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Research article

Reliability assessment and remaining useful life prediction based on the unit Gompertz distribution

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Abstract: In this paper, we mainly consider reliability inference and remaining useful life prediction for the unit Gompertz distribution and the related stress-strength model. The exact confidence interval for the model parameter β is derived. The generalized confidence intervals for model parameter α and some commonly used reliability metrics such as the failure rate function, the quantile, and the reliability function are explored. Based on the observed failure data set, the prediction intervals for the remaining useful life and the future failure times are developed. In addition, when the stress and strength variables follow the unit Gompertz distributions with different parameters, the generalized confidence interval for the reliability of the stress-strength model is also proposed. A Monte Carlo simulation study is implemented to evaluate the accuracy of the proposed inferential procedures and compared with the Wald and bootstrap methods. Finally, two examples are provided to illustrate the applicability of the proposed methods.

Keywords: unit Gompertz distribution; stress-strength model; generalized confidence interval; generalized prediction interval; reliability metrics
Mathematics Subject Classification: 62F30

Acronyms and abbreviations

UG: Unit-Gompertz; GCI: Generalized confidence interval; PI: Prediction interval; PDF: Probability density function; CDF: Cumulative distribution function; MLE: Maximum likelihood estimator; CI: Confidence interval; FFT: Future failure time; RUL: Remaining useful lifetime; GPQ: Generalized pivot quantity; GCI: Generalized confidence interval; GPI: Generalized prediction interval; CP: Coverage probability

1. Introduction

The Gompertz distribution was first proposed by Benjamin Gompertz in 1825. Because this distribution has some well-known characteristics such as exponential distribution, Weibull distribution, and extreme value distribution, it is more useful in medical and actuarial research due to its exponential failure rate of lifetimes [1]. Consequentially, some generalizations about the Gompertz distribution have been studied by many researchers. Notable, by exponentially transforming the variables of Gompertz distribution, Mazucheli et al. [2] proposed the unit-Gompertz (UG) distribution, which is more available than some well-known lifetime distributions.

The probability density function (PDF) and cumulative distribution function (CDF) of the UG distribution are given by

$$f(x \mid \alpha,\beta) = \alpha\beta x^{-(\beta+1)} \exp\left(-\alpha(x^{-\beta} - 1)\right), 0 < x < 1, \alpha > 0, \beta > 0,$$

and

$$F(x \mid \alpha,\beta) = \exp\left(-\alpha(x^{-\beta} - 1)\right), 0 < x < 1, \alpha > 0, \beta > 0,$$

respectively. For more properties of the UG distribution, one can refer to [2, 3] and the references therein. As explained by Arshad et al. [1], the UG distribution can be used for the analysis of skew data, which can be applied to survival analysis and environmental science.

Since the UG distribution appeared, some work has been done by researchers on this distribution. Here, we list some of the more notable works. On the basis of dual generalized order statistics, Arshad et al. [1] proposed Bayesian inference of the UG distribution. When the stress and strength variables follow UG distributions, Jha et al. [4] estimated multicomponent stress-strength reliability by using frequentist and Bayesian approaches. Kumar et al. [5] considered inter-record times and lower recorded values to obtain the single and product moments' recurrence relation. The authors also considered parameter estimation by obtaining maximum likelihood estimators (MLEs) and approximate Bayes estimators, with the result concerning the prediction of future recorded values. Jha et al. [6] proposed Bayes estimates to estimate multi-component reliability by assuming the UG distribution. Bayes estimates of the system's reliability are obtained by using Lindley's approximation and the Metropolis Hastings algorithm methods when all the parameters are unknown. Dey and Mosawi [7] studied estimation methodologies for parameters of an UG distribution based on two frequentist methods and a Bayesian method based on progressively Type II censored data.

Considering the stress-strength model has important applications in reliability engineering, we will study the stress-strength model in which both the stress variable X and the strength variable Y are independent UG distributions, and develop the interval estimation of the stress strength-reliability $R = P(X < Y)$. The stress-strength model was first proposed by Birnbaum [8]. Under the assumption of different distributions of stress and strength, the inference methods of the reliability R have been studied by many authors, for example, the power Lindley distribution [9, 10], the Weibull distribution [11–13], the Burr XII distribution [14], the generalized inverted exponential distribution [15], the generalized exponential distribution [16], and the inverse Gaussian distribution [17, 18], among others. For results and applications of the stress-strength model, one can refer to [19].

To the best of our knowledge, due to the complexity of the UG distribution, there is limited literature on the reliability inference procedures for the stress-strength model based on this distribution. Compared with point estimation, interval estimation offers a more effective means of quantifying uncertainty. Therefore, this paper focuses primarily on investigating the interval estimation inference procedures for the UG distribution and its associated stress-strength model.

Moreover , due to constraints on experimental costs, the data obtained from reliability tests are generally small-sample data, which exhibit greater uncertainty compared with large-sample data. Consequently, it remains challenging to derive satisfactory confidence intervals (CIs) for certain reliability metrics of the UG distribution and its associated stress-strength model when relying on small-sample data. Furthermore, accurate reliability assessments are critical for ensuring the system's safety, among other applications. These factors collectively motivate us to develop statistical methods that are suitable for the UG distribution and its associated stress-strength model under small-sample scenarios. The main contributions of this paper are summarized as follows:

(1) Accurate CIs for the parameters and some reliability metrics of UG distribution are presented.

(2) Based on small-sample scenarios, accurate CIs for the stress-strength reliability R are deduced.

(3) For engineering applications and decisions, accurate prediction intervals (PIs) for future failure times (FFT) and remaining useful life (RUL) are proposed, based on small-sample scenarios.

The objective of this paper is to derive accurate confidence intervals (CIs) for the UG distribution and its associated stress-strength model's intervals that remain effective even under small-sample scenarios. To achieve this, the paper carefully constructs generalized pivot quantities (GPQs) and then develops the corresponding CI inference procedures. For a detailed discussion of the GPQ method, please refer to [20–22]. The GPQs of some reliability metrics can be obtained via the substitution method based on the GPQs of the model's parameters [20, 22].

