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Research article

A joint design method of interval function observer for discrete-continuous nonlinear interconnected systems

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Abstract: The interval function observer designed to reconstructs only a linear function of the state variable. This specialized structure significantly reduces the order and complexity of the traditional state observer, making it particularly suitable for interval estimation of states—an increasingly prominent topic in control theory. First, unlike the interval state observer, the interval function observer is formally defined and constructed for nonlinear interconnected systems that exhibit discrete or continuous state changes over time. Second, sufficient conditions for the existence of such an observer are derived. Using the solution method for the generalized Sylvester equation, the gain matrix for nonlinear interconnected systems is calculated. The design method is further extended from simple systems to more complex interconnected systems. Finally, two models of interconnected systems are introduced, and the proposed design method is validated through two numerical examples, confirming its effectiveness and feasibility.

Keywords: nonlinear interconnected systems; interval function observer; generalized Sylvester systems; state variable reconstruction
Mathematics Subject Classification: 15A09, 93B07, 93B17, 93C10

1. Introduction

State observers are crucial in practical control systems, with applications in fault-tolerant control, fault estimation, and diagnosis. Recently, designing system state observers has witnessed significant progress. References [1–3] explored methods for nonlinear systems and extended these methods to interconnected large-scale systems. In Reference [1], an observer design method was presented for a class of nonlinear control systems with global Lipschitz distribution, without requiring the hypothesis of a full relative degree. In Reference [2], a Hamilton–Jacobi–Bellman state observer design method was proposed for a class of nonlinear systems. Interconnected systems are widely used in transportation, power, and biology. In Reference [3], a class of nonlinear interconnected systems was investigated. A state observer design method was proposed to achieve robust estimation of the system's unknown states, accounting for its nonlinear and interconnected characteristics. In Reference [4], a solar thermal heating system was studied, and the unmeasured state variables of the solar heating system were estimated in real time using a state observer. Reference [5] presented a state observer designed for a modular multilevel converter, considering the number of sensors, reliability, and cost. The observer estimates capacitance and voltage and provides the solution method for the gain matrix. Reference [6] presented a design method for a state observer using discrete systems with perturbations and nonlinear terms as research objects. By introducing performance indices, the influence of perturbations on state estimation is mitigated, achieving robust estimation. Reference [7] investigated a class of switching systems with discrete time characteristics. Unlike the design ideas of traditional observers, a new state observer was designed by minimizing a nonsmooth $\ell 2$-norm-based weighted cost functional, achieving better robust estimation. Reference [8] investigated electro-hydraulic actuator systems and designed an extended state observer using the Lyapunov method, highlighting its fast convergence in an adaptive frame compared with existing observers. Reference [9] proposed an adaptive extended state observer based on vehicle disturbance and velocity recovery, enabling synchronous estimation of unknown input gain, lumped disturbance, and unknown velocity, ensuring convergence of estimation errors. Reference [10] classified existing observer design methods, reviewed their core technologies, and discussed future challenges for state observers.

In the observer design method, the concept of a function observer is introduced to make the designed observer more targeted. References [11–15] studied the design method of a system function observer. Reference [11] presented a design method for a function observer tailored to generalized systems, validated through discrete-time and continuous systems. Reference [12] introduced a design method for a linear compensation function observer and evaluated it using a small unmanned aerial vehicle system that reflects the high-performance characteristics of the observer. Reference [13] proposed a design method for a function observer for linear description systems, enabling accurate real-time estimation of the objective function. Observer-based system fault estimation is a commonly used approach for fault diagnosis. In Reference [14], a function observer was proposed and system fault detection was achieved based on the constructed function observer. The function observer demonstrated superior performance compared with the state observer. Reference [15] described a design method for a functional observer of dynamical systems, enriching observer design theory in modern control theory.

The interval observer has more relaxed preconditions for the system, ensuring its universal applicability. Rich research results have been achieved regarding the design of interval observers in control systems. References [16–19] proposed a new design method for an interval observer for a continuous system with external interference and measurement noise, a discrete system, and a nonlinear system with Lipschitz nonlinear properties. It improved the degrees of freedom of the observer design and demonstrated the superiority of the interval observer. References [20–22] extended the design method of the interval observer to generalized systems. Reference [22] studied a class of similar generalized interconnected systems and proposed a design method for interval observation based on the characteristics of similar structures of interconnected systems, further enriching the design theory of interval observers. References [23–26] explored the linear variable parameter system, steady exponential stable linear system, and networked control system and proposed a design method for the interval observer. The internal observer's rationality and scientific validity were tested through simulation examples, including vehicle state estimation, confirming the effectiveness of the proposed method.

Considering the advantages of the state observer, function observer, and interval observer, the interval function observer, which can realize interval estimation of the state function, has gradually become a research focus. Additionally, the design of the observer was more relaxed. Research on interval function observers has yielded significant results. References [27,28] introduced the concept of an interval function observer for time-varying linear systems with perturbation delay, providing sufficient conditions for its existence and achieving the preset convergence rate and observation error. References [29] and [30] focused on a discrete switching singular system as a research object, proposing an improved performance estimation method for the interval function observer and testing performance through examples. References [31–33] extended the research object of the interval function observer from linear and generalized systems to time-delay and fractional interconnected systems, providing the design method for the interval function observer.

Given the complexity, order, and constraints of observer design, interval function observers offer certain advantages over traditional observers. They can support system fault estimation and robust control. While interval function observers have been extensively studied for simple linear systems, research on their application to large-scale interconnected systems—such as those in biology and power—remains limited. To advance the study of complex nonlinear interconnected large systems, the design method for interval function observer is introduced, offering significant practical implications.

Based on the above analysis and current research results, the interval function observer offers some advantages; however, the research results are relatively limited. We present a design method for an interval function observer for a class of discrete and continuous nonlinear interconnected systems, along with a method for solving the observer gain matrix. Compared with existing methods, we extend the study object of the interval function observer from a simple system to an interconnected system with nonlinear, discrete, and continuous characteristics, further enriching the design theory of interval function observers. Finally, the proposed method is tested using discrete and continuous nonlinear interconnected system models. The feasibility and effectiveness of the interval-function observer design method are verified through simulation results.

The main contributions of this study are as follows:

1) Nonlinear interconnected large systems have strong application prospects in mathematical economics and artificial intelligence. This study proposes a design method of interval function observer for a class of nonlinear interconnected large systems, extending the research of interval function observer from general control systems to nonlinear interconnected large systems, and further enriching the design theory of interval function observer.

2) Considering the structure of the interval function observer, two forms of interconnected large-scale systems, including discrete systems and continuous systems, are described through a model. For the two different forms of nonlinear interconnected systems, the interval function observer is uniformly designed, thereby making the design method of the interval function observer more inclusive and convenient.

3) Owing to the existence of interconnection terms among the subsystems of nonlinear interconnected large systems, the difficulty of designing the interval function observer increases

further. In the design process of the interval function observer for the interconnected large-scale system, considering the interconnection structure of the nonlinear interconnection system, the constraints on the coefficient matrix of the interval function observer are added. This addition eliminates the influence of the interconnection term in the nonlinear interconnection system on the design of the interval function observer. The interval estimation of the state functions in each subsystem is achieved.

The influence of interconnections on the observer design makes the design of the interval function observer more independent.

The proposed design method of the interval function observer realizes the interval estimation of the state function in discrete and continuous interconnected systems, and the application of this method is more extensive.

