Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 1085–1115.
DOI:10.3934/math.2025052
Received: 12 September 2024
Revised: 25 December 2024
Accepted: 08 January 2025
Published: 20 January 2025

Research article

Entropy evaluation in inverse Weibull unified hybrid censored data with application to mechanical components and head-neck cancer patients

Refah Alotaibi[1], **Mazen Nassar**[2], **Zareen A. Khan**[1], **Wejdan Ali Alajlan**[1] **and Ahmed Elshahhat**[3,*]

[1] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[2] Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[3] Faculty of Technology and Development, Zagazig University, Zagazig 44519, Egypt

* **Correspondence:** Email: aelshahhat@ftd.zu.edu.eg.

Abstract: Entropy is a scientific term that finds applications in various domains, such as the laws of thermodynamics, where it was initially discovered, as well as statistical physics and information theory. We used unified hybrid censored data to investigate some inverse Weibull distribution entropy metrics. Entropy is defined using three measures: Rényi, Shannon, and Tsallis entropy. The classical estimates of the entropy measures were developed using the unified hybrid censored data, which included both point and approximation confidence intervals. The Bayesian method utilized the Markov Chain Monte Carlo sampling technique to develop Bayesian estimations. This was done by employing two loss functions, namely squared error and general entropy loss functions. Additionally, we delved into the investigation of Bayes credible intervals. Monte Carlo simulations were applied to explain how the estimates functioned at different sample sizes and censoring strategies via some accuracy criteria. Several observations were made in light of the simulation results. To provide a clear explanation of the offered methodologies, two applications using mechanical and cancer data sets were investigated.

Keywords: entropy; inverse Weibull; censored data; likelihood estimation; Bayesian estimation; MCMC process; Monte Carlo simulation
Mathematics Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05

1. Introduction

Entropy is a useful metric in a variety of fields, including insurance, physics, statistics, artificial intelligence, and economics. Entropy in information theory specifies the level of uncertainty for a random variable (RV). Its earliest application was in physics, specifically in terms of the second law of thermodynamics. Clausius [1] first established the notion of entropy in thermodynamics to measure the volume of energy in a system that cannot do work.

Shannon [2] outlined entropy as a quantitative indicator of uncertainty integrating probability and statistics. Shannon entropy (SE) is regarded as one of the cornerstones of information theory. This notion was supplemented by multiple entropy indicators acquired from real-world scenarios; see Amigó et al. [3] for an in-depth review. In this work, we concentrate on three important entropy measures, mainly Rényi entropy (REn) by Rényi [4], the Tsallis entropy (TEn) by Tsallis [5], and Shannon entropy (SEn) by [2]. Consider a non-negative continuous RV Z with probability density function (PDF) $g(z;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ represents the unknown parameter vector. Subsequent to this, the REn, TEn, and SEn of the RV Z can be determined, respectively:

$$\mathcal{E}_R = \frac{1}{1-\xi}\log\left\{\int_0^\infty [g(z;\boldsymbol{\theta})]^\xi dz\right\}, \quad \xi \geq 0, \quad \xi \neq 1, \tag{1.1}$$

$$\mathcal{E}_T = \frac{1}{1-\zeta}\left\{\int_0^\infty [g(z;\boldsymbol{\theta})]^\zeta dz - 1\right\}. \quad \zeta \geq 0 \quad \zeta \neq 1, \tag{1.2}$$

and

$$\mathcal{E}_S = -\int_0^\infty g(z;\boldsymbol{\theta})\log\left[g(z;\boldsymbol{\theta})\right] dz. \tag{1.3}$$

Data with a large entropy measure suggests that there is a more heterogeneous dataset, whereas a small entropy measure indicates a more homogeneous collection of data. In practice, the entropy indices and the parameter vector $\boldsymbol{\theta}$ are unknown and need to be estimated. Estimating the entropy of an RV has frequently been a topic that attracts attention in information theory. Many publications, including those by Bantan et al. [6], Al-Babtain et al. [7], Brochet et al. [8], Okasha and Nassar [9], Ren and Hu [10], Alam and Nassar [11], and Ahmed et al. [12] involve evaluating entropy indices and unknown parameters. See also the review paper by Feutrill and Roughan [13].

From a parametric standpoint, we assume in this work that the data come from the inverse Weibull (IW) model. This approach involves estimating the unknown parameters of the IW model and then utilizing them directly to assess the entropy indices, a process known as plug-in estimation. The IW distribution is a desirable statistical model for describing numerous kinds of data, including engineering, medical, and actuarial sciences data, since its failure rate can be decreasing or upside-down bathtub subject to the value of the shape parameter. Keller et al. [14] employed the IW distribution to study mechanical failures of parts due to degradation. Following that, several researchers investigated the IW distribution with various forms of data, see for example Xiuyun and Zaizai [15], Lee [16], Hassan and Zaky [17], Jana and Bera [18], Maswadah [19], Kayid and Alshehri [20], and Mou et al. [21].

It is said that the RV Z follows the IW distribution, written as IW($\boldsymbol{\theta}$), where $\boldsymbol{\theta} = (\gamma, \alpha)^\top$, if the associated PDF and cumulative distribution function (CDF) are expressed, respectively, by

$$g(z;\boldsymbol{\theta}) = \gamma\alpha z^{-(\alpha+1)} e^{-\gamma z^{-\alpha}}, \quad z \geq 0, \quad \gamma, \alpha > 0, \tag{1.4}$$

and

$$G(z;\boldsymbol{\theta}) = e^{-\gamma z^{-\alpha}}, \tag{1.5}$$

where γ is the scale parameter and α is the shape parameter. Assume that $Z \sim IW(\boldsymbol{\theta})$ with the PDF presented in (1.4), then the REn, TEn, and SEn of the RV Z are given, respectively, by

$$\mathcal{E}_R = \frac{\log(\xi\gamma)}{\alpha} - \log(\alpha) + \frac{1}{1-\xi}\left\{\log\left[\Gamma\left[\varphi(\alpha;\xi)\right]\right] - \xi\log(\xi)\right\}, \xi \neq 1, \xi \geq \frac{1}{1+\alpha}, \tag{1.6}$$

$$\mathcal{E}_T = \frac{1}{1-\zeta}\left[\frac{\alpha^{\zeta-1}\Gamma\left[\varphi(\alpha;\zeta)\right]}{\gamma^{\frac{\zeta-1}{\alpha}}\zeta^{\varphi(\alpha;\zeta)}} - 1\right], \zeta \neq 1, \zeta \geq \frac{1}{1+\alpha} \tag{1.7}$$

and

$$\mathcal{E}_S = \left(1 + \frac{1}{\alpha}\right)\left[\gamma^* + \log(\gamma)\right] - \log(\gamma\alpha) + 1, \tag{1.8}$$

where $\varphi(\alpha, v) = \frac{v-1}{\alpha} + v$ and γ^* are Euler constants.

In life testing experiments, it is impractical to gather data on every single item being studied due to the extensive time and cost involved. As a result, it is common practice to use censored samples. This approach enables researchers to end the experiment once a predetermined number of failures or a set time has been reached. In the statistical literature, numerous censoring plans have been proposed, each with its own merits. For the RV Z, suppose that we have a random sample of size n with ordered failure times $Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n}$. Epstein [22] proposed a hybrid censored (HC) sampling plan, where the life-testing experiment ends at a random time given by $\tau_1^* = min(T, Z_{m:n})$, where $T \in (0, \infty)$ and $1 \leq m < n$ are predetermined before starting the test. Childs et al. [23] termed this plan as a Type-I HC scheme (Type-I HCS) mainly because the duration of the experiment will not be higher than T. In this Type-I HCS, however, there is a possibility that very few failures may occur. For this reason, Childs et al. [23] proposed a new HCS, referred to as the Type-II hybrid censoring scheme (Type-II HCS), which guarantees a fixed number of failures. In this case, the termination point is determined by $\tau_2^* = max(T, Z_{m:n})$. Despite that Type-II HCS assures a certain number of failures, it comes with the problem of taking a while to record m failures and finish the life test. Chandrasekar et al. [24] modified these censored sampling plans through the development of two extensions: Generalized Type-I and Type-II HCSs. Although these two new censoring plans have more advantages than the previous ones, they continue to possess certain flaws. For instance, in the generalized Type-I HCS, we are not able to ensure a specific number of failures. On the other hand, in the generalized Type-II HCS, there is a chance of either getting no failures at all or only a few. To address these shortcomings, Balakrishnan et al. [25] developed a unified HCS (Unified-HCS) by combining the last two censoring schemes. A detailed description of the Unified-HCS is presented in the next section.

We are motivated to do this work because no researcher, to the best of our knowledge, has investigated the estimations of the mentioned three entropy measures of the IW distribution based on Unified-HCS data. In the literature, some researchers used different censoring schemes, including progressive First-Failure, generalized Type-I progressive hybrid censoring, adaptive Type-II progressive hybrid censoring, and improved adaptive Type-II progressive censoring, to investigate entropy estimations. For more details, refer to Yu et al. [26], Xu and Gui [27], and Lee [28]. Some of these researchers solely evaluated classical estimations, while others considered both classical and Bayesian estimations for a single entropy measure. Thus, no effort is made to estimate the entropy measures REn, TEn, and SEn for the IW distribution using both classical and Bayesian estimation methods. As a result, our goal is to compare the maximum likelihood and Bayesian estimation methodologies for the entropy metrics REn, TEn, and SEn. The IW distribution is used in this study for a variety of reasons. It is widely used for modeling life data, especially when failure rates are decreasing or exhibit an upside-down bathtub form. This distribution accurately captures scenarios where usual Weibull or exponential distributions may fail. It has numerous practical applications in various domains, such as reliability engineering, survival analysis, and risk assessment. We are also interested in using the Unified-HCS scheme because it helps balance cost and efficiency in experimental design by ensuring a specified number of observed failures. This scheme is particularly beneficial in reliability studies, where other single-stage censoring methods may yield less efficient results.

In this study, the maximum likelihood estimates (MLEs) and approximate confidence intervals (ACIs) for these metrics are described. Furthermore, in Bayesian estimation, the Markov Chain Monte Carlo (MCMC) sampling approach is used to derive point Bayes estimates using both the squared error loss (SEL) and general entropy loss (GEL) functions. The Bayesian credible intervals (BCIs) are also investigated. Simulation research is used to assess the efficiency of alternative methodologies by varying sample sizes and censoring strategies. Finally, a couple of applications are explored to demonstrate the use of the described approaches.

The remaining part of this work is structured in the following manner: In Section 2, we describe the Unified-HCS and investigate maximum likelihood estimations for the entropy metrics REn, TEn, and SEn. In Section 3, we provide Bayesian estimations for entropy metrics. In Section 4, a simulation study is performed and some insights are presented. In Section 5, we investigate two real data sets. In Section 6, we conclude the paper to a close.

2. Maximum likelihood estimation

In this part, we first present the Unified-HCS, and then look at the MLEs of the entropy measures for the IW distribution and the related ACIs.

2.1. Unified-HCS

Assume we run a life-testing experiment using n identical items. Let $Z_{1:n}, \ldots, Z_{n:n}$ represent the associated lifetimes from the IW distribution with PDF and CDF as given by in (1.4) and (1.5), respectively. Before starting the experiment, fix $m_1, m_2 \in \{1, 2, \ldots, n\}$, where $m_1 < m_2$ and $T_1 < T_2 \in (0, \infty)$. Balakrishnan et al. [25] described the Unified-HCS as follows. If $Z_{m_1:n} < T_1$, then the test is terminated at $min[max(Z_{m_2:n}, T_1), T_2]$. On the other hand, if $T_1 < Z_{m_1:n} < T_2$, then the test is stopped at $min(Z_{m_2:n}, T_2)$. In addition, if the $m_1 - th$ failure happens after time T_2, end the experiment at $Z_{m_1:n}$. Following this censoring plan, we can ensure that the test will be concluded upon reaching T_2 with at least m1 failures and that there will be exactly m_1 failures otherwise. Let τ denote the termination point, then under the Unified-HCS one can observed one of the following six cases (C):

- C-1: $0 < Z_{m_1:n} < Z_{m_2:n} < T_1 < T_2$, terminate the test at $\tau = T_1$.
- C-2: $0 < Z_{m_1:n} < T_1 < Z_{m_2:n} < T_2$, terminate the test at $\tau = Z_{m_2:n}$.