The remainder of the paper is organized as follows. In Section 2, the generalized confidence intervals (GCIs) are proposed for some quantities of interest based on the UG distribution. In Section 3, the GCIs for the stress-strength reliability R are provided. In Section 4, the generalized prediction intervals (GPIs) for the FFT and the RUL are obtained. In Section 5, the performance of the proposed GCIs and GPIs is assessed via a Monte Carlo simulation. In Section 6, the proposed inferential procedures are illustrated through two examples.

2. CIs for the UG distribution

2.1. CI for the parameter β

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed random variables of the UG distribution, and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the corresponding order statistics.

The following two lemmas are neeeded for obtaining the interval estimation method.

Lemma 2.1. *Suppose that $X_1, X_2, \cdots, X_n$ are defined as above. Let*

$$V_{(j)} = X_{(n-j+1)}^{-\beta} - 1, j = 1, 2, \cdots, n,$$

and $S_k = \sum_{j=1}^{k} V_{(j)} + (n-k)V_{(k)}$ for $k = 1, 2, \cdots, n$.

Assume that

$$W(\beta) = 2 \sum_{k=1}^{n-1} \log(S_n/S_k).$$

Then (1) $W(\beta) \sim \chi^2(2n-2)$, (2) $2\alpha S_n \sim \chi^2(2n)$, and (3) $W(\beta)$ and S_n are independent.

Proof. Note that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the corresponding order statistics from the $F(x \mid \alpha,\beta)$. Then

$$V_{(j)} = -\frac{1}{\alpha}\log(F(X_{(n-j+1)} \mid \alpha,\beta)) = X_{(n-j+1)}^{-\beta} - 1,$$

are the order statistics from the exponential distribution with mean $1/\alpha$. We also have

$$\tau_1 = n\alpha V_{(1)}, \tau_i = n\alpha[n-i+1](V_{(i)} - V_{(i-1)}), i = 2, \cdots, n.$$

From [23], we know that $\tau_1, \tau_2, \cdots, \tau_n$ are independent standard exponential random variables. Notice that $\alpha S_k = \sum_{j=1}^{k} \tau_j, k = 1, 2, \cdots, n$. Let

$$\nu_{(k)} = (\alpha S_k)/(\alpha S_n) = S_k/S_n, k = 1, 2, \cdots, n-1.$$

Then $\nu_{(1)} < \cdots < \nu_{(n-1)}$ are order statistics from the uniform (0,1) distribution with a sample size of $n-1$. Notice that $W(\beta) = \sum_{k=1}^{n-1}(-2\log \nu_{(k)}) = 2\sum_{k=1}^{n-1}\log(S_n/S_k)$, and thus $W(\beta) \sim \chi^2(2n-2)$. From [24] , we can obtain $2\alpha S_n \sim \chi^2(2n)$, and $W(\beta)$ and S_n are independent.

Lemma 2.2. *The function $W(\beta)$ in Lemma 1 is a strictly increasing function of β.*

Proof.

$$S_n/S_k = 1 + \frac{S_n - S_k}{S_k} \tag{2.1}$$

$$= \frac{\sum_{j=k+1}^{n} Q_{j,k} - (n-k)}{\sum_{j=1}^{k} Q_{j,k} + (n-k)}, \tag{2.2}$$

where $Q_{j,k} = \frac{X_{(n-j+1)}^{-\beta} - 1}{X_{(n-k+1)}^{-\beta} - 1}$. Notice that, for $j > k$, $0 < X_{(n-j+1)} < X_{(n-k+1)} < 1$. Let

$$t = X_{(n-j+1)}^{-\beta}, \theta = \log_t X_{(n-k+1)}^{-\beta}.$$

We take

$$f(t) = \frac{t-1}{t^\theta - 1}, \theta > 1.$$

We can prove that $f(t)$ is a strictly decreasing function of t, and $Q_{j,k}$ is a strictly decreasing function of β. From [24], we know that $W(\beta)$ is a strictly increasing function of β.

Based on the two Lemmas above, the exact CI of parameter β is obtained via the following Theorem 1.

Theorem 2.1. *Suppose that $X_1, X_2, \cdots, X_n$ are defined as above. Then for any $0 < \gamma < 1$,*

$$\left[W^{-1}\left(\chi^2_{1-\gamma/2}(2n-2)\right), W^{-1}\left(\chi^2_{\gamma/2}(2n-2)\right)\right],$$

is a $(1-\gamma)$ level CI for the parameter β, where $\chi^2_\gamma(\upsilon)$ is the upper γ percentile of the χ^2 distribution with degrees of freedom υ, and for $t > 0$, $W^{-1}(t)$ is the solution of the equation $W(\beta) = t$.

Proof. Notice that $W(\beta) \sim \chi^2(2n-2)$. In this case

$$p\left(\chi^2_{1-\gamma/2}(2n-2) \leq W(\beta) \leq \chi^2_{\gamma/2}(2n-2)\right) = 1 - \gamma.$$

As $W(\beta)$ is a strictly increasing function of β,

$$\left[W^{-1}\left(\chi^2_{1-\gamma/2}(2n-2)\right), W^{-1}\left(\chi^2_{\gamma/2}(2n-2)\right)\right]$$

is a $(1-\gamma)$ level CI for the parameter β, where $\chi^2_\gamma(\upsilon)$ is the upper γ percentile of the χ^2 distribution with degrees of freedom υ, and for $t > 0$, $W^{-1}(t)$ is the solution of the equation $W(\beta) = t$.

2.2. GCIs for the parameter α and other quantities

Now we derive the GCIs for the parameter α and some quantities of the UG distribution, such as its failure rate function, quantiles, and reliability function.

Notice that $W(\beta)$ is a strictly increasing function of β. Generate a random sample V_1^* from the distribution $\chi^2(2n-2)$. Let $W(\beta) = V_1^*$, in which case, there is a unique solution of β to the equation $V_1^* = W(\beta)$, and denote this as $\beta = g_1(V_1^*, \mathbf{X})$.