The remainder of this paper is organized as follows: Section 2 describes the structure of a class of nonlinear interconnected systems and presents the related concepts of interval function observers. Section 3 details the main results of the study, including the design method and sufficient conditions of the interconnectable system interval function observer, as well as the solution method and design steps for the observer coefficient matrix. Section 4 tests the proposed method using both discrete and continuous interconnected system models. Finally, Section 5 summarizes the advantages of the interval function observer design method and future research directions.

Notation:

For any matrix A, A^{T}represents the transpose of matrix A. 0 represents a scalar or matrix of an appropriate dimension. $\|\cdot\|$ is the Euclidean norm. Vector and matrix relations are understood in terms of elements, $x_1=[x_{11}\quad x_{12}\quad x_{13}]\le x_2=[x_{21}\quad x_{22}\quad x_{23}]$that is$x_{11}\le x_{21}, x_{12}\le x_{22}$, and $x_{13}\le x_{23.}$ $A=\left[a_{ij}\right]_{m\times n}\le B=\left[b_{ij}\right]_{m\times n}$ that is $a_{ij}\le b_{ij}$. $eig(A)$ denotes the set of all the eigenvalues of matrix A. deg denotes the degree of a polynomial matrix. $r(A)$ represents the rank of the matrix A.

Matrix $T\in\mathrm{R}^{m\times n}$ can be expressed as:

$$T=T^+-T^-, \tag{1.1}$$

where

$$\begin{cases}T^+=max\{T,0\},\\T^-=max\{-T,0\}.\end{cases}$$

Based on the above relationship, if there are vectors x, x^-, and $x^+\in\mathrm{R}^n$ that satisfy $x^-\le x\le x^+$, then the following equation is true:

$$T^+x^--T^-x^+\le Tx\le T^+x^+-T^-x^-. \tag{1.2}$$

2. Problem description and related concepts

Consider the following nonlinear interconnected systems:

$$\begin{cases}\sigma x_i(t)=A_ix_i(t)+B_iu_i(t)+g_i[x_i(t)]+\sum\limits_{\substack{j=1\\j\neq i}}^{\mathrm{N}}A_{ij}x_j(t),\\y_i(t)=C_ix_i(t), i=1,2,\cdots,\mathrm{N},\end{cases} \tag{2.1}$$

where $\sigma x_i(t)$ represents $x_i(t+1)$ ($i.e.$ $\sigma x_i(t) = x_i(t+1)$) when (2.1) is a nonlinear discrete interconnected system. And, $\sigma x_i(t)$ represents $\dot{x}_i(t)$, i.e., ($\sigma x_i(t) = \dot{x}_i(t)$) when (2.1) is a nonlinear continuous interconnected system.

In interconnected system (2.1), $x_i(t) \in \mathrm{R}^{n_i}, x_j(t) \in \mathrm{R}^{n_j}, y_i(t) \in \mathrm{R}^{p_i}$, and $u_i(t) \in \mathrm{R}^{m_i}$ represent the state vector of the i-th and j-th subsystem, and the output vector and control input of the first sub-system, respectively. $g_i[x_i(t)]$ denotes the unknown external disturbances or system uncertainties in the i-th subsystem. $A_i \in \mathrm{R}^{n_i \times n_i}, B_i \in \mathrm{R}^{n_i \times m_i}, and\ C_i \in \mathrm{R}^{p_i \times n_i}$ are the coefficient matrix with appropriate dimension in the i-th subsystem. $A_{ij} \in \mathrm{R}^{n_i \times n_j}$ is the interconnection matrix between the i-th and j-th subsystems in the interconnected system.

Definition 1. For a square matrix $A = [a_{ij}]_{n \times n}$, there are the following definitions:

(1) When all the elements of the matrix are nonnegative, that is $a_{ij} \geq 0, 1 \leq i, j \leq n$, the matrix is termed nonnegative.

(2) When the Euclidean norm of all eigenvalues of the matrix is less than 1, that is $\|\gamma_i(A)\| < 1, 1 \leq i \leq n$, the matrix is termed Schur.

(3) When the nondiagonal elements of the matrix are nonnegative, that is $a_{ij} \geq 0, 1 \leq i \neq j \leq n$, the matrix is termed Metzler.

(4) When the spectral radius of the matrix is negative, that is $\gamma_{max}(A) < 0$, the matrix is called Hurwitz.

Lemma 1. [34] provides a discrete system of the following form:

$$x(t+1) = Ax(t) + f^+(t),\ f^+(t): \mathrm{R}^+ \to \mathrm{R}^{n^+}. \tag{2.2}$$

If $A \in \mathrm{R}^n$ is nonnegative, function $f^+(t)$ indicates that the value of the nonlinear function is nonnegative, initial condition $x(t_0) = x_0 > 0$. Following monotonicity theory, all solutions of system (2.2) are nonnegative, that is $x(t) \geq 0, \forall t_0 \geq 0$.

Lemma 2. [27] provides a continuous system of the following form:

$$\dot{x}(t) = Ax(t) + f^+(t), f^+(t): \mathrm{R}^+ \to \mathrm{R}^{n^+}. \tag{2.3}$$

If $A \in \mathrm{R}^n$ is Metzler, function $f^+(t)$ indicates that the value of the nonlinear function is nonnegative, initial condition $x(t_0) = x_0 > 0$. According to monotonicity theory, all solutions of system (2.3) are nonnegative, that is $x(t) \geq 0, \forall t_0 \geq 0$.

Hypothesis 1. The unknown external disturbances or system uncertainties $g[x_i(t)]$ in each subsystem of nonlinear interconnected system (2.1) are bounded, and the following conditions are satisfied:

$$g_i^-[x_i(t)] \leq g_i[x_i(t)] \leq g_i^+[x_i(t)], i = 1,2,\cdots,\mathrm{N},$$

where $g_i^-[x_i(t)]$ and $g_i^+[x_i(t)]$ represent the lower and upper bounds of $g_i[x_i(t)]$, respectively.

To realize the interval estimation of the states in interconnected system (2.1), the following linear function is constructed:

$$v_i(t) = F_i x_i(t), i = 1,2,\cdots,\mathrm{N}, \tag{2.4}$$

where $F_i \in \mathrm{R}^{r_i \times n_i}$ is a constant matrix, and the interval observer is constructed for $v_i(t)$.

Definition 2. If the initial states of each subsystem of a nonlinear interconnected system (2.1) are bounded and satisfy $x_i^-(0) \leq x_i(0) \leq x_i^+(0)$, the equation of the following form is given:

$$\sigma \xi_i(t) = \phi(\xi_i(t), y_i(t), u_i(t), g_i^+[x_i(t)], g_i^-[x_i(t)]). \tag{2.5}$$

Output is

$$\begin{cases} v_i^+(t) = \phi_i^+(\xi_i(t), y_i(t), t), \\ v_i^-(t) = \phi_i^-(\xi_i(t), y_i(t), t), \end{cases} \tag{2.6}$$

where $\xi_i(t) \in \mathrm{R}^{r_i}, v_i^+(t), v_i^-(t) \in \mathrm{R}^{r_i}, i = 1,2, \cdots, \mathrm{N}$.

When (2.5) and (2.6) meet the following conditions, (2.5) is the interval observer of the state function (2.4) in the nonlinear interconnected system (2.1), that is, the interval function observer. Additionally, (2.6) is the output of the interval function observer.

