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Research article

Improved convergence analysis on the accelerated modulus-based matrix splitting iteration method for nonlinear complementarity problems

Yanmei Chen[1], **Yihang Lin**[1] **and Jianwei Dong**[2,*]

[1] School of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, China
[2] College of Medical Information Engineering, Guangdong Pharmaceutical University, Guangzhou, China

* **Correspondence:** Email: dongjianwei@gpdu.edu.cn, djw8026@163.com.

Abstract: In this paper, we focused on the accelerated modulus-based matrix splitting iteration method for solving nonlinear complementarity problems. A thorough analysis of convergence conditions for the method was conducted. Compared to the work "B. H. Huang, C. F. Ma, Accelerated modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems, *Comp. Appl. Math.*, **37** (2018), 3053–3076", our results achieved three significant improvements: Relaxing the assumptions on matrix splittings, providing an expanded convergence domain for parameter matrix, and simplifying conditions for relaxation parameters. The validity of the theoretical findings was verified by numerical examples.

Keywords: nonlinear complementarity problem; modulus-based method; H_+-matrix; H-splitting
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1. Introduction

Consider a class of nonlinear complementarity problems (NCP) defined as follows: a matrix $A \in \mathbb{R}^{n\times n}$ and a nonlinear mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, find a vector $z \in \mathbb{R}^n$ such that

$$z \geq 0,\ \ r = Az + \phi(z) \geq 0 \text{ and } z^T r = 0.$$

The NCP has widespread applications in scientific computing, economics, engineering, and many other fields (see [1, 2]).

When $\phi(z) = q \in \mathbb{R}^n$, the NCP reduces to the classical linear complementarity problem (LCP), which arises in numerous areas; see [3, 4]. Modulus-based matrix splitting (MMS) iteration methods have emerged as an efficient class of solvers for the LCP, attracting significant research interest. These methods are derived from the equivalent fixed-point formulation of the LCP. Bai [5] first introduced the MMS iteration method, and subsequent works proposed acceleration techniques to improve its convergence rate (e.g., [6–16]). Compared to projected-type methods [3], the key advantage of MMS methods is that each iteration requires solving only a linear system than a projected LCP. When $\phi(z)$ is a general nonlinear function, the resulting NCP is said to exhibit weak nonlinearity. The weakly linear equations with weakly nonlinearity was first considered in [17], which is also equivalently called the mildly nonlinear system. In recent years, MMS iteration methods have been successfully extended to solve the NCP. The foundational MMS schemes for the NCP were developed in [18, 19]. Building on these works, improved techniques, analogous to those applied in the LCP case, have been adapted for the NCP and other related complementarity problems (see [20–27]).

To enhance the convergence rate of the MMS iteration method for solving the NCP, we focus particularly on the accelerated techniques. In the study of numerical methods for complementarity problems, the accelerated technique was first introduced by [13] for solving the LCP. The core idea is analogous to the construction principle of the Gauss-Seidel iteration method for solving linear systems [28], namely the strategy of partial updates and immediate utilization of the latest information. During the iteration process, once a new value for a component is computed, it immediately replaces the old value and participates in subsequent calculations, thereby accelerating convergence. Beyond the LCP, this technique has also been successfully applied in the MMS iteration methods for solving the NCP [23], horizontal LCP [29] and the generalized complex-valued horizontal LCP [30]. In this paper, we focus on the accelerated modulus-based matrix splitting (AMMS) iteration method for solving the NCP [23]. It is worth noting that in the convergence conditions established by [23], all matrix splitting assumptions are required to be H-compatible splittings. However, as is well-known, this assumption cannot always be guaranteed for the commonly used AOR splitting. Therefore, we conduct an in-depth investigation into the convergence analysis of the AMMS iteration method, with three major innovative contributions:

- Establishing new convergence results that relax the assumptions on matrix splittings;
- Obtaining a larger convergence domain of the parameter matrix;
- Performing comprehensive analysis of the widely-used AOR splitting method, thereby obtaining simpler conditions for relaxation parameters.

Next, after necessary preliminaries in Section 2, we present the new convergence analysis of the AMMS iteration method in Section 3. Subsequently, in Section 4, numerical experiments are conducted to validate the correctness of the theoretical results obtained. Finally, in Section 5, we conclude the paper with a summary.

2. Preliminaries

We begin by introducing some necessary notations, definitions, and lemmas.