Let

$$T(\beta) = \sum_{j=1}^{n} (X_{(n-j+1)}^{-\beta} - 1), V_2 = 2\alpha T(\beta).$$

We can prove that $V_2 \sim \chi^2(2n)$, and hence $\alpha = V_2/(2T(\beta))$. Generate a random sample V_2^* from the distribution $\chi^2(2n)$. The following GPQ of the parameter α is obtained via the substitution method [20]

$$W_1 = \frac{V_2^*}{2\sum_{j=1}^{n}(X_{(n-j+1)}^{-g_1(V_1^*,\mathbf{x})} - 1)}, \tag{2.3}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is the observed value of $\mathbf{X} = (X_1, X_2, \cdots, X_n)$.

Notice that the failure rate function at t_0, $r(t_0)$; the p quantile $Q(p)$; and the reliability function at t_0, $R(t_0)$, are given by

$$r(t_0) = r(t_0|\alpha,\beta) = \frac{\alpha\beta t_0^{-(\beta+1)} \exp\left(-\alpha(t_0^{-\beta} - 1)\right)}{1 - \exp\left(-\alpha(t_0^{-\beta} - 1)\right)},$$
$$Q(p) = Q(p|\alpha,\beta) = \exp\left(-\frac{1}{\beta}\log(\alpha - \log(p)) - \log(\alpha)\right),$$
$$R(t_0) = 1 - F(t_0|\alpha,\beta).$$

Applying the substitution method again, the GPQs for $r(t_0), Q(p)$, and $R(t_0)$ are obtained by replacing (α,β) with $(W_1, g_1(V_1^*, \mathbf{x}))$. Hence, the GPQs for $r(t_0)$, $Q(p)$, and $R(t_0)$ are given by

$$W_2 = r(t_0|W_1, g_1(V_1^*, \mathbf{x})), \tag{2.4}$$
$$W_3 = Q(p|W_1, g_1(V_1^*, \mathbf{x})), \tag{2.5}$$
$$W_4 = 1 - F(t_0|W_1, g_1(V_1^*, \mathbf{x})), \tag{2.6}$$

respectively.

Let $W_{i,\gamma}$ be the γ quantile of W_i. Then the $100(1-\gamma)\%$ GCIs for α, $r(t_0)$, $Q(p)$, and $R(t_0)$ are given by $[W_{i,\gamma/2}, W_{i,1-\gamma/2}], i = 1, \ldots, 4$, respectively. The following Monte Carlo algorithm (Algorithm 1) can be used to obtain the quantiles of W_i.

Algorithm 1 Quantiles of $W_i, i = 1, 2, 3, 4$.

Input: Observation data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Output: The quantiles of W_i.

Initialization: Let $W_{i,j}$ be the jth value of $W_i, i = 1, 2, 3, 4$, and let $W_{i,(j)}$ denote the ordered value of W_i .

for j *in* $1 : B$ **do**

1) Generate $V_1^* \sim \chi^2(2n-2)$. Then, compute $g_1(V_1^*, \mathbf{x})$ via the equation $W(\beta) = V_1^*$.

2) Generate $V_2^* \sim \chi^2(2n)$. Using the Eq (2.3), compute $W_{1,j}$.

3) Using the Eq (2.4) to Eq (2.6), compute $W_{2,j}$, $W_{3,j}$, and $W_{4,j}$.

end

$W_{i,(\gamma B)}$ can be used to estimate the γ quantile of W_i.

On the basis of the MLEs $\hat{\alpha}, \hat{\beta}$ of the model's parameters, the bootstrap-p CIs for α, β, $r(t_0)$, $Q(p)$, and $R(t_0)$ can be obtained by the following algorithm (Algorithm 2).

Algorithm 2 The bootstrap-p CIs for α, β, $r(t_0)$, $Q(p)$, and $R(t_0)$.

Input: Observation data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Output: The bootstrap-p CIs for α, β, $r(t_0)$, $Q(p)$, and $R(t_0)$.

Initialization: Let $\vartheta_{i,j}, i = 1, 2, \ldots, 5$ be the MLEs for the quantities above based on the jth bootstrap sample, and let $\vartheta_{i,(j)}$ denote the ordered value of $\vartheta_{i,j}$.

For given $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, calculate the MLEs $\hat{\alpha}, \hat{\beta}$.

for j *in* $1 : B$ **do**

1) For a given $(n, \hat{\alpha}, \hat{\beta})$, generate the bootstrap sample $\mathbf{x}^*$. Then calculate the bootstrap MLEs $\hat{\alpha}^*, \hat{\beta}^*$. Define $\vartheta_{1,j} = \hat{\alpha}^*, \vartheta_{2,j} = \hat{\beta}^*$.

2) Compute $\vartheta_{3,j}, \vartheta_{4,j}, \vartheta_{5,j}$ as follows:

$\vartheta_{3,j} = r(t_0|\vartheta_{1,j}, \vartheta_{2,j})$, $\vartheta_{4,j} = Q(p|\vartheta_{1,j}, \vartheta_{2,j})$, $\vartheta_{5,j} = 1 - F(t_0|\vartheta_{1,j}, \vartheta_{2,j})$

end

Then $[\vartheta_{i,(\gamma B/2)}, \vartheta_{i,(B-\gamma B/2)}], i = 1, \ldots, 5$ are the $(1-\gamma)$ bootstrap-p CIs for α, β, $r(t_0)$, $Q(p)$, and $R(t_0)$, respectively.

3. GCI for the stress-strength reliability R

3.1. General case: $\beta_1 \neq \beta_2$

Let Z be the stress variable, and let Y be the strength variable, where Z and Y are independent, and $Z \sim F(z|\alpha_1, \beta_1), Y \sim F(y|\alpha_2, \beta_2)$. Then the reliability of the stress-strength model can be defined as

$$\begin{aligned} R &= P(Z < Y) \\ &= \int_0^1 P(Z < Y \mid Y = y) f(y \mid \alpha_2, \beta_2) dy \\ &= \int_0^1 F(y \mid \alpha_1, \beta_1) f(y \mid \alpha_2, \beta_2) dy. \end{aligned}$$

Let $Z_1, Z_2, \cdots, Z_{n_1}$ be the random sample from $F(z|\alpha_1, \beta_1)$, and $Y_1, Y_2, \cdots, Y_{n_2}$ be the random sample from $F(y|\alpha_2, \beta_2)$. Let $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n_1)}$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n_2)}$ be the corresponding order statistics, respectively.