(1) The system (2.5) is input-state stable.

(2) State function (2.4) and output (2.6) satisfy the following relation:

$$v_i^-(t) \le v_i(t) \le v_i^+(t), \forall t \ge 0, i = 1,2, \cdots, \mathrm{N}. \tag{2.7}$$

(3) If $\|g_i^+[x_i(t)] - g_i^-[x_i(t)]\|$ is uniformly bounded, $\|v_i^+(t) - v_i^-(t)\|$ is uniformly bounded. When $\|g_i^+[x_i(t)] - g_i^-[x_i(t)]\|$ converges to zero at $t \ge 0$, then, $\|v_i^+(t) - v_i^-(t)\|$ converges to zero.

3. Main results

3.1. Design of interval function observer

For nonlinear interconnected system (2.1), we can design an interval function observer as follows:

$$\begin{cases} \sigma\xi_i^+(t) = M_i\xi_i^+(t) + J_i y_i(t) + \sum\limits_{\substack{j=1 \\ j\neq i}}^{\mathrm{N}} J_{ij} y_j(t) + H_i u_i(t) + G_i^+ g_i^+[x_i(t)] - G_i^- g_i^-[x_i(t)], \\ \sigma\xi_i^-(t) = M_i\xi_i^-(t) + J_i y_i(t) + \sum\limits_{\substack{j=1 \\ j\neq i}}^{\mathrm{N}} J_{ij} y_j(t) + H_i u_i(t) + G_i^+ g_i^-[x_i(t)] - G_i^- g_i^+[x_i(t)]. \end{cases} \tag{3.1}$$

The system output is:

$$\begin{cases} v_i^+(t) = L_i^+\xi_i^+(t) - L_i^-\xi_i^-(t) + E_i y_i(t), \\ v_i^-(t) = L_i^+\xi_i^-(t) - L_i^-\xi_i^+(t) + E_i y_i(t), \end{cases} \tag{3.2}$$

where $\xi_i^+(t), \xi_i^-(t), v_i^+(t), v_i^-(t) \in \mathrm{R}^{r_i}, i = 1,2, \cdots, \mathrm{N}$.

Matrices M_i,J_i,J_{ij},H_i,E_i,G_i^+, G_i^-, L_i^+, L_i^- respectively are observer unknown coefficient matrices to be solved.

The purpose of the interval function observer constructed above is to perform interval estimation for the state vector function in system (2.1) and ensure that the output of the observer satisfies the condition:$v_i^-(t) \le v_i(t) \le v_i^+(t), \forall t \ge 0, i = 1,2, \cdots, \mathrm{N}$.

The conditions for the existence of an observer are presented in the following theorem.

3.2. Sufficient conditions for the presence of the observer

Theorem 3.1. For the interval function observer in (3.1), we obtain the following conditions:

$$M_i \text{ is a Nonnegative and Schur matrix,} \tag{3.3}$$

$$M_i \text{ is a Metzler and Hurwitz matrix,} \tag{3.4}$$

$$G_iA_i - M_iG_i = J_iC_i, \tag{3.5}$$

$$J_{ij}C_j - G_iA_{ij} = 0, \tag{3.6}$$

$$F_i - L_iG_i - E_iC_i = 0, \tag{3.7}$$

$$H_i = G_iB_i, \tag{3.8}$$

where $i, j = 1,2,\cdots,\mathrm{N}$ and $i \neq j$.

Based on conditions (3.3)–(3.8), the following conclusions can be drawn.

(a) When the nonlinear interconnected system (2.1) is a discrete interconnected system (i.e., $\sigma x_i(t) \triangleq x_i(t+1)$), if (10) satisfies conditions (3.3), (3.5)–(3.8), then (3.1) is the interval function observer of the linear function (2.4) of the i-*th* subsystem ($i = 1,2,\cdots,\mathrm{N}$) of system (2.1).

(b) When the nonlinear interconnected system (2.1) is a continuously interconnected system (i.e., $\sigma x_i(t) \triangleq \dot{x}_i(t)$), if (3.1) satisfies the conditions (3.4)–(3.8), then (3.1) is the interval function observer of the linear function (2.4) of the i-*th* subsystem ($i = 1,2,\cdots,\mathrm{N}$) of system (2.1).

Proof: We present the following notation

$$\begin{cases} \varepsilon_i^+(t) = \xi_i^+(t) - G_ix_i(t), \\ \varepsilon_i^-(t) = G_ix_i(t) - \xi_i^-(t). \end{cases} \tag{3.9}$$

$$\begin{cases} e_i^+(t) = v_i^+(t) - F_ix_i(t), \\ e_i^-(t) = F_ix_i(t) - v_i^-(t). \end{cases} \tag{3.10}$$

Subsequently, we combine the three conditions in the definition of an interval function observer to provide the proof of Theorem 3.1.

According to (2.1), (3.1), and (3.9), we obtain

$$\begin{aligned} \sigma\varepsilon_i^+(t) &= \sigma\xi_i^+(t) - G_i\sigma x_i(t) \\ &= M_i\varepsilon_i^+(t) + (M_iG_i + J_iC_i - G_iA_i)x_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{\mathrm{N}} (J_{ij}C_j - G_iA_{ij})x_j(t) + (H_i - G_iB_i)u_i(t) + \\ &\quad [G_i^+g_i^+[x_i(t)] - G_i^-g_i^-[x_i(t)] - G_ig_i[x_i(t)]]. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \sigma\varepsilon_i^-(t) &= G_i\sigma x_i(t) - \sigma\xi_i^-(t) \\ &= M_i\varepsilon_i^-(t) + (G_iA_i - M_iG_i - J_iC_i)x_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{\mathrm{N}} (G_iA_{ij} - J_{ij}C_j)x_j(t) + (G_iB_i - H_i)u_i(t) + \\ &\quad [G_ig_i[x_i(t)] - G_i^+g_i^-[x_i(t)] + G_i^-g_i^+[x_i(t)]]. \end{aligned} \tag{3.12}$$

From (2.4), (3.2), and (3.10), we obtain

$$
\begin{aligned}
e_i^+(t) &= v_i^+(t) - F_i x_i(t) \\
&= L_i^+ \xi_i^+(t) - L_i^- \xi_i^-(t) + E_i y_i(t) - F_i x_i(t) \\
&= L_i^+ \varepsilon_i^+(t) + L_i^- \varepsilon_i^-(t) + (E_i C_i + L_i G_i - F_i) x_i(t).
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
e_i^-(t) &= F_i x_i(t) - v_i^-(t) \\
&= F_i x_i(t) - L_i^+ \xi_i^-(t) + L_i^- \xi_i^+(t) - E_i y_i(t) \\
&= L_i^- \varepsilon_i^+(t) + L_i^+ \varepsilon_i^-(t) + (F_i - E_i C_i - L_i G_i) x_i(t).
\end{aligned} \tag{3.14}
$$

