{1i};\mathbf{\Phi}_j\right)^{a_j}}\\&+\sum_{i=1}^{n}\frac{2\gamma\alpha_j\theta_ja_j\left(1-2\left(1+\theta_k\left(\frac{\overline{G}(x_{ki};\mathbf{\Phi}_k)}{G(x_{ki};\mathbf{\Phi}_k)}\right)^{a_k}\right)^{-\alpha_k}\right)\left(1+\theta_j\left(\frac{\overline{G}\left(x_{ji};\mathbf{\Phi}_j\right)}{G\left(x_{ji},\mathbf{\Phi}_j\right)}\right)^{a_j}\right)^{-\alpha_j-1}\left(\frac{\overline{G}\left(x_{ji};\mathbf{\Phi}_j\right)}{G\left(x_{ji},\mathbf{\Phi}_j\right)}\right)^{a_j-1}\Omega\left(x_{ji};\mathbf{\Phi}_j\right)}{1+\gamma\left(1-2\left(1+\theta_1\left(\frac{\overline{G}(x_{1i};\mathbf{\Phi}_1)}{G(x_{1i},\mathbf{\Phi}_1)}\right)^{a_1}\right)^{-\alpha_1}\right)\left(1-2\left(1+\theta_2\left(\frac{\overline{G}(x_{2i};\mathbf{\Phi}_2)}{G(x_{2i},\mathbf{\Phi}_2)}\right)^{a_2}\right)^{-\alpha_2}\right)},\end{aligned} \tag{3.26}$$

where $\Omega(x;\mathbf{\Phi})=\frac{\partial}{\partial\mathbf{\Phi}}\left(\frac{\overline{G}(x;\mathbf{\Phi})}{G(x,\mathbf{\Phi})}\right)$ and $j,k=1,2;\ j\neq k$. The *FindArgMax* function in Mathematica and the *optim* function in R can be utilized for optimization to derive the maximum likelihood estimators. Mathematica was utilized for parameter estimation, whereas R was applied for visualization in this study.

4. Estimator performance: A simulation study

Simulation is a robust statistical method employed to mimic real-world processes and perform experiments in a controlled and reproducible manner. In statistics, simulations are often used for a number of things, such as hypothesis testing, Monte Carlo integration, risk assessment, statistical power analysis, model validation, parameter estimation, and more. Statistical simulations are often done with computer languages and software tools like R, Python, MATLAB, and others. These tools have libraries and functions that let you make random numbers, run tests, and look at the results of simulations. In this study, we used the R program with two packages: *copula* for modeling general dependence structures and *extraDistr* for generating special continuous distributions. The key to successful statistical simulation is to properly plan the procedure, set the assumptions and parameters, then run enough iterations to get useful results. In this section, a simulation analysis is conducted to evaluate the maximum likelihood (ML) estimators for the BMOODaEx and

BFGMODaEx distributions using the R software tool across various sample sizes. There are many schemes for simulations that are covered below:

(1) Generate $N = 1000$ samples of various sample sizes n_i; i =1, 2, 3, 4, 5, 6 ($n_1 = 20$, $n_2 = 50$, $n_3 = 100$, $n_4 = 200$, $n_5 = 350$, $n_6 = 500$) from the BMOODaEx and BFGMODaEx distributions as follows:

- Scheme I: $\alpha_1 = 1.3$, $\alpha_2 = 1.5$, $\alpha_3 = 1.7$, $\theta = 0.8$, $a = 0.5$, $\delta = 0.9$.
- Scheme II: $\alpha_1 = 0.8$, $\alpha_2 = 0.6$, $\alpha_3 = 0.5$, $\theta = 1.2$, $a = 1.3$, $\delta = 1.5$.
- Scheme III: $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 0.9$, $\theta = 1.5$, $a = 1.7$, $\delta = 1.8$.
- Scheme IV: $\alpha_1 = 7$, $\alpha_2 = 6$, $\theta_1 = 0.1$, $\theta_2 = 0.6$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\gamma = -0.2$.
- Scheme V: $\alpha_1 = 7$, $\alpha_2 = 6$, $\theta_1 = 0.1$, $\theta_2 = 0.3$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\gamma = -0.2$.
- Scheme VI: $\alpha_1 = 3$, $\alpha_2 = 4$, $\theta_1 = 0.8$, $\theta_2 = 0.1$, $a_1 = 0.4$, $a_2 = 0.5$, $\delta_1 = 0.5$, $\delta_2 = 0.8$, $\gamma = -0.7$.

(2) Compute the ML estimators for the 1000 samples, say $\widehat{\mathbf{\Delta}}_j$ for $j = 1, 2, ..., 1000$.

(3) Calculate bias and mean squared errors (MSE) for $N = 1000$ samples.

The simulation outcomes, detailed in Tables 1–6 for schemes I–VI, offer a thorough evaluation of the efficacy of the ML estimate method under diverse scenarios. In both approaches, it is apparent that the bias of the estimators diminishes consistently as the sample size n increases, converging to zero for bigger samples. Likewise, the MSE demonstrates a declining tendency as n increases, ultimately approaching zero in large samples. These observations validate that the ML estimators are unbiased, consistent, and efficient as the sample size increases. The findings demonstrate that the ML technique exhibits consistent performance across various schemes and sample sizes, underscoring its robustness and appropriateness for actual applications in parameter estimation.

Table 1. Simulation results for the BMOODaEx parameters under scheme I.

Parameter	n	$\|Bias\|$	MSE	Parameter	n	$\|Bias\|$	MSE
α_1	20	0.37455180	0.31266739	θ	20	0.56489364	0.43645487
	50	0.25376856	0.21323452		50	0.43256355	0.31353780
	100	0.16756884	0.14385691		100	0.32894565	0.26458411
	200	0.10748634	0.07489761		200	0.19469500	0.14378642
	350	0.02295765	0.00847490		350	0.12330247	0.08845722
	500	0.00063784	0.00011337		500	0.02284655	0.00438562
α_2	20	0.66048761	0.43854910	a	20	0.39568371	0.27356453
	50	0.43977764	0.31694055		50	0.31238695	0.17489665
	100	0.29465531	0.22849572		100	0.23057658	0.10476656
	200	0.21465632	0.16684962		200	0.14956842	0.04437851
	350	0.11837478	0.09347665		350	0.10565985	0.00438545
	500	0.05648785	0.00835641		500	0.04659941	0.00032264
α_3	20	0.20587762	0.17655490	δ	20	0.22396946	0.16496925
	50	0.18549876	0.13994624		50	0.16386354	0.11263574
	100	0.13334865	0.10487660		100	0.11397059	0.04374598
	200	0.10046156	0.08846819		200	0.07735423	0.00765185
	350	0.05547831	0.00845275		350	0.00394582	0.00053746
	500	0.00128456	0.00054861		500	0.00014496	0.00003284

Table 2. Simulation results for the BMOODaEx parameters under scheme II.

Parameter	n	$\|Bias\|$	MSE	Parameter	n	$\|Bias\|$	MSE
α_1	20	0.72541478	0.54853234	θ	20	0.33254707	0.25485571
	50	0.53484510	0.39568365		50	0.26486931	0.17548931
	100	0.33268491	0.26465401		100	0.15649478	0.11294719
	200	0.20468495	0.15438465		200	0.10468536	0.05479013
	350	0.12678459	0.08846831		350	0.06649635	0.00338566
	500	0.04438641	0.00468508		500	0.00335984	0.00049567
α_2	20	0.47859901	0.36965491	a	20	0.27458903	0.17459491
	50	0.37159540	0.28856836		50	0.21463873	0.14047983
	100	0.23856815	0.19565488		100	0.14375631	0.09454753
	200	0.18565936	0.12475689		200	0.10365945	0.00437864
	350	0.12957641	0.06659863		350	0.06639531	0.00054873
	500	0.08856784	0.00336471		500	0.00294683	0.00003408
α_3	20	0.56377345	0.34745891	δ	20	0.19468465	0.13864791
	50	0.38946610	0.25487891		50	0.16398645	0.11494865
	100	0.27404657	0.18654806		100	0.12843584	0.08569461
	200	0.18845810	0.13349649		200	0.10465478	0.02756849
	350	0.12648961	0.06648530		350	0.04438647	0.00659643
	500	0.04385459	0.0006598		500	0.00054781	0.00009075

Table 3. Simulation results for the BMOODaEx parameters under scheme III.