Let

$$V_{(z,j)} = Z_{(n_1-j+1)}^{-\beta_1} - 1, S_{z,k} = \sum_{j=1}^{k} V_{(z,j)} + (n_1 - k)V_{(z,k)}, W_z(\beta_1) = 2\sum_{k=1}^{n_1-1} \log(S_{z,n_1}/S_{z,k}).$$

$$V_{(y,j)} = Y_{(n_2-j+1)}^{-\beta_2} - 1, S_{y,k} = \sum_{j=1}^{k} V_{(y,j)} + (n_2 - k)V_{(y,k)}, W_y(\beta_2) = 2\sum_{k=1}^{n_2-1} \log(S_{y,n_2}/S_{y,k}).$$

From Lemma 2.1, we have the following findings:

$$V_3 = W_z(\beta_1) \sim \chi^2(2n_1 - 2), V_4 = 2\alpha_1 S_{z,n_1} \sim \chi^2(2n_1).$$

$$V_5 = W_y(\beta_2) \sim \chi^2(2n_2 - 2), V_6 = 2\alpha_2 S_{y,n_2} \sim \chi^2(2n_2).$$

Generate a series of random samples V_3^*, V_4^*, V_5^*, and V_6^* from the distributions $\chi^2(2n_1 - 2), \chi^2(2n_1)$, $\chi^2(2n_2 - 2)$, and $\chi^2(2n_2)$, respectively. Let $g_2(V_3^*, \mathbf{Z})$ be the solution of the equation $W_z(\beta_1) = V_3^*$, and thus $\beta_1 = g_2(V_3^*, \mathbf{Z})$. Let $g_3(V_5^*, \mathbf{Y})$ be the solution of the equation $W_y(\beta_2) = V_5^*$, and thus $\beta_2 = g_3(V_5^*, \mathbf{Y})$.

We assume that

$$T_2(\beta_1) = \sum_{j=1}^{n_1}(Z_{(n_1-j+1)}^{-\beta_1} - 1), T_3(\beta_2) = \sum_{j=1}^{n_2}(Y_{(n_2-j+1)}^{-\beta_2} - 1),$$

then

$$T_2(\beta_1) = S_{z,n_1}, T_3(\beta_2) = S_{y,n_2}$$

Generate a series of random samples V_3^*, V_4^*, V_5^*, and V_6^* from the distributions $\chi^2(2n_1 - 2), \chi^2(2n_1)$, $\chi^2(2n_2 - 2)$, and $\chi^2(2n_2)$, respectively. Using the substitution method, the GPQs of the parameters α_1 and α_2 are given by

$$W_5 = V_4^*/(2T_2(g_2(V_3^*, \mathbf{z}))), \tag{3.1}$$

$$W_6 = V_6^*/(2T_3(g_3(V_5^*, \mathbf{y}))), \tag{3.2}$$

respectively, where $\mathbf{z} = (z_1, z_2, \cdots, z_{n_1})$ and $\mathbf{y} = (y_1, y_2, \cdots, y_{n_2})$ are the observed values of $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_{n_1})$ and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_{n_2})$, respectively.

Using the substitution method, the GPQ of the stress-strength reliability R is given by

$$W_7 = \int_0^1 F(y \mid W_5, g_2(V_3^*, \mathbf{z}))f(y \mid W_6, g_3(V_5^*, \mathbf{y}))dy. \tag{3.3}$$

Let $W_{7,\gamma}$ represent the γ-th quantile of W_7. Then the $100(1-\gamma)\%$ GCI for R is given by $[W_{7,\gamma/2}, W_{7,1-\gamma/2}]$. The following Monte Carlo algorithm (Algorithm 3) can be used to obtain the quantiles of W_7.

Algorithm 3 Quantiles of W_7.

Input: Observation data $\mathbf{z} = (z_1, z_2, \ldots, z_{n_1})$ and $\mathbf{y} = (y_1, y_2, \ldots, y_{n_2})$.

Output: The quantiles of W_7.

Initialization: Let $W_{7,j}$ be the jth value of W_7, and let $W_{7,(j)}$ denote the ordered value of W_7 .

for j *in* $1 : B$ **do**

1) Generate $V_3^* \sim \chi^2(2n_1 - 2)$ and $V_5^* \sim \chi^2(2n_2 - 2)$. Then compute $g_2(V_3^*, \mathbf{z})$ and $g_3(V_5^*, \mathbf{y})$ from the equations $W_z(\beta_1) = V_3^*$ and $W_y(\beta_2) = V_5^*$, respectively.

2) Generate $V_4^* \sim \chi^2(2n_1)$ and $V_6^* \sim \chi^2(2n_2)$. Using Eqs (3.1) and (3.2), compute W_5 and W_6.

3) Using Eq (3.3), compute $W_{7,j}$.

end

$W_{7,(\gamma B)}$ can be used to estimate the γ-th quantile of W_7.

3.2. *Special case: $\beta_1 = \beta_2 \hat{=} \beta$*

Clearly, $\beta_1 = \beta_2$ is a special case of the stress-strength model R. Let $\beta_1 = \beta_2 \hat{=} \beta$, and thus the expression of the stress-strength model R^* is given as

$$\begin{aligned} R^* &= P(Z < Y) \\ &= \int_0^1 P(Z < Y \mid Y = y) f(y \mid \alpha_2, \beta) dy \\ &= \int_0^1 F(y \mid \alpha_1, \beta) f(y \mid \alpha_2, \beta) dy = \frac{\alpha_2}{\alpha_1 + \alpha_2}. \end{aligned}$$

Let $V = W_z(\beta) + W_y(\beta)$, and thus $V \sim \chi^2(2(n_1 + n_2 - 2))$. Generate a random sample V^* from the distribution $\chi^2(2(n_1 + n_2 - 2))$. Let $g(V^*, \mathbf{Z})$ be the solution of the equation $W_z(\beta) + W_y(\beta) = V^*$, and thus $\beta = g(V^*, \mathbf{Z})$. Assume the following:

$$T_2^*(\beta) = \sum_{j=1}^{n_1} (Z_{(n_1-j+1)}^{-\beta} - 1), T_3^*(\beta) = \sum_{j=1}^{n_2} (Y_{(n_2-j+1)}^{-\beta} - 1).$$