For a matrix $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, use a_{ij} to represent the element in the i-th row and j-th column of A and x_i to represent the i-th component of x. Let $A = D_A - B_A = D_A - L_A - U_A$, where $D_A, -B_A, -L_A$, and $-U_A$ denote the diagonal, off-diagonal, strictly lower-triangular, and strictly upper-triangular parts of A, respectively. Same as [3], the modulus of A is defined by $|A| = (|a_{ij}|)$ and its comparison matrix $\langle A\rangle = (\langle a_{ij}\rangle)$ is given by $\langle a_{ij}\rangle = |a_{ij}|$ for $i = j$ and $\langle a_{ij}\rangle = -|a_{ij}|$ for $i \neq j$. Following [32], matrix A is called

- A Z-matrix if $a_{ij} \leq 0$ for all $i \neq j$;
- A nonsingular M-matrix if it is a nonsingular Z-matrix with $A^{-1} \geq 0$;
- An H-matrix if $\langle A \rangle$ is a nonsingular M-matrix;
- A strictly diagonally dominant (s.d.d.) matrix if $|a_{ii}| > \sum_{j=1, j\neq i} |a_{ij}|$ for all $1 \leq i \leq n$.

An H_+-matrix is an H-matrix with $a_{ii} > 0$ for every i; see [33]. The splitting $A = M - N$ is termed:

- An M-splitting if M is a nonsingular M-matrix and $N \geq 0$;
- An H-splitting if $\langle M \rangle - |N|$ is an M-matrix;
- An H-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$.

A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a uniform P-function on a subset $X \subseteq \mathbb{R}^n$ if there exists a constant $\eta > 0$ such that for every pair of distinct vectors x and y in X, the following inequality holds:

$$\max_{1\leq i\leq n} (x_i - y_i)(F_i(x) - F_i(y)) \geq \eta(x - y)^{\mathrm{T}}(x - y),$$

see [35].

Lemma 2.1. *Let A be an H-matrix. Then, we have $|A^{-1}| \leq \langle A \rangle^{-1}$.*

Lemma 2.2. *[31] Let $B \in \mathbf{R}^{n\times n}$ be a s.d.d. matrix. Then $\forall C \in \mathbf{R}^{n\times n}$,*

$$\|B^{-1}C\|_\infty \leq \max_{1\leq i\leq n} \frac{(|C|e)_i}{(\langle B \rangle e)_i},$$

where $e = (1, 1, \ldots, 1)^T \in \mathbf{R}^n$ and $\|\cdots\|_\infty$ denotes the infinity norm of matrix.

Lemma 2.3. *[32] $\rho(D_A^{-1}|B_A|) < 1$ holds if $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix.*

Lemma 2.4. *[32] Let A, B be two Z-matrices, A be a nonsingular M-matrix, and $B \geq A$.Then, B is a nonsingular M-matrix.*

Lemma 2.5. *[32] Let A be a nonsingular M-matrix. If $A = M - N$ is an M-splitting, then $\rho(M^{-1}N) < 1$.*

Lemma 2.6. *[32] Let A be a Z-matrix. Then, A is a nonsingular M-matrix if and only if there exists a positive diagonal matrix D making AD is an s.d.d. matrix.*

Lemma 2.7. *[32] $A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if A can be expressed as $A = sI - B$, where $s > 0$, $B \geq 0$, and $s > \rho(B)$.*

3. New results

We begin by reviewing the established AMMS method from [23], followed by the presentation of our new results.

Let $\Omega \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix, $\sigma > 0$ be a constant scalar parameter, and $A = M_1 - N_1 = M_2 - N_2$ represent two matrix splittings of A. Through the introduction of an auxiliary variable x and the transformation $z = \frac{1}{\sigma}(|x| + x)$, solving the NCP becomes equivalent to solving the following modulus equation system:

$$(\Omega + M_1)x = N_1 x + (\Omega - M_2)|x| + N_2|x| - \sigma\phi(z). \tag{3.1}$$

Based on (3.1), the AMMS method was established in [23].

Method 3.8. *[23] Given A, ϕ, Ω, σ and $x^{(1)}$, for $k = 1, 2, \ldots$, calculate $z^{(k)} = \frac{1}{\sigma}(x^{(k)} + |x^{(k)}|)$ until the sequences $\{z^{(k)}\}_{k=1}^{\infty}$ is convergent, where $x^{(k+1)}$ is solved from the following linear equation*

$$(\Omega + M_1)x^{(k+1)} = N_1 x^{(k)} + (\Omega - M_2)|x^{(k)}| + N_2|x^{(k+1)}| - \sigma\phi(z^{(k)}). \tag{3.2}$$

