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Research article

A new aftertreatment technique for the Mittag-Leffler function of a fractional nonlinear SIR-epidemic model

Laila F. Seddek[1,*], **Abdelhalim Ebaid**[2], **Essam R. El-Zahar**[1] **and Mona D. Aljoufi**[2]

[1] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia

* **Correspondence:** Email: l.morad@psau.edu.sa.

Abstract: This paper introduced a new aftertreatment technique for solving a fractional nonlinear Susceptible-Infectious-Recovered (SIR)-epidemic model. The proposed approach reformulated the power series solution via incorporating the Laplace transform and its inverse. Basically, it applied the Laplace transform to convert the series solution into different Padé-approximants involving Laplace's parameter with arbitrary order. This procedure facilitated the method of deriving the inverse Laplace transform explicitly, as a final step, using basic special functions in fractional calculus. Our analysis was capable of obtaining a sequence of closed-form approximations in terms of the Mittag-Leffler functions. The current results may be provided for the first time regarding the Mittag-Leffler solution of the fractional SIR-model. Additionally, the outcome of this analysis revealed that our approach was not only effective but also applicable to a wide range of fractional differential equations and systems.

Keywords: fractional ordinary differential equation; epidemic; Mittag-Leffler; initial value problem
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1. Introduction

Fractional calculus (FC) is a generalization of classical calculus to non-integer order derivatives and integrals [1,2]. It has gained prominence in modeling complex dynamical systems that exhibit memory and hereditary properties [3–5]. One of the foundational functions in the FC is the Mittag-Leffler function (MLF), which plays a critical role in the solutions of fractional differential equations (FDEs). Its relevance has been increasingly highlighted in the modeling of epidemic dynamics [6], particularly in infectious diseases [7–9], the SIR framework extended to fractional-order systems [10–12] and the asymptomatic transmission [13] in addition to COVID-19 modeling in India [14, 15] using variable-order operators [16, 17] and fractal-fractional operators [18, 19]. Fractional SIR models replace the integer-order time derivative with a fractional Caputo or Riemann-Liouville derivative, allowing for more flexible and realistic descriptions. Although classical calculus is basically used to model infectious diseases [20, 21] including COVID-19, Sars-Cov-2 in Europe [22] and other countries [23–25], the fractional modeling is viewed as a generalization of such results. The present fractional SIR-model generalizes the ordinary dimensionless model [26, 27], and it takes the form:

$$ {}_0^C D_\tau^\alpha R(\tau) = I(\tau), \tag{1.1} $$

$$ {}_0^C D_\tau^\alpha I(\tau) = \sigma\,[1 - R(\tau) - I(\tau)]\, I(\tau) - I(\tau), \tag{1.2} $$

where $\tau = t/T$ is the normalized time, t is the time in days and T is the time of transmission of the virus. $I(\tau)$ and $R(\tau)$ denote the infected and the recovered individuals, respectively, while $S(\tau)$ represents the susceptible individuals $S(\tau) = 1 - R(\tau) - I(\tau)$ and σ is the transmission rate (physical contact number between susceptible and infected individuals). The present fractional model implements the Caputo fractional definition. As $\alpha \to 1$, the system (1.1)-(1.2) reduces to the classical form.

The initial conditions (ICs) are

$$ R(0) = A, \quad I(0) = B. \tag{1.3} $$

The study of FDEs relies on both analytical and numerical approaches due to the challenges associated with obtaining closed-form solutions. Analytical methods involve finding exact or approximate closed-form solutions using special functions like the MLF [28, 29]. The Laplace transform (LT) is commonly employed in conjunction with fractional derivatives, where the transform of a Caputo derivative leads to algebraic equations in the Laplace domain. The inverse LT often introduces the MLFs into the solution [30, 31]. The objective of this paper is to introduce a new aftertreatment technique to treat the fractional SIR-system (1.1)-(1.3). A first-step of this approach is to obtain the standard series solution, then applying the LT to establish different diagonal Padé-approximants in terms of Laplace's parameter. By inverting such Padé-approximants one can finally get different closed-form approximations in terms of the MLFs. The main advantage of the proposed aftertreatment is that it gives better analytical approximations in closed forms in terms of the MLFs. Also, it overcomes the slow convergence of the standard power series solution and consequently, the aftertreatment is found applicable in a wider domain. To the author's knowledge, the present aftertreatment may be suggested for the first time for obtaining the MLFs of the fractional SIR-model.

2. Basic definitions and rules in FC

The Riemann-Liouville fractional integral (RLFI) of order α is defined by [31]:

$$ {}_0 I_\tau^\alpha y(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{y(z)}{(\tau - z)^{1-\alpha}} dz, \qquad \alpha > 0, \quad \tau > 0. \tag{2.1} $$

Assume $\alpha \neq 0$ denotes the order of the derivative such that $n - 1 < \alpha \leq n$. The Caputo fractional derivative (CFD) of a function $y(t)$ is defined by [31]

$$ {}_0^C D_\tau^\alpha y(\tau) = \frac{d^\alpha y(\tau)}{d\tau^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau - z)^{n-\alpha-1} y^{(n)}(z) dz, & \text{if } n - 1 < \alpha < n, \\ \frac{d^n y(\tau)}{d\tau^n}, & \text{if } \alpha = n. \end{cases} \tag{2.2} $$

Basic properties of the CFD and the RLFI are

$$\begin{cases} {}_0^C D_\tau^\alpha(\text{constant}) = 0, \\ {}_0^C D_\tau^\alpha \tau^r = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} r^{r-\alpha}, \\ {}_0 I_\tau^\alpha \tau^r = \frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} r^{r+\alpha}. \end{cases} \tag{2.3}$$