- C-3: $0 < Z_{m_1:n} < T_1 < T_2 < Z_{m_2:n}$, terminate the test at $\tau = T_2$.
- C-4: $0 < T_1 < Z_{m_1:n} < Z_{m_2:n} < T_2$, terminate the test at $\tau = Z_{m_2:n}$.
- C-5: $0 < T_1 < Z_{m_1:n} < T_2 < Z_{m_2:n}$, terminate the test at $\tau = T_2$.
- C-6: $0 < T_1 < T_2 < Z_{m_1:n} < Z_{m_2:n}$, terminate the test at $\tau = Z_{m_1:n}$.

For the various cases mentioned above, we can rewrite the unified likelihood function of the observed data as

$$L(\boldsymbol{\theta}; \underline{z}) \propto \prod_{i=1}^{\nu} g(z_i; \boldsymbol{\theta}) \, [1 - G(\tau; \boldsymbol{\theta})]^{n-\nu}, \tag{2.1}$$

where $\underline{z} = (z_{1:n}, \ldots, z_{\nu:n})$,

$$\nu = \begin{cases} d_1, & \text{for C-1} \\ m_2, & \text{for C-2 and C-4} \\ d_2, & \text{for C-3 and C-5} \\ m_1, & \text{for C-6} \end{cases},$$

and d_1 and d_2 are the observed number of failures recorder before T_1 and T_2, respectively. Many authors considered the Unified-HCS in their studies, see for example Ateya [29], Jeon and Kang [30] and Alrashidi et al. [31].

2.2. Point estimation of the entropy measures

Based on a Unified-HCS sample $\underline{z} = (z_{1:n}, \ldots, z_{\nu:n})$, we can write the likelihood function based on (1.4), (1.5), and (2.1) as given below

$$L(\boldsymbol{\theta}; \underline{z}) = (\gamma\alpha)^{\nu} \exp\left[-\gamma \sum\nolimits_{i=1}^{\nu} z_i^{-\alpha} - (\alpha - 1) \sum\nolimits_{i=1}^{\nu} \log(z_i) + (n - \nu)\phi(\tau; \boldsymbol{\theta})\right], \tag{2.2}$$

where $z_i = z_{i:n}$, $i = 1, \ldots, \nu$ and $\phi(\tau; \boldsymbol{\theta}) = \log\left(1 - e^{-\gamma\tau^{-\alpha}}\right)$. In order to calculate the MLEs for the entropy indices REn, TEn, and SEn, we must first calculate the MLEs for the IW distribution parameters γ and α. Once we have obtained the MLEs for these parameters, we can then use the invariance property of the MLEs to calculate the required estimates for the entropy measures. From (2.2), the log-likelihood function follows

$$\log L(\boldsymbol{\theta}; \underline{z}) = \nu \log(\gamma\alpha) - \gamma \sum\nolimits_{i=1}^{\nu} z_i^{-\alpha} - (\alpha - 1) \sum\nolimits_{i=1}^{\nu} \log(z_i) + (n - \nu)\phi(\tau; \boldsymbol{\theta}). \tag{2.3}$$

The MLEs of γ and α, represented by $\hat{\gamma}$ and $\hat{\alpha}$, respectively, are determined by solving the next system simultaneously

$$\frac{\partial \log L(\boldsymbol{\theta}; \underline{z})}{\partial \gamma} = \frac{\nu}{\gamma} - \sum\nolimits_{i=1}^{\nu} z_i^{-\alpha} + (n - \nu)\phi_1(\tau; \boldsymbol{\theta}) = 0 \tag{2.4}$$

and

$$\frac{\partial \log L(\boldsymbol{\theta}; \underline{z})}{\partial \alpha} = \frac{\nu}{\alpha} + \gamma \sum\nolimits_{i=1}^{\nu} z_i^{-\alpha} \log(z_i) - \sum\nolimits_{i=1}^{\nu} \log(z_i) + (n - \nu)\phi_2(\tau; \boldsymbol{\theta}) = 0, \tag{2.5}$$

where $\phi_1(\tau; \boldsymbol{\theta}) = \tau^{-\alpha}\eta(\tau; \boldsymbol{\theta})$ and $\phi_2(\tau; \boldsymbol{\theta}) = -\gamma\tau^{-\alpha} \log(\tau)\eta(\tau; \boldsymbol{\theta})$, with $\eta(\tau; \boldsymbol{\theta}) = e^{-\gamma\tau^{-\alpha}}(1 - e^{-\gamma\tau^{-\alpha}})^{-1}$. Following (2.4) and (2.5), the MLEs $\hat{\gamma}$ and $\hat{\alpha}$ clearly lack closed forms. Therefore, one can use any numerical approach to compute them numerically. Once $\hat{\gamma}$ and $\hat{\alpha}$ are obtained, the MLEs of REn, TEn, and SEn can be computed from Eqs (1.6), (1.7), and (1.8), respectively, as

$$\widehat{\mathcal{E}_R} = \frac{\log(\xi\hat{\gamma})}{\hat{\alpha}} - \log(\hat{\alpha}) + \frac{1}{1-\xi}\left\{\log\left[\Gamma\left[\varphi(\hat{\alpha};\xi)\right]\right] - \xi\log(\xi)\right\}, \xi \neq 1, \xi \geq \frac{1}{1+\hat{\alpha}},$$

$$\widehat{\mathcal{E}_T} = \frac{1}{1-\zeta}\left[\frac{\hat{\alpha}^{\zeta-1}\Gamma\left[\varphi(\hat{\alpha};\zeta)\right]}{\hat{\gamma}^{\frac{\zeta-1}{\hat{\alpha}}}\zeta^{\varphi(\hat{\alpha};\zeta)}} - 1\right], \zeta \neq 1, \zeta \geq \frac{1}{1+\hat{\alpha}}$$

and

$$\widehat{\mathcal{E}_S} = \left(1 + \frac{1}{\hat{\alpha}}\right)\left[\gamma^* + \log(\hat{\gamma})\right] - \log(\hat{\gamma}\hat{\alpha}) + 1.$$

2.3. Interval estimation of the entropy measures

In order to calculate the $100(1-\epsilon)\%$ ACIs for the entropy measurements REn, TEn, and SEn, we must determine their respective variances. Here, we use the delta method to approximate the needed variances; for more details, see Greene [32] and Elshahhat and Abu El Azm [33]. Using this approach requires obtaining the variance-covariance matrix, which we estimate by taking the inverse of the observed Fisher information matrix as described below

$$\boldsymbol{J}^{-1}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} -\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\gamma^2} & -\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\gamma\partial\alpha} \\ -\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\alpha\partial\gamma} & -\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\alpha^2} \end{pmatrix}^{-1}_{(\gamma,\alpha)=(\hat{\gamma},\hat{\alpha})} = \begin{pmatrix} \widehat{var}(\hat{\gamma}) & \widehat{cov}(\hat{\gamma},\hat{\alpha}) \\ \widehat{cov}(\hat{\alpha},\hat{\gamma}) & \widehat{var}(\hat{\alpha}) \end{pmatrix},$$

where $\hat{\boldsymbol{\theta}} = (\hat{\gamma},\hat{\alpha})^\top$ and

$$\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\gamma^2} = -\frac{\nu}{\gamma^2} - (n-\nu)\phi_{11}(\tau;\boldsymbol{\theta}),$$

$$\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\alpha^2} = -\frac{\nu}{\alpha^2} - \gamma\sum_{i=1}^{\nu} z_i^{-\alpha}\log^2(z_i) + (n-\nu)\phi_{22}(\tau;\boldsymbol{\theta})$$

and

$$\frac{\partial^2 \log L(\boldsymbol{\theta};\underline{z})}{\partial\gamma\partial\alpha} = \sum_{i=1}^{\nu} z_i^{-\alpha}\log(z_i) + (n-\nu)\phi_{12}(\tau;\boldsymbol{\theta}),$$

where $\phi_{11}(\tau;\boldsymbol{\theta}) = \phi_1(\tau;\boldsymbol{\theta})[\phi_1(\tau;\boldsymbol{\theta}) + \tau^{-\alpha}]$,

$$\phi_{22}(\tau;\boldsymbol{\theta}) = \phi_2(\tau;\boldsymbol{\theta})\tau^{-\alpha}\log(\tau)(1-e^{-\gamma\tau^{-\alpha}})^{-1}\left[\gamma + \tau^{-\alpha}(e^{-\gamma\tau^{-\alpha}} - 1)\right]$$

and

$$\phi_{12}(\tau;\boldsymbol{\theta}) = \phi_1(\tau;\boldsymbol{\theta})\tau^{-\alpha}\log(\tau)(1-e^{-\gamma\tau^{-\alpha}})^{-1}\left[\gamma + \tau^{-\alpha}(e^{-\gamma\tau^{-\alpha}} - 1)\right].$$

Then, by applying the delta method, we can approximate the estimated variances of REn, TEn, and SEn. These approximations are given below

$$\widehat{var}(\widehat{\mathcal{E}_R}) \approx \begin{pmatrix} \widehat{\mathcal{E}}_{R.1} & \widehat{\mathcal{E}}_{R.2} \end{pmatrix}\begin{pmatrix} \widehat{var}(\hat{\gamma}) & \widehat{cov}(\hat{\gamma},\hat{\alpha}) \\ \widehat{cov}(\hat{\alpha},\hat{\gamma}) & \widehat{var}(\hat{\alpha}) \end{pmatrix}\begin{pmatrix} \widehat{\mathcal{E}}_{R.1} \\ \widehat{\mathcal{E}}_{R.2} \end{pmatrix}$$

$$\widehat{var}(\widehat{\mathcal{E}_T}) \approx \begin{pmatrix} \widehat{\mathcal{E}}_{T.1} & \widehat{\mathcal{E}}_{T.2} \end{pmatrix} \begin{pmatrix} \widehat{var}(\hat{\gamma}) & \widehat{cov}(\hat{\gamma},\hat{\alpha}) \\ \widehat{cov}(\hat{\alpha},\hat{\gamma}) & \widehat{var}(\hat{\alpha}) \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{E}}_{T.1} \\ \widehat{\mathcal{E}}_{T.2} \end{pmatrix}$$

and

$$\widehat{var}(\widehat{\mathcal{E}_S}) \approx \begin{pmatrix} \widehat{\mathcal{E}}_{S.1} & \widehat{\mathcal{E}}_{S.2} \end{pmatrix} \begin{pmatrix} \widehat{var}(\hat{\gamma}) & \widehat{cov}(\hat{\gamma},\hat{\alpha}) \\ \widehat{cov}(\hat{\alpha},\hat{\gamma}) & \widehat{var}(\hat{\alpha}) \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{E}}_{S.1} \\ \widehat{\mathcal{E}}_{S.2} \end{pmatrix},$$

where

$$\widehat{\mathcal{E}}_{R.1} = \frac{1}{\hat{\gamma}\hat{\alpha}}, \ \widehat{\mathcal{E}}_{R.2} = -\frac{\hat{\alpha} + \log(\xi\hat{\gamma}) - \psi\left[\varphi(\hat{\alpha};\xi)\right]}{\hat{\alpha}^2},$$