The iterative scheme (3.2) was designed with the idea of splitting the matrix A in front of the modulus term $|x^{(k)}|$ in the MMS iteration method into two parts through a second matrix splitting $A = M_2 - N_2$. Once a new value for any component is computed, it immediately replaces the corresponding old value in the latter part of the modulus term. Here, N_2 can be configured as a matrix with special structure (such as a strictly lower triangular matrix) to achieve this effect. In practical applications, the accelerated overrelaxation (AOR) splitting is commonly employed given by

$$\begin{cases} M_1 = \frac{1}{\alpha}(D_A - \beta L_A), N_1 = M_1 - A, \\ M_2 = D_A - U_A, N_2 = L_A, \end{cases} \tag{3.3}$$

where α and β are relaxation parameters. In this context, we refer to Method 3.8 as the AMAOR (accelerated modulus-based accelerated overrelaxation) iteration method. When $\alpha = \beta$, $\alpha = \beta = 1$ and $(\alpha,\beta) = (1, 0)$, it is called the AMSOR (accelerated modulus-based successive overrelaxation), AMGS (accelerated modulus-based Gauss-Seidel), and AMJ (accelerated modulus-based Jacobi) iteration methods, respectively.

Before presenting the new convergence analysis, we state the following assumptions on the nonlinear function ϕ of the NCP, which are analogous to those adopted in [18, 23]. Let

$$\phi(z) = (\phi_1(z_1), \phi_2(z_2), \ldots, \phi_n(z_n))^T,$$

where $z_i \in \mathbb{R}$ and $\phi_i(z_i)$ is differentiable with respect to $z_i, i = 1, 2, \ldots n$. Assume that $0 \leq \frac{\mathrm{d}\phi_i(z_i)}{\mathrm{d}z_i} \leq \psi_i$, where $\psi_i \in \mathbb{R}, i = 1, 2, \ldots, n$. By the Lagrange theorem [34], we have

$$\phi_i(z_i^{(k)}) - \phi_i(z_i^*) = \frac{\mathrm{d}\phi_i(\zeta_i^{(k)})}{\mathrm{d}z_i}(z_i^{(k)} - z_i^*), i = 1, 2, \ldots, n,$$

where $\zeta_i^{(k)}$ is selected as an intermediate value lying between $z_i^{(k)}$ and z_i^*. Denote

$$\Psi^{(k)} = \mathrm{diag}(\frac{\mathrm{d}\phi_1(\zeta_1^{(k)})}{\mathrm{d}z_1}, \frac{\mathrm{d}\phi_2(\zeta_2^{(k)})}{\mathrm{d}z_2}, \ldots, \frac{\mathrm{d}\phi_n(\zeta_n^{(k)})}{\mathrm{d}z_n})$$

and

$$\Psi = \mathrm{diag}(\psi_1, \psi_2, \cdots, \psi_n). \tag{3.4}$$

Then, one can get

$$\phi(z^{(k)}) - \phi(z^*) = \Psi^{(k)}(z^{(k)} - z^*) = \frac{1}{\sigma}\Psi^{(k)}[(x^{(k)} - x^*) + (|x^{(k)}| - |x^*|)] \tag{3.5}$$

and

$$\Psi^{(k)} \leq \Psi.$$

We now present the new convergence theorem for Method 3.8 under the assumption that A is an H_+-matrix. Recall that the aforementioned conditions imposed on the nonlinear function ϕ guarantee that $f(z) = Az + \phi(z)$ becomes a uniform P-function, which consequently ensures the existence and uniqueness of the solution to the NCP, as established in [35].

Theorem 3.9. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix. Assume that $A = M_1 - N_1 = M_2 - N_2$ are two H-splittings satisfying $\langle M_1\rangle \geq \langle M_2\rangle$. Then, $\forall x^{(1)} \in \mathbb{R}^n$, $\{z^{(k)}\}_{k=1}^{\infty}$ produced by Method 3.8 converges to the solution of the NCP provided*

$$\Omega \geq D_{M_2} + \Psi, \tag{3.6}$$

or

$$\frac{1}{2}(|M_2| + |N_2| - \langle M_1\rangle + |N_1| + 2\Psi)De < \Omega De \leq D_{M_2}De, \tag{3.7}$$

where Ψ is defined in (3.4) and D is a positive diagonal matrix satisfying $(\langle M_2\rangle - |N_2|)D$ being an s.d.d. matrix.