The MLFs of one parameter and two parameters are defined by

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad (\alpha > 0,\ \beta > 0). \tag{2.4}$$

For a straightforward analysis of the present aftertreatment technique, the following equality is essential:

$$\mathcal{L}^{-1}\Big(\frac{s^{\alpha-\gamma}}{s^\alpha + \omega^2}\Big) = t^{\gamma-1} E_{\alpha,\gamma}(-\omega^2 t^\alpha), \quad Re(s) > |\omega^2|^{\frac{1}{\alpha}}, \tag{2.5}$$

for the inverse LT of some expressions in terms of the MLFs, given as

$$\begin{cases} \mathcal{L}^{-1}\Big(\frac{s^{\alpha-1}}{s^\alpha+1}\Big) = E_\alpha(-t^\alpha), \\ \mathcal{L}^{-1}\Big(\frac{1}{s^\alpha+\omega^2}\Big) = t^{\alpha-1} E_{\alpha,\alpha}(-\omega^2 t^\alpha), \qquad Re(s) > |\omega^2|^{\frac{1}{\alpha}}, \\ \mathcal{L}^{-1}\Big(\frac{s^{-1}}{s^\alpha+\omega^2}\Big) = t^{\alpha} E_{\alpha,\alpha+1}(-\omega^2 t^\alpha), \qquad Re(s) > |\omega^2|^{\frac{1}{\alpha}}. \end{cases} \tag{2.6}$$

3. Power series solution (PSS)

This section derives the PSS of the governing system (1.1)-(1.3). Our approach is based on assuming $R(\tau)$ and $I(\tau)$ in the forms:

$$R(\tau) = \sum_{n=0}^{\infty} \frac{a_n \tau^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad I(\tau) = \sum_{n=0}^{\infty} \frac{b_n \tau^{\alpha n}}{\Gamma(\alpha n + 1)}. \tag{3.1}$$

Employing the properties (2.3) yields

$${}_0^C D_\tau^\alpha R(\tau) = \sum_{n=0}^{\infty} \frac{a_{n+1} \tau^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad {}_0^C D_\tau^\alpha I(\tau) = \sum_{n=0}^{\infty} \frac{b_{n+1} \tau^{\alpha n}}{\Gamma(\alpha n + 1)}, \tag{3.2}$$

and

$$\left(R(\tau) + I(\tau)\right) I(\tau) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(a_k + b_k) b_{n-k} \tau^{\alpha n}}{\Gamma(\alpha k + 1)\Gamma(\alpha(n-k) + 1)}. \tag{3.3}$$

Substituting Eqs (3.1)–(3.3) into Eqs (1.1) and (1.2) and collecting like powers, then

$$\sum_{n=0}^{\infty} \frac{\tau^{\alpha n}}{\Gamma(\alpha n + 1)} (a_{n+1} - b_n) = 0, \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{\tau^{\alpha n}}{\Gamma(\alpha n + 1)} \left(b_{n+1} - (\sigma - 1) b_n + \sigma \Gamma(\alpha n + 1) \sum_{k=0}^{n} \frac{(a_k + b_k) b_{n-k}}{\Gamma(\alpha k + 1)\Gamma(\alpha(n-k) + 1)} \right) = 0. \tag{3.5}$$

Accordingly, we have the recurrence scheme:

$$\begin{aligned} a_{n+1} &= b_n, \\ b_{n+1} &= (\sigma-1)b_n - \sigma\Gamma(\alpha n+1)\sum_{k=0}^{n}\frac{(a_k+b_k)b_{n-k}}{\Gamma(\alpha k+1)\Gamma(\alpha(n-k)+1)}, \quad n \geq 0. \end{aligned} \tag{3.6}$$

Applying the ICs (1.3) on Eq (3.1) requires $a_0 = A$ and $b_0 = B$. From algorithm (3.6) at $n = 1$, we get

$$a_1 = B, \quad b_1 = B\mu_1, \quad \mu_1 = \sigma(1-A-B)-1. \tag{3.7}$$

At $n = 1$, we obtain

$$a_2 = B\mu_1, \quad b_2 = B\mu_2, \tag{3.8}$$

where

$$\mu_2 = \mu_1^2 - \sigma B(\mu_1+1). \tag{3.9}$$

Similarly, one can obtain

$$a_3 = B\mu_2, \quad b_3 = B\mu_3, \tag{3.10}$$

where

$$\mu_3 = \mu_1\mu_2 - \sigma B\mu_1(\mu_1+1)\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} - \sigma B\left(\mu_1+\mu_2\right). \tag{3.11}$$