$$\widehat{\mathcal{E}}_{T.1} = -\frac{\hat{\alpha}^{\zeta-3}\Gamma\left[\varphi(\hat{\alpha};\zeta)\right]}{\hat{\gamma}^{\frac{\zeta-1}{\hat{\alpha}}}\zeta^{\varphi(\hat{\alpha};\zeta)}}\left\{\hat{\alpha} + \log(\zeta\hat{\gamma}) + \psi\left[\varphi(\hat{\alpha};\zeta)\right]\right\}, \ \widehat{\mathcal{E}}_{T.2} = \frac{\hat{\alpha}^{\zeta-2}\Gamma\left[\varphi(\hat{\alpha};\zeta)\right]}{\hat{\gamma}^{\frac{\zeta-1}{\hat{\alpha}}+1}\zeta^{\varphi(\hat{\alpha};\zeta)}},$$

and

$$\widehat{\mathcal{E}}_{S.1} = -\frac{\hat{\alpha} + \gamma^* + \log(\hat{\gamma})}{\hat{\alpha}^2}, \ \widehat{\mathcal{E}}_{S.2} = \frac{1}{\hat{\gamma}\hat{\alpha}},$$

where $\psi(.)$ is the digamma function. After calculating $\widehat{var}(\widehat{\mathcal{E}_R})$, $\widehat{var}(\widehat{\mathcal{E}_T})$ and $\widehat{var}(\widehat{\mathcal{E}_S})$, one can compute the $100(1-\epsilon)\%$ ACIs of REn, TEn, and SEn, respectively, as

$$\widehat{\mathcal{E}}_R \pm y_{\frac{\epsilon}{2}}\sqrt{\widehat{var}(\widehat{\mathcal{E}_R})}, \ \widehat{\mathcal{E}}_T \pm y_{\frac{\epsilon}{2}}\sqrt{\widehat{var}(\widehat{\mathcal{E}_T})} \text{ and } \widehat{\mathcal{E}}_S \pm y_{\frac{\epsilon}{2}}\sqrt{\widehat{var}(\widehat{\mathcal{E}_S})},$$

where $y_{\frac{\epsilon}{2}}$ is the upper $(\epsilon/2)^{th}$ percentile point of the standard normal distribution.

3. Bayesian estimation

Bayesian estimation is an approach for using previous information to address decision challenges. The benefit of this method is that it can incorporate prior knowledge into statistical deduction, which improves the precision of the results. Bayesian estimation is frequently used in statistics and other domains like machine learning and data mining. Because of Bayesian estimation's outstanding effectiveness, many scholars now employ it for estimating unknown parameters and some related functions. See for example, Zhou et al. [34]. In this section, we investigate the Bayesian estimation of the entropy indices REn, TEn, and SEn based on two loss functions, namely SEL and GEL functions. Due to the fact that there are no conjugate prior distributions associated with the unknown parameters, we assume that the two parameters γ and α are independent and follow the gamma distribution with the joint prior distribution

$$\pi(\boldsymbol{\theta}) \propto \gamma^{a_1-1}\alpha^{a_2-1}e^{-(b_1\gamma+b_2\alpha)}, \tag{3.1}$$

where the hyper-parameters $a_j, b_j > 0, j = 1,2$ and are known. The gamma distribution is used for a variety of causes. Its flexibility enables it to produce a wide range of shapes. Furthermore, it provides the same support as the unknown parameters of the IW distribution. The gamma distribution is frequently used in practice, enabling both analytical and numerical Bayesian inference. Furthermore, it provides closed-form formulas for both mean and variance, making it easier to determine hyperparameter values during simulations or empirical computations in data analysis. The posterior distribution is obtained by combining the likelihood function in (2.2) with the joint prior distribution given by (3.1) as given below

$$Q(\boldsymbol{\theta}|\underline{z}) = \frac{1}{A}\gamma^{\nu+a_1-1}\alpha^{\nu+a_2-1}\exp\left[-\gamma\left(\sum\nolimits_{i=1}^{\nu} z_i^{-\alpha}+b_1\right)-\alpha\left(\sum\nolimits_{i=1}^{\nu}\log(z_i)+b_2\right)+(n-\nu)\phi(\tau;\boldsymbol{\theta})\right], \quad (3.2)$$

where A is the normalized constant. In our study, we use two loss functions to obtain Bayes estimators for the entropy indices REn, TEn, and SEn. The first is the SEL function, in which the Bayes estimator represents the posterior mean. The second is the GEL function, which produces the Bayes estimator as follows

$$\tilde{\mu}_{GEL} = \left[E(\mu^{-\rho})\right]^{-1/\rho}, \rho \neq 0,$$

in which ρ is a factor that regulates the degree of asymmetry and $\tilde{\mu}_{GEL}$ is the Bayes estimator of the parameter μ using the GEL function. Now, for any entropy index, say for example REn, its Bayes estimators using both SEL and GEL functions can be obtained, respectively, as

$$\widetilde{\mathcal{E}}_{R.SEL} = \frac{\int_0^\infty\int_0^\infty \mathcal{E}_R\pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\underline{z})d\gamma d\alpha}{\int_0^\infty\int_0^\infty \pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\underline{z})d\gamma d\alpha} \quad (3.3)$$

and

$$\widetilde{\mathcal{E}}_{R.GEL} = \left\{\frac{\int_0^\infty\int_0^\infty [\mathcal{E}_R]^{-\rho}\pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\underline{z})d\gamma d\alpha}{\int_0^\infty\int_0^\infty \pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\underline{z})d\gamma d\alpha}\right\}^{-1/\rho}. \quad (3.4)$$

The complexity of the ratio of integrals, as shown in (3.3) and (3.4) means that the Bayes estimators for the REn cannot be computed in closed forms. The same issue arises when attempting to obtain the Bayes estimators for the other entropy indices TEn and SEn. To address this issue, we propose using the MCMC technique to construct a posterior distribution of $Q(\boldsymbol{\theta}|\underline{z})$ depending on the produced samples, which may then be used to make additional statistical inferences. To apply the MCMC technique, we first derive the fully conditional distributions of γ and α from (3.2) as follows, respectively,

$$Q_1(\gamma|\alpha,\underline{z}) \propto \gamma^{\nu+a_1-1}\exp\left[-\gamma\left(\sum\nolimits_{i=1}^{\nu} z_i^{-\alpha}+b_1\right)+(n-\nu)\phi(\tau;\boldsymbol{\theta})\right], \quad (3.5)$$

and

$$\begin{aligned} Q_2(\alpha|\gamma,\underline{z}) &\propto \alpha^{\nu+a_2-1} \\ &\times \exp\left[-\gamma\sum\nolimits_{i=1}^{\nu} z_i^{-\alpha}-\alpha\left(\sum\nolimits_{i=1}^{\nu}\log(z_i)+b_2\right)+(n-\nu)\phi(\tau;\boldsymbol{\theta})\right]. \end{aligned} \quad (3.6)$$

By generating one Unified-HCS random sample from IW(0.5, 1.5) when $n = 100$, $(m_1, m_2) = (40, 80)$ and $(T_1, T_2) = (2.5, 5.5)$, Figure 1 shows that the fully conditional distributions of γ (3.5) and α (3.6) using this collected sample behave similarly to the normal density.

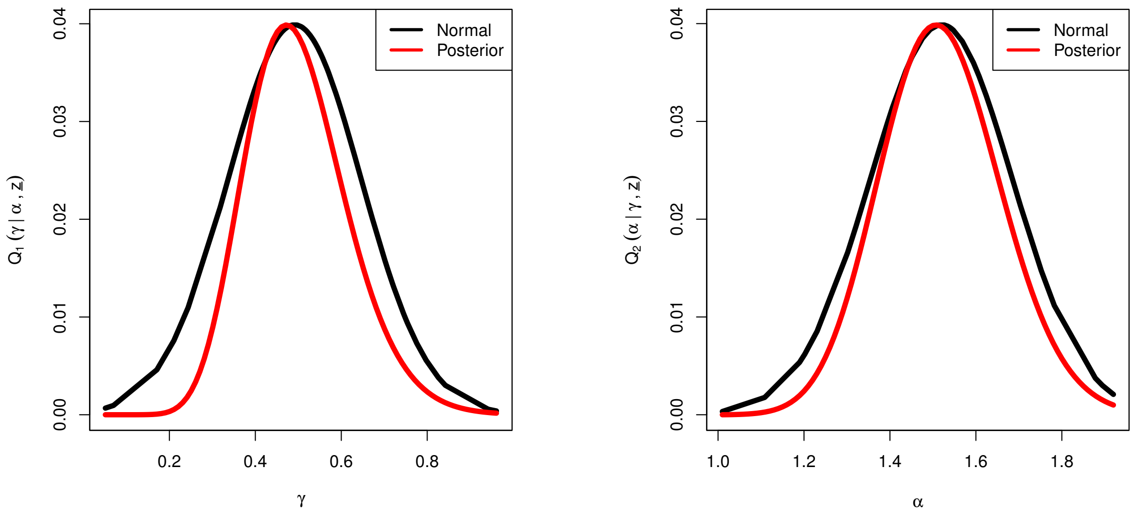


Figure 1. The normal and posterior density curves of γ (left) and α (right).

We use Metropolis-Hastings (M-H) sampling to acquire samples from the posterior distribution due to the nonstandard fully posterior distributions of γ and α as seen from (3.5) and (3.6). To utilize the M-H procedure, we take the normal (N) distribution as the proposal distribution and then apply the next steps for getting the requisite samples

Step 1. Set $c = 1$ and determine the starting values $(\gamma^{(0)}, \alpha^{(0)}) = (\hat{\gamma}, \hat{\alpha})$.

Step 2. Get $\gamma^{(c)}$ from (3.5) via the M-H algorithm with $N(\hat{\gamma}, \widehat{var}(\hat{\gamma}))$.

Step 3. Simulate $\alpha^{(c)}$ from (3.6) based on the M-H steps with $N(\hat{\alpha}, \widehat{var}(\hat{\alpha}))$.

Step 4. Set $c = c + 1$.

Step 5. Redo steps 2–4, $\mathbb{M}$ times to acquire.

$$\left\{\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(\mathbb{M})}\right\} \text{ and } \left\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\mathbb{M})}\right\}.$$

Before obtaining the Bayes estimates and the BCIs, we remove the first $\mathbb{B}$ samples as a burn-in period to eliminate the impact of the initial values. Then, the Bayes estimates for the entropy measures, for example, the REn, based on both SEL and GEL functions can be computed as

$$\widetilde{\mathcal{E}}_{R.SEL} = \frac{1}{\mathbb{M} - \mathbb{B}} \sum_{c=\mathbb{B}+1}^{\mathbb{M}} \mathcal{E}_R^{(c)}$$

and

$$\widetilde{\mathcal{E}}_{R.GEL} = \left\{\frac{1}{\mathbb{M} - \mathbb{B}} \sum_{c=\mathbb{B}+1}^{\mathbb{M}} \left[\mathcal{E}_R^{(c)}\right]^{-\rho}\right\}^{-1/\rho},$$

where

$$\mathcal{E}_R^{(c)} = \frac{\log\left(\xi\gamma^{(c)}\right)}{\alpha^{(c)}} - \log(\alpha^{(c)}) + \frac{\log\left[\Gamma\left[\varphi(\alpha^{(c)};\xi)\right]\right] - \xi\log(\xi)}{1-\xi}.$$

On the other hand, to compute the BCI of the REn, for example, we first sort the MCMC sample $\mathcal{E}_R^{(c)}, c = \mathbb{B}+1,\dots,\mathbb{M}$, indicated by $\mathcal{E}_R^{[\mathbb{B}+1]},\dots,\mathcal{E}_R^{[\mathbb{M}]}$. Thus, the $100(1-\epsilon)\%$ BCI of the REn is given by

$$\left\{\mathcal{E}_R^{[\epsilon(\mathbb{M}-\mathbb{B})/2]},\ \mathcal{E}_R^{[(1-\epsilon/2)(\mathbb{M}-\mathbb{B})]}\right\}.$$

4. Monte Carlo simulations

In this part, we offer some simulation findings that compare the effectiveness of point and interval estimates of $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ developed by various strategies proposed in terms of root mean squared errors (RMSEs), mean absolute biases (MABs), average interval lengths (AILs), and coverage percentages (CPs). To achieve this objective, utilizing different options of n(size of full experimental items), $m_1(m_2)$(lower (upper) failure data size), and $T_1(T_2)$(thresholds), we simulate large 2,000 Unified-HCS samples based on two different sets from IW(γ,α) population, such as IW$(0.5, 1.5)$ (say, Set-1) and IW$(1.5, 2.5)$ (say, Set-2). It is better to mention here that the values used are chosen based on their theoretical domains. Subsequently, the plausible value of ($\mathcal{E}_R$, $\mathcal{E}_Q$, $\mathcal{E}_S$) is taken as (2.0216,3.1123,1.0944) for Set-1, while for Set-2, it is taken as (1.5571,2.1605,1.0539). Without loss of generality, for sets 1 and 2, all fitted estimates of $\mathcal{E}_R$ and $\mathcal{E}_T$ are obtained when $\xi = \zeta = 0.6$. In Table 1, different simulation tests are provided.