Parameter	n	$\|Bias\|$	MSE	Parameter	n	$\|Bias\|$	MSE
α_1	20	0.38755698	0.32548575	θ	20	0.54857859	0.39548857
	50	0.30857550	0.26947508		50	0.39569671	0.30567471
	100	0.21894758	0.15547580		100	0.26649590	0.18160466
	200	0.12947569	0.08595654		200	0.15538956	0.10947665
	350	0.09459587	0.00159573		350	0.07759734	0.00537581
	500	0.00085651	0.00000597		500	0.00065175	0.00003568
α_2	20	0.33068618	0.26658592	a	20	0.41048575	0.38857920
	50	0.25374585	0.20576960		50	0.33859657	0.27904566
	100	0.14436486	0.11068793		100	0.20585806	0.16508367
	200	0.05547589	0.00638463		200	0.13306759	0.08697793
	350	0.00058057	0.00006387		350	0.00465926	0.00065846
	500	0.00000587	0.00000071		500	0.00008576	0.00000028
α_3	20	0.28465872	0.23256773	δ	20	0.37760374	0.29958760
	50	0.17749576	0.14427460		50	0.28058604	0.21109868
	100	0.13289465	0.08365485		100	0.17505738	0.10876509
	200	0.05759730	0.00648515		200	0.07495682	0.00596763
	350	0.00058765	0.00007451		350	0.00158756	0.00008797
	500	0.00000873	0.00000046		500	0.00006857	0.00000598

Table 4. Simulation results for the BFGMODaEx parameters under scheme IV.

Parameter	n	$\|Bias\|$	MSE	Parameter	n	$\|Bias\|$	MSE
α_1	20	0.64727462	0.37485291	a_1	20	0.39456641	0.32259385
	50	0.49465671	0.26436840		50	0.28465541	0.24386410
	100	0.34356701	0.18454781		100	0.21404876	0.18464820
	200	0.21478492	0.12468305		200	0.14385413	0.11234865
	350	0.13947614	0.07745831		350	0.07748623	0.02238751
	500	0.08845712	0.00235371		500	0.00094375	0.00033561
α_2	20	0.59946731	0.33748955	a_2	20	0.44384605	0.31548831
	50	0.42483678	0.23845714		50	0.32169510	0.25548839
	100	0.31730937	0.15434784		100	0.23326063	0.17495414
	200	0.18458361	0.10455731		200	0.16649510	0.12385304
	350	0.11047656	0.07745831		350	0.08846482	0.01135754
	500	0.02248641	0.00084965		500	0.00043875	0.00008459
θ_1	20	0.23496411	0.19994165	δ_1	20	0.53687821	0.39943681
	50	0.19454378	0.15038459		50	0.41326187	0.26553489
	100	0.14376384	0.12046851		100	0.31793594	0.17648094
	200	0.10464645	0.07745813		200	0.23846539	0.12896574
	350	0.03325741	0.00649351		350	0.14268409	0.08947631
	500	0.00094647	0.00033284		500	0.09436811	0.00436573
θ_2	20	0.26486914	0.18465835	δ_2	20	0.51738538	0.41046374
	50	0.22946810	0.15385349		50	0.39464187	0.29047651
	100	0.17745824	0.11380452		100	0.31654783	0.21543756
	200	0.13283504	0.06534821		200	0.20468316	0.13337458
	350	0.07486351	0.00443789		350	0.13838904	0.09458351
	500	0.00174589	0.00033256		500	0.08568381	0.00540623
γ	20	0.21934640	0.16349932				
	50	0.16639462	0.12253726				
	100	0.12146803	0.08864541				
	200	0.07846510	0.00954779				
	350	0.01993467	0.00053795				
	500	0.00328462	0.00002783				

Table 5. Simulation results for the BFGMODaEx parameters under scheme V.

Parameter	n	$\|Bias\|$	MSE	Parameter	n	$\|Bias\|$	MSE
α_1	20	0.71538384	0.66348741	a_1	20	0.21947640	0.17453894
	50	0.52947689	0.47836891		50	0.16649531	0.13334868
	100	0.31784665	0.28865457		100	0.13698841	0.10454839
	200	0.14473908	0.11037646		200	0.11849740	0.06453891
	350	0.11847574	0.07730748		350	0.05487719	0.00204683
	500	0.04938663	0.00394681		500	0.00184681	0.00069027
α_2	20	0.81965241	0.56093581	a_2	20	0.28496854	0.21538963
	50	0.58956649	0.38745920		50	0.24674946	0.17245825
	100	0.37458910	0.23783496		100	0.17749532	0.13084375
	200	0.21946793	0.16499718		200	0.12643854	0.08469322
	350	0.13547898	0.12946841		350	0.06454930	0.00284764
	500	0.06495286	0.00843678		500	0.00343791	0.00019476
θ_1	20	0.36608461	0.30375146	δ_1	20	0.45473889	0.32876544
	50	0.27459301	0.23186541		50	0.34679810	0.25934566
	100	0.19345785	0.16485913		100	0.23047661	0.17496028
	200	0.13327593	0.10468532		200	0.17746892	0.12183845
	350	0.07495489	0.01046841		350	0.12056942	0.08457809
	500	0.00016457	0.00006455		500	0.04437982	0.00649531
θ_2	20	0.28857619	0.18457288	δ_2	20	0.39946714	0.28496754
	50	0.23946871	0.14931764		50	0.31638934	0.24438619
	100	0.16586630	0.11846794		100	0.26649691	0.18936547
	200	0.12045681	0.03372694		200	0.20474265	0.14386589
	350	0.07595531	0.00144783		350	0.13304762	0.11009843
	500	0.00332745	0.00025438		500	0.08547521	0.03467821
γ	20	0.18451069	0.15583610				
	50	0.15549761	0.13062064				
	100	0.12075417	0.10084671				
	200	0.04437595	0.02504387				
	350	0.00283574	0.00105264				
	500	0.00005372	0.00003975				

Table 6. Simulation results for the BFGMODaEx parameters under scheme VI.

Parameter	n	$\|Bias\|$	MSE	Parameter	n	$\|Bias\|$	MSE
α_1	20	0.45758820	0.36647851	a_1	20	0.27756585	0.21435674
	50	0.32957694	0.25547528		50	0.19967693	0.13304757
	100	0.23385690	0.15248593		100	0.11285648	0.04375478
	200	0.14468096	0.07495752		200	0.07594076	0.00058367
	350	0.07758753	0.00066585		350	0.00857104	0.00009571
	500	0.00008678	0.00000037		500	0.00000687	0.00000006
α_2	20	0.39575970	0.31579506	a_2	20	0.14304855	0.12047675
	50	0.31108677	0.24865857		50	0.11936464	0.07495756
	100	0.25538565	0.16584976		100	0.03367509	0.00849572
	200	0.13850630	0.06658563		200	0.00648461	0.00008670
	350	0.01746558	0.00967653		350	0.00008475	0.00000258
	500	0.00038759	0.00008675		500	0.00000058	0.00000001
θ_1	20	0.48856901	0.37596757	δ_1	20	0.26389467	0.23058667
	50	0.39857576	0.29967530		50	0.21108475	0.18859646
	100	0.28859576	0.18068655		100	0.14375485	0.11049757
	200	0.19586963	0.16869593		200	0.06484689	0.00354875
	350	0.07769873	0.00305852		350	0.00048752	0.00000968
	500	0.00094372	0.00000957		500	0.00000585	0.00000003
θ_2	20	0.65487798	0.54058562	δ_2	20	0.32874685	0.28847655
	50	0.55649402	0.38595766		50	0.25474649	0.21058657
	100	0.39957691	0.26658462		100	0.16384695	0.10206875
	200	0.27955062	0.18859573		200	0.08486421	0.00328658
	350	0.11900586	0.03485065		350	0.00058756	0.00004301
	500	0.00856874	0.00005857		500	0.00005852	0.00000392
γ	20	0.28850967	0.22146780				
	50	0.24068475	0.17659475				
	100	0.17769745	0.10059857				
	200	0.10987653	0.00860583				
	350	0.05775832	0.00043840				
	500	0.00075992	0.00003095				

5. Data analysis: Statistical criteria and fitting

In this section, we illustrate the experimental relevance of the BMOODa-G family and the BFGMODa-G family through two applications to real datasets. The tested distributions are assessed using several criteria, including negative maximum log-likelihood ($-L$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC).