Following the same analysis, we obtain

$$a_i = B\mu_{i-1}, \quad b_i = B\mu_i, \quad i \geq 1. \tag{3.12}$$

As a summary, the magnitudes μ_i for $i = 1, 2, 3, \ldots, 6$, take the form:

$$\begin{cases} \mu_1 = \sigma(1-A-B)-1, \\ \mu_2 = \mu_1^2 - \sigma B(\mu_1+1) \\ \mu_3 = \mu_1\mu_2 - \sigma B\mu_1(\mu_1+1)\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} - \sigma B\left(\mu_1+\mu_2\right), \\ \mu_4 = \mu_1\mu_3 - \sigma B[\mu_2(\mu_1+1)+\mu_1(\mu_1+\mu_2)]\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \sigma B\left(\mu_2+\mu_3\right), \\ \mu_5 = \mu_1\mu_4 - \sigma B\mu_2(\mu_1+\mu_2)\frac{\Gamma(1+4\alpha)}{(\Gamma(1+2\alpha))^2} - \sigma B[\mu_3(\mu_1+1)+\mu_1(\mu_2+\mu_3)]\frac{\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} - \sigma B\left(\mu_3+\mu_4\right), \\ \mu_6 = \mu_1\mu_5 - \sigma B[\mu_3(\mu_1+\mu_2)+\mu_2(\mu_2+\mu_3)]\frac{\Gamma(1+5\alpha)}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} - \\ \qquad \sigma B[\mu_4(\mu_1+1)+\mu_1(\mu_3+\mu_4)]\frac{\Gamma(1+5\alpha)}{\Gamma(1+\alpha)\Gamma(1+4\alpha)} - \sigma B\left(\mu_4+\mu_5\right). \end{cases} \tag{3.13}$$

Thus, $R(\tau)$, $I(\tau)$ can be approximated as

$$R(\tau) = A + \frac{B\tau^{\alpha}}{\Gamma(1+\alpha)} + \frac{B\mu_1\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{B\mu_2\tau^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{B\mu_3\tau^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{B\mu_4\tau^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{B\mu_5\tau^{6\alpha}}{\Gamma(1+6\alpha)} + \ldots, \tag{3.14}$$

and

$$I(\tau) = B + \frac{B\mu_1\tau^{\alpha}}{\Gamma(1+\alpha)} + \frac{B\mu_2\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{B\mu_3\tau^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{B\mu_4\tau^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{B\mu_5\tau^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{B\mu_6\tau^{6\alpha}}{\Gamma(1+6\alpha)} + \ldots, \tag{3.15}$$

where μ_i, $i = 1, 2, 3, \ldots, 6$, are already defined. Other higher terms can be derived as desired through the algorithm (3.6). However, the first few terms of the series (3.14) and (3.15) are sufficient to construct three different approximations in terms of the MLFs through the developed aftertreatment technique (Section 4), given later by Eqs (4.15) and (4.30).

3.1. Exact solution via MLF: special case

This section shows that the PSS (3.14), and (3.15) can be converted to exact forms under the condition $A + B = 1$ (the number of susceptibles is zero at initial time). Although this condition seems rare (or invalid), it can be occurred under specific rare occasions such as the lockdowns imposed by some countries during the spread of COVID-19. Under such extreme conditions, the number of susceptible individuals may sharply decrease, perhaps even reaching zero. Even if this condition is not met, this section introduces the exact solution from a pure mathematical perspective. The above condition implies $\mu_1 = \mu_3 = \mu_4 = -1$ while $\mu_2 = \mu_4 = \mu_6 = 1$. Therefore, the series (3.14) can be expressed as

$$R(\tau) = A - B\left[\sum_{m=0}^{\infty} \frac{(-\tau^{\alpha})^m}{\Gamma(m\alpha + 1)} - 1\right] = 1 - BE_{\alpha}\left(-\tau^{\alpha}\right). \tag{3.16}$$

Also, the series (3.15) becomes

$$I(\tau) = B\sum_{m=0}^{\infty} \frac{(-\tau^{\alpha})^m}{\Gamma(m\alpha + 1)} = BE_{\alpha}\left(-\tau^{\alpha}\right). \tag{3.17}$$

One can directly see from (3.16) and (3.17) that $I(\tau)$ is the CFD of $R(\tau)$ which satisfies the first equation of the system (1.1)-(1.3). The second equation can also be verified via direct substitution under the constrain $A+B = 1$, while the ICs (1.3) are automatically valid. Additionally, as $\alpha \to 1$, the expressions (3.16) and (3.17) reduce to $R(\tau) = 1 - B\, e^{-\tau}$ and $I(\tau) = B\, e^{-\tau}$, respectively. This agrees with the corresponding solutions in [32] for the ordinary SIR-model.

4. A new aftertreatment technique

This section proposes a new aftertreatment technique to treat the series solutions (3.14) and (3.15). The idea is based on applying the LT on the series, then establishing the Padé approximants of the transformed series, and finally applying the inverse LT to express the approximate solution in terms of the MLFs in different forms.

4.1. Diagonal Laplace-Padé approximants for $R(\tau)$

Assume that $\overline{R}(s) = \mathcal{L}\{R(\tau)\}$, then the LT for the series solutions (3.14) is

$$\overline{R}(s) = \frac{A}{s} + \frac{B}{s^{\alpha+1}} + \frac{B\mu_1}{s^{2\alpha+1}} + \frac{B\mu_2}{s^{3\alpha+1}} + \frac{B\mu_3}{s^{4\alpha+1}} + \frac{B\mu_4}{s^{5\alpha+1}} + \frac{B\mu_5}{s^{6\alpha+1}} + \dots, \tag{4.1}$$

or equivalently

$$\overline{R}(s) = \frac{1}{s}\phi(s), \tag{4.2}$$

where

$$\phi(s) = A + \frac{B}{s^{\alpha}} + \frac{B\mu_1}{s^{2\alpha}} + \frac{B\mu_2}{s^{3\alpha}} + \frac{B\mu_3}{s^{4\alpha}} + \frac{B\mu_4}{s^{5\alpha}} + \frac{B\mu_5}{s^{6\alpha}} + \dots. \tag{4.3}$$