Table 1. Seven simulation scenarios for each n.

Test↓ n →	40				80				
	m_1	m_2	T_1	T_2		m_1	m_2	T_1	T_2
1	10	30	0.5	1.5	1	20	50	0.5	1.5
2	15	30	0.5	1.5	2	40	50	0.5	1.5
3	10	35	0.5	1.5	3	20	60	0.5	1.5
4	18	38	0.5	1.5	4	45	70	0.5	1.5
5	10	30	1.0	1.5	5	40	50	1.0	1.5
6	10	30	0.5	2.5	6	40	50	0.5	2.5
7	10	30	1.4	2.8	7	40	50	1.4	2.8

After collecting 2,000 Unified-HCS samples, we install the 'maxLik' package (by Henningsen and Toomet [35]) and 'coda' package (by Plummer et al. [36]) to evaluate the likelihood and Bayes' MCMC estimates of the same unknown entropies. From SEL and GEL (from $\rho(= -2, 2)$), all Bayes' computations of $\mathcal{E}_R$, $\mathcal{E}_Q$, or $\mathcal{E}_S$ are implemented when the first 2,000 (of 12,000) MCMC iterations are ignored as burn-in. To assign values to the hyper-parameters a_i and b_i for $i = 1, 2$, following the past sample idea, we generate 10,000 complete samples (with $n = 50$) from each given set of the IW population. As a result, the values of (a_1, a_2, b_1, b_2) are taken as (27.4089,76.4457,54.6755,49.5563) for Set-1 and (44.1015,76.4458,28.6406,29.7338) for Set-2. For more information on the idea of past sample information, based on different types of censoring techniques, one can refer to Elshahhat and Elemary [37] and Alotaibi et al. [38] (for Type-II adaptive progressive); Nassar et al. [39] (for adaptive Type-I progressive) and Alotaibi et al. [40] (for progressive first-failure); Elbatal et al. [41], Alotaibi et al. [42], Nassar and Elshahhat [43] (for improved adaptive Type-II progressive), among others.

From Tables 2–7, in terms of the lowest RMSEs, MABs, and AILs as well as the highest CPs, we report the following comments:

- The general aspect is that the offered estimates of $\mathcal{E}_R$, $\mathcal{E}_T$, or $\mathcal{E}_S$ perform well.
- As expected, Bayes' findings of all unknown IW entropies $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ outperform frequentist estimates due to gamma knowledge. One can see that the Bayes estimates of the entropy measures have the smallest RMSEs in all the cases when compared with those acquired based on the classical approach. A similar observation is found with BCI estimates when compared to ACI estimates.
- As n increases, all point (or interval) estimates of $\mathcal{E}_R$, $\mathcal{E}_T$, or $\mathcal{E}_S$ are pretty satisfactory. The same behavior is also noted when T_i (or m_i) for $i = 1, 2$ grow.
- All estimates developed from the Bayes approach using GEL-based are superior compared to those developed from the Bayes approach using SEL-based; moreover, both are better compared to those developed from the likelihood approach.
- As m_1 increases (for fixed m_2 and $T_i,\ i = 1, 2$), the RMSEs, MABs, and AILs of $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ decrease while their CPs increase. This observation is also achieved when m_2 increases (for fixed m_1 and $T_i,\ i = 1, 2$).
- As T_1 increases (for fixed T_2 and $m_i,\ i = 1, 2$), the RMSEs, MABs, and AILs of $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ decrease while their CPs increase. This observation is also achieved when T_2 increases (for fixed T_1 and $m_i,\ i = 1, 2$).
- As $m_i,\ i = 1, 2$ increase (for fixed $T_i,\ i = 1, 2$), the RMSEs, MABs, and AILs of $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ decrease while their CPs increase. This observation is also achieved when $T_i,\ i = 1, 2$ increase (for fixed $m_i,\ i = 1, 2$).
- In summary, the Bayesian procedure is recommended to achieve more accurate estimates of REn, SEn, or TEn measures of the IW model in the context of uniform hybrid censoring.

Table 2. The RMSEs (left) and MABs (right) of $\mathcal{E}_R$.

n	Test	MLE		SEL		GEL($\rho=-2$)		GEL($\rho=2$)	
				Set-1					
40	1	0.6694	0.6693	0.6579	0.6252	0.6154	0.6102	0.4690	0.3691
	2	0.6608	0.6608	0.6507	0.6160	0.6041	0.6003	0.4640	0.3631
	3	0.6402	0.6401	0.6311	0.6067	0.5963	0.5961	0.4617	0.3600
	4	0.6218	0.6218	0.6111	0.5789	0.5681	0.5678	0.4432	0.3505
	5	0.6278	0.6278	0.6161	0.5831	0.5723	0.5719	0.4502	0.3529
	6	0.6316	0.6314	0.6234	0.5969	0.5861	0.5858	0.4591	0.3573
	7	0.6177	0.6175	0.6091	0.5764	0.5642	0.5620	0.4291	0.3416
80	1	0.6136	0.6134	0.6064	0.5724	0.5622	0.5604	0.3587	0.2730
	2	0.6069	0.6068	0.6015	0.5701	0.5574	0.5573	0.3503	0.2703
	3	0.5990	0.5990	0.5942	0.5624	0.5497	0.5495	0.3399	0.2689
	4	0.4894	0.4823	0.4825	0.4543	0.4456	0.4453	0.3122	0.2505
	5	0.5150	0.4877	0.4878	0.4659	0.4498	0.4496	0.3174	0.2533
	6	0.5830	0.5899	0.5659	0.5518	0.5386	0.5385	0.3290	0.2641
	7	0.4695	0.4615	0.4617	0.4347	0.4264	0.4261	0.3096	0.2474
				Set-2					
40	1	0.6787	0.6219	0.6221	0.5975	0.6108	0.5913	0.5917	0.5725
	2	0.6397	0.6134	0.6135	0.5974	0.6074	0.5895	0.5899	0.5686
	3	0.6397	0.6106	0.6109	0.5857	0.5997	0.5774	0.5780	0.5618
	4	0.5986	0.5860	0.5788	0.5632	0.5553	0.5547	0.5358	0.5444
	5	0.6252	0.5894	0.5840	0.5632	0.5788	0.5547	0.5553	0.5479
	6	0.6381	0.5894	0.5897	0.5707	0.5810	0.5647	0.5652	0.5546
	7	0.5819	0.5815	0.5709	0.5544	0.5463	0.5456	0.5253	0.5238
80	1	0.5722	0.5718	0.5616	0.5443	0.5382	0.5378	0.5036	0.4891
	2	0.5711	0.5707	0.5602	0.5430	0.5261	0.5353	0.4850	0.4691
	3	0.5565	0.5563	0.5528	0.5425	0.5141	0.5314	0.4643	0.4255
	4	0.5350	0.5348	0.5319	0.5216	0.4697	0.5165	0.4375	0.3641
	5	0.5524	0.5522	0.5485	0.5380	0.4852	0.5232	0.4425	0.3767
	6	0.5524	0.5522	0.5485	0.5380	0.5033	0.5274	0.4540	0.3952
	7	0.5104	0.5101	0.5079	0.4974	0.4593	0.4924	0.4227	0.3399

Table 3. The RMSEs (left) and MABs (right) of $\mathcal{E}_T$.

n	Test	MLE		SEL		GEL($\rho = -2$)		GEL($\rho = 2$)	
					Set-1				
40	1	1.3267	1.3267	1.2797	1.2270	1.1922	1.1920	1.1431	0.8394
	2	1.3121	1.3120	1.2670	1.2100	1.1747	1.1744	1.1307	0.8284
	3	1.2747	1.2746	1.2372	1.1986	1.1673	1.1673	1.1280	0.8221
	4	1.2372	1.2371	1.1987	1.1485	1.1232	1.1226	1.0202	0.7843
	5	1.2472	1.2471	1.2120	1.1559	1.1306	1.1430	1.1130	0.8068
	6	1.2599	1.2597	1.2229	1.1807	1.1554	1.1548	1.1181	0.8174
	7	1.2358	1.2315	1.1927	1.1409	1.1166	1.1106	0.9912	0.7675
80	1	1.2286	1.2283	1.1834	1.1388	1.1107	1.1063	0.7877	0.6288
	2	1.2172	1.2171	1.1704	1.1311	1.1006	1.1003	0.7677	0.6184
	3	1.2034	1.2033	1.1552	1.1173	1.0866	1.0863	0.7566	0.6084
	4	0.9919	0.9916	0.9742	0.9219	0.9106	0.8995	0.7162	0.5642
	5	1.0019	1.0017	0.9836	0.9302	0.9574	0.9271	0.7249	0.5769
	6	1.1876	1.1875	1.0987	1.0976	1.0653	1.0651	0.7492	0.5920
	7	0.9532	0.9529	0.9385	0.8854	0.8644	0.8639	0.7125	0.5577
					Set-2				
40	1	1.2519	1.0289	1.1104	1.0091	1.0292	0.9987	0.9762	0.9856
	2	1.2338	1.0156	1.0158	0.9860	1.0095	0.9747	0.9725	0.9706
	3	1.2282	1.0126	1.0130	0.9684	0.9872	0.9654	0.9545	0.9496
	4	1.0251	0.9755	0.9757	0.9393	0.9359	0.9348	0.9203	0.8900
	5	1.1988	0.9816	0.9820	0.9493	0.9456	0.9398	0.9265	0.9208
	6	1.2108	0.9816	0.9820	0.9563	0.9622	0.9476	0.9370	0.9343
	7	1.0140	0.9702	0.9707	0.9275	0.9439	0.9254	0.9064	0.8659
80	1	0.9631	0.9558	0.9563	0.9217	0.9298	0.9165	0.8973	0.8267
	2	0.9548	0.9543	0.9275	0.9109	0.8908	0.9070	0.8790	0.7667
	3	0.9315	0.9312	0.9218	0.9084	0.8859	0.8988	0.8627	0.7383
	4	0.8993	0.8990	0.8892	0.8874	0.8451	0.8643	0.8285	0.6491
	5	0.9215	0.9252	0.9040	0.8989	0.8577	0.8766	0.8378	0.6716
	6	0.9255	0.9252	0.9160	0.9019	0.8789	0.8881	0.8426	0.7147
	7	0.8620	0.8862	0.8530	0.8638	0.8328	0.8577	0.8216	0.6272

Table 4. The RMSEs (left) and MABs (right) of $\mathcal{E}_S$.

n	Test	MLE		SEL		GEL($\rho=-2$)		GEL($\rho=2$)	
					Set-1				
40	1	0.3716	0.3716	0.3593	0.3371	0.3170	0.3116	0.2784	0.2160
	2	0.3648	0.3647	0.3515	0.3216	0.3105	0.3103	0.2724	0.2140
	3	0.3541	0.3540	0.3442	0.3105	0.3085	0.3044	0.2705	0.2132
	4	0.3336	0.3330	0.3269	0.3007	0.2906	0.2903	0.2602	0.2078
	5	0.3367	0.3365	0.3300	0.3041	0.2981	0.2937	0.2628	0.2098
	6	0.3471	0.3469	0.3390	0.3085	0.3037	0.3033	0.2642	0.2108
	7	0.3282	0.3302	0.3227	0.2931	0.2776	0.2828	0.2548	0.2025
80	1	0.3302	0.3271	0.3207	0.2912	0.2776	0.2775	0.2241	0.1861
	2	0.3227	0.3227	0.3190	0.2876	0.2714	0.2743	0.2126	0.1789
	3	0.3170	0.3169	0.3140	0.2823	0.2692	0.2690	0.2011	0.1621
	4	0.2472	0.2385	0.2309	0.2142	0.2063	0.2060	0.1839	0.1478
	5	0.2669	0.2541	0.2342	0.2366	0.2174	0.2272	0.1876	0.1495
	6	0.3106	0.3106	0.3087	0.2759	0.2461	0.2610	0.1934	0.1552
	7	0.2306	0.2236	0.2238	0.2009	0.1930	0.1927	0.1823	0.1459
					Set-2				
40	1	0.4937	0.4788	0.4845	0.4705	0.4684	0.4843	0.4478	0.4563
	2	0.4749	0.4671	0.4735	0.4532	0.4623	0.4464	0.4468	0.4319
	3	0.4671	0.4567	0.4574	0.4464	0.4525	0.4373	0.4378	0.4188
	4	0.4401	0.4350	0.4351	0.4276	0.4189	0.4183	0.4172	0.3734
	5	0.4457	0.4457	0.4416	0.4276	0.4285	0.4265	0.4219	0.3846
	6	0.4569	0.4525	0.4519	0.4308	0.4463	0.4302	0.4273	0.3987
	7	0.4351	0.4247	0.4275	0.4203	0.4114	0.4108	0.3970	0.3607
80	1	0.4274	0.4142	0.4122	0.4120	0.4092	0.4079	0.3828	0.3459
	2	0.4263	0.4104	0.4091	0.4054	0.4030	0.4041	0.3777	0.3359
	3	0.4168	0.4082	0.4062	0.4006	0.4013	0.3971	0.3720	0.3231
	4	0.3992	0.3903	0.3855	0.3764	0.3638	0.3691	0.3521	0.2742
	5	0.4027	0.4014	0.3910	0.3851	0.3854	0.3782	0.3585	0.2860
	6	0.4127	0.4034	0.3980	0.3976	0.3934	0.3820	0.3682	0.3068
	7	0.3786	0.3694	0.3647	0.3676	0.3582	0.3582	0.3451	0.2499

Table 5. The AILs (left) and CPs (right) of $\mathcal{E}_R$.

n	Test	Set-1				Set-2			
		ACI		BCI		ACI		BCI	
40	1	1.735	0.919	0.805	0.941	2.235	0.927	1.623	0.951
	2	1.718	0.921	0.788	0.944	2.123	0.930	1.461	0.955
	3	1.700	0.922	0.766	0.946	1.922	0.933	1.256	0.958
	4	1.639	0.927	0.714	0.952	1.348	0.943	0.946	0.964
	5	1.652	0.926	0.736	0.951	1.562	0.940	1.005	0.962
	6	1.685	0.924	0.756	0.948	1.723	0.937	1.145	0.961
	7	1.608	0.930	0.698	0.954	1.147	0.945	0.884	0.967
80	1	1.485	0.935	0.674	0.956	0.929	0.948	0.741	0.970
	2	1.428	0.939	0.658	0.958	0.896	0.951	0.680	0.973
	3	1.376	0.944	0.648	0.960	0.875	0.953	0.597	0.978
	4	1.236	0.953	0.617	0.964	0.821	0.961	0.438	0.986
	5	1.285	0.950	0.628	0.962	0.836	0.958	0.485	0.984
	6	1.330	0.947	0.633	0.961	0.852	0.955	0.525	0.981
	7	1.199	0.955	0.611	0.965	0.809	0.962	0.398	0.989

Table 6. The AILs (left) and CPs (right) of $\mathcal{E}_T$.

n	Test	Set-1				Set-2			
		ACI		BCI		ACI		BCI	
40	1	4.277	0.895	1.438	0.927	2.260	0.915	0.724	0.956
	2	4.161	0.901	1.387	0.931	2.126	0.918	0.717	0.957
	3	4.076	0.904	1.368	0.933	2.059	0.920	0.691	0.959
	4	3.636	0.914	1.282	0.940	1.710	0.926	0.657	0.963
	5	3.872	0.911	1.303	0.938	1.888	0.925	0.668	0.962
	6	3.938	0.908	1.328	0.937	1.932	0.923	0.679	0.961
	7	3.534	0.916	1.246	0.941	1.696	0.928	0.636	0.965
80	1	3.395	0.920	1.226	0.943	1.662	0.930	0.624	0.966
	2	3.195	0.924	1.209	0.944	1.639	0.932	0.620	0.966
	3	2.953	0.927	1.189	0.945	1.613	0.933	0.615	0.967
	4	2.598	0.937	1.139	0.947	1.545	0.937	0.580	0.971
	5	2.675	0.934	1.143	0.947	1.560	0.936	0.593	0.970
	6	2.795	0.931	1.165	0.946	1.593	0.935	0.605	0.968
	7	2.396	0.940	1.127	0.949	1.524	0.939	0.561	0.973

Table 7. The AILs (left) and CPs (right) of $\mathcal{E}_S$.

n	Test	Set-1				Set-2			
		ACI		BCI		ACI		BCI	
40	1	1.107	0.927	0.755	0.948	0.960	0.931	0.388	0.962
	2	0.997	0.931	0.735	0.950	0.946	0.933	0.380	0.963
	3	0.965	0.933	0.715	0.952	0.939	0.935	0.375	0.964
	4	0.946	0.936	0.689	0.960	0.929	0.938	0.370	0.966
	5	0.921	0.938	0.653	0.956	0.900	0.941	0.367	0.965
	6	0.913	0.937	0.629	0.955	0.875	0.938	0.363	0.964
	7	0.906	0.940	0.603	0.962	0.838	0.946	0.357	0.967
80	1	0.885	0.941	0.578	0.964	0.814	0.948	0.344	0.969
	2	0.865	0.942	0.547	0.967	0.796	0.950	0.323	0.971
	3	0.847	0.944	0.535	0.969	0.764	0.953	0.312	0.972
	4	0.818	0.948	0.515	0.977	0.759	0.960	0.305	0.976
	5	0.796	0.946	0.495	0.974	0.742	0.957	0.286	0.975
	6	0.777	0.945	0.473	0.972	0.729	0.955	0.276	0.973
	7	0.768	0.949	0.431	0.980	0.703	0.962	0.269	0.978

5. Real Applications

In this section, we examine two distinct applications from engineering and medical to demonstrate the value of the supplied estimating methodologies and the applicability of the offered estimators in real-world scenarios.

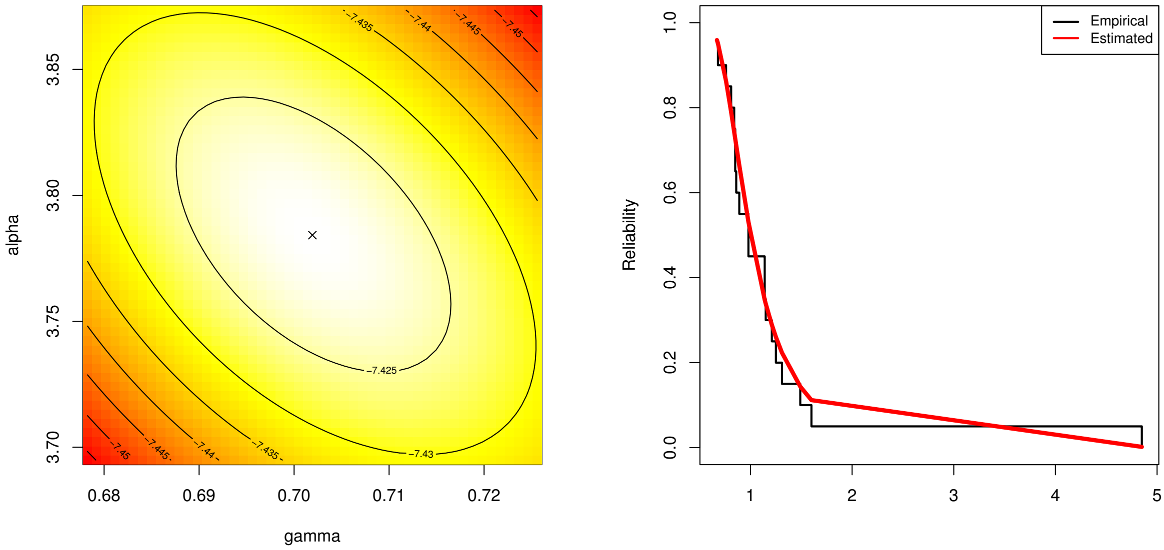
5.1. Mechanical data

This application examines the failure time points of 20 mechanical components (MCs) reported by Murthy et al. [44] and re-discussed by Mohammed et al. [45]. Table 8 simplifies the MCs data by multiplying each point by ten. The Kolmogorov–Smirnov (KS) distance with its P-value is used to determine whether the IW distribution's fit to the MCs data is adequate. To achieve this purpose, use Table 8 to fit the MLEs (with their standard errors (Std.Errs)) of γ and α as 0.7019 (0.1821) and 3.7841 (0.6867), respectively. Futhermore, the KS (P-value) is 0.1141 (0.957). Clearly, the IW model fits the MCs satisfactorily.

Table 8. Times to failure of 20 MCs.

0.67	0.68	0.76	0.81	0.84	0.85	0.85	0.86	0.89	0.98
0.98	1.14	1.14	1.15	1.21	1.25	1.31	1.49	1.60	4.85

Figure 2(a) provides the contour diagram of the log-likelihood for γ and α and indicates that the calculated MLEs of γ and α exist and are unique. It also stands that the best starting points of γ and α are 0.7019 and 3.7841, respectively. Figure 2(b) represents the empirical and estimated reliability lines, which supports the same fitting result.



(a) Contour (b) Reliability

Figure 2. Contour and empirical/fitted RF from MCs data.

Using the full MCs times, by fixing $(m_1, m_2) = (5, 10)$, we generate six Unified-HCS samples based on various options of T_i, $i = 1, 2$; see Table 9. For each data set in Table 9, the maximum likelihood and MCMC estimates (with their Std.Errs) as well as 95% ACI/BCI estimates (with their interval lengths (ILs)) of the unknown IW parameters (γ, α) as well as of the unknown IW entropies ($\mathcal{E}_R$ (at $\xi(= 0.4, 0.8)$), $\mathcal{E}_T$ (at $\zeta(= 0.4, 0.8)$), $\mathcal{E}_S$) are obtained.

Via the suggested MCMC sampler, fusing noninformative joint prior, from 50,000 MCMC samples with 10,000 burn-in, the Bayes results of γ, α, $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ are developed under SEL and GEL (for $\rho(= -3, 3)$) functions. The more frequent values of γ and α are selected as initial guesses to run the M-H algorithm. In Tables 10 and 11, the point and interval findings of all unknown quantities are listed. It is clear, from Tables 10 and 11, that the calculated point (or interval) estimates of γ, α, $\mathcal{E}_R$, $\mathcal{E}_T$, or $\mathcal{E}_S$ obtained by likelihood and Bayes methods are close to being similar. It also shows that the Bayes MCMC estimates behave better compared to others in terms of minimum Std.Errs. A similar observation is also reached when comparing ACIs and BCIs in terms of minimum ILs.

Table 9. Artificial Unified-HCS samples from MCs data.

Sample	m_1	m_2	$T_1(d_1)$	$T_2(d_2)$	τ	ν
1	5	10	1.18(14)	1.28(16)	1.18	14
2	5	10	0.85(7)	0.95(9)	0.95	9
3	5	10	0.87(8)	1.17(14)	0.98	10
4	5	10	0.78(3)	0.88(8)	0.88	8
5	5	10	0.78(3)	1.28(16)	0.98	10
6	5	10	0.72(2)	0.82(4)	0.84	5

Table 10. Point evaluations for parameters and entropies of IW model from MCs data.

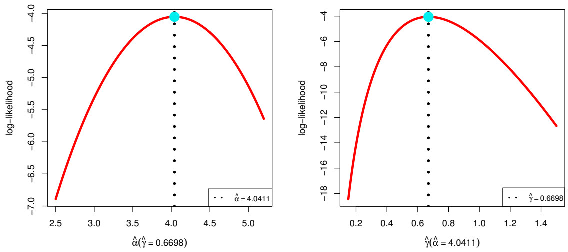
Sample	Par.	MLE		SEL		GEL			
$\rho \rightarrow$						-3		-3	
1	γ	0.6698	0.1901	0.6782	3.3646	0.6946	3.3465	0.6432	3.3979
	α	4.0411	0.8368	4.0399	3.3732	4.0452	3.3754	4.0291	3.3593
	$\mathcal{E}_S$	0.2244	0.2651	0.2255	0.0576	0.2394	0.0150	0.0037	0.2206
	$\mathcal{E}_R(\xi = 0.4)$	1.0244	0.4019	1.0271	0.0775	1.0329	0.0085	1.0153	0.0091
	$\mathcal{E}_R(\xi = 0.8)$	0.3553	0.2794	0.3566	0.0595	0.3663	0.0110	0.3342	0.0211
	$\mathcal{E}_T(\zeta = 0.4)$	1.4150	0.7430	1.4233	0.1443	1.4379	0.0228	1.3944	0.0207
	$\mathcal{E}_T(\zeta = 0.8)$	0.3683	0.3000	0.3700	0.0640	0.3808	0.0125	0.3452	0.0230
2	γ	0.6106	0.2141	0.6207	3.7446	0.6372	3.7267	0.5858	3.7781
	α	4.3639	1.1236	4.3626	3.7550	4.3677	3.7571	4.3525	3.7419
	$\mathcal{E}_S$	0.1231	0.3279	0.1252	0.0537	0.1452	0.0221	0.0076	0.1230
	$\mathcal{E}_R(\xi = 0.4)$	0.8752	0.4758	0.8780	0.0704	0.8836	0.0084	0.8666	0.0086
	$\mathcal{E}_R(\xi = 0.8)$	0.2489	0.3439	0.2507	0.0563	0.2627	0.0138	0.0329	0.2160
	$\mathcal{E}_T(\zeta = 0.4)$	1.1511	0.8044	1.1584	0.1198	1.1707	0.0196	1.1337	0.0174
	$\mathcal{E}_T(\zeta = 0.8)$	0.2552	0.3615	0.2574	0.0592	0.2704	0.0152	0.0329	0.2223
3	γ	0.6128	0.2042	0.6228	3.7350	0.6392	3.7172	0.5881	3.7683
	α	4.3564	1.0587	4.3552	3.7453	4.3602	3.7474	4.3451	3.7322
	$\mathcal{E}_S$	0.1257	0.3086	0.1273	0.0546	0.1474	0.0217	0.0008	0.1249
	$\mathcal{E}_R(\xi = 0.4)$	0.8788	0.4485	0.8816	0.0704	0.8872	0.0083	0.8702	0.0086
	$\mathcal{E}_R(\xi = 0.8)$	0.2516	0.3237	0.2533	0.0562	0.2652	0.0136	0.1227	0.1290
	$\mathcal{E}_T(\zeta = 0.4)$	1.1572	0.7599	1.1644	0.1200	1.1767	0.0195	1.1398	0.0174
	$\mathcal{E}_T(\zeta = 0.8)$	0.2580	0.3404	0.2602	0.0592	0.2730	0.0150	0.1231	0.1349
4	γ	0.4995	0.2068	0.5048	4.5283	0.5157	4.5167	0.4814	4.5510
	α	5.0324	1.3385	5.0317	4.5334	5.0337	4.5342	5.0278	4.5284
	$\mathcal{E}_S$	0.0619	0.3315	0.0629	0.0347	0.0789	0.0170	0.0046	0.0619
	$\mathcal{E}_R(\xi = 0.4)$	0.6174	0.4501	0.6179	0.0424	0.6208	0.0034	0.6120	0.0055
	$\mathcal{E}_R(\xi = 0.8)$	0.0558	0.3451	0.0560	0.0373	0.0739	0.0181	0.0042	0.0557
	$\mathcal{E}_T(\zeta = 0.4)$	0.7473	0.6519	0.7489	0.0615	0.7539	0.0065	0.7385	0.0088
	$\mathcal{E}_T(\zeta = 0.8)$	0.0561	0.3490	0.0565	0.0377	0.0746	0.0186	0.0000	0.0560
5	γ	0.6128	0.2042	0.6228	3.7350	0.6392	3.7172	0.5881	3.7683
	α	4.3564	1.0587	4.3552	3.7453	4.3602	3.7474	4.3451	3.7322
	$\mathcal{E}_S$	0.1257	0.3086	0.1273	0.0546	0.1474	0.0217	0.0008	0.1249
	$\mathcal{E}_R(\xi = 0.4)$	0.8788	0.4485	0.8816	0.0704	0.8872	0.0083	0.8702	0.0086
	$\mathcal{E}_R(\xi = 0.8)$	0.2516	0.3237	0.2533	0.0562	0.2652	0.0136	0.1227	0.1290
	$\mathcal{E}_T(\zeta = 0.4)$	1.1572	0.7599	1.1644	0.1200	1.1767	0.0195	1.1398	0.0174
	$\mathcal{E}_T(\zeta = 0.8)$	0.2580	0.3404	0.2602	0.0592	0.2730	0.0150	0.1231	0.1349
6	γ	0.6673	0.3112	0.6739	3.4963	0.6878	3.4811	0.6444	3.5245
	α	4.1689	1.4668	4.1676	3.5025	4.1713	3.5040	4.1601	3.4928
	$\mathcal{E}_S$	0.1910	0.4662	0.1917	0.0492	0.2035	0.0125	0.0040	0.1870
	$\mathcal{E}_R(\xi = 0.4)$	0.9708	0.6872	0.9725	0.0639	0.9767	0.0059	0.9640	0.0068
	$\mathcal{E}_R(\xi = 0.8)$	0.3198	0.4897	0.3206	0.0506	0.3284	0.0086	0.3026	0.0172
	$\mathcal{E}_T(\zeta = 0.4)$	1.3175	1.2305	1.3227	0.1149	1.3326	0.0152	1.3028	0.0147
	$\mathcal{E}_T(\zeta = 0.8)$	0.3303	0.5221	0.3314	0.0539	0.3399	0.0097	0.3116	0.0186

Table 11. Interval evaluations for parameters and entropies of IW model from MCs data.

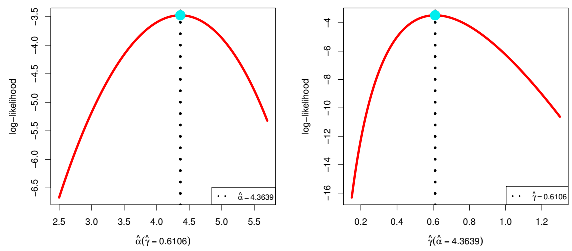
Sample	Par.	95% ACI			95% BCI		
		Low.	Upp.	IL	Low.	Upp.	IL
1	γ	0.2972	1.0425	0.7452	0.4770	0.8938	0.4169
	α	2.4010	5.6812	3.2801	3.7503	4.3263	0.5760
	$\mathcal{E}_S$	0.0000	0.7440	0.7440	0.1131	0.3384	0.2253
	$\mathcal{E}_R(\xi = 0.4)$	0.2368	1.8121	1.5752	0.8805	1.1835	0.3030
	$\mathcal{E}_R(\xi = 0.8)$	0.0000	0.9029	0.9029	0.2407	0.4735	0.2328
	$\mathcal{E}_T(\zeta = 0.4)$	0.0000	2.8714	2.8714	1.1601	1.7236	0.5635
	$\mathcal{E}_T(\zeta = 0.8)$	0.0000	0.9562	0.9562	0.2465	0.4967	0.2501
2	γ	0.1910	1.0301	0.8391	0.4315	0.8312	0.3998
	α	2.1616	6.5662	4.4046	4.0703	4.6549	0.5847
	$\mathcal{E}_S$	0.0000	0.7657	0.7657	0.0204	0.2324	0.2120
	$\mathcal{E}_R(\xi = 0.4)$	0.0000	1.8077	1.8077	0.7430	1.0201	0.2771
	$\mathcal{E}_R(\xi = 0.8)$	0.0000	0.9229	0.9229	0.1398	0.3618	0.2220
	$\mathcal{E}_T(\zeta = 0.4)$	0.0000	2.7276	2.7276	0.9362	1.4070	0.4708
	$\mathcal{E}_T(\zeta = 0.8)$	0.0000	0.9636	0.9636	0.1418	0.3752	0.2334
3	γ	0.2125	1.0131	0.8006	0.4337	0.8329	0.3992
	α	2.2813	6.4315	4.1502	4.0631	4.6474	0.5842
	$\mathcal{E}_S$	0.0000	0.7305	0.7305	0.0195	0.2348	0.2154
	$\mathcal{E}_R(\xi = 0.4)$	0.0000	1.7578	1.7578	0.7467	1.0236	0.2769
	$\mathcal{E}_R(\xi = 0.8)$	0.0000	0.8861	0.8861	0.1428	0.3641	0.2214
	$\mathcal{E}_T(\zeta = 0.4)$	0.0000	2.6465	2.6465	0.9420	1.4135	0.4715
	$\mathcal{E}_T(\zeta = 0.8)$	0.0000	0.9253	0.9253	0.1448	0.3777	0.2329
4	γ	0.0942	0.9048	0.8106	0.3619	0.6577	0.2958
	α	2.4089	7.6559	5.2470	4.8379	5.2241	0.3862
	$\mathcal{E}_S$	0.0000	0.7117	0.7117	0.0046	0.1361	0.1315
	$\mathcal{E}_R(\xi = 0.4)$	0.0000	1.4996	1.4996	0.5334	0.6996	0.1662
	$\mathcal{E}_R(\xi = 0.8)$	0.0000	0.7322	0.7322	-0.0193	0.1267	0.1461
	$\mathcal{E}_T(\zeta = 0.4)$	0.0000	2.0251	2.0251	0.6286	0.8693	0.2407
	$\mathcal{E}_T(\zeta = 0.8)$	0.0000	0.7401	0.7401	-0.0193	0.1284	0.1476
5	γ	0.2125	1.0131	0.8006	0.4337	0.8329	0.3992
	α	2.2813	6.4315	4.1502	4.0631	4.6474	0.5842
	$\mathcal{E}_S$	0.0000	0.7305	0.7305	0.0195	0.2348	0.2154
	$\mathcal{E}_R(\xi = 0.4)$	0.0000	1.7578	1.7578	0.7467	1.0236	0.2769
	$\mathcal{E}_R(\xi = 0.8)$	0.0000	0.8861	0.8861	0.1428	0.3641	0.2214
	$\mathcal{E}_T(\zeta = 0.4)$	0.0000	2.6465	2.6465	0.9420	1.4135	0.4715
	$\mathcal{E}_T(\zeta = 0.8)$	0.0000	0.9253	0.9253	0.1448	0.3777	0.2329
6	γ	0.0575	1.2772	1.2197	0.4906	0.8695	0.3789
	α	1.2940	7.0438	5.7498	3.9240	4.4131	0.4890
	$\mathcal{E}_S$	0.0000	1.1048	1.1048	0.0937	0.2870	0.1933
	$\mathcal{E}_R(\xi = 0.4)$	0.0000	2.3178	2.3178	0.8477	1.0991	0.2514
	$\mathcal{E}_R(\xi = 0.8)$	0.0000	1.2797	1.2797	0.2199	0.4188	0.1989
	$\mathcal{E}_T(\zeta = 0.4)$	0.0000	3.7292	3.7292	1.1049	1.5562	0.4513
	$\mathcal{E}_T(\zeta = 0.8)$	0.0000	1.3535	1.3535	0.2248	0.4368	0.2121

To show the optimal starting points and highlight the existence and uniqueness features of the offered estimates of $\hat{\alpha}$ and $\hat{\gamma}$, Figure 3 depicts the profile log-likelihoods of α and γ based on all datasets Si for $i = 1, 2, ..., 6$, presented in Table 9. It supports the point estimation results provided in Table 10 and indicates that the calculated estimates of $\hat{\alpha}$ and $\hat{\gamma}$ from Si, $i = 1, 2, ..., 6$, exist and are unique. As a result, we advocate adopting these starting points as the basis for all adhering to computational iterations.

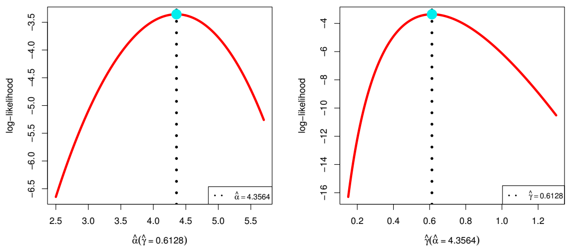
To highlight the convergence of the staying 40,000 MCMC outputs, using sample 1 as an example, trace and Gaussian kernel plots of γ, α, $\mathcal{E}_R$, $\mathcal{E}_T$, or $\mathcal{E}_S$ are displayed in Figure 4. This shows that the MCMC strategy converges fairly effectively and demonstrates that removing the first 10,000 samples is a proper amount to ignore the influence of the beginning values.



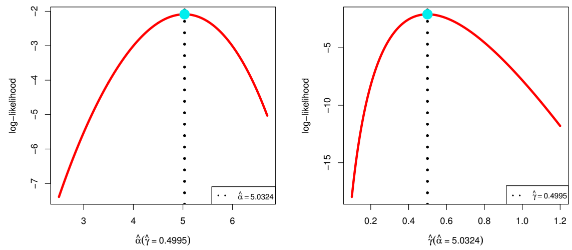
(a) Sample 1



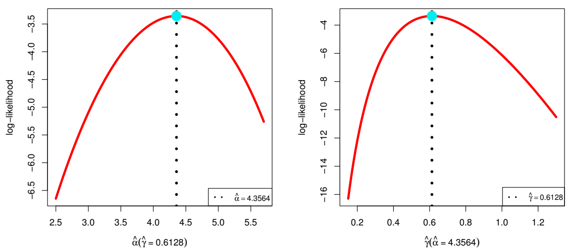
(b) Sample 2



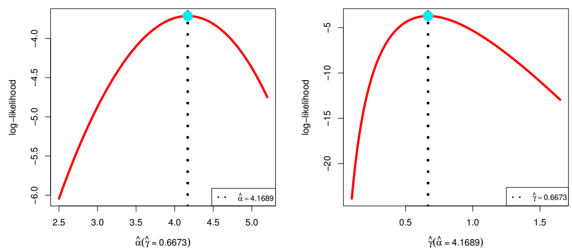
(c) Sample 3



(d) Sample 4

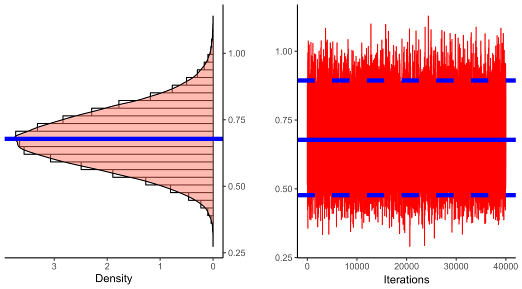


(e) Sample 5

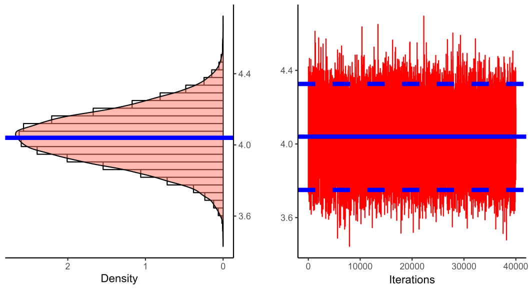


(f) Sample 6

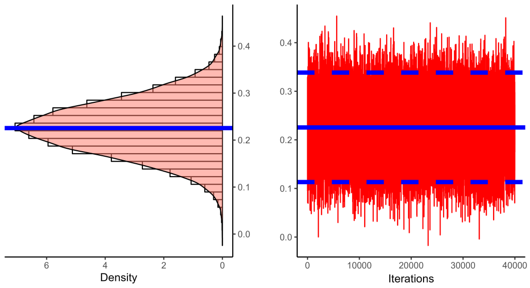
Figure 3. Profile log-likelihoods for α (left) and γ (right) from MCs data.



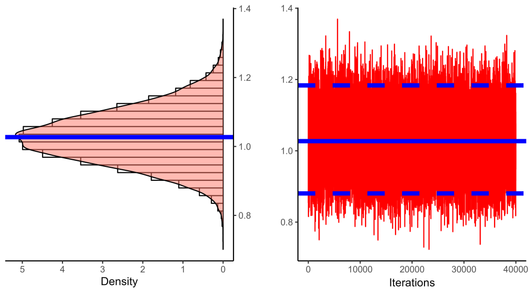
(a) γ



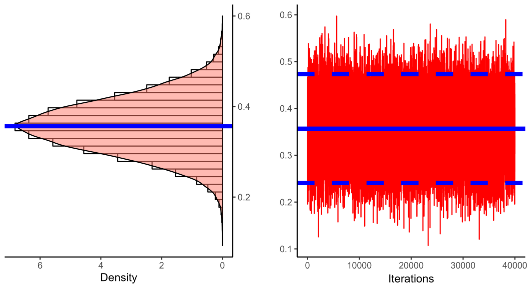
(b) α



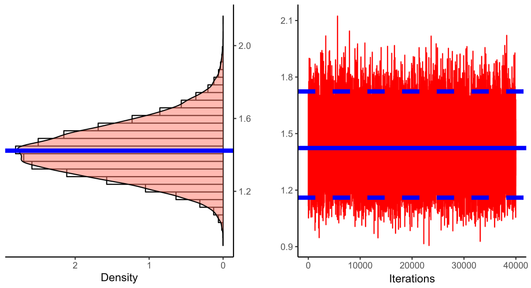
(c) $\mathcal{E}_S$



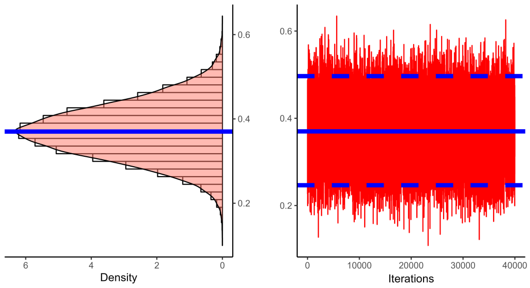
(d) $\mathcal{E}_R(\xi = 0.4)$



(e) $\mathcal{E}_R(\xi = 0.8)$



(f) $\mathcal{E}_T(\zeta = 0.4)$



(g) $\mathcal{E}_Q(\zeta = 0.8)$

Figure 4. The MCMC plots for parameters and entropies of IW model from MCs data.

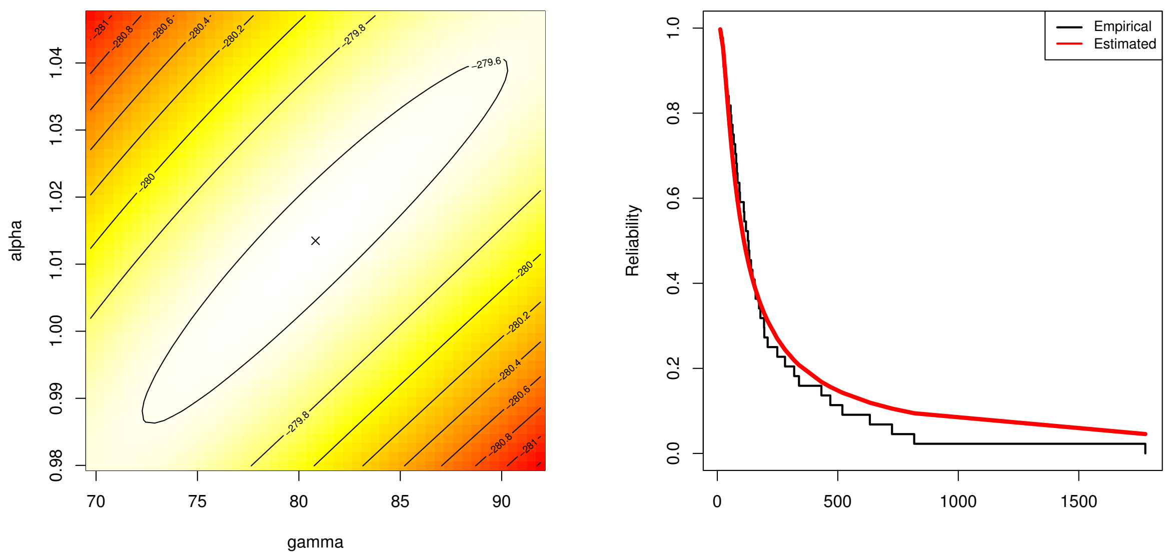
5.2. *Head-neck cancer data*

To demonstrate the importance of the offered estimators in real-world medical scenarios, we use a real data set provided by Efron [46] that depicts the survival times (in days) for forty-four head and neck cancer patients (HNCPs) who were treated with radiotherapy and chemotherapy. Table 12 displays the HNCPs data. Elshahhat and Nassar [47] have also analyzed this data.

Table 12. Survival time points of 44 HNCPs data.

12.20	23.56	23.74	25.87	31.98	37.00	41.35	47.38	55.46	58.36	63.47
68.46	78.26	74.47	81.43	84.00	92.00	94.00	110.0	112.0	119.0	127.0
130.0	133.0	140.0	146.0	155.0	159.0	173.0	179.0	194.0	195.0	209.0
249.0	281.0	319.0	339.0	432.0	469.0	519.0	633.0	725.0	817.0	1776

From Table 12, the MLEs (along with their Std.Errs) of γ and α are 80.821 (36.973) and 1.0134 (0.1119), respectively, while the KS (P-value) is 0.0901 (0.836). This result shows that the IW model fits the HNCPs data adequately. To highlight this fitting, using the full HNCPs data points, Figure 5(a) displays the contour plot of the log-likelihood as well as Figure 5(b) provides the empirical and estimated RF lines. Consequently, Figure 5(a) shows that the acquired MLEs $\hat{\gamma} \simeq 0.80.821$ and $\hat{\alpha} \simeq 1.0134$ existed and are also unique.



(a) Contour (b) Reliability

Figure 5. Contour and empirical/fitted RF from HNCPs data.

From the complete HNCPs data, by fixing $(m_1, m_2) = (10, 20)$, different Unified-HCS samples are created; see Table 13. For each Unified-HCS data set, the point estimations (with their Std.Errs) as well as the interval estimations (with their ILs) of γ, α, $\mathcal{E}_R$ (at $\xi(= 0.7, 0.9)$), $\mathcal{E}_T$ (at $\zeta(= 0.7, 0.9)$), $\mathcal{E}_S$) are calculated; see Tables 14 and 15.

Table 13. Artificial Unified-HCS samples from HNCPs data.

Sample	m_1	m_2	$T_1(d_1)$	$T_2(d_2)$	τ	ν
1	10	20	145(25)	205(32)	145	25
2	10	20	70(12)	90(16)	90	16
3	10	20	70(12)	175(29)	112	20
4	10	20	50(8)	100(18)	100	18
5	10	20	50(8)	175(29)	112	20
6	10	20	30(4)	55(8)	58.36	10

Table 14. Point evaluations for parameters and entropies of IW model from HNCPs data.

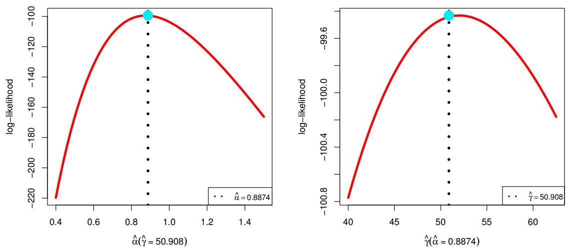
Sample	Par.	MLE		SEL		GEL			
$\rho \rightarrow$						-3		-3	
1	γ	50.908	8.7074	50.902	50.015	50.903	50.015	50.900	50.013
	α	0.8874	0.0554	0.8902	50.018	0.8918	50.016	0.8870	50.021
	$\mathcal{E}_S$	6.7759	0.2724	6.7664	0.2578	6.7762	0.0003	6.7468	0.0291
	$\mathcal{E}_R(\xi = 0.4)$	7.9778	0.3969	7.9683	0.3503	7.9837	0.0059	7.9376	0.0402
	$\mathcal{E}_R(\xi = 0.8)$	7.0226	0.2918	7.0129	0.2727	7.0235	0.0009	6.9918	0.0308
	$\mathcal{E}_T(\zeta = 0.4)$	33.166	4.3455	33.265	3.8837	33.720	0.5533	32.395	0.7715
	$\mathcal{E}_T(\zeta = 0.8)$	10.183	0.5889	10.171	0.5506	10.201	0.0178	10.112	0.0713
2	γ	37.744	5.8044	37.748	36.947	37.749	36.948	37.746	36.945
	α	0.8013	0.0552	0.8034	36.941	0.8053	36.939	0.7997	36.945
	$\mathcal{E}_S$	7.0504	0.3284	7.0470	0.3000	7.0598	0.0094	7.0215	0.0288
	$\mathcal{E}_R(\xi = 0.4)$	8.5064	0.5124	8.5077	0.4378	8.5303	0.0239	8.4630	0.0434
	$\mathcal{E}_R(\xi = 0.8)$	7.3366	0.3551	7.3336	0.3204	7.3476	0.0109	7.3056	0.0310
	$\mathcal{E}_T(\zeta = 0.4)$	39.439	6.5752	39.830	5.7853	40.674	1.2351	38.269	1.1695
	$\mathcal{E}_T(\zeta = 0.8)$	10.827	0.7396	10.831	0.6690	10.873	0.0457	10.749	0.0776
3	γ	43.234	11.618	43.238	42.398	43.239	42.398	43.236	42.395
	α	0.8408	0.0757	0.8431	42.391	0.8448	42.390	0.8398	42.395
	$\mathcal{E}_S$	6.9167	0.3242	6.9115	0.2739	6.9223	0.0057	6.8898	0.0269
	$\mathcal{E}_R(\xi = 0.4)$	8.2442	0.5308	8.2414	0.3855	8.2594	0.0152	8.2055	0.0388
	$\mathcal{E}_R(\xi = 0.8)$	7.1833	0.3549	7.1783	0.2911	7.1901	0.0067	7.1547	0.0287
	$\mathcal{E}_T(\zeta = 0.4)$	36.204	6.2957	36.437	4.6643	37.036	0.8319	35.304	0.9001
	$\mathcal{E}_T(\zeta = 0.8)$	10.510	0.7279	10.508	0.5979	10.542	0.0323	10.441	0.0694
4	γ	40.981	13.984	40.985	40.160	40.986	40.160	40.983	40.157
	α	0.8255	0.0924	0.8277	40.153	0.8294	40.151	0.8242	40.156
	$\mathcal{E}_S$	6.9659	0.3645	6.9621	0.2850	6.9738	0.0079	6.9388	0.0271
	$\mathcal{E}_R(\xi = 0.4)$	8.3405	0.6378	8.3402	0.4067	8.3600	0.0195	8.3008	0.0397
	$\mathcal{E}_R(\xi = 0.8)$	7.2399	0.4046	7.2363	0.3036	7.2490	0.0091	7.2109	0.0290
	$\mathcal{E}_T(\zeta = 0.4)$	37.362	7.7870	37.665	5.0896	38.355	0.9929	36.374	0.9884
	$\mathcal{E}_T(\zeta = 0.8)$	10.626	0.8345	10.629	0.6274	10.666	0.0392	10.555	0.0714
5	γ	43.234	11.618	43.238	42.398	43.239	42.398	43.236	42.395
	α	0.8408	0.0757	0.8431	42.391	0.8448	42.390	0.8398	42.395
	$\mathcal{E}_S$	6.9167	0.3242	6.9115	0.2739	6.9223	0.0057	6.8898	0.0269
	$\mathcal{E}_R(\xi = 0.4)$	8.2442	0.5308	8.2414	0.3855	8.2594	0.0152	8.2055	0.0388
	$\mathcal{E}_R(\xi = 0.8)$	7.1833	0.3549	7.1783	0.2911	7.1901	0.0067	7.1547	0.0287
	$\mathcal{E}_T(\zeta = 0.4)$	36.204	6.2957	36.437	4.6643	37.036	0.8319	35.304	0.9001
	$\mathcal{E}_T(\zeta = 0.8)$	10.510	0.7279	10.508	0.5979	10.542	0.0323	10.441	0.0694
6	γ	33.890	9.4816	43.238	42.469	43.239	42.469	43.236	42.466
	α	0.7699	0.0860	0.8431	33.047	0.8448	33.045	0.8398	33.050
	$\mathcal{E}_S$	7.1646	0.4354	6.9115	0.3728	6.9223	0.2422	6.8898	0.2748
	$\mathcal{E}_R(\xi = 0.4)$	8.7407	0.7712	8.2414	0.6308	8.2594	0.4813	8.2055	0.5353
	$\mathcal{E}_R(\xi = 0.8)$	7.4685	0.4822	7.1783	0.4111	7.1901	0.2785	7.1547	0.3139
	$\mathcal{E}_T(\zeta = 0.4)$	42.554	10.617	36.437	7.6892	37.036	5.5184	35.304	7.2505
	$\mathcal{E}_T(\zeta = 0.8)$	11.103	1.0176	10.508	0.8436	10.542	0.5610	10.441	0.6627

Table 15. Interval evaluations for parameters and entropies of IW model from HNCPs data.

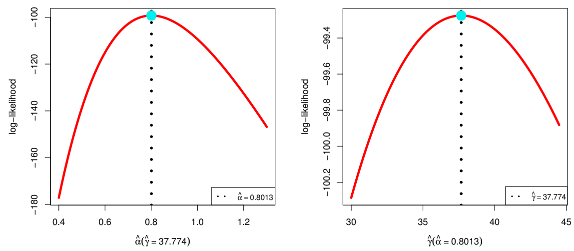
Sample	Par.	95% ACI			95% BCI		
		Low.	Upp.	IL	Low.	Upp.	IL
1	γ	33.842	67.974	34.133	50.499	51.292	0.7933
	α	0.7787	0.9960	0.2173	0.8195	0.9665	0.1470
	$\mathcal{E}_S$	6.2420	7.3097	1.0677	6.2756	7.2752	0.9996
	$\mathcal{E}_R(\xi = 0.4)$	7.1999	8.7556	1.5556	7.3101	8.6690	1.3590
	$\mathcal{E}_R(\xi = 0.8)$	6.4507	7.5945	1.1438	6.4946	7.5525	1.0579
	$\mathcal{E}_T(\zeta = 0.4)$	24.649	41.683	17.034	26.541	41.578	15.037
	$\mathcal{E}_T(\zeta = 0.8)$	9.0288	11.337	2.3086	9.1450	11.281	2.1364
2	γ	26.368	49.121	22.753	37.369	38.144	0.7745
	α	0.6932	0.9094	0.2162	0.7294	0.8812	0.1519
	$\mathcal{E}_S$	6.4068	7.6939	1.2871	6.4792	7.6639	1.1848
	$\mathcal{E}_R(\xi = 0.4)$	7.5020	9.5107	2.0087	7.6955	9.4251	1.7296
	$\mathcal{E}_R(\xi = 0.8)$	6.6406	8.0326	1.3920	6.7283	7.9939	1.2656
	$\mathcal{E}_T(\zeta = 0.4)$	26.551	52.326	25.774	30.203	53.012	22.809
	$\mathcal{E}_T(\zeta = 0.8)$	9.3774	12.277	2.8991	9.5977	12.2418	2.6441
3	γ	20.463	66.006	45.542	42.860	43.634	0.7740
	α	0.6925	0.9892	0.2967	0.7714	0.9186	0.1472
	$\mathcal{E}_S$	6.2812	7.5521	1.2710	6.3904	7.4672	1.0768
	$\mathcal{E}_R(\xi = 0.4)$	7.2039	9.2845	2.0806	7.5203	9.0366	1.5163
	$\mathcal{E}_R(\xi = 0.8)$	6.4878	7.8789	1.3912	6.6253	7.7702	1.1449
	$\mathcal{E}_T(\zeta = 0.4)$	23.864	48.543	24.679	28.486	46.813	18.327
	$\mathcal{E}_T(\zeta = 0.8)$	9.0835	11.937	2.8533	9.3970	11.750	2.3528
4	γ	13.573	68.388	54.816	40.606	41.381	0.7756
	α	0.6444	1.0067	0.3623	0.7548	0.9043	0.1494
	$\mathcal{E}_S$	6.2515	7.6803	1.4288	6.4246	7.5428	1.1182
	$\mathcal{E}_R(\xi = 0.4)$	7.0904	9.5906	2.5002	7.5869	9.1862	1.5993
	$\mathcal{E}_R(\xi = 0.8)$	6.4470	8.0328	1.5858	6.6649	7.8560	1.1912
	$\mathcal{E}_T(\zeta = 0.4)$	22.100	52.625	30.525	29.128	49.116	19.987
	$\mathcal{E}_T(\zeta = 0.8)$	8.9909	12.262	3.2710	9.4738	11.937	2.4635
5	γ	20.463	66.006	45.542	42.8602	43.634	0.7740
	α	0.6925	0.9892	0.2967	0.7714	0.9186	0.1472
	$\mathcal{E}_S$	6.2812	7.5521	1.2710	6.3904	7.4672	1.0768
	$\mathcal{E}_R(\xi = 0.4)$	7.2039	9.2845	2.0806	7.5203	9.0366	1.5163
	$\mathcal{E}_R(\xi = 0.8)$	6.4878	7.8789	1.3912	6.6253	7.7702	1.1449
	$\mathcal{E}_T(\zeta = 0.4)$	23.864	48.543	24.679	28.486	46.813	18.327
	$\mathcal{E}_T(\zeta = 0.8)$	9.0835	11.937	2.8533	9.3970	11.750	2.3528
6	γ	15.306	52.473	37.167	42.860	43.634	0.7740
	α	0.6014	0.9384	0.3369	0.7714	0.9186	0.1472
	$\mathcal{E}