5.1. Dataset I

These data were generated by the US National Climatic Data Center (NCDC) and are available at *https://www.ncdc.noaa.gov*, which includes data from 51 of the largest cities in the USA, where X_1 stands for average precipitation (in millimeters) and X_2 to average maximum temperature (in degrees Celsius). To discuss the behavior of data set I, a scatter plot can be used. A scatter plot is a fundamental tool for visualizing the relationship between two continuous variables. It allows you to quickly assess patterns, correlations, and outliers within your data, making it a valuable tool in data analysis and data-driven decision-making. Figure 9 shows scatter and box plots for dataset I, showing that the dataset is asymmetric and has extreme points.

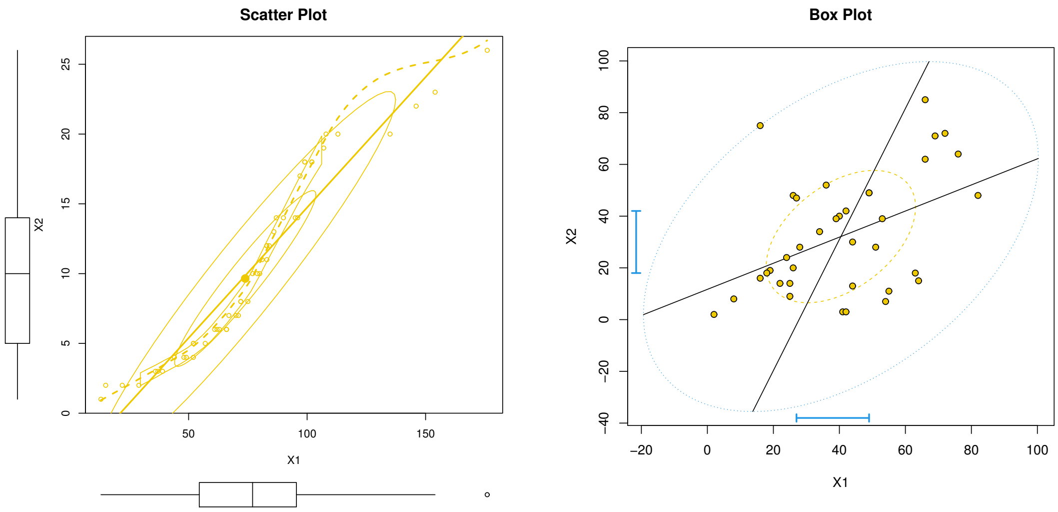

Figure 9. Scatter and box plots of Dataset I.

We consider the BFGMODaEx model to analyze the first dataset, and compare it with other popular bivariate models, such as the bivariate FGM exponential (BFGMEx), bivariate FGM inverted Top Leone (BFGMITL), and FGM generalized half-logistic (BFGMGHL) distributions. We fit at first the marginals X_1 and X_2 separately. The MLEs of the parameters α, θ, a, and δ of the corresponding ODaEx distribution for X_1 and X_2 are (0.438, 0.117, 4.756, 0.00525), and (1.244, 3.918, 1.039, 0.1896), respectively. The $-L$, Kolmogorov-Smirnov (KS) test, and its corresponding P-value for the marginals are listed in Table 7. The non-parametric visualization plots for the marginals X_1 and X_2 of dataset I are given in Figures 10 and 11.

Table 7. The goodness-of-fit test of the marginals for Dataset I.

		X_1			X_2	
Model	$-L$	KS	P-value	$-L$	KS	P-value
ODaEx	248.347	0.052	0.999	163.011	0.094	0.759

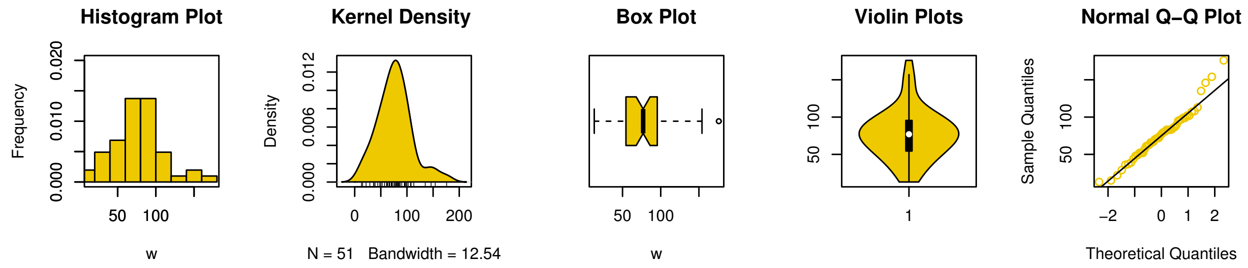

Figure 10. Nonparametric visualization plots for the marginal X_1 of Dataset I.

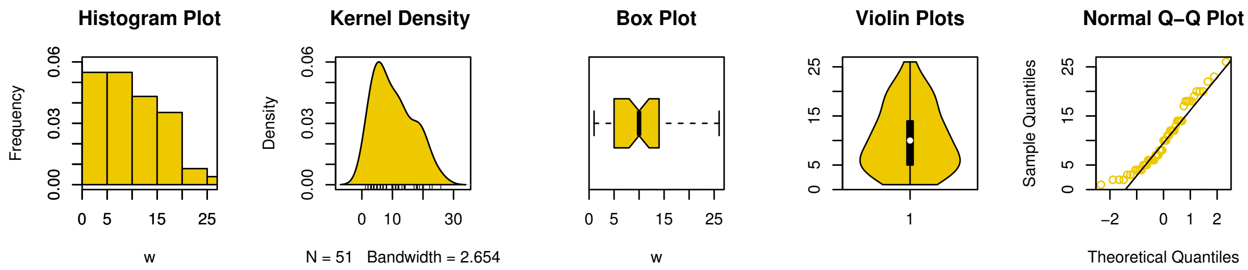

Figure 11. Nonparametric visualization plots for the marginal X_2 of Dataset I.

The KS is a statistical hypothesis test used to determine if dataset follows a specified probability distribution. It is a nonparametric test, which means it does not assume a particular distribution for the data. The KS test is particularly useful when you want to assess whether a sample of data is consistent with a theoretical distribution or when comparing two samples to see if they come from the same distribution. It is clear that the BFGMODaEx model fits the marginal data. The estimated CDF and probability probability (PP) plots are shown in Figures 12 and 13, which support our results in Table 7.