According to (3.5)–(3.8), we can obtain

$$
\begin{cases}
\sigma\varepsilon_i^+(t) = M_i \varepsilon_i^+(t) + [G_i^+ g_i^+[x_i(t)] - G_i^- g_i^-[x_i(t)] - G_i g_i[x_i(t)]], \\
\sigma\varepsilon_i^-(t) = M_i \varepsilon_i^-(t) + [G_i g_i[x_i(t)] - G_i^+ g_i^-[x_i(t)] + G_i^- g_i^+[x_i(t)]].
\end{cases} \tag{3.15}
$$

$$
\begin{cases}
e_i^+(t) = L_i^+ \varepsilon_i^+(t) + L_i^- \varepsilon_i^-(t), \\
e_i^-(t) = L_i^- \varepsilon_i^+(t) + L_i^+ \varepsilon_i^-(t).
\end{cases} \tag{3.16}
$$

According to (1.1) and (1.2), we can obtain

$$
G_i^+ g_i^+[x_i(t)] - G_i^- g_i^-[x_i(t)] - G_i g_i[x_i(t)] \geq 0. \tag{3.17}
$$

$$
G_i g_i[x_i(t)] - G_i^+ g_i^-[x_i(t)] + G_i^- g_i^+[x_i(t)] \geq 0. \tag{3.18}
$$

When system (2.1) is a discrete interconnected system, according to (3.3), (3.15), and Lemma 1, $t \geq 0$, $\varepsilon_i^+(t) \geq 0$, $\varepsilon_i^-(t) \geq 0$ and system (3.1) is asymptotically stable. When system (2.1) is a continuously connected system, according to Eqs (3.4), (3.15), and Lemma 2, $t \geq 0$, $\varepsilon_i^+(t) \geq 0$, $\varepsilon_i^-(t) \geq 0$, and system (3.1) is asymptotically stable. Therefore, the first condition of the interval function observer is proven.

According to (3.16), combined with the first condition of the above proof, we know $e_i^+(t) \geq 0, e_i^-(t) \geq 0$, and therefore the second condition of the $v_i^-(t) \leq v_i(t) \leq v_i^+(t), \forall t \geq 0$ interval observer is proven.

Let $e_{i,m}(t) = \xi_i^+(t) - \xi_i^-(t)$, based on the first condition of completion of the above proof, according to (3.11), (3.12) and (3.15), we can obtain

$$
\begin{aligned}
\sigma e_{i,m}(t) &= \sigma\xi_i^+(t) - \sigma\xi_i^-(t) \\
&= \sigma\varepsilon_i^+(t) + \sigma\varepsilon_i^-(t) \\
&= M_i(\varepsilon_i^+(t) + \varepsilon_i^-(t)) + (G_i^+ g_i^+[x_i(t)] - G_i^- g_i^-[x_i(t)] - G_i^+ g_i^-[x_i(t)] + G_i^- g_i^+[x_i(t)]) \\
&= M_i e_{i,m}(t) + \omega_i(t),
\end{aligned} \tag{3.19}
$$

where

$$
\omega_i(t) = G_i^+ g_i^+[x_i(t)] - G_i^- g_i^-[x_i(t)] - G_i^+ g_i^-[x_i(t)] + G_i^- g_i^+[x_i(t)]
$$

$$= (G_i^+ + G_i^-)(g_i^+[x_i(t)] - g_i^-[x_i(t)]).$$

When system (2.1) is a nonlinear discrete interconnected system, since N_i is nonnegative and Schur matrix, when $\|g_i^+[x_i(t)] - g_i^-[x_i(t)]\|$ is uniformly bounded, $e_{i,m}(t) = \xi_i^+(t) - \xi_i^-(t)$ is uniformly bounded and convergent. When system (2.1) is a continuously interconnected system, since N_i is a Metzler and Hurwitz matrix, $e_{i,m}(t) = \xi_i^+(t) - \xi_i^-(t)$ is uniformly bounded and convergent when $\|g_i^+[x_i(t)] - g_i^-[x_i(t)]\|$ is uniformly bounded.

Let $\tau_{i,m}(t) = v_i^+(t) - v_i^-(t)$, we can obtain

$$\begin{aligned} \tau_{i,m}(t) &= v_i^+(t) - v_i^-(t) \\ &= (L_i^+ + L_i^-)(\xi_i^+(t) - \xi_i^-(t)) \\ &= (L_i^+ + L_i^-)e_{i,m}(t). \end{aligned} \tag{3.20}$$

When $\|g_i^+[x_i(t)] - g_i^-[x_i(t)]\|$ converges uniformly to 0 and $e_{i,m}(t)$ converges uniformly to 0, that is, $\|v_i^+(t) - v_i^-(t)\|$ converges uniformly to 0, the third condition of the interval observer is proven. Theorem 3.1 is completely proven.

Remark 1. Theorem 3.1 provides sufficient conditions for the existence of an interval function observer for nonlinear systems (2.1), enabling target estimation of the interval function of the system state function. Compared with the state reconstruction and estimation of a traditional system observer, the design of the interval function observer and its state-estimation method offer more flexibility.

Solving the undetermined coefficient matrix in the interval function observer in (3.1) involves solving a generalized Sylvester equation. Subsequently, we provide relevant information for solving the generalized Sylvester equation and the method for determining the interval function observer coefficient matrix in the form of a theorem.

Considering the generalized Sylvester equation of the following form, Reference [35] provides a solution to the equation:

$$\mathcal{A}\mathcal{V} - \mathcal{E}\mathcal{V}\mathcal{F} = \mathcal{B}\mathcal{W}, \tag{3.21}$$

where $\mathcal{A} \in \mathrm{R}^{n\times n}, \mathcal{V} \in \mathrm{R}^{n\times p}, \mathcal{E} \in \mathrm{R}^{n\times n}, \mathcal{F} \in \mathrm{R}^{p\times p}, \mathcal{B} \in \mathrm{R}^{n\times r}, \mathcal{W} \in \mathrm{R}^{r\times p}$.

When $\mathrm{rank}[\lambda\mathcal{E} - \mathcal{A} \quad \mathcal{B}] = n$, $\forall\lambda \in eig(\mathcal{F})$, if exists $\mathcal{K}(s) = [\kappa_{ij}(s)]_{n\times r} = \sum_{i=0}^{\omega} \mathcal{K}_i s^i, \mathcal{K}_i \in \mathrm{R}^{n\times r}$,

$\mathcal{D}(s) = [d_{ij}(s)]_{r\times r} = \sum_{i=0}^{\omega} \mathcal{D}_i s^i, \mathcal{D}_i \in \mathrm{R}^{r\times r}$, and satisfy the following equation

$$(\lambda\mathcal{E} - \mathcal{A})\mathcal{K}(s) + \mathcal{B}\mathcal{D}(s) = 0, \tag{3.22}$$

where

$\omega_1 = max(deg(\kappa_{ij}(s)), i, j = 1,2, \cdots, r)$,

$\omega_2 = max(deg(d_{ij}(s)), i = 1,2, \cdots, n, j = 1,2, \cdots, r)$,

$\omega = max(\omega_1, \omega_2)$.

We can obtain:

$$\begin{cases} \mathcal{V} = \sum_{i=0}^{\omega} \mathcal{K}_i Z \mathcal{F}^i, \\ \mathcal{W} = \sum_{i=0}^{\omega} \mathcal{D}_i Z \mathcal{F}^i, \end{cases} \tag{3.23}$$

where $Z \in \mathrm{R}^{r\times p}$ is an arbitrary matrix.