Proof. Since $\langle M_1\rangle \geq \langle M_1\rangle - |N_1|$, by Lemma 2.4, $\langle M_1\rangle$ is a nonsingular M-matrix. Note that A is an H_+-matrix, which implies that $D_A = D_{M_1} - D_{N_1} > 0$. If $D_{M_1} \leq 0$, we have $D_{N_1} \leq 0$ and $|D_{N_1}| \geq |D_{M_1}|$, resulting $D_{\langle M_1\rangle - |N_1|} \leq 0$, where contradicts $\langle M_1\rangle - |N_1|$ being a nonsingular M-matrix. Hence, we have $D_{M_1} > 0$. Therefore, $\Omega + M_1$ is an H_+-matrix. Then, by Lemma 2.1,we have

$$0 \leq |(\Omega + M_1)^{-1}| \leq (\Omega + \langle M_1\rangle)^{-1}.$$

Let (z^*, r^*) be the solution of the NCP. Then, $x^* = \frac{1}{2}(z^* - \Omega^{-1}r^*)$ is the solution of (3.1) and

$$(\Omega + M_1)x^* = N_1x^* + (\Omega - M_2)|x^*| + N_2|x^*| - \sigma\phi(z^*). \tag{3.8}$$

Together with (3.5) and (3.2) gives

$$\begin{aligned} & x^{(k+1)} - x^* \\ = \; & N_1(x^{(k)} - x^*) + (\Omega - M_2)(|x^{(k)}| - |x^*|) + N_2(|x^{(k+1)}| - |x^*|) - \sigma(\phi(z^{(k)}) - \phi(z^*)) \\ = \; & (N_1 - \Psi^{(k)})(x^{(k)} - x^*) + (\Omega - M_2 - \Psi^{(k)})(|x^{(k)}| - |x^*|) + N_2(|x^{(k+1)}| - |x^*|). \end{aligned}$$

Denote

$$\delta^{(k)} = |x^{(k)} - x^*|, \bar{\delta}^{(k)} = \left||x^{(k)}| - |x^*|\right|.$$

We have

$$\begin{aligned} \delta^{(k+1)} & \leq |(\Omega + M_1)^{-1}|(|N_1 - \Psi^{(k)}|\delta^{(k)} + |N_2|\bar{\delta}^{(k+1)} + |\Omega - M_2 - \Psi^{(k)}|\bar{\delta}^{(k)}) \\ & \leq (\Omega + \langle M_1\rangle)^{-1}[(|\Omega - M_2 - \Psi^{(k)}| + |N_1 - \Psi^{(k)}|)\delta^{(k)} + |N_2|\delta^{(k+1)}], \end{aligned}$$

and

$$[I - (\Omega + \langle M_1\rangle)^{-1}|N_2|]\delta^{(k+1)} \leq (\Omega + \langle M_1\rangle)^{-1}(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)\delta^{(k)}. \tag{3.9}$$

Since $\Omega + \langle M_2\rangle - |N_2|$ is a nonsingular M-matrix and $\langle M_1\rangle \geq \langle M_2\rangle$, it follows from Lemma 2.4 that both $\Omega + \langle M_1\rangle - |N_2|$ and $\Omega + \langle M_1\rangle$ are nonsingular M-matrices. Consequently, the splitting $\Omega + \langle M_1\rangle - |N_2|$ constitutes an M-splitting, and Lemma 2.5 guarantees that $\rho((\Omega + \langle M_1\rangle)^{-1}|N_2|) < 1$. This implies that $I - (\Omega + \langle M_1\rangle)^{-1}|N_2|$ is a nonsingular M-matrix with a nonnegative inverse by Lemma 2.7. Then, we can deduce (3.9) to

$$(\Omega + \langle M_1\rangle)^{-1}(\Omega + \langle M_1\rangle - |N_2|)\delta^{(k+1)} \leq (\Omega + \langle M_1\rangle)^{-1}(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)\delta^{(k)}$$

$$\Rightarrow (\Omega + \langle M_1\rangle - |N_2|)\delta^{(k+1)} \le (|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)\delta^{(k)}$$
$$\Rightarrow \delta^{(k+1)} \le (\Omega + \langle M_1\rangle - |N_2|)^{-1}(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)\delta^{(k)}.$$

Given that $\langle M_1\rangle \ge \langle M_2\rangle$ and $(\langle M_2\rangle - |N_2|)D$ is an s.d.d. matrix, it follows that $(\Omega + \langle M_1\rangle - |N_2|)D$ must also be s.d.d., due to the fact that

$$(\Omega + \langle M_1\rangle - |N_2|)De \ge (\Omega + \langle M_2\rangle - |N_2|)De > (\langle M_2\rangle - |N_2|)De > 0.$$

Let

$$\mathcal{L}^{(k)} = (\Omega + \langle M_1\rangle - |N_2|)^{-1}(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|).$$