Using the substitution method, the GPQs of the parameters α_1 and α_2 are , respectively, given by

$$W_5^* = V_4^* / (2T_2(g(V^*, \mathbf{Z}))), \tag{3.4}$$

$$W_6^* = V_6^* / (2T_3(g(V^*, \mathbf{Z}))). \tag{3.5}$$

Similarly, the GPQ of the stress-strength reliability R^* is given by

$$W_7^* = \frac{W_6^*}{W_5^* + W_6^*}. \tag{3.6}$$

Let $W_{7,\gamma}^*$ represent the γ-th quantile of W_7^*. Then the $100(1 - \gamma)\%$ GCI for R^* is given by $[W_{7,\gamma/2}^*, W_{7,1-\gamma/2}^*]$. The quantiles of W_7^* can be obtained using a method similar to Algorithm 3.

4. GPIs for the FFT and RUL

Since the FFT and RUL of a product play an important role in engineering decisions.. For purposes like maintenance, prognosis, and health management, developing prediction procedures for the FFT and RUL is crucial.

Let $T_1, T_2, \cdots, T_m$ be the future sample from the UG distribution with the sample size m, and the corresponding order statistics are $T_{(1)} < T_{(2)} < \cdots < T_{(m)}$. In this section, the PI for $T_{(k)}$ is proposed.

Clearly, the PDF and CDF of $T_{(k)}$ are given, respectively, by

$$f_{T_{(k)}}(t|\alpha,\beta) = \frac{m!}{(k-1)!(m-k)!} F^{k-1}(t|\alpha,\beta)\,(1 - F(t|\alpha,\beta))^{m-k}\, f(t|\alpha,\beta), t > 0, \tag{4.1}$$

and

$$F_{T_{(k)}}(t|\alpha,\beta) = \int_0^t f_{T_{(k)}}(s|\alpha,\beta)ds, t > 0. \tag{4.2}$$

Let $H_1 = F_{T_{(k)}}(t_{(k)}|\alpha,\beta)$, and then $H_1 \sim U(0,1)$. Generate a random sample H_1^* from the distribution $U(0,1)$. Obviously, the equation $F_{T_{(k)}}(t_{(k)}|\alpha,\beta) = H_1^*$ is uniquely resolved. Define $\kappa_1(H_1^*, \alpha,\beta)$ as the solution to this equation, and thus $T_{(k)} = \kappa_1(H_1^*, \alpha,\beta)$. By employing the substitution method, we can construct the GPQ pertaining to $T_{(k)}$ as follows:

$$W_8 = \kappa_1(H_1^*, W_1, g_1(V_1^*, \mathbf{x})). \tag{4.3}$$

Suppose that the lifetime of a unit T follows the UG distribution, and its elapsed time is t_u; naturally, its RUL can be defined as $T_{RL} = T - t_u$. It is easy to deduce that the CDF of T_{RL} based on $T_{RL} \geq 0$ is

$$\begin{aligned} F_{RL}(t \mid T > t_u) &= P(T_{RL} \leq t \mid T > t_u) \\ &= \frac{F(t_u + t|\alpha,\beta) - F(t_u|\alpha,\beta)}{\overline{F}(t_u|\alpha,\beta)}. \end{aligned}$$

Clearly

$$H_2 = \frac{F(t_u + T_{RL}|\alpha,\beta) - F(t_u|\alpha,\beta)}{\overline{F}(t_u|\alpha,\beta)} \sim U(0,1).$$

Generate a random sample H_2^* from the distribution $U(0,1)$. Hence, the equation

$$H_2^* = \frac{F(t_u + T_{RL}|\alpha,\beta) - F(t_u|\alpha,\beta)}{\overline{F}(t_u|\alpha,\beta)}$$

has a unique solution. If we let $\kappa_2(t_u, H_2^*, \alpha,\beta)$ be the solution of this equation, then $T_{RL} = \kappa_2(t_u, H_2^*, \alpha,\beta)$. Using the substitution method, the GPQ of T_{RL} is given by

$$W_9 = \kappa_2(t_u, H_2^*, W_1, g_1(V_1^*, \mathbf{x})), \tag{4.4}$$

Let $W_{8,\gamma}$ and $W_{9,\gamma}$ be the γ-th quantiles of W_8 and W_9, respectively. Then the $100(1-\gamma)\%$ GPIs for $T_{(k)}$ and T_{RL} are given by $[W_{8,\gamma/2}, W_{8,1-\gamma/2}]$ and $[W_{9,\gamma/2}, W_{9,1-\gamma/2}]$, respectively. Below is a Monte Carlo method that can be utilized to calculate the quantiles of W_8 and W_9 (Algorithm 4).

Algorithm 4 Quantiles of W_8 and W_9.

Input: Observation data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Output: The quantiles of W_8 and W_9.

Initialization: Let $W_{i,j}$ be the jth value of $W_i, i = 8, 9$, and let $W_{i,(j)}$ denote the ordered value of W_i .

for j *in* $1 : B$ **do**

(1) Determine W_1 and $g_1(V_1^*, \mathbf{x})$ according to the procedure outlined in Algorithm 1.

(2) Generate two independent random samples H_1^* and H_2^* from the uniform distribution $U(0, 1)$, and subsequently calculate $W_{8,j}$ and $W_{9,j}$ using the formulas (4.3) and (4.4).

end

$W_{8,(\gamma B)}$ and $W_{9,(\gamma B)}$ can be used to estimate the γ-th quantile of W_8 and W_9, respectively .

Remark 1: It is evident that $T_{(k)}$ equals T_{RL} under the conditions that $m = 1$ and $t_u = 0$. Moreover, to simplify matters, we set $m = 1$ for the subsequent simulation and case study.

Remark 2: The bootstrap-p CIs for R, R^*, $T_{(k)}$, and T_{RL} can be obtained using a method that is similar to Algorithm 2. For the details of Wald CIs, one can refer to [25].