If $s_1 = 1/s^{\alpha}$, then

$$\phi(s_1) = A + Bs_1 + B\mu_1 s_1^2 + B\mu_2 s_1^3 + B\mu_3 s_1^4 + B\mu_4 s_1^5 + B\mu_5 s_1^6 + \dots. \tag{4.4}$$

The $[L/L]$-diagonal Padé approximants for $\phi(s_1)$ are denoted by $\phi_{[L/L]}(s_1)$ and defined by

$$\phi_{[L/L]}(s_1) = \frac{P_{L0} + P_{L1}s_1 + P_{L2}s_1^2 + \cdots + P_{LL}s_1^L}{1 + Q_{L1}s_1 + Q_{L2}s_1^2 + \cdots + Q_{LL}s_1^L}. \tag{4.5}$$

For $L = 1, 2, 3$, we have

$$\begin{cases} \phi_{[1/1]}(s_1) = \frac{P_{10}+P_{11}s_1}{1+Q_{11}s_1}, \\ \phi_{[2/2]}(s_1) = \frac{P_{20}+P_{21}s_1+P_{22}s_1^2}{1+Q_{21}s_1+Q_{22}s_1^2}, \\ \phi_{[3/3]}(s_1) = \frac{P_{30}+P_{31}s_1+P_{32}s_1^2+P_{33}s_1^3}{1+Q_{31}s_1+Q_{32}s_1^2+Q_{33}s_1^3}. \end{cases} \tag{4.6}$$

The quantities P_{L0}, P_{L1}, P_{L2}, Q_{L1}, and Q_{L2}, $(L = 1, 2, 3)$ can be determined through concepts of Padé-approximants. In terms of s, we can write

$$\begin{cases} \phi_{[1/1]}(s) = \frac{P_{10}s^\alpha+P_{11}}{s^\alpha+Q_{11}}, \\ \phi_{[2/2]}(s) = \frac{P_{20}s^{2\alpha}+P_{21}s^\alpha+P_{22}}{s^{2\alpha}+Q_{21}s^\alpha+Q_{22}}, \\ \phi_{[3/3]}(s) = \frac{P_{30}s^{3\alpha}+P_{31}s^{2\alpha}+P_{32}s^\alpha+P_{33}}{s^{3\alpha}+Q_{31}s^{2\alpha}+Q_{32}s^\alpha+Q_{33}}, \end{cases} \tag{4.7}$$

or, equivalently,

$$\begin{cases} \phi_{[1/1]}(s) = P_{10} + \frac{P_{11}-P_{10}Q_{11}}{s^\alpha+Q_{11}}, \\ \phi_{[2/2]}(s) = P_{20} + \frac{(P_{21}-P_{20}Q_{21})s^\alpha+(P_{22}-P_{20}Q_{22})}{s^{2\alpha}+Q_{21}s^\alpha+Q_{22}} = P_{20} + \frac{\sigma_1 s^\alpha+\sigma_2}{(s^\alpha-r_1)(s^\alpha-r_2)}, \\ \phi_{[3/3]}(s) = P_{30} + \frac{(P_{31}-P_{30}Q_{31})s^{2\alpha}+(P_{32}-P_{30}Q_{32})s^\alpha+(P_{33}-P_{30}Q_{33})}{s^{3\alpha}+Q_{31}s^{2\alpha}+Q_{32}s^\alpha+Q_{33}} = P_{30} + \frac{\lambda_1 s^{2\alpha}+\lambda_2 s^\alpha+\lambda_3}{(s^\alpha-\rho_1)(s^\alpha-\rho_2)(s^\alpha-\rho_3)}, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} &\sigma_1 = P_{21} - P_{20}Q_{21}, \quad \sigma_2 = P_{22} - P_{20}Q_{22}, \\ &\lambda_1 = P_{31} - P_{30}Q_{31}, \quad \lambda_2 = P_{32} - P_{30}Q_{32}, \quad \lambda_3 = P_{33} - P_{30}Q_{33}, \end{aligned} \tag{4.9}$$

and r_1 and r_2 are two distinct roots of the quadratics equation:

$$r^2 + Q_{21}r + Q_{22} = 0, \tag{4.10}$$

while ρ_1, ρ_2, and ρ_3 are three distinct roots of the cubic equation:

$$\rho^3 + Q_{31}\rho^2 + Q_{32}\rho + Q_{33} = 0. \tag{4.11}$$

In view of (4.8), we have

$$\begin{cases} \phi_{[1/1]}(s) = P_{10} + \frac{P_{11}-P_{10}Q_{11}}{s^\alpha+Q_{11}}, \\ \phi_{[2/2]}(s) = P_{20} + \frac{h_1}{s^\alpha-r_1} + \frac{h_2}{s^\alpha-r_2}, \\ \phi_{[3/3]}(s) = P_{30} + \frac{k_1}{s^\alpha-\rho_1} + \frac{k_2}{s^\alpha-\rho_2} + \frac{k_3}{s^\alpha-\rho_3}, \end{cases} \tag{4.12}$$

where

$$\begin{aligned} &h_1 = \frac{\sigma_1 r_1 + \sigma_2}{r_1 - r_2}, \quad h_2 = \frac{\sigma_1 r_2 + \sigma_2}{r_2 - r_1}, \\ &k_1 = \frac{\lambda_1\rho_1^2 + \lambda_2\rho_1 + \lambda_3}{(\rho_1 - \rho_2)(\rho_1 - \rho_3)}, \quad k_2 = \frac{\lambda_1\rho_2^2 + \lambda_2\rho_2 + \lambda_3}{(\rho_2 - \rho_1)(\rho_2 - \rho_3)}, \quad k_3 = \frac{\lambda_1\rho_3^2 + \lambda_2\rho_3 + \lambda_3}{(\rho_3 - \rho_1)(\rho_3 - \rho_2)}. \end{aligned} \tag{4.13}$$