Then, by Lemma 2.3, we have

$$\begin{aligned} & \|D^{-1}\mathcal{L}^{(k)}D\|_\infty \\ = & \|((\Omega + \langle M_1\rangle - |N_2|)D)^{-1}(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)D\|_\infty \\ \le & \max_{1\le i\le n} \frac{[(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)De]_i}{[(\Omega + \langle M_1\rangle - |N_2|)De]_i}. \end{aligned} \tag{3.10}$$

Furthermore, for every $1 \le i \le n$, we get

$$\begin{aligned} & [(\Omega + \langle M_1\rangle - |N_2|)De]_i - [(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)De]_i \\ \ge & (\Omega + \langle M_1\rangle - |N_2| - |N_1| - \Psi^{(k)} - |\Omega - D_{M_2} - \Psi^{(k)}| - |B_{M_2}|)De]_i. \end{aligned}$$

If (3.6) hold, we can further bound the above inequality to obtain

$$\begin{aligned} & [(\Omega + \langle M_1\rangle - |N_2|)De]_i - [(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)De]_i \\ \ge & [(\langle M_1\rangle - |N_1| + \langle M_2\rangle - |N_2|)De]_i \\ > & 0. \end{aligned}$$

On the other hand, if (3.7) is satisfied, we have

$$\begin{aligned} & [(\Omega + \langle M_1\rangle - |N_2|)De]_i - [(|N_1 - \Psi^{(k)}| + |\Omega - M_2 - \Psi^{(k)}|)De]_i \\ \ge & [(2\Omega + \langle M_1\rangle - |N_1| - |M_2| - |N_2| - 2\Psi)De]_i \\ > & 0. \end{aligned}$$

Consequently, with (3.10), we obtain

$$\rho(\mathcal{L}^{(k)}) = \rho(D^{-1}\mathcal{L}^{(k)}D) \le \|D^{-1}\mathcal{L}^{(k)}D\|_\infty < 1,$$

which immediately establishes the convergence of the iteration.

Remark 3.10. *The positive diagonal matrix D in Theorem 3.9 can be obtained by solving the system $(\langle M_2\rangle - |N_2|)x = p$ for a positive vector x, where p is any positive vector, and then setting $D = \mathrm{diag}(x)$.*

In Theorem 4.1 of [23], the convergence results for Method 3.8 are given as follows.

Lemma 3.11. *[23] Assume that:*
(A1) A is an H_+-matrix and $\Omega + M_1 - |N_2|$ is a nonsingular M-matrix;
(A2) $A = M_1 - N_1 = M_2 - N_2$ are two H-compatible splittings;
(A3) $\langle A\rangle - G$ is a nonsingular M-matrix, where G is a nonnegative matrix such that

$$|\phi(z_1) - \phi(z_2)| \le G|z_1 - z_2| \tag{3.11}$$

for any $z_1, z_2 \in \mathbb{R}^n$;
(A4) $\Omega \ge D_{M_2}$.
Then, Method 3.8 is convergent for any $x^{(1)} \in \mathbb{R}^n$.

Comparing with the assumptions in Theorem 3.9, we make the following remarks:

- From the proof of Theorem 3.9, it is evident that Assumption (A1) is required by both theorems.
- Regarding Assumption (A2), while it is well-known that an H-compatible splitting is necessarily an H-splitting, the converse does not hold. Theorem 3.9 thus relaxes the matrix splitting assumptions, enabling us to explore more matrix splittings for either algorithmic acceleration or deeper theoretical analysis.
- Assumption (A3) serves as an additional condition. In this comparison, Theorem 3.9 proves superior.
- Turn to Assumption (A4). The condition (3.7) formally permits Ω to take values smaller than D_{M_2} flexibility not afforded by Assumption (A4). This demonstrates that condition (3.7) effectively expands the convergence domain of the parameter matrix Ω. It is worth noting that, when $\Omega \ge D_{M_2}$, the parameter range in Assumption (A4) is broader than that in (3.6). However, according to Theorem 4.5 in [23], the optimal Ω is precisely $\Omega = D_A$, which corresponds to $\Omega = D_{M_2}$ in the case AMAOR iteration method. Therefore, the parameter range in (3.6) we provide can include the optimal choice.