5. Simulation study

In this section, a simulation study is conducted to evaluate the finite sample properties of the proposed GCIs and GPIs in terms of their coverage probability (CP). As is known to all, the parametric bootstrap method and the Wald method are two classical approaches to obtain confidence intervals for a model's parameters. In order to fully evaluate the advantages of the proposed GCIs/GPIs, we compare them with the classical Wald CIs and bootstrap-p CIs. Since the CI for the parameter β is exact, a simulation for evaluating the CP of the CI of β is unnecessary. The simulation study uses $(\alpha, \beta) = (0.02, 4), (0.2, 0.5), (0.5, 2), (1, 3), (1.5, 1)$, with a simulation budget of $B = 5000$. In the simulation, the number of Wald CI and bootstrap-p CI replicates is also 5000. The simulation's results are presented in Tables 1–6. Subsequently, given Z ~UG(0.02,4) and Y ~UG(0.5,8), a simulation study is carried out for the GCI concerning the stress-strength reliability R with $B = 5000$. The simulation's results are listed in Table 7.

Table 1. The simulation CPs of the parameter α.

		GCIs		Wald CIs		Bootstrap-p CIs	
Parameter	n	0.9000	0.9500	0.9000	0.9500	0.9000	0.9500
$(\alpha,\beta) = (0.02, 4)$	15	0.9048	0.9500	0.7538	0.7852	0.8422	0.9002
	20	0.9020	0.9474	0.7770	0.8090	0.8540	0.9052
	25	0.9052	0.9526	0.8030	0.8326	0.8664	0.9242
	30	0.9004	0.9474	0.8080	0.8396	0.8620	0.9168
$(\alpha,\beta) = (1, 3)$	15	0.9022	0.9508	0.7198	0.7554	0.8138	0.8834
	20	0.8978	0.9482	0.7430	0.7790	0.8302	0.8870
	25	0.8952	0.9476	0.7678	0.8094	0.8504	0.9078
	30	0.9000	0.9472	0.7790	0.8160	0.8498	0.9098
$(\alpha,\beta) = (0.5, 2)$	15	0.8968	0.9494	0.7416	0.7740	0.8212	0.8838
	20	0.9002	0.9464	0.7722	0.8042	0.8438	0.8992
	25	0.8950	0.9484	0.7950	0.8352	0.8598	0.9178
	30	0.9008	0.9468	0.8042	0.8404	0.8582	0.9162
$(\alpha,\beta) = (0.2, 0.5)$	15	0.9076	0.9528	0.7634	0.8048	0.8393	0.9024
	20	0.9056	0.9548	0.7882	0.8238	0.8524	0.9176
	25	0.9014	0.9530	0.8048	0.8428	0.8632	0.9198
	30	0.8984	0.9486	0.8256	0.8578	0.8706	0.9232
$(\alpha,\beta) = (1.5, 1)$	15	0.9070	0.9548	0.6764	0.7112	0.8030	0.8718
	20	0.9078	0.9550	0.7072	0.7460	0.8150	0.8874
	25	0.8998	0.9536	0.7422	0.7772	0.8338	0.8976
	30	0.8965	0.9506	0.7650	0.8048	0.8526	0.9076

Table 2. The simulation CPs for reliability metrics with $(\alpha, \beta, t_u) = (0.02, 4, 0.1)$.

		GCIs/GPIs		Bootstrap-p CIs/PIs	
n	Parameters	0.9000	0.9500	0.9000	0.9500
15	$r(0.3)$	0.8932	0.9480	0.8250	0.8812
	$Q(0.1)$	0.9048	0.9524	0.8122	0.8622
	$R(0.32)$	0.9106	0.9544	0.8234	0.8752
	FFT	0.9118	0.9554	0.8590	0.9244
	T_{RL}	0.8968	0.9478	0.8448	0.9134
20	$r(0.3)$	0.8870	0.9434	0.8332	0.8980
	$Q(0.1)$	0.8984	0.9476	0.8266	0.8806
	$R(0.32)$	0.8986	0.9474	0.8344	0.8886
	FFT	0.8926	0.9448	0.8528	0.9172
	T_{RL}	0.9010	0.9492	0.8670	0.9226
25	$r(0.3)$	0.9020	0.9516	0.8610	0.9090
	$Q(0.1)$	0.9084	0.9546	0.8460	0.8942
	$R(0.32)$	0.9052	0.9527	0.8572	0.9036
	FFT	0.9004	0.9548	0.8676	0.9322
	T_{RL}	0.9006	0.9500	0.8704	0.9270
30	$r(0.3)$	0.8938	0.9470	0.8542	0.9114
	$Q(0.1)$	0.9006	0.9498	0.8456	0.8984
	$R(0.32)$	0.8994	0.9486	0.8522	0.9060
	FFT	0.8988	0.9524	0.8732	0.9332
	T_{RL}	0.8904	0.9444	0.8676	0.9290

Table 3. The simulation CPs for reliability metrics with $(\alpha, \beta, t_u) = (1, 3, 0.2)$.

		GCIs/GPIs		Bootstrap-p CIs/GPIs	
n	Parameters	0.9000	0.9500	0.9000	0.9500
15	$r(0.3)$	0.8994	0.9478	0.7642	0.8290
	$Q(0.1)$	0.9002	0.9498	0.7732	0.8276
	$R(0.43)$	0.8984	0.9470	0.7490	0.8145
	FFT	0.9048	0.9534	0.8542	0.9186
	T_{RL}	0.8910	0.9454	0.8388	0.9062
20	$r(0.3)$	0.9014	0.9486	0.7796	0.8426
	$Q(0.1)$	0.9032	0.9494	0.7978	0.8456
	$R(0.43)$	0.8984	0.9468	0.7726	0.8292
	FFT	0.8996	0.9496	0.8618	0.9204
	T_{RL}	0.8902	0.9458	0.8504	0.9150
25	$r(0.3)$	0.8978	0.9470	0.8063	0.8642
	$Q(0.1)$	0.9018	0.9472	0.8166	0.8682
	$R(0.43)$	0.8984	0.9468	0.7996	0.8524
	FFT	0.9048	0.9536	0.8736	0.9292
	T_{RL}	0.8964	0.9424	0.8682	0.9198
30	$r(0.3)$	0.8978	0.9470	0.8158	0.8680
	$Q(0.1)$	0.8986	0.9486	0.8120	0.8728
	$R(0.43)$	0.8930	0.9460	0.8088	0.8594
	FFT	0.8988	0.9522	0.8688	0.9316
	T_{RL}	0.8900	0.9450	0.8624	0.9254

Table 4. The simulation CPs for reliability metrics with $(\alpha, \beta, t_u) = (0.5, 2, 0.2)$.