Inserting (4.12) into (4.2) yields

$$\begin{cases}\overline{R}_{[1/1]}(s) = \frac{P_{10}}{s} + \frac{(P_{11}-P_{10}Q_{11})s^{-1}}{s^{\alpha}+Q_{11}},\\ \overline{R}_{[2/2]}(s) = \frac{P_{20}}{s} + \frac{h_1 s^{-1}}{s^{\alpha}-r_1} + \frac{h_2 s^{-1}}{s^{\alpha}-r_2},\\ \overline{R}_{[3/3]}(s) = \frac{P_{30}}{s} + \frac{k_1 s^{-1}}{s^{\alpha}-\rho_1} + \frac{k_2 s^{-1}}{s^{\alpha}-\rho_2} + \frac{k_3 s^{-1}}{s^{\alpha}-\rho_3}.\end{cases} \tag{4.14}$$

Applying the inverse LT on Eq (4.14) gives

$$\begin{aligned} R_{[1/1]}(\tau) &= P_{10} + (P_{11} - P_{10}Q_{11})\, t^{\alpha} E_{\alpha,\alpha+1}\left(-Q_{11}t^{\alpha}\right),\\ R_{[2/2]}(\tau) &= P_{20} + h_1 t^{\alpha} E_{\alpha,\alpha+1}\left(r_1 t^{\alpha}\right) + h_2 t^{\alpha} E_{\alpha,\alpha+1}\left(r_2 t^{\alpha}\right),\\ R_{[3/3]}(\tau) &= P_{30} + k_1 t^{\alpha} E_{\alpha,\alpha+1}\left(\rho_1 t^{\alpha}\right) + k_2 t^{\alpha} E_{\alpha,\alpha+1}\left(\rho_2 t^{\alpha}\right) + k_3 t^{\alpha} E_{\alpha,\alpha+1}\left(\rho_3 t^{\alpha}\right). \end{aligned} \tag{4.15}$$

Equations (4.15) can be viewed as analytic approximations for the recovered individuals $R(\tau)$ using the diagonal Padé approximants $[L/L]$, $L = 1, 2, 3$. Such approximations are explicitly obtained in terms of the MLFs for the first time.

4.2. Diagonal Laplace-Padé approximants for $I(\tau)$

Assume that $\overline{I}(s) = \mathcal{L}\{I(\tau)\}$, then the LT for the series solutions (3.15) is

$$\overline{I}(s) = \frac{A}{s} + \frac{B\mu_1}{s^{\alpha+1}} + \frac{B\mu_2}{s^{2\alpha+1}} + \frac{B\mu_3}{s^{3\alpha+1}} + \frac{B\mu_4}{s^{4\alpha+1}} + \frac{B\mu_5}{s^{5\alpha+1}} + \frac{B\mu_6}{s^{6\alpha+1}} + \dots, \tag{4.16}$$

or equivalently

$$\overline{I}(s) = \frac{1}{s}\psi(s), \tag{4.17}$$

where

$$\psi(s) = A + \frac{B\mu_1}{s^{\alpha}} + \frac{B\mu_2}{s^{2\alpha}} + \frac{B\mu_3}{s^{3\alpha}} + \frac{B\mu_4}{s^{4\alpha}} + \frac{B\mu_5}{s^{5\alpha}} + \frac{B\mu_6}{s^{6\alpha}} + \dots. \tag{4.18}$$

Employing $s_1 = 1/s^{\alpha}$ leads to

$$\psi(s_1) = A + B\mu_1 s_1 + B\mu_2 s_1^2 + B\mu_3 s_1^3 + B\mu_4 s_1^4 + B\mu_5 s_1^5 + B\mu_6 s_1^6 + \dots. \tag{4.19}$$

Similar to the previous section, the $[L/L]$-diagonal Padé approximants for $\psi(s_1)$ are defined by $\psi_{[L/L]}(s_1)$ as follows:

$$\psi_{[L/L]}(s_1) = \frac{U_{L0} + U_{L1}s_1 + U_{L2}s_1^2 + \dots + U_{LL}s_1^L}{1 + V_{L1}s_1 + V_{L2}s_1^2 + \dots + V_{LL}s_1^L}. \tag{4.20}$$

For $L = 1, 2, 3$, we have

$$\begin{cases}\psi_{[1/1]}(s_1) = \frac{U_{10}+U_{11}s_1}{1+V_{11}s_1},\\ \psi_{[2/2]}(s_1) = \frac{U_{20}+U_{21}s_1+U_{22}s_1^2}{1+V_{21}s_1+V_{22}s_1^2},\\ \psi_{[3/3]}(s_1) = \frac{U_{30}+U_{31}s_1+U_{32}s_1^2+U_{33}s_1^3}{1+V_{31}s_1+V_{32}s_1^2+V_{33}s_1^3}.\end{cases} \tag{4.21}$$