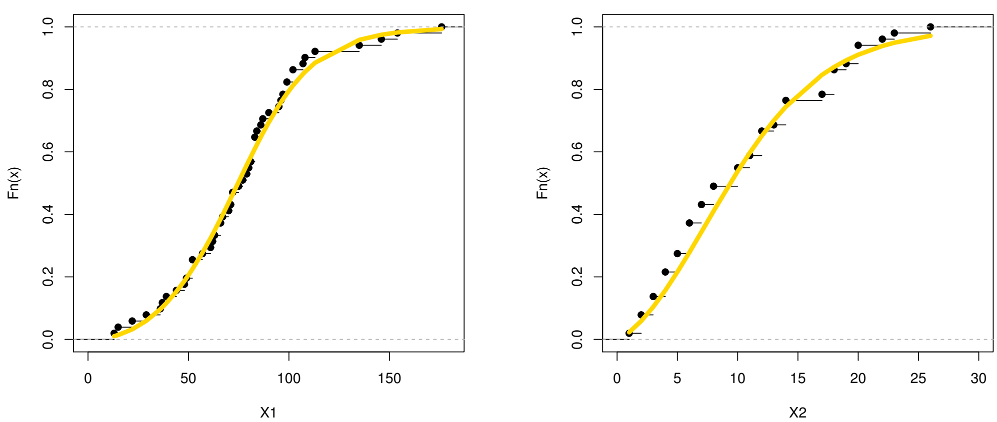

Figure 12. The estimated CDF for the marginals X_1 and X_2 using Dataset I.

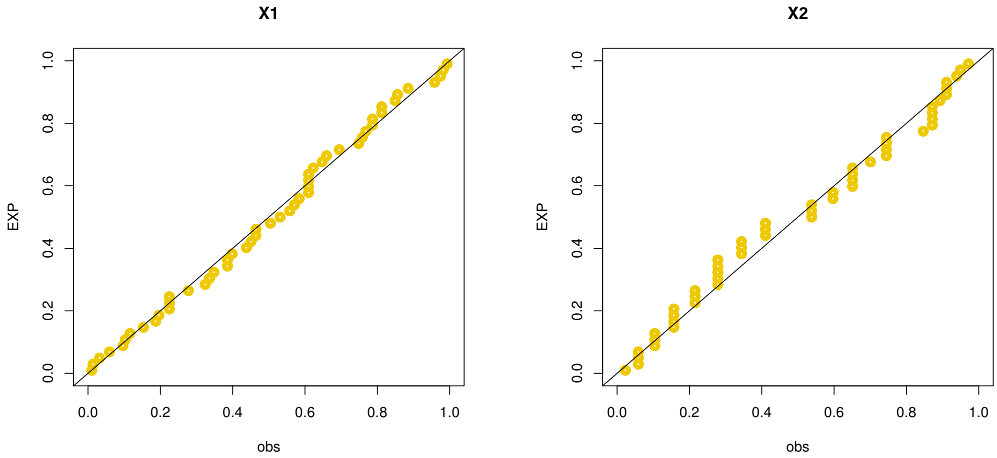

Figure 13. The PP plots for the marginals X_1 and X_2 using Dataset I.

After discussing the fit of the margins of the BFGMODaEx distribution to the first dataset, we are now able to test the BFGMODaEx model for fit to the bivariate data. Table 8 shows some statistical criteria for competitive models.

Table 8. The goodness-of-fit test of the competitive models for Dataset I.

Model	MLEs	$-L$	AIC	CAIC	BIC	HQIC
BFGMEx	$\widehat{\alpha}_1 = 0.014, \widehat{\alpha}_2 = 0.111, \widehat{\gamma} = -0.9999$	441.714	889.428	889.938	895.223	891.642
BFGMITL	$\widehat{\alpha}_1 = 0.338, \widehat{\alpha}_2 = 0.755, \widehat{\gamma} = 0.9999$	523.228	1052.46	1052.97	1058.25	1054.67
BFGMGHL	$\widehat{\alpha}_1 = 37.738, \widehat{\alpha}_2 = 5.683, \widehat{\gamma} = 0.2744$	431.267	872.534	873.868	882.193	876.225
	$\widehat{\beta}_1 = 0.677, \widehat{\beta}_2 = 0.762$					
BFGMODaEx	$\widehat{\alpha}_1 = 2.751, \widehat{\alpha}_2 = 5.386, \widehat{\gamma} = -0.307,$	411.972	841.945	846.335	859.331	848.589
	$\widehat{\theta}_1 = 5.206, \widehat{\theta}_2 = 0.767, \widehat{\delta}_1 = 0.059$					
	$\widehat{\delta}_2 = 0.426, \widehat{a}_1 = 0.678, \widehat{a}_2 = 0.436$					

It is noted that the BFGMODaEx model is the best among all tested models. To demonstrate the uniqueness property of each estimator, the log-likelihood profile is plotted in two and three dimensions. The results are shown in Figures 14 and 15.

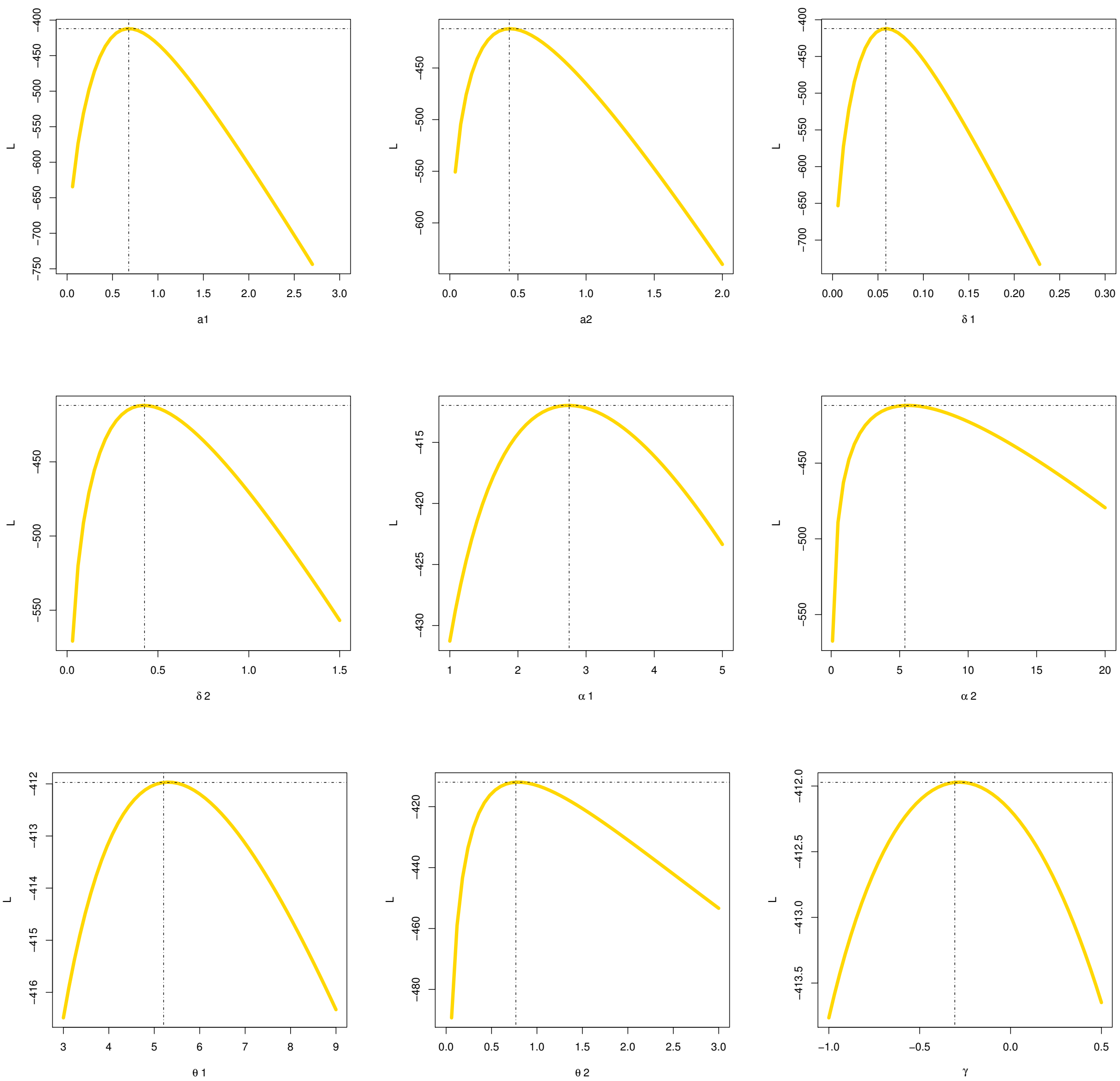

Figure 14. The 2D log-likelihood profiles of BFGMODaEx parameter estimators for Dataset I.