		GCIs/GPIs		Bootstrap-p CIs/GPIs	
n	Parameters	0.9000	0.9500	0.9000	0.9500
15	$r(0.3)$	0.9090	0.9528	0.7334	0.7904
	$Q(0.1)$	0.9000	0.9500	0.7722	0.8252
	$R(0.43)$	0.8988	0.9502	0.7772	0.8314
	FFT	0.9044	0.9510	0.8588	0.9138
	T_{RL}	0.8926	0.9450	0.8458	0.9102
20	$r(0.3)$	0.9040	0.9568	0.7670	0.8226
	$Q(0.1)$	0.9072	0.9524	0.8078	0.8544
	$R(0.43)$	0.9078	0.9530	0.8110	0.8584
	FFT	0.9024	0.9520	0.8630	0.9224
	T_{RL}	0.8916	0.9462	0.8520	0.9160
25	$r(0.3)$	0.9053	0.9502	0.7924	0.8442
	$Q(0.1)$	0.9053	0.9486	0.8256	0.8760
	$R(0.43)$	0.9062	0.9494	0.8366	0.8782
	FFT	0.9072	0.9548	0.8736	0.9308
	T_{RL}	0.8970	0.9418	0.8688	0.9206
30	$r(0.3)$	0.9038	0.9512	0.7978	0.8510
	$Q(0.1)$	0.8988	0.9506	0.8206	0.8810
	$R(0.43)$	0.9018	0.9506	0.8224	0.8834
	FFT	0.9020	0.9534	0.8694	0.9324
	T_{RL}	0.8898	0.9450	0.8638	0.9246

Table 5. The simulation CPs for reliability metrics with $(\alpha, \beta, t_u) = (0.2, 0.5, 0.5)$.

		GCIs/GPIs		Bootstrap-p CIs/PIs	
n	Parameters	0.9000	0.9500	0.9000	0.9500
15	$r(0.3)$	0.9075	0.9560	0.8338	0.8940
	$Q(0.1)$	0.9078	0.9570	0.7870	0.8398
	$R(0.32)$	0.9072	0.9522	0.8600	0.9166
	FFT	0.8966	0.9472	0.8450	0.9092
	T_{RL}	0.8986	0.9506	0.8980	0.9494
20	$r(0.3)$	0.9072	0.9562	0.8430	0.9066
	$Q(0.1)$	0.9046	0.9506	0.7982	0.8562
	$R(0.32)$	0.9060	0.9492	0.8746	0.9322
	FFT	0.8988	0.9500	0.8548	0.9200
	T_{RL}	0.8964	0.9478	0.8958	0.9468
25	$r(0.3)$	0.9000	0.9566	0.8564	0.9146
	$Q(0.1)$	0.9032	0.9532	0.8338	0.8801
	$R(0.32)$	0.9026	0.9508	0.8778	0.9326
	FFT	0.9040	0.9538	0.8732	0.9324
	T_{RL}	0.8986	0.9539	0.8988	0.9510
30	$r(0.3)$	0.8976	0.9500	0.8662	0.9220
	$Q(0.1)$	0.9006	0.9506	0.8418	0.8938
	$R(0.32)$	0.8962	0.9446	0.8802	0.9329
	FFT	0.9060	0.9588	0.8816	0.9400
	T_{RL}	0.8950	0.9482	0.8962	0.9480

Table 6. The simulation CPs for reliability metrics with $(\alpha, \beta, t_u) = (1.5, 1, 0.2)$.

		GCIs/GPIs		Bootstrap-p CIs/PIs	
n	Parameters	0.9000	0.9500	0.9000	0.9500
15	$r(0.6)$	0.9072	0.9536	0.8580	0.9182
	$Q(0.1)$	0.8906	0.9574	0.7536	0.8088
	$R(0.5)$	0.8842	0.9500	0.8198	0.8754
	FFT	0.8896	0.9398	0.8450	0.9092
	T_{RL}	0.8802	0.9322	0.8410	0.9062
20	$r(0.6)$	0.9058	0.9512	0.8692	0.9258
	$Q(0.1)$	0.8916	0.9460	0.7708	0.8306
	$R(0.5)$	0.8914	0.9416	0.8338	0.8889
	FFT	0.8882	0.9416	0.8508	0.9160
	T_{RL}	0.8818	0.9370	0.8552	0.9132
25	$r(0.6)$	0.9010	0.9514	0.8756	0.9298
	$Q(0.1)$	0.8968	0.9486	0.8056	0.8616
	$R(0.5)$	0.8998	0.9460	0.8566	0.9166
	FFT	0.8968	0.9502	0.8698	0.9276
	T_{RL}	0.8958	0.9462	0.8612	0.9220
30	$r(0.6)$	0.8982	0.9462	0.8808	0.9336
	$Q(0.1)$	0.8952	0.9506	0.8216	0.8428
	$R(0.5)$	0.8926	0.9488	0.8928	0.9196
	FFT	0.9016	0.9562	0.8770	0.9388
	T_{RL}	0.8962	0.9469	0.8678	0.9276

Table 7. The CPs of the simulation's CIs for the stress-strength reliability R.