The quantities U_{L0}, U_{L1}, U_{L2}, V_{L1}, and V_{L2}, $L = 1, 2, 3$, are well-defined from Padé-approximants. In terms of s, we can write

$$\begin{cases}\psi_{[1/1]}(s) = \frac{U_{10}s^{\alpha}+U_{11}}{s^{\alpha}+V_{11}},\\ \psi_{[2/2]}(s) = \frac{U_{20}s^{2\alpha}+U_{21}s^{\alpha}+U_{22}}{s^{2\alpha}+V_{21}s^{\alpha}+V_{22}},\\ \psi_{[3/3]}(s) = \frac{U_{30}s^{3\alpha}+U_{31}s^{2\alpha}+U_{32}s^{\alpha}+U_{33}}{s^{3\alpha}+V_{31}s^{2\alpha}+V_{32}s^{\alpha}+V_{33}},\end{cases} \tag{4.22}$$

or equivalently

$$\begin{cases}\psi_{[1/1]}(s) = U_{10} + \frac{U_{11}-U_{10}V_{11}}{s^{\alpha}+V_{11}},\\ \psi_{[2/2]}(s) = U_{20} + \frac{(U_{21}-U_{20}V_{21})s^{\alpha}+(U_{22}-U_{20}V_{22})}{s^{2\alpha}+V_{21}s^{\alpha}+V_{22}} = U_{20} + \frac{\gamma_1 s^{\alpha}+\gamma_2}{(s^{\alpha}-z_1)(s^{\alpha}-z_2)},\\ \psi_{[3/3]}(s) = U_{30} + \frac{(U_{31}-U_{30}V_{31})s^{2\alpha}+(U_{32}-U_{30}V_{32})s^{\alpha}+(P_{33}-U_{30}V_{33})}{s^{3\alpha}+V_{31}s^{2\alpha}+V_{32}s^{\alpha}+V_{33}} = U_{30} + \frac{\eta_1 s^{2\alpha}+\eta_2 s^{\alpha}+\eta_3}{(s^{\alpha}-\xi_1)(s^{\alpha}-\xi_2)(s^{\alpha}-\xi_3)},\end{cases} \tag{4.23}$$

where

$$\begin{aligned}&\gamma_1 = U_{21} - U_{20}V_{21}, \quad \gamma_2 = U_{22} - U_{20}V_{22},\\ &\eta_1 = U_{31} - U_{30}V_{31}, \quad \eta_2 = U_{32} - U_{30}V_{32}, \quad \eta_3 = U_{33} - U_{30}V_{33},\end{aligned} \tag{4.24}$$

and z_1 and z_2 are two distinct roots of the quadratics equation:

$$z^2 + V_{21}z + V_{22} = 0, \tag{4.25}$$

while ξ_1, ξ_2, and ξ_3 are three distinct roots of the cubic equation:

$$\xi^3 + V_{31}\xi^2 + V_{32}\xi + V_{33} = 0. \tag{4.26}$$

In view of (4.23), we have

$$\begin{cases}\psi_{[1/1]}(s) = U_{10} + \frac{U_{11}-U_{10}V_{11}}{s^{\alpha}+V_{11}},\\ \psi_{[2/2]}(s) = U_{20} + \frac{l_1}{s^{\alpha}-z_1} + \frac{l_2}{s^{\alpha}-z_2},\\ \psi_{[3/3]}(s) = U_{30} + \frac{m_1}{s^{\alpha}-\xi_1} + \frac{m_2}{s^{\alpha}-\xi_2} + \frac{m_3}{s^{\alpha}-\xi_3},\end{cases} \tag{4.27}$$

where

$$\begin{aligned}&l_1 = \frac{\gamma_1 z_1 + \gamma_2}{z_1 - z_2}, \quad l_2 = \frac{\gamma_1 z_2 + \gamma_2}{z_2 - z_1},\\ &m_1 = \frac{\eta_1\xi_1^2 + \eta_2\xi_1 + \eta_3}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}, \quad m_2 = \frac{\eta_1\xi_2^2 + \eta_2\xi_2 + \eta_3}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)}, \quad m_3 = \frac{\eta_1\xi_3^2 + \eta_2\xi_3 + \eta_3}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)}.\end{aligned} \tag{4.28}$$

Substituting (4.27) into (4.17), it then follows:

$$\begin{cases}\bar{I}_{[1/1]}(s) = \frac{U_{10}}{s} + \frac{(U_{11}-U_{10}V_{11})s^{-1}}{s^{\alpha}+V_{11}},\\ \bar{I}_{[2/2]}(s) = \frac{U_{20}}{s} + \frac{l_1 s^{-1}}{s^{\alpha}-z_1} + \frac{l_2 s^{-1}}{s^{\alpha}-z_2},\\ \bar{I}_{[3/3]}(s) = \frac{U_{30}}{s} + \frac{m_1 s^{-1}}{s^{\alpha}-\xi_1} + \frac{m_2 s^{-1}}{s^{\alpha}-\xi_2} + \frac{m_3 s^{-1}}{s^{\alpha}-\xi_3}.\end{cases} \tag{4.29}$$

Taking the inverse LT on Eq (4.29) we obtain

$$\begin{aligned}&I_{[1/1]}(\tau) = U_{10} + (U_{11} - U_{10}V_{11})\, t^{\alpha}E_{\alpha,\alpha+1}(-V_{11}t^{\alpha}),\\ &I_{[2/2]}(\tau) = U_{20} + l_1 t^{\alpha}E_{\alpha,\alpha+1}(z_1 t^{\alpha}) + l_2 t^{\alpha}E_{\alpha,\alpha+1}(z_2 t^{\alpha}),\\ &I_{[3/3]}(\tau) = U_{30} + m_1 t^{\alpha}E_{\alpha,\alpha+1}(\xi_1 t^{\alpha}) + m_2 t^{\alpha}E_{\alpha,\alpha+1}(\xi_2 t^{\alpha}) + m_3 t^{\alpha}E_{\alpha,\alpha+1}(\xi_3 t^{\alpha}).\end{aligned} \tag{4.30}$$

Equations (4.30) represent the analytic approximations of the infected individuals $I(\tau)$ using the diagonal Pad*é* approximants $[L/L]$, $L = 1, 2, 3$, in terms of the MLFs.