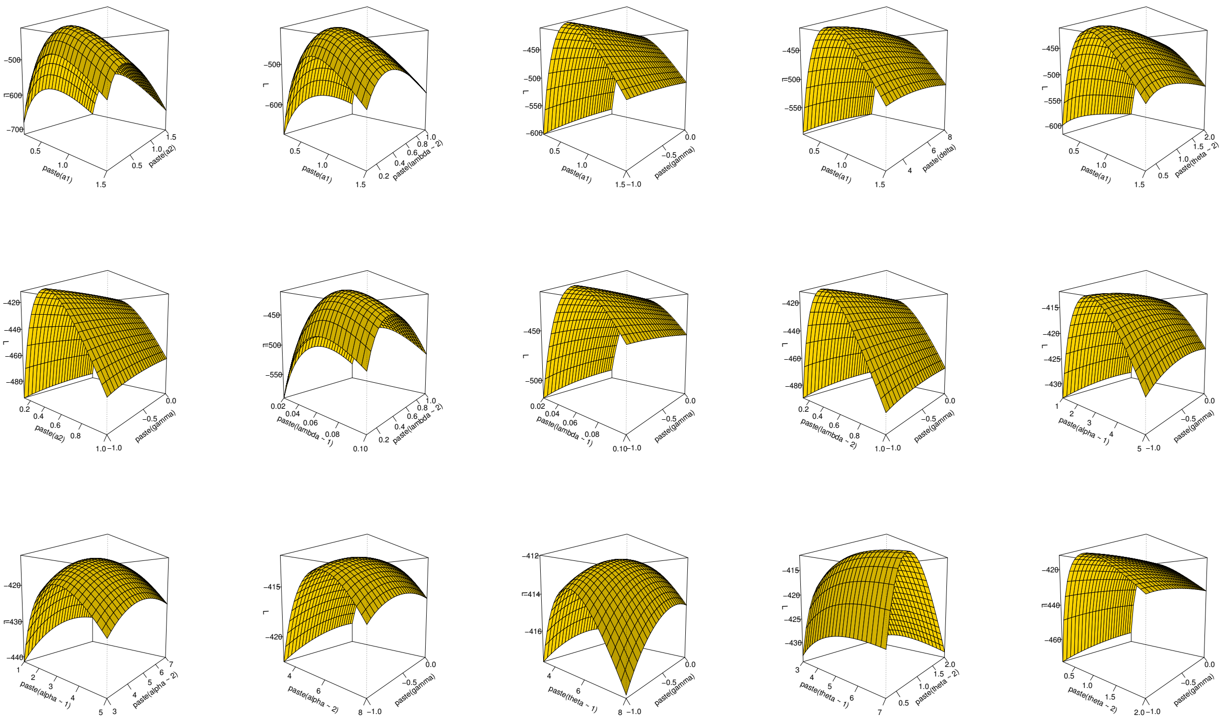

Figure 15. The log-likelihood profiles of BFGMODaEx parameter estimators for Dataset I.

Another way to prove the uniqueness property of estimators is to use contour plots. Figure 16 proves our claim.

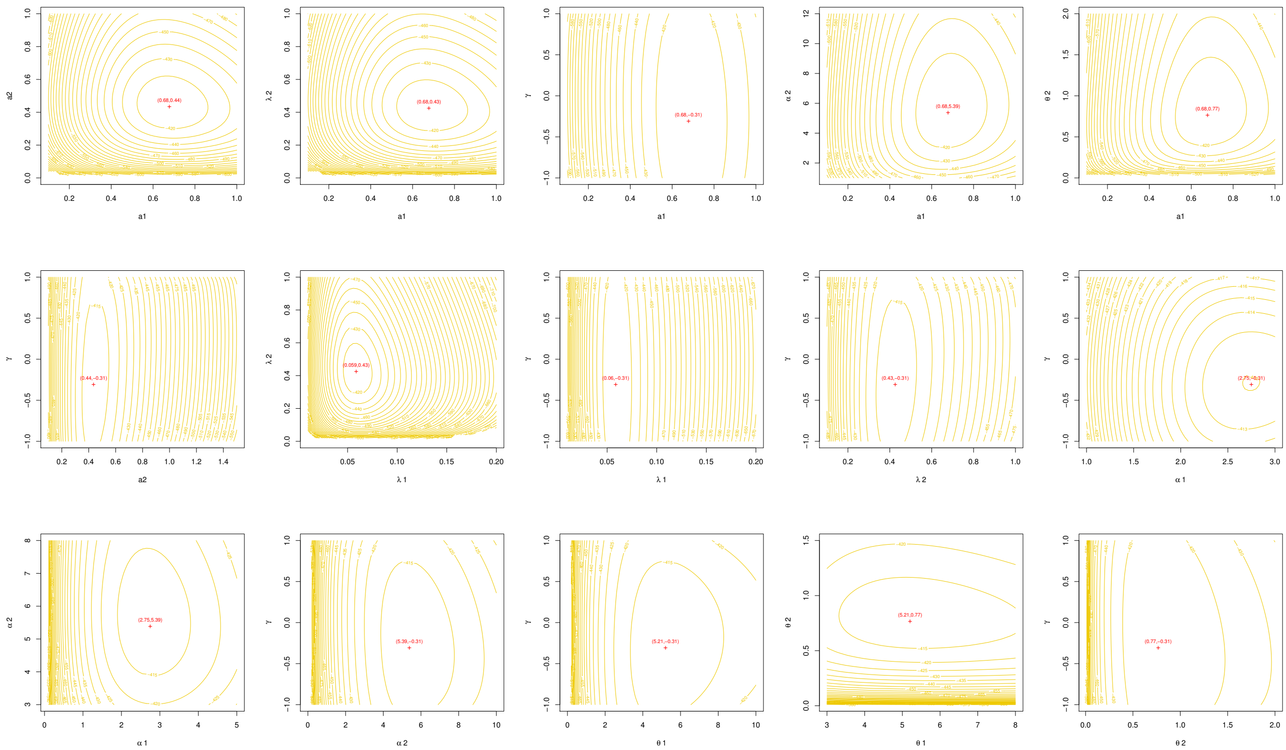

Figure 16. Contour diagrams of BFGMODaEx parameter estimators for Dataset I.

A contour plot, also known as a contour chart or contour map, is a graphical representation of a three-dimensional surface in two dimensions. It is a way to visualize differences in a continuous function of two variables. Contour charts are especially useful for displaying data that depends on two independent variables.