Based on theolution of the generalized Sylvester equation [34], we obtain the following theorem to complete the solution of the interval observer's undetermined matrix:

Theorem 3.2. For a discrete interconnected system, let M_i be a nonnegative Schur matrix. For a continuously interconnected, let M be a Metzler and Hurwitz matrix. In a nonlinear interconnected system (2.1) with each subsystem satisfying condition $x_i^-(t_0) \le x_i(t_0) \le x_i^+(t_0)$, the undetermined matrix of the interval observer (3.1) for the function $z_i(t) = F_i x_i(t), i = 1,2,\cdots,N,$ can be solved as follows:

$$G_i = \sum_{k=0}^{\omega} (M_i^{\mathrm{T}})^k Z_i^{\mathrm{T}} \mathcal{K}_k^{\mathrm{T}}. \tag{3.24}$$

$$J_i = \sum_{k=0}^{\omega} (M_i^{\mathrm{T}})^k Z_i^{\mathrm{T}} \mathcal{D}_k^{\mathrm{T}}. \tag{3.25}$$

$$\chi_i = Y_i X_i^{\dagger} + \Phi_i (I_{\theta_i} - X_i X_i^{\dagger}). \tag{3.26}$$

Where

$\chi_i = \begin{bmatrix} E_i & J_{ij_1} & J_{ij_2} & \cdots & J_{ij_k} & \cdots & J_{ij_N} \end{bmatrix}$,

$X_i = diag(\, C_i,\, C_{ij_1},\, C_{ij_2}, \cdots,\, C_{ij_k}, \cdots,\, C_{ij_N})$,

$Y_i = \begin{bmatrix} F_i - L_i G_i & G_i A_{ij_1} & G_i A_{ij_2} & \cdots & G_i A_{ij_k} & \cdots & G_i A_{ij_N} \end{bmatrix}$,

and $i \ne j_k = k, k = 1,2,\cdots,\mathrm{N}$.

Z_i *and* L_i are any matrices that satisfies the conditions.

Proof: By transposing both sides of Eq (3.5) in Theorem 3.1, we obtain

$$A_i^{\mathrm{T}} G_i^{\mathrm{T}} - G_i^{\mathrm{T}} M_i^{\mathrm{T}} = C_i^{\mathrm{T}} J_i^{\mathrm{T}}, \tag{3.27}$$

where G_i, J_i are the matrices to be solved.

By replacing $\mathcal{E}, \mathcal{A}, \mathcal{F}, \mathcal{B}$ in (3.21) with $I_{n_i}, A_i^{\mathrm{T}}, N_i^{\mathrm{T}}, C_i^{\mathrm{T}}$ in (3.27) and according to (3.23), we can obtain

$$\begin{cases} G_i^{\mathrm{T}} = \sum_{k=0}^{\omega} \mathcal{K}_k Z_i M_i^k, \\ J_i^{\mathrm{T}} = \sum_{k=0}^{\omega} \mathcal{D}_k Z_i M_i^k. \end{cases} \tag{3.28}$$

We transposed the matrices on either side of the equation of (3.28) to obtain (3.24) and (3.25) from the theorem.

Considering that system (2.1) is an interconnected system, the subscripts of (3.6) and (3.7) in Theorem 3.1 are $i, j = 1,2, \cdots, \mathrm{N}, i \neq j.$ For the correlation matrix in Eqs (3.6) and (3.7), according to Reference [32], the following notation is given

$$\chi_i = \begin{bmatrix} E_i & J_{ij_1} & J_{ij_2} & \cdots & J_{ij_k} & \cdots & J_{ij_N} \end{bmatrix}. \tag{3.29}$$

$$X_i = diag(\, C_i,\, C_{ij_1},\, C_{ij_2}, \cdots,\, C_{ij_k}, \cdots, C_{ij_N}). \tag{3.30}$$

$$Y_i = \begin{bmatrix} F_i - L_i G_i & G_i A_{ij_1} & G_i A_{ij_2} & \cdots & G_i A_{ij_k} & \cdots & G_i A_{ij_N} \end{bmatrix}. \tag{3.31}$$

Instructions:

(1) The second subscript of the matrices $J_{ij_k}, C_{ij_k}, and\ A_{ij_k}$ satisfies the conditions:$i \neq j_k = k, k = 1,2, \cdots, \mathrm{N}.$

(2) The $N-1$ second subscripts $j_1, \cdots, j_k, \cdots, j_N$ in the matrix $J_{ij_k}, C_{ij_k},$ and A_{ij_k} are arranged in ascending order.

The matrix relationship in terms of (3.29)–(3.31), (3.6), and (3.7) can be described by the following equation:

$$\chi_i X_i = Y_i. \tag{3.32}$$

The solution of the undetermined matrices in Eqs (3.6) and (3.7) is transformed into the solution of Eq (3.32).

When the constant matrices X_i and Y_i in (3.32) satisfy the condition $\mathrm{rank}[X_i] = \mathrm{rank}\begin{bmatrix} X_i \\ Y_i \end{bmatrix}$, (3.32) has a feasible solution.

By calculating $\chi_i = Y_i X_i^{\dagger} + \Phi_i (I_{\theta_i} - X_i X_i^{\dagger})$, the matrix $E_i, J_{ij}, i, j = 1,2, \cdots, \mathrm{N}, i \neq j$ can be solved, where $X_i^{\dagger}$ is the Moore–Penrose inverse of X_i. Φ_i is an arbitrary matrix to be determined. Theorem 3.2 is proven.

Remark 2. Based on the solution method of the generalized Sylvester equation, Theorem 3.2 provides the solution method of the interval function observer gain matrix of the i-*th* subsystem in the nonlinear interconnected system. Similar to the i-*th* subsystem, it can solve the interval function observer gain matrix of other subsystems in the interconnected system.

Algorithm 1.

Step 1: Construct matrix $\mathcal{A}(s) = sI_{n_i} - A_i^{\mathrm{T}}, \mathcal{B}(s) = -C_i^{\mathrm{T}}$ based on the known matrices $A_i \in \mathrm{R}^{n_i \times n_i}\ and\ C_i \in \mathrm{R}^{p_i \times n_i}$ in system (2.1).

Step 2: Based on the matrices $\mathcal{A}(s)$ and $\mathcal{B}(s)$ obtained in Step 1, matrices

$$\begin{cases} \mathcal{K}(s) = \left[\kappa_{ij}(\mathrm{s})\right]_{n \times r} = \sum\limits_{i=0}^{\omega} \mathcal{K}_i s^i, \mathcal{K}_i \in \mathrm{R}^{n \times r}, \\ \mathcal{D}(s) = \left[d_{ij}(\mathrm{s})\right]_{r \times r} = \sum\limits_{i=0}^{\omega} \mathcal{D}_i s^i, \mathcal{D}_i \in \mathrm{R}^{r \times r}, \end{cases}$$

are solved to satisfy the condition: $\mathcal{A}(s)\mathcal{K}(s) - \mathcal{B}(s)\mathcal{D}(s) = 0.$

Step 3: If system (2.1) is a discrete interconnected system, the nonnegative and Schur matrices N_i are selected. If system (2.1) is a continuously interconnected system, the Metzler and Hurwitz matrix N_i.
Step 4: Given an unknown matrix $Z_i \in \mathrm{R}^{\gamma_i \times r_i}$, matrices $G_i\ and\ J_i$ are solved by expression

$$\begin{cases} G_i = \sum_{k=0}^{\omega} (M_i^{\mathrm{T}})^k Z_i^{\mathrm{T}} \mathcal{K}_k^{\mathrm{T}}, \\ J_i = \sum_{k=0}^{\omega} (M_i^{\mathrm{T}})^k Z_i^{\mathrm{T}} \mathcal{D}_k^{\mathrm{T}}. \end{cases}$$