5. Numerical example

In order to demonstrate the effectiveness and practicality of the current method, a concrete numerical example is examined in this section. Also, this example demonstrates the efficiency of the aftertreatment technique over the standard PSS. For numerical purposes, the N-term approximate solutions for $R(\tau)$ and $I(\tau)$ are defined as

$$\chi_N(\tau) = \sum_{n=0}^{N-1} R_n(\tau), \quad \Omega_N(\tau) = \sum_{n=0}^{N-1} I_n(\tau), \tag{5.1}$$

respectively. To examine the convergence of the approximate solutions (5.1), the approximations $\chi_2(\tau)$, $\chi_3(\tau)$, and $\chi_4(\tau)$ are displayed in Figure 1 while $\chi_5(\tau)$, $\chi_6(\tau)$, and $\chi_7(\tau)$ are represented in Figure 2 at certain selections of the physical parameter A, B, σ, and α. In view of these figures, it can be noticed that the domain of convergence is enlarged as the number of terms increases as usual. Additionally, Figures 3 and 4 confirm this point for the approximations $\Omega_2(\tau)$, $\Omega_3(\tau)$, $\Omega_4(\tau)$ (Figure 3) and $\Omega_5(\tau)$, $\Omega_6(\tau)$, $\Omega_7(\tau)$ (Figure 4). Consequently, the number of terms N of the PSS can be increased to achieve the desired domain of convergence.

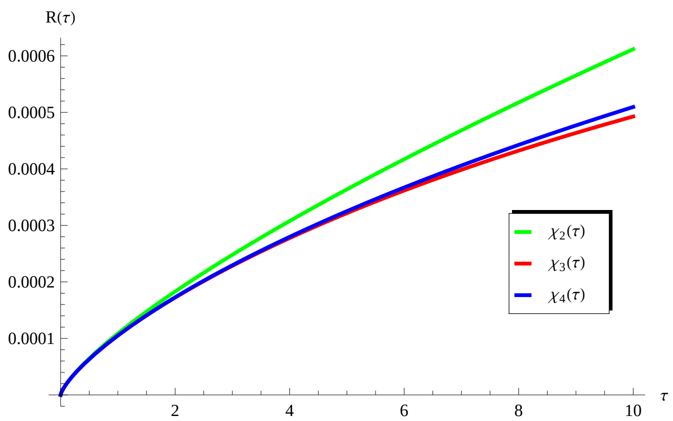

Figure 1. Plots of the approximations $\chi_N(\tau)$ ($N = 2, 3, 4$) for $R(\tau)$ at $\alpha = 0.75$, $\sigma = 0.95$, $A = 0$, and $B = 0.0001$.

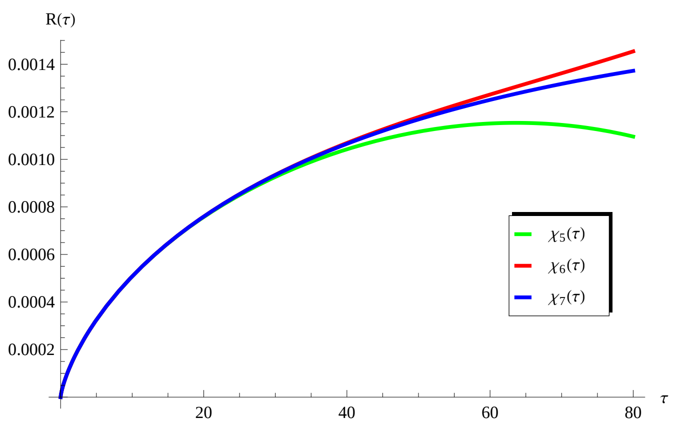

Figure 2. Plots of the approximations $\chi_N(\tau)$ ($N = 5, 6, 7$) for $R(\tau)$ at $\alpha = 0.75$, $\sigma = 0.95$, $A = 0$, and $B = 0.0001$.

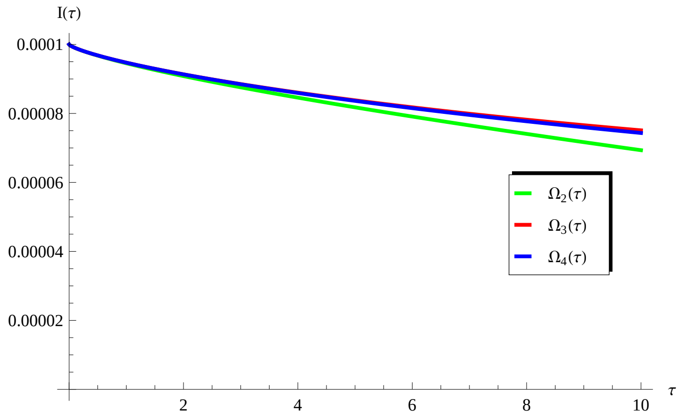

Figure 3. Plots of the approximations $\Omega_N(\tau)$ ($N = 2, 3, 4$) for $I(\tau)$ at $\alpha = 0.75$, $\sigma = 0.95$, $A = 0$, and $B = 0.0001$.