5.2. Dataset II

These data were obtained by Meintanis [25], and represent football matches in which at least one goal was scored by any team from a penalty kick, and the home team must score at least one goal. Figure 17 shows scatter and box plots of such soccer score data. We consider the BMOOGaEx model to analyze these dataset, and compare it with other popular bivariate models, such as bivariate generalized exponential (BGEx), bivariate exponential (BEx), bivariate exponential Gumbel exponential (BGuEx), bivariate generalized linear failure rate (BGLFR), bivariate Weibull (BW), bivariate exponential Weibull (BEW), bivariate generalized Weibull distributions (BGPW), and Gompertz (BGz) distributions. We fit at first the marginals X_1, and X_2 separately as well as $\min(X_1, X_2)$. The MLEs of the parameters α, θ, a, and δ of the corresponding ODaEx distribution for X_1, X_2, and $\min(X_1, X_2)$ are (0.176, 0.055, 7.14, 0.0077), (4.619, 0.623, 0.498, 0.101), and (30.467, 0.096, 0.391, 0.148), respectively. The $-L$, KS test and its corresponding P-value for the marginals are reported in Table 9. The nonparametric visualization plots for the marginals X_1, X_2, and $\min(X_1, X_2)$ of dataset II are given in Figures 18–20.

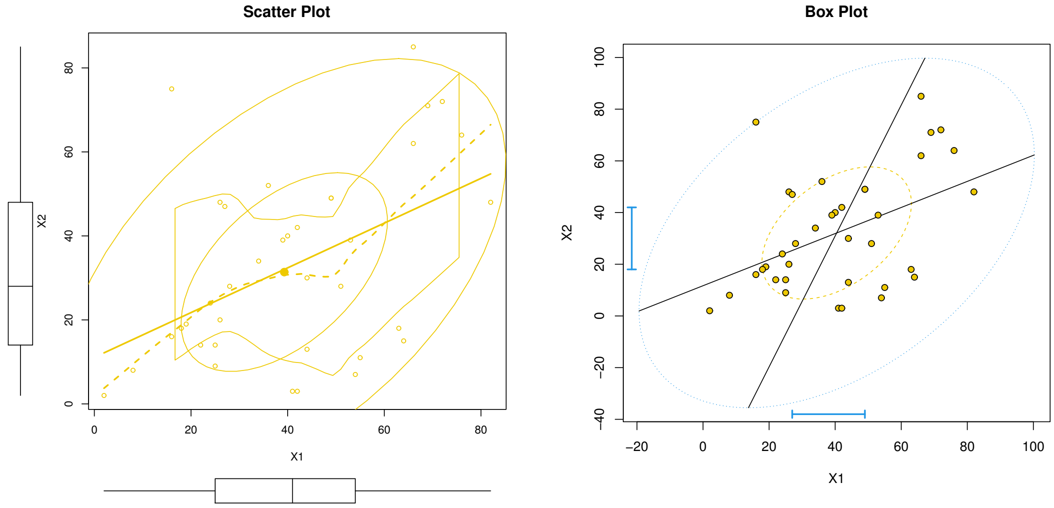

Figure 17. Scatter and box plots of Dataset II.

Table 9. The goodness-of-fit test of the marginals for Dataset II.

	X_1			X_2			$\min(X_1, X_2)$		
Model	$-L$	KS	P-value	$-L$	KS	P-value	$-L$	KS	P-value
ODaEx	162.587	0.099	0.858	163.303	0.106	0.803	158.193	0.062	0.999

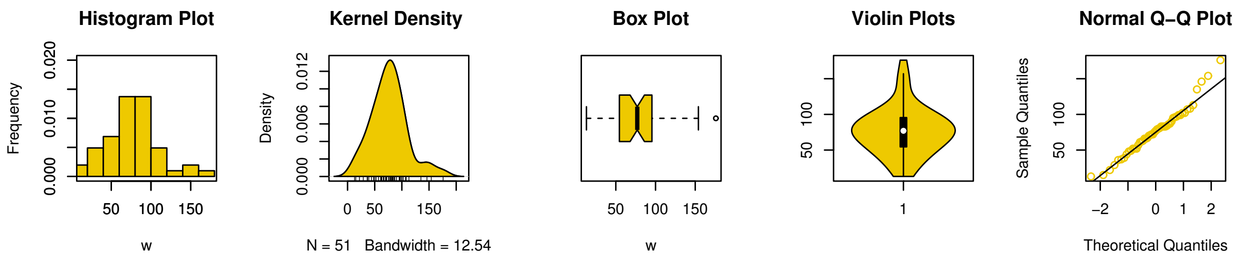

Figure 18. Nonparametric visualization plots for the marginal X_1 of Dataset II.

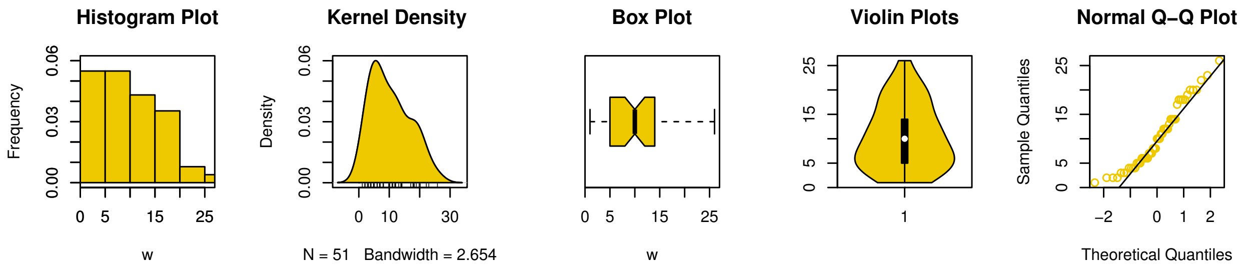

Figure 19. Nonparametric visualization plots for the marginal X_2 of Dataset II.

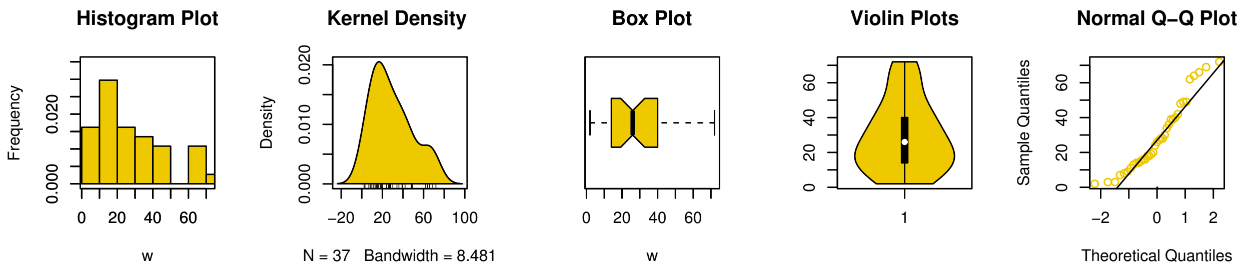

Figure 20. Nonparametric visualization plots for $\min(X_1, X_2)$ of Dataset II.

It is clear that the BMOODaEx model fits the marginal data. The estimated CDF and PP plots are displayed in Figures 21 and 22, which support our results in Table 9.

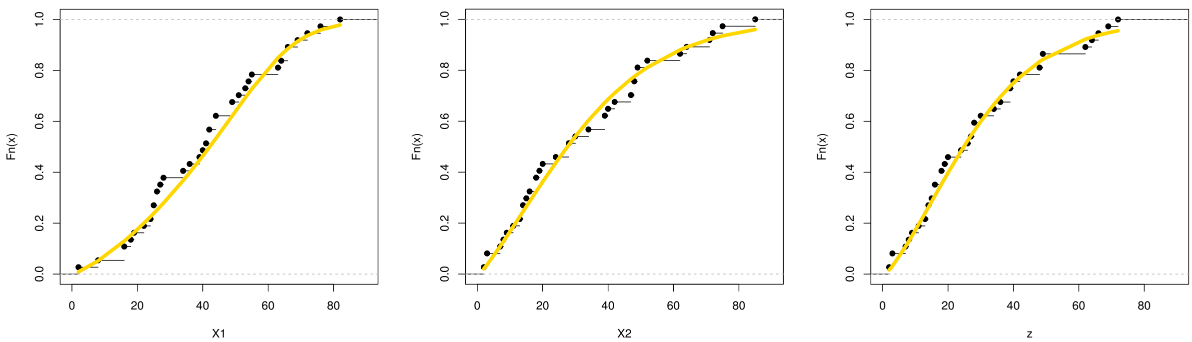

Figure 21. The estimated CDF for the marginals using Dataset II.