Step 5: Take the unknown matrix $L_i \in \mathrm{R}^{r_i \times r_i}$, use L_i and the matrix G_i solved in Step 4 to represent Y_i, and select the undetermined matrices $Z_i\ and\ L_i$ to satisfy the condition: $\mathrm{rank}[X_i] = \mathrm{rank}\begin{bmatrix} X_i \\ Y_i \end{bmatrix}$.
Step 6: Based on Steps 4 and 5, $\chi_i = Y_i X_i^{\dagger} + \Phi_i(I_{\theta_i} - X_i X_i^{\dagger})$ is solved. Finally, the matrices $E_i, J_{ij}, i, j = 1,2,\cdots,\mathrm{N}, i \neq j$ is obtained.
Step 7: Based on the obtained matrices G_i and L_i, we can obtain: $G_i^{+}, G_i^{-}, L_i^{+}, L_i^{-}\ and\ H_i = G_i B_i$; the interval function observer design is complete.

4. Simulation examples

To verify the validity of the interval function observer design method proposed in this study, discrete and continuous nonlinear interconnected systems were simulated and analyzed.

4.1. Interval estimation of nonlinear discrete interconnected system state function

Discrete interconnected system with two subsystems with the following parameters:

$$A_1 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.005 & 0.005 \\ 0.002 & 0.002 \\ -0.005 & -0.005 \end{bmatrix},$$

$$C_1 = [0.5 \quad 1 \quad 1], g_1[x_1(t)] = \begin{bmatrix} 0 \\ 0.006\sin x_{11}(t) \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.001 & 0.002 & 0.002 \\ -0.001 & -0.002 & -0.002 \end{bmatrix},$$

$$C_1 = [1 \quad 1], g_2[x_2(t)] = \begin{bmatrix} 0.004\sin x_{21}(t) \\ 0 \\ 0 \end{bmatrix}.$$

Initial states:

$$x_1(0) = [0.3 \quad -0.3 \quad 0.4]^{\mathrm{T}}, \qquad x_2(0) = [0.3 \quad -0.2]^{\mathrm{T}}.$$

Nonlinear function satisfying

$$\begin{bmatrix} 0 \\ -0.006 \\ 0 \end{bmatrix} \leq g_1[x_1(t)] \leq \begin{bmatrix} 0 \\ 0.006 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -0.004 \\ 0 \\ 0 \end{bmatrix} \leq g_2[x_2(t)] \leq \begin{bmatrix} 0.004 \\ 0 \\ 0 \end{bmatrix}.$$

In the discrete interconnected system above, the coefficient matrix in (2.4) is defined as

$$\begin{cases} F_1 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \\ F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \end{cases}$$

The first subsystem is obtained according to the first step of the algorithm.

$$\begin{cases} \mathcal{A}(s) = \begin{bmatrix} s-1 & 0 & 0 \\ -4 & s+1 & 0 \\ -3 & -2 & s+1 \end{bmatrix}, \\ \mathcal{B}(s) = \begin{bmatrix} -0.5 \\ -1 \\ -1 \end{bmatrix}. \end{cases}$$

According to Step 2 of the algorithm, we can obtain

$$\begin{cases} \mathcal{K}(s) = s^2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \\ \mathcal{D}(s) = -2s^3 - 2s^2 + 2s + 2. \end{cases}$$

According to Steps 3 and 4 of the algorithm, the following Schur and nonnegative matrices are selected:

$$M_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Let $Z_1 = [z_1 \quad z_2], L_1 = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$, and then we can obtain

$$\begin{cases} G_1 = \begin{bmatrix} 1.44z_1 & 2.88z_1 & 6.48z_1 \\ 1.69z_2 & 3.38z_2 & 7.28z_2 \end{bmatrix}, \\ J_1 = \begin{bmatrix} 2.304z_1 \\ 2.366z_2 \end{bmatrix}. \end{cases}$$

According to Step 5 of the algorithm, $rank \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = rank(X_1)$ is equivalent to the following equation:

$$\begin{cases} 3.6z_1 l_{11} + 3.9z_2 l_{12} = 0, \\ 2 - 3.6z_1 l_{21} + 3.9z_2 l_{22} = 0. \end{cases}$$

Here, matrices Z_1, L_1 satisfying the above formula are respectively

$$\begin{cases} Z_1 = [-1 \quad 1], \\ L_1 = \begin{bmatrix} 13/12 & 1 \\ 1 & 16/19 \end{bmatrix}. \end{cases}$$

According to Step 6 of the algorithm, we obtain

$$\begin{cases} G_1 = \begin{bmatrix} -1.44 & -2.88 & -6.48 \\ 1.69 & 3.38 & 7.28 \end{bmatrix}, \\ E_1 = \begin{bmatrix} 1.7400 \\ 3.4933 \end{bmatrix}, \\ J_1 = \begin{bmatrix} -2.304 \\ 2.3660 \end{bmatrix}, \\ J_{12} = \begin{bmatrix} -2.304 \\ 2.366 \end{bmatrix}. \end{cases}$$

According to Step 7 of the algorithm, we obtain

$$\begin{cases} G_1^+ = \begin{bmatrix} 0 & 0 & 0 \\ 1.69 & 3.38 & 7.28 \end{bmatrix}, \\ G_1^- = \begin{bmatrix} 1.44 & 2.88 & 6.48 \\ 0 & 0 & 0 \end{bmatrix}. \end{cases}$$

$$\begin{cases} L_1^+ = \begin{bmatrix} 13/12 & 1 \\ 1 & 16/19 \end{bmatrix}, \\ L_1^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{cases}$$

and

$$H_1 = \begin{bmatrix} -6.48 \\ 7.28 \end{bmatrix}.$$

We use a similar method to obtain the interval observer gain matrix of the second subsystem, as follows:

$$M_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\begin{cases} L_2^+ = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/3 \end{bmatrix}, \\ L_2^- = \begin{bmatrix} 0 & 1/6 \\ 1/3 & 0 \end{bmatrix}. \end{cases}$$

$$\begin{cases} J_2 = \begin{bmatrix} 4.16 \\ -3.71 \end{bmatrix}, \\ J_{21} = \begin{bmatrix} 0.06 \\ -0.06 \end{bmatrix}. \end{cases}$$

$$\begin{cases} G_2^+ = \begin{bmatrix} 5.2 & 2.2 \\ 0 & 0 \end{bmatrix}, \\ G_2^- = \begin{bmatrix} 0 & 0 \\ 5.3 & 2.3 \end{bmatrix}, \end{cases}$$

and

$$\begin{cases} H_2 = \begin{bmatrix} 2.2 \\ -2.3 \end{bmatrix}, \\ E_2 = \begin{bmatrix} -0.75 \\ 3.53 \end{bmatrix}. \end{cases}$$

Figures 1 and 2 show that, when the initial states of the two subsystems in a nonlinear discrete interconnected system are respectively $x_1(0) = [0.3 \quad -0.3 \quad 0.4]^{\mathrm{T}}$ and $x_2(0) = [0.3 \quad -0.2]^{\mathrm{T}}$, the constructed interval observer can realize the interval estimation of the linear function $v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) + 2x_{12}(t) + 2x_{13}(t) \\ x_{11}(t) + 2x_{12}(t) \end{bmatrix}$ and $v_2(t) = \begin{bmatrix} v_{21}(t) \\ v_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{21}(t) \\ 2x_{22}(t) \end{bmatrix}$ in Subsystems 1 and 2.