Moreover, we consider the convergence of the AMAOR iteration method given by (3.3) following the proof framework of Theorem 3.9. When A is an H_+-matrix, we have $\rho(D_A^{-1}|B_A|) < 1$ by Lemma 2.3. To guarantee that the assumptions of Theorem 3.9 are satisfied, it is sufficient to make $A = M_1 - N_1$ be an H-splitting. In fact, by (3.3), if $0 < \beta \le \alpha$, we have

$$\begin{aligned}
&\langle M_1\rangle - |N_1| \\
= \;& \langle \frac{1}{\alpha}D_A - \frac{\beta}{\alpha}L_A\rangle - |\frac{1-\alpha}{\alpha}D_A + (1-\frac{\beta}{\alpha})L_A + U_A| \\
= \;& \frac{1}{\alpha}D_A - \frac{\beta}{\alpha}|L_A| - \frac{|1-\alpha|}{\alpha}D_A - (1-\frac{\beta}{\alpha})|L_A| - |U_A| \\
= \;& \frac{1-|1-\alpha|}{\alpha}D_A - |B_A| \\
= \;& \begin{cases} \langle A\rangle, & \text{if } \; 0 < \alpha \le 1; \\ (\frac{2}{\alpha} - 1)D_A - |B_A|, & \text{if } \; \alpha > 1. \end{cases}
\end{aligned}$$

For the case where $0 < \alpha \le 1$, $\langle M_1\rangle - |N_1| = \langle A\rangle$ is a nonsingular M-matrix when A is an H_+-matrix. In contrast, when $\alpha > 1$, by further restricting the strict upper bound of α to $\frac{2}{1+\rho(D_A^{-1}|B_A|)}$, i.e.,

$$1 < \alpha < \frac{2}{1 + \rho(D_A^{-1}|B_A|)},$$

we can obtain $\frac{2}{\alpha} - 1 > \rho(D_A^{-1}|B_A|)$, which implies that $(\frac{2}{\alpha} - 1)I - D_A^{-1}|B_A|$ is a nonsingular M-matrix by Lemma 2.7. Then, we obtain the fact that

$$\langle M_1 \rangle - |N_1| = (\frac{2}{\alpha} - 1)D_A - |B_A| = D_A[(\frac{2}{\alpha} - 1) - D_A^{-1}|B_A|]$$

is a nonsingular M-matrix by Lemma 2.6.

In summary, $\langle M_1 \rangle - |N_1|$ is a nonsingular M-matrix provided

$$0 < \beta \leq \alpha < \frac{2}{1 + \rho(D_A^{-1}|B_A|)}. \tag{3.12}$$

Therefore, we get the following theorem.

Theorem 3.12. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix and M_1, N_1, M_2, N_2 be given by (3.3). Assume that Ω satisfied (3.6) or (3.7). Then, $\forall x^{(1)} \in \mathbb{R}^n$, the AMAOR iteration method converges provided that (3.12) holds.*

In Theorem 4.2 of [23], the convergence results of the AMAOR iteration method are listed below.

Lemma 3.13. *[23] Assume that:*
(B1) A is an H_+-matrix;
(B2) $\rho(\langle A \rangle^{-1}G) < 1$;
(B3) $\alpha < \beta < \frac{1}{\rho(\langle A \rangle^{-1}G)}, \beta \in [0, \alpha] \cup [\alpha, \alpha\theta_\alpha]$, where $\theta_\alpha \in [1, +\infty)$ such that

$$\rho(D^{-1}(\theta|L_A| + |U_A| + G)) = \frac{\alpha + 1 - |1 - \alpha|}{2\alpha}.$$

(B4) $\Omega \geq D_A$.
Then, AMAOR iteration method is convergent for any $x^{(1)} \in \mathbb{R}^n$.

In comparison with our results, the following observations can be made.

- Assumption (B1) is also present in Theorem 3.12.
- Assumption (B2) imposes an additional constraint, requiring a mutual dependence between the system matrix A and the nonnegative matrix G associated with the nonlinear function ϕ. This restricts the applicability of Lemma 3.13. For example, when

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}, \quad \phi(z) = 4sin(z),$$

  where $G = 4I$ can satisfy (3.11), we have $\rho(\langle A \rangle^{-1}G) = 2 > 1$. This implies that Assumption (B2) fails to hold.
- Assumption (B3) involves a highly intricate relationship among α,β, and A, making it significantly more difficult to verify than condition (3.12). In fact, the relaxation parameters α used in the AMSOR iteration numerical examples of [23] were presented without any theoretical justification related to Assumption (B3).
- Comments on Assumption (B4) are similar to those on Assumption (A4) in Lemma 3.11.

In summary, Theorems 3.9 and 3.12 demonstrate clear advantages over Lemmas 3.11 and 3.13, respectively.

4. Numerical examples

This section, we present numerical experiments validate the effectiveness of the theoretical results obtained.