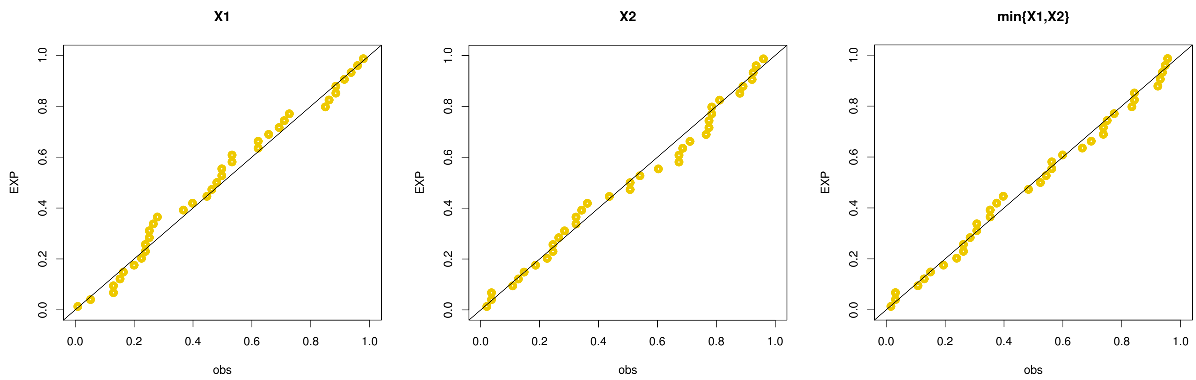

Figure 22. The PP plots for the marginals using Dataset II.

Now, we fit the BMOODaEx model on dataset II. The attached Table 10 lists some statistical criteria for competitive models.

Table 10. The goodness-of-fit test of the competitive models for Dataset II.

Model	MLEs	$-L$	AIC	CAIC	BIC	HQIC
BGz	$\widehat{\alpha}_1 = 0.033, \widehat{\alpha}_2 = 0.002, \widehat{\alpha}_3 = 0.021, \widehat{\rho} = 0.040$	303.48	614.97	616.22	621.41	617.24
BGPW	$\widehat{\alpha}_1 = 3.229, \widehat{\alpha}_2 = 1.983, \widehat{\alpha}_3 = 4.084, \widehat{\epsilon} = 0.037$	344.76	697.53	698.78	703.97	699.79
BEW	$\widehat{\alpha}_1 = 1.227, \widehat{\alpha}_2 = 0.382, \widehat{\alpha}_3 = 0.661,$ $\widehat{a} = 0.012, \widehat{b} = 1.268$	298.93	607.86	609.79	615.91	610.69
BW	$\widehat{\alpha}_1 = 0.397, \widehat{\alpha}_2 = 0.274, \widehat{\alpha}_3 = 0.339, \widehat{\beta} = 0.083$	346.00	700.00	701.25	706.44	702.27
BGLFR	$\widehat{\alpha}_1 = 0.452, \widehat{\alpha}_2 = 0.156, \widehat{\alpha}_3 = 0.360,$ $\widehat{\eta} = 0.0002, \widehat{\tau} = 0.0008$	296.84	603.68	605.62	611.73	606.52
BGuEx	$\widehat{\alpha}_1 = 2.678, \widehat{\alpha}_2 = 0.962, \widehat{\alpha}_3 = 2.065,$ $\widehat{c} = 5.011, \widehat{\sigma} = 4.081$	297.77	605.55	607.48	613.60	608.39
BEx	$\widehat{\alpha}_1 = 0.012, \widehat{\alpha}_2 = 0.014, \widehat{\alpha}_3 = 0.022$	298.93	607.86	609.79	615.91	610.69
BGEx	$\widehat{\alpha}_1 = 1.553, \widehat{\alpha}_2 = 0.499, \widehat{\alpha}_3 = 1.156, \widehat{\rho} = 0.039$	299.86	607.72	608.97	614.16	609.99
BMOODaEx	$\widehat{\alpha}_1 = 0.0802, \widehat{\alpha}_2 = 0.0299, \widehat{\alpha}_3 = 0.074,$ $\widehat{\theta} = 0.855, \widehat{a} = 7.434, \widehat{\delta} = 0.0095$	292.24	596.49	599.286	606.152	599.89

It is noted that the BMOODaEx model is the best among all tested models. To test the uniqueness property of each estimator, the log-likelihood profile is plotted in two and three dimensions as well as the contour diagrams. The results are reported in Figures 23–25.

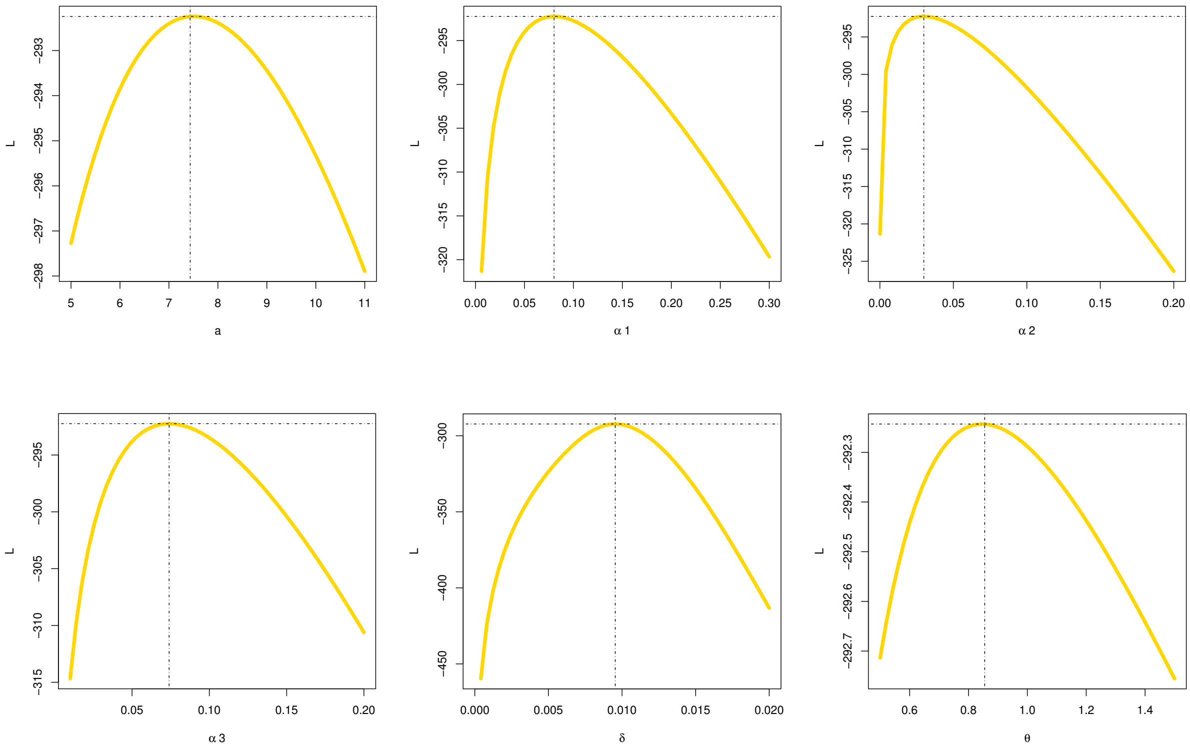

Figure 23. The 2D log-likelihood profiles of BMOODaEx parameter estimators for Dataset II.