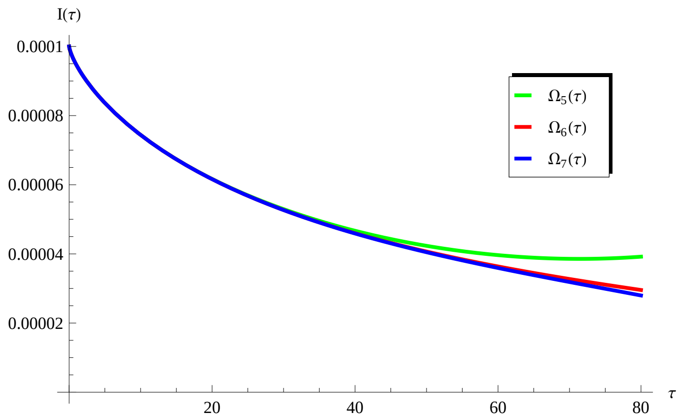

Figure 4. Plots of the approximations $\Omega_N(\tau)$ ($N = 5, 6, 7$) for $I(\tau)$ at $\alpha = 0.75$, $\sigma = 0.95$, $A = 0$, and $B = 0.0001$.

On the other hand, Figure 5 shows the behavior of the Mittag-Leffler approximate solutions $R_{[L/L]}(\tau)$ ($L = 1, 2, 3$), Eq (4.15), at the same set of the parameters values. Comparing Figure 5 with the PSS in Figures 1 and 2, one can detect that the domain of convergence is enhanced through the developed aftertreatment technique. This conclusion can also be confirmed when comparing Figure 6 for $I_{[L/L]}(\tau)$ ($L = 1, 2, 3$), Eq (4.30), with the PSS in Figures 3 and 4. This numerical example would provide a direct evidence of the method's capability to handle complex models in physical or engineering contexts.

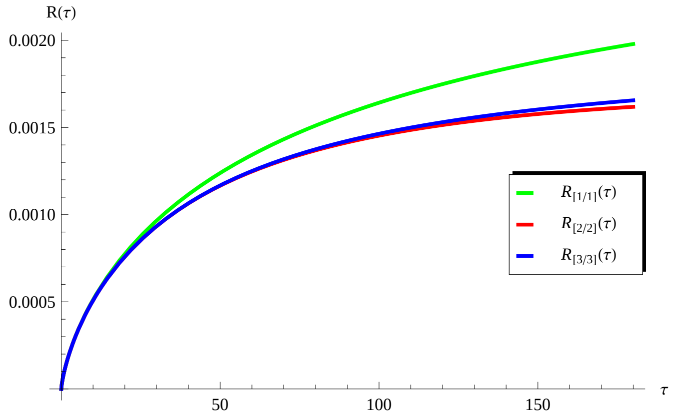

Figure 5. Plots of the Mittag-Leffler approximate solutions (4.15) for $R(\tau)$ at $\alpha = 0.75$, $\sigma = 0.95$, $A = 0$, and $B = 0.0001$.

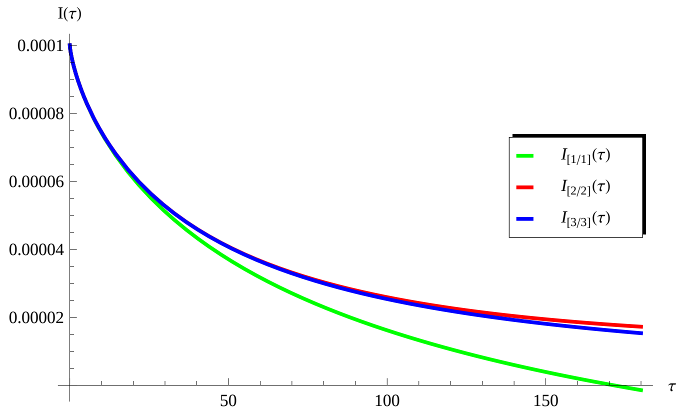

Figure 6. Plots of the Mittag-Leffler approximate solutions (4.30) for $I(\tau)$ at $\alpha = 0.75$, $\sigma = 0.95$, $A = 0$, and $B = 0.0001$.

6. Conclusions

A new aftertreatment technique was developed to solving a fractional nonlinear SIR-epidemic model. The suggested approach converted the standard series form into different equivalent approximations by means of the Laplace transform and its properties via different Pad*é*-approximants. The obtained approximations were given as closed-forms in terms of the MLFs. The proposed analysis can be viewed as a new approach for solving the fractional SIR-model. Moreover, the present analysis can be effectively extended to include a wide range of other fractional differential equations/systems. The main advantage of the developed approach is its capability of obtaining different approximations in terms of the MLFs in a straightforward manner. Once the standard power series solution is established, the new approximations via MLFs are easy to construct by the help of the series coefficients. This may deserve further extension to include other complex models with vaccinations such as [33].

Moreover, the aftertreatment technique transforms a possibly slowly convergent or structurally complex power series into a compact closed-form expression composed of a few MLFs. Conceptually, this is analogous to a filtering process in signal processing as potential applications, where Pad*é* approximation effectively smooths the series. For instance, this method may be applied to process noisy experimental data and fit it to fractional-order models.

Author contributions

Laila F. Seddek: Methodology, validation, formal analysis, funding acquisition, investigation, writing-original draft preparation, writing-review and editing; Abdelhalim Ebaid: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing; Essam R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing, visualization; Mona D. Aljoufi: Conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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