Example 4.14. *[36–39] Consider an NCP derived from the discretization of a boundary value problem. Take the domain* $\Omega = (0,1) \times (0,1)$ *with boundary conditions specified by the function* $g(x_1, x_2)$*, where* $g(0, x_2) = x_2(1 - x_2), g(x_1, x_2) = 0$ *on* $x_2 = 0, x_2 = 1$*, or* $x_1 = 1$*. The problem requires finding u, satisfying the following system:*

$$\begin{cases} u \geq 0, & in\ \Omega, \\ -\Delta u + \phi(u, x_1, x_2) - 8(x_2 - 0.5) \geq 0, & in\ \Omega, \\ u(-\Delta u + \phi(u, x_1, x_2) - 8(x_2 - 0.5)) = 0, & in\ \Omega, \\ u = g, & on\ \partial\Omega, \end{cases}$$

where $\frac{\partial \phi}{\partial u} \geq 0$ *on* $\bar{\Omega} \times \{u : u \geq 0\}$.

Using the standard five-point finite difference discretization scheme, one can obtain the corresponding NCP.

Example 4.15. *[5] Let*

$$A = \begin{bmatrix} S & -0.5I & & \\ -1.5I & S & \ddots & \\ & \ddots & \ddots & -0.5I \\ & & -1.5I & S \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where $n = m^2$, $S = \text{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m\times m}$, $I \in \mathbb{R}^{m\times m}$ *is the identity matrix.*

The numerical experiments are conducted on a PC equipped with a 12th Gen Intel Core i7-12700 processor (2.10 GHz) using MATLAB. The stopping criterion is chosen as $\|min\{Az^{(k)} + \phi(z^{(k)}), z^{(k)}\}\|_\infty < 10^{-7}$.

4.1. Experiment 1

In this subsection, we focus on the computational performance of Method 3.8 when neither Assumption (A3) in Lemma 3.11 nor Assumption (B2) in Lemma 3.13 is satisfied, while all assumptions of Theorem 3.9 are met.

We first set $\Omega = D_A + 4I$. For Example 4.14, selecting the nonlinear function $\phi(u, x_1, x_2)$ as $4\sin u, 4\arctan u$ and $4\ln(1+u)$. As for Example 4.15, the nonlinear function $\phi(z)$ is chosen as $4\sin z + q, 4\arctan z + q$, and $4\ln(1+z) + q$, where the vector $q = (-1, 1, -1, 1, \ldots)^T \in \mathbb{R}^n$. For the three nonlinear functions considered here, we can choose $G = 4I \in \mathbb{R}^{4\times 4}$ to satisfy (3.11). By utilizing the "eigs" function in MATLAB, we can estimate the $\rho(\langle A\rangle^{-1}G)$. For $m = 10, 20$, and 30, the spectral radius values are 89.53, 194.90, and 340.80 in Example 4.14, and 6.96, 7.22, and 7.32 in Example 4.15, respectively. For these two examples, neither Assumption (A3) in Lemma 3.11 nor Assumption (B2) in Lemma 3.13 is satisfied, thereby rendering the convergence theorem in [23] inapplicable. In contrast, Theorem 3.9 and Theorem 3.12 proposed in our work satisfy all required conditions.

For both methods, we employ the AMJ and AMGS iteration methods. Note that in Theorem 4.4 of [23], by the proof process of the AMAOR iteration method, the case where the spectral radius of the amplified iteration matrix reaches its minimum corresponds precisely to the AMGS iteration method. Consequently, we exclusively focus on the AMGS iteration method without considering other AMAOR variants. Numerical results are presented for three different sizes: $m = 20, 30, 40$, as shown in Table 1 (for Example 4.14) and Table 2 (for Example 4.15), where "IT" and "CPU" denote the number of iteration steps and the computation time (in seconds), respectively. The numerical results demonstrate that AMJ and AMGS methods successfully satisfy the stopping criterion for all test cases. Comparatively, the AMGS requires fewer iteration steps and less computation time than the AMJ, which aligns with the performance trends observed in the literature. These experimental findings validate the correctness of Theorems 3.9 and 3.12.

Table 1. Results of Example 4.14 when $\Omega = D_A + 4I$.