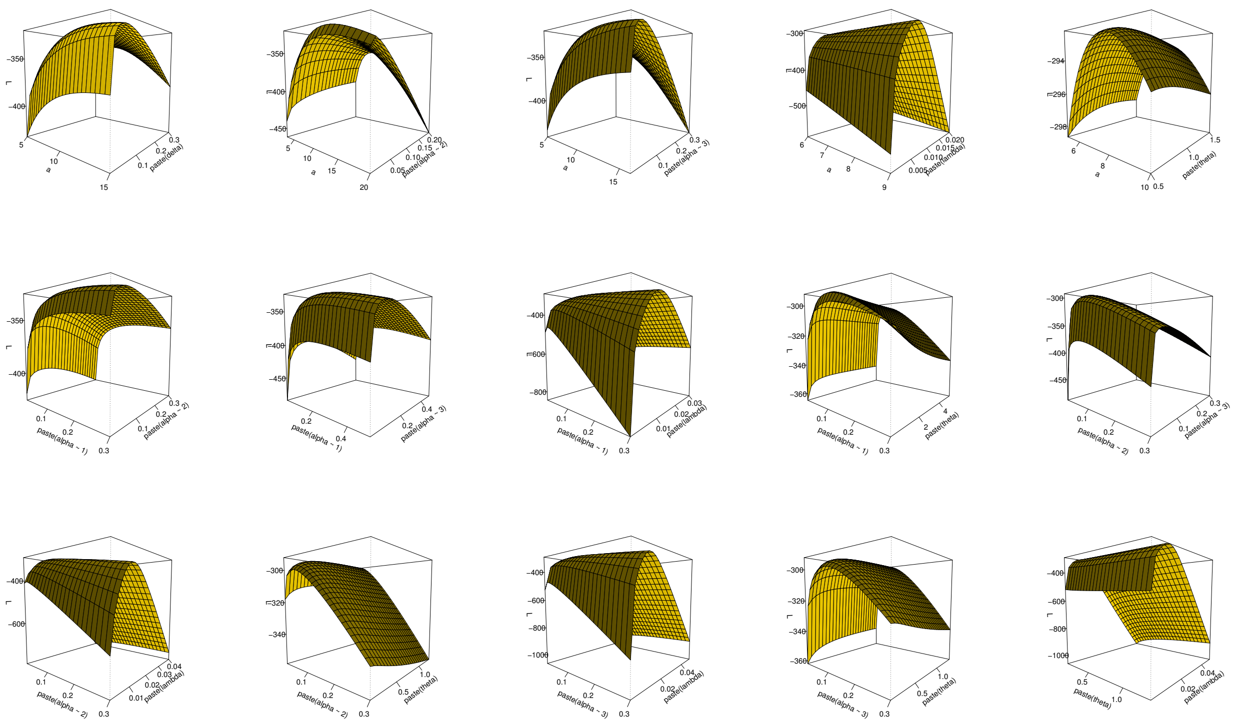

Figure 24. The log-likelihood profiles of BMOODaEx parameter estimators for Dataset II.

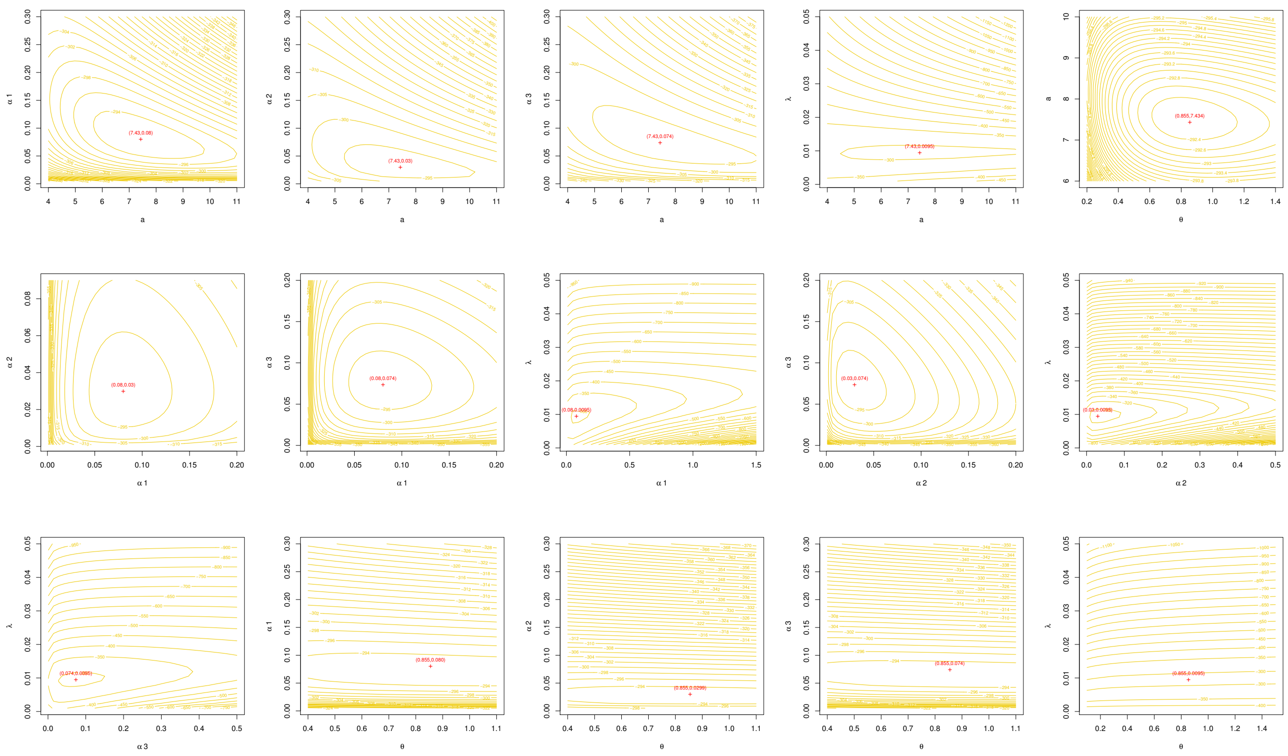

Figure 25. The log-likelihood profiles of BMOODaEx parameter estimators for Dataset II.

6. Concluding remarks

We have developed two new bivariate elastic generators in this article: the bivariate Marshall-Olkin odd Dagum-G (BMOOGE-G) and the bivariate Farlie-Gumbel-Morgenstern odd Dagum-G (BFGMOGE-G) families. The marginal distributions of these newly proposed generator classes are consistent with the OGE-G family. We have also shown that the joint cumulative distribution function and joint probability density function of the BMOOGE-G and BFGMOGE-G families can be written down in a clear way. This makes them easy to use for modeling bivariate data in the $(0, \infty)$ range. Our investigation encompassed the extraction and thorough examination of distribution statistics, underscoring the efficacy of the BMOOGE-G and BFGMOGE-G families in modeling asymmetric data characterized by varying failure rates. These generators are also quite good at modeling extreme data. Additionally, we underscored that the stress-strength model's dependency is exclusively on the parameters of the bivariate generator inside the BMOOGE-G class, rather than on fundamental functions. The maximum likelihood estimation method was used to figure out the parameters that are common to these families. The simulation results showed that this method worked well for estimating parameters. After that, we looked at three real datasets, not only to save computing power but also to show how well our new generator works. Based on the empirical data, these new sub-models are strong competitors in the statistical literature, competing with existing well-known bivariate models. In the future, our research will focus on the complicated multivariate extension of the OGE-G family. This area has several uses, including life span analysis, environmental sciences, economics, engineering, and medical sciences. In conclusion, we expect that our new bivariate families will get more attention and be used in a wide range of domains, including engineering, survival data analysis, economics, and more.

Author contributions

The author confirms sole responsibility for the conception of the study, design of methodology, data curation, formal analysis, software coding, presentation of results, and preparation of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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