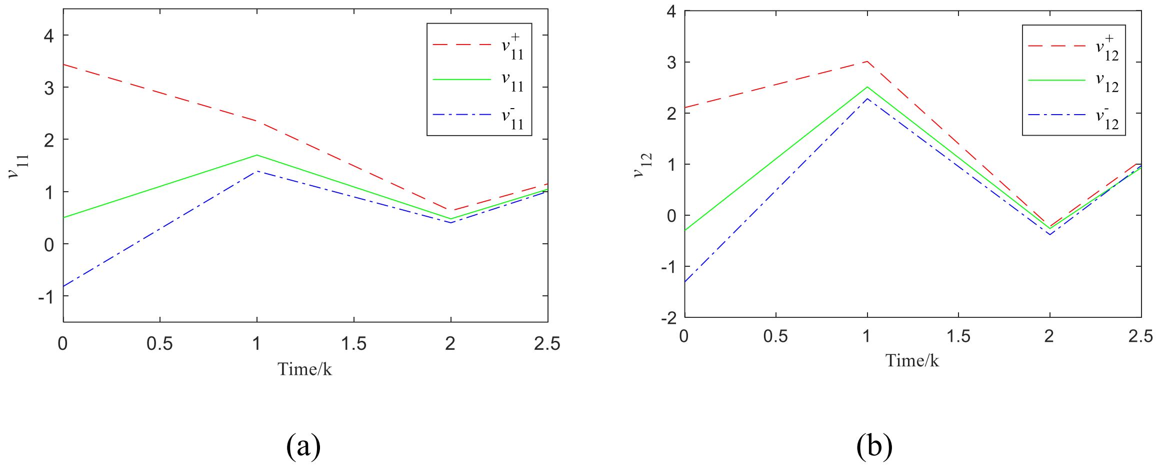


Figure 1. Interval estimation of state function $v_1(t)$ in discrete subsystem 1.

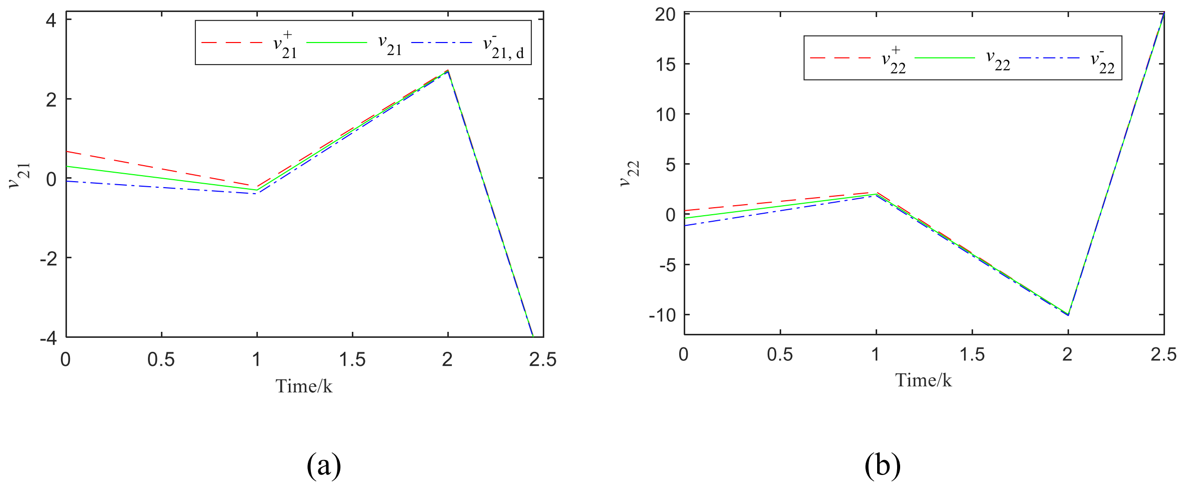


Figure 2. Interval estimation of state function $v_2(t)$ in discrete subsystem 2.

Remark 3. Figures 1 and 2 show that the interval estimation error $e_i^+(t) = v_i^+(t) - F_i x_i(t)$ and $e_i^-(t) = F_i x_i(t) - v_i^-(t)$ asymptotically converges to zero in the process of interval estimation of the state functions in the two subsystems of the interconnected system, which guarantees the estimation effect of the observer; the validity of the proposed method in discrete interconnected systems is verified.

4.2. Interval estimation of nonlinear continuously interconnected system state function

A continuously interconnected system consisting of two subsystems with the following parameters:

$$A_1=\begin{bmatrix}-2&0&1\\-1&-2&0\\0&1&5\end{bmatrix},B_1=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix},A_{12}=\begin{bmatrix}0.1&-0.1&0.1\\0&0&0.1\\0.1&0.1&0\end{bmatrix},$$

$$C_1=\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix},g_1[x_1(t)]=\begin{bmatrix}0\\0.004\sin x_{11}(t)\\0\end{bmatrix},d_1(t)=\begin{bmatrix}0.001\\0.001\\0.001\end{bmatrix},$$

$$A_2=\begin{bmatrix}-1&0&0\\0&3&-1\\0&-1&-2\end{bmatrix},B_2=\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix},A_{21}=\begin{bmatrix}0&0&0.1\\0.1&0.1&0.2\\0.1&0&0.1\end{bmatrix},$$

$$C_2=\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix},g_2[x_2(t)]=\begin{bmatrix}0\\0.006\sin x_{21}(t)\\0\end{bmatrix},d_2(t)=\begin{bmatrix}0.001\\0.001\\0.001\end{bmatrix}.$$

Initial states:

$$x_1(0)=[0.1\quad -0.2\quad 0.3]^{\mathrm{T}},\qquad x_2(0)=[0.2\quad -0.1\quad 0.4]^{\mathrm{T}}.$$

Nonlinear function satisfying

$$\begin{bmatrix}0\\-0.004\\0\end{bmatrix}\le g_1[x_1(t)]\le\begin{bmatrix}0\\0.004\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\-0.006\\0\end{bmatrix}\le g_2[x_2(t)]\le\begin{bmatrix}0\\0.006\\0\end{bmatrix}.$$

In the above continuously interconnected system, the coefficient in (2.4) is defined as

$$\begin{cases}F_1=\begin{bmatrix}1&0&2\\0&1&0\end{bmatrix},\\ F_2=\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}.\end{cases}$$

The solution to the interval function observer's undetermined matrix in this example is similar to the previous numerical example.