	GCIs		Bootstrap-p CIs	
(n_1, n_2)	0.9000	0.9500	0.9000	0.9500
$(15, 15)$	0.9092	0.9528	0.8726	0.9253
$(20, 20)$	0.9088	0.9550	0.8818	0.9334
$(25, 25)$	0.9132	0.9576	0.8866	0.9374
$(30, 30)$	0.9116	0.9572	0.8878	0.9434

The results displayed in Tables 1–7 indicate that the proposed methods achieve CP levels that are very close to the nominal levels across all scenarios. Comparatively, the CPs of the Wald CIs and bootstrap-p CIs consistently fall below the nominal levels, particularly in small-sample cases. As the sample size increases, the CPs of the Wald CIs and bootstrap CIs become closer to the nominal levels. When the sample size is large enough, the CPs of the Wald CIs and bootstrap-p CIs may reach the nominal levels. These results demonstrate superior CP of our GCIs/GPIs in comparison with the Wald CIs and bootstrap-p CIs. However, in the reliability field, due to the destructiveness of the test and the limitation of the test's cost, the data obtained are mostly small-sample data, so the proposed method is suitable for application in engineering practice. Therefore, we recommend the proposed GCIs/GPIs for the UG distribution and related stress-strength model, especially in small-sample cases.

6. Case study

Example 1

In this example, we use the data reported in Dumonceaux and Antle [26] to illustrate the proposed method. This data-set contains 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania. These observations are given as follows:

0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.

Mazucheli et al. [2] have shown that employing the UG distribution to model flood data is justifiable. Utilizing the proposed method, the exact CIs for the parameter β with the confidence levels $\gamma = (0.1, 0.05)$ are given by (2.7145, 5.4133) and (2.4703, 5.6929), respectively. The simulated CIs for α, $r(0.3)$, $t_{0.1}$, $R(0.3)$, FFT, and T_{RL} are given in Table 8. For example, the 90% GPI for the FFT is given by $(0.2690, 0.7181)$, the 95% bootstrap-p PI for the FFT is given by $(0.2621, 0.7863)$. On the basis of the flood data, we are 90% confident that the true FFT lies within $(0.2690, 0.7181)$, and 95% confident that the true FFT lies within $(0.2621, 0.7863)$.

Table 8. The CIs for α and some reliability metrics for flood data with $t_u = 0.1$ based on $B = 5000$ replications.

Levels	Parameters	GCIs/GPIs	Bootstrap-p CIs/PIs
0.9000	α	$(0.0033, 0.0731)$	$(0.0017, 0.0506)$
	$r(0.3)$	$(0.1927, 0.5884)$	$(0.1472, 0.5876)$
	$Q(0.1)$	$(0.2590, 0.3179)$	$(0.2713, 0.3282)$
	$R(0.3)$	$(0.7447, 0.9471)$	$(0.7836, 0.9729)$
	FFT	$(0.2690, 0.7181)$	$(0.2757, 0.6901)$
	T_{RL}	$(0.1663, 0.6298)$	$(0.1771, 0.5946)$
0.9500	α	$(0.0023, 0.0946)$	$(0.0010, 0.0654)$
	$r(0.3)$	$(0.1701, 0.6393)$	$(0.1093, 0.6516)$
	$Q(0.1)$	$(0.2509, 0.3226)$	$(0.2671, 0.3348)$
	$R(0.3)$	$(0.7154, 0.9553)$	$(0.7612, 0.9836)$
	FFT	$(0.2523, 0.8026)$	$(0.2621, 0.7863)$
	T_{RL}	$(0.1505, 0.7224)$	$(0.1634, 0.6766)$

Example 2

In this example, we use the data reported in Alsadat et al. [27] to illustrate the proposed methods. This study's dataset comprises two cohorts of head and neck cancer patients: The first cohort documents survival times for 58 individuals treated solely with radiation therapy, while the second cohort includes survival data for 44 patients who received combined radiotherapy and chemotherapy. The information is as follows:

Data-set I (X): 523, 583, 594, 14.48, 16.1, 22.7, 34, 41.55, 42, 45.28, 49.4, 84, 91, 160, 160, 165, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 6.53, 7, 10.42, 225, 241, 248, 273, 277, 297, 405, 417, 53.62, 63, 64, 83, 420, 440, 1101, 1146, 1417.

Data-set II (Y): 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 319, 339, 432, 469, 68.46, 78.26, 173, 179, 194, 195, 74.47, 81.43, 84, 92, 519, 633, 725, 94, 110, 112, 119, 127, 130, 133, 140, 146, 12.2, 23.56, 23.74, 155, 159, 209, 249, 281, 817, 1776.

Following Alsadat et al. [27], the data are divided by 2000 to obtain values between 0 and 1. Alsadat et al. [27] showed that employing the UG distribution to model the two data-sets is justifiable, and the assumption of $\beta_1 = \beta_2$ is justified. Utilizing the proposed method, the 90% and 95% GCIs for R^* are given by $(0.5123, 0.6772)$ and $(0.4976, 0.6930)$, respectively. The 90% and 95% bootstrap-p CIs for R^* are given by $(0.4492, 0.6170)$ and $(0.4322, 0.6332)$, respectively. Note that the results presented in Sections 5 and 6 were generated using MATLAB software.

7. Conclusions

This study focuses on interval estimation for the UG distribution and its associated stress-strength model via the GPQ approach, which involves deriving GCIs for the parameters of the UG distribution and key reliability metrics. Additionally, GPIs for the FFT and RUL are developed. Furthermore, we explore the construction of GCIs for the reliability of the stress-strength model under UG distributions with non-identical parameter configurations. The performance of the proposed GCIs and GPIs is evaluated through Monte Carlo simulations. The results of these simulations demonstrate that the CPs of the recommended GCIs and GPIs closely approximate the nominal confidence levels, even in small-sample settings. To illustrate the implementation of GCIs and GPIs based on the UG distribution, two numerical examples involving small-sample UG distribution scenarios are provided.

In future studies, the interval estimation procedure proposed herein can be extended to other generalized Gompertz lifetime distributions, such as the beta-Gompertz distribution or beta-generalized Gompertz distribution. Furthermore, the methodology for developing the prediction framework for the UG distribution, as described in this paper, can be adapted to other distributions within the UG family. These research topics are both meaningful and worthy of further exploration.

Author contributions

Xiaofei Wang: Conceptualization, methodology, validation, formal analysis, writing-original draft; Peihua Jiang: Supervision, funding acquisition, methodology, simulation, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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