$\phi(u)$	m	AMJ		AMGS	
		IT	CPU	IT	CPU
$4\sin u$	20	36	0.0824	22	0.0489
	30	37	0.2608	22	0.1585
	40	37	0.6271	22	0.4125
$4\arctan u$	20	35	0.0850	21	0.0474
	30	35	0.2416	21	0.1459
	40	35	0.5946	21	0.3896
$4\ln(1+u)$	20	27	0.0662	17	0.0384
	30	28	0.1972	18	0.1280
	40	28	0.4811	18	0.3223

Table 2. Results of Example 4.15 when $\Omega = D_A + 4I$

$\phi(z)$	m	AMJ		AMGS	
		IT	CPU	IT	CPU
$4\sin z$	20	34	0.0800	19	0.0431
	30	35	0.2552	19	0.1319
	40	35	0.5937	19	0.3432
$4\arctan z$	20	33	0.0796	18	0.0420
	30	34	0.2366	18	0.1276
	40	34	0.5794	18	0.3403
$4\ln(1+z)$	20	29	0.0704	14	0.0318
	30	30	0.2075	14	0.0994
	40	30	0.5103	15	0.2698

4.2. Experiment 2

In this subsection, we examine the performance of Method 3.8 in cases where the parameter matrix Ω falls outside the range required by Assumptions (A4) of Lemmas 3.11 and Assumption (B4) in Lemma 3.13 but remain within the range specified by Theorems 3.9 and 3.12.

Here, we take $\Omega = 0.9D_A$. In Example 4.14, we employ the nonlinear functions $\phi(u, x_1, x_2)$ as $\sin u, \arctan u$, and $\ln(1+u)$. Correspondingly, in Example 4.15, the function $\phi(z)$ is chosen to be $\sin z + q, \arctan z + q$ and $\ln(1+z)+q$, with the vector $q = (-1, 1, -1, 1, \ldots)^T \in \mathbb{R}^n$. A direct examination reveals that while Assumptions (A4) and (B4) in Lemmas 3.11 and 3.13, respectively, fail to hold for both test cases, while all requisite conditions are completely satisfied for the proposed Theorems 3.9 and 3.12.

Table 3. Results of Example 4.14 when $\Omega = 0.9D_A$.

$\phi(u)$	m	AMJ		AMGS	
		IT	CPU	IT	CPU
$\sin u$	20	49	0.1071	22	0.0503
	30	50	0.3413	23	0.1629
	40	50	0.8523	23	0.4273
$\arctan u$	20	46	0.1058	22	0.0505
	30	47	0.3246	22	0.1543
	40	47	0.8053	22	0.4100
$\ln(1+u)$	20	43	0.1033	20	0.0459
	30	44	0.2998	20	0.1403
	40	44	0.7520	20	0.3752

Table 4. Results of Example 4.15 when $\Omega = 0.9D_A$.

$\phi(z)$	m	AMJ		AMGS	
		IT	CPU	IT	CPU
$\sin z$	20	36	0.0808	17	0.0375
	30	37	0.2547	17	0.1198
	40	37	0.6200	17	0.3073
$\arctan z$	20	35	0.0835	16	0.0365
	30	35	0.2422	17	0.1210
	40	36	0.6048	17	0.3125
$\ln(1+z)$	20	32	0.0744	15	0.0335
	30	32	0.2222	15	0.1056
	40	33	0.5609	15	0.2714

We also show the numerical results of the AMJ and AMGS iteration methods for different sizes; see Tables 3 and 4. The numerical results demonstrate that AMJ and AMGS iteration methods continue to satisfy the stopping criterion across all test cases, exhibiting performance characteristics consistent with Experiment 1. This observation further validates the effectiveness of the proposed theorems.

Remark 4.16. *It is worth noting that in this subsection, we have simply chosen a parameter matrix* Ω *smaller than* D_A *to test the effectiveness of the AMMS iteration method, aiming to demonstrate that the proposed convergence range is larger than that in [23]. However, due to the presence of the modulus term in the equivalent modulus equation of the complementarity problem, a theoretical analysis of the parameter matrix* Ω *is rather challenging. In recent literature [40, 41], the authors conducted theoretical analyses on the selection of* Ω *in the MMS iteration method for the LCP and second-order cone LCP, respectively, and under certain conditions, derived improved strategies for choosing* Ω*. For the MMS and AMMS methods used to solve the NCP, a corresponding theoretical analysis of the selection of* Ω *can also be anticipated, which represents a meaningful direction for future research.*

5. Conclusions

We present a rigorous analysis of convergence conditions for the AMMS iteration method in solving the NCP, establishing two novel convergence theorems (Theorems 3.9 and 3.12). Compared with corresponding results in [23], the proposed convergence conditions are theoretically weaker, offering three significant advantages: Greater flexibility in matrix splitting selection, broader admissible ranges for positive diagonal parameter matrix, and more easily verifiable assumptions regarding relaxation parameters in the AOR splitting. The numerical experiments further confirm the accuracy and effectiveness of these theoretical advancements.

Author contributions

Yanmei Chen: Writing-original draft; Yihang Lin: Writing-review and editing; Jianwei Dong: Software, validation, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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