For the first subsystem, we can obtain

$$\begin{cases}\mathcal{A}(s)=\begin{bmatrix}s+2&1&0\\0&s+2&-1\\-1&0&s-5\end{bmatrix},\\ \mathcal{B}(s)=\begin{bmatrix}0&0\\-1&0\\0&-1\end{bmatrix}.\end{cases}$$

According to Step 2 of the algorithm, we obtain

$$\begin{cases} \mathcal{K}(s) = s\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 0 \\ 0 & 1 \end{bmatrix}, \\ \mathcal{D}(s) = s^2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix}. \end{cases}$$

According to Steps 3 and 4 of the algorithm, the following Metzler and Hurwitz matrix is selected:

$$M_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Let $Z_1 = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}, L_1 = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$, then, we obtain:

$$\begin{cases} G_1 = \begin{bmatrix} z_{11} & -z_{11} & z_{21} \\ z_{12} & 0 & z_{22} \end{bmatrix}, \\ J_1 = \begin{bmatrix} z_{11} + z_{21} & z_{11} + 6z_{21} \\ z_{22} & z_{12} + 7z_{22} \end{bmatrix}. \end{cases}$$

When $rank\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = rank(X_1)$, it is equivalent to the following equation:

$$\begin{cases} 1 - z_{11}l_{11} - z_{12}l_{12} = 0, \\ -z_{11}l_{21} - z_{12}l_{22} = 0, \\ z_{11} + z_{21} = 0, \\ z_{12} + z_{22} = 0. \end{cases}$$

Here, matrices Z_1, L_1 satisfying the above formula are respectively

$$\begin{cases} Z_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\ L_1 = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}. \end{cases}$$

We can then calculate

$$\begin{cases} G_1 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \\ E_1 = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}, \\ J_1 = \begin{bmatrix} 0 & -5 \\ 1 & 6 \end{bmatrix}, \\ J_{12} = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}. \end{cases}$$

Therefore, we have

$$\begin{cases} G_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ G_1^- = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \end{cases}$$

$$\begin{cases} L_1^+ = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \\ L_1^- = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \end{cases}$$

and

$$H_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Below, we use a similar method to obtain the interval observer gain matrix of the second subsystem, as follows:

$$M_2 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix},$$

$$\begin{cases} L_2^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ L_2^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{cases}$$

$$\begin{cases} J_2 = \begin{bmatrix} 6 & -1 \\ -7 & 2 \end{bmatrix}, \\ J_{21} = \begin{bmatrix} 0.1 & 0.1 \\ -0.1 & -0.1 \end{bmatrix}. \end{cases}$$

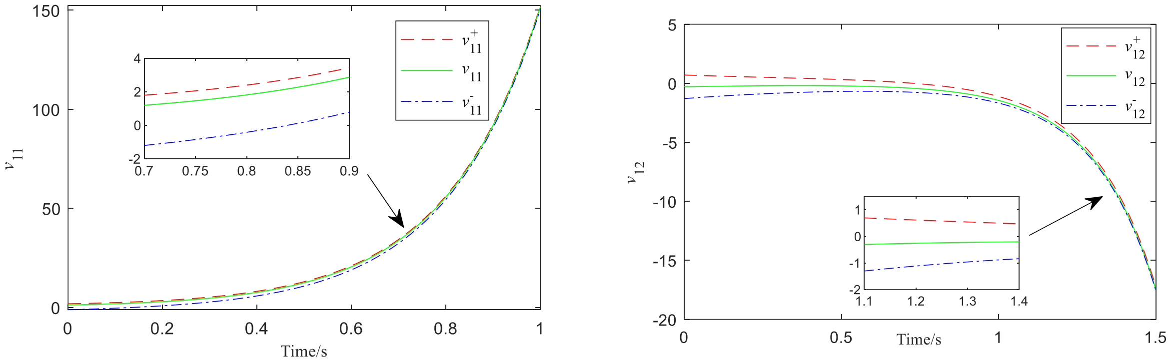
$$\begin{cases} G_2^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ G_2^- = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \end{cases}$$

and

$$\begin{cases} H_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\ E_2 = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}. \end{cases}$$

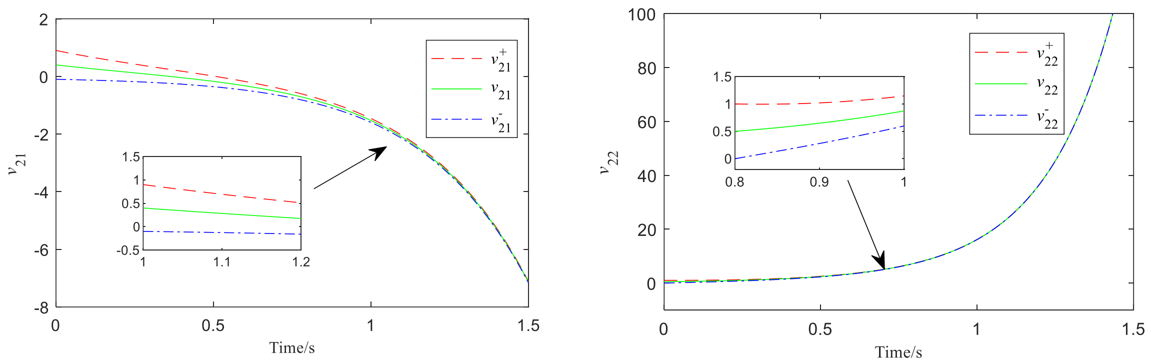
Figures 3 and 4 show that, when the initial states of the two subsystems in a continuously interconnected system are respectively $x_1(0) = [0.1 \quad -0.2 \quad 0.3]^{\mathrm{T}}$ and $x_2(0) = [0.2 \quad -0.1 \quad 0.4]^{\mathrm{T}}$, the interval observer constructed can realize the interval estimation of the linear function $v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) + 2x_{13}(t) \\ x_{12}(t) \end{bmatrix}$ and $v_2(t) = \begin{bmatrix} v_{21}(t) \\ v_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{23}(t) \\ x_{22}(t) \end{bmatrix}$ in Subsystems 1 and 2.

As shown in Figures 3 and 4, the interval estimation error $e_i^+(t) = v_i^+(t) - F_i x_i(t)$ and $e_i^-(t) = F_i x_i(t) - v_i^-(t)$ asymptotically converges to zero fleetingly, which guarantees the estimation effect of the observer, the validity of the proposed method in nonlinear continuous interconnected systems is verified.



(a) (b)

Figure 3. Interval estimation of state function $v_1(t)$ in continuous subsystem 1.



(a) (b)

Figure 4. Interval estimation of state function $v_2(t)$ in continuous subsystem 2.

Remark 4. In nonlinear continuously interconnected system, considering factors such as uncertainties or external disturbances existing in the actual control system, the external disturbance terms $d_1(t)$ and $d_2(t)$ exist in the state equations of nonlinear, continuously interconnected systems. Considering the perturbation part existing in the system, in this study, the nonlinear and perturbation terms in the interconnected system are combined as a nonlinear component. An interval estimation method is employed to estimate the state functions of the interconnected system.

Numerical simulations for both discrete and continuous interconnected systems are presented. As shown in Figures 1–4, the proposed interval function observer effectively estimates the state intervals in both cases, demonstrating the method's feasibility and robustness in the presence of external disturbances.

5. Conclusions

We presented a design method for an interval function observer for nonlinear interconnected systems. During the observer design process, discrete and continuous interconnected systems were unified into a single model, the interval function observer was designed, and the sufficient conditions for its existence were provided using monotone system theory. Based on the Sylvester equation, we proposed a method for solving the undetermined matrix of an interval function observer. Finally, the concrete steps for constructing an interval function observer were presented. The proposed observer design method can be applied to discrete and continuous interconnected systems. It can provide a reference for designing interval function observers for more complex control systems.

Author contributions

Yanxiu Sun: Investigation, methodology, analysis, validations, software, original manuscript; Ying Du and Yuping Zhang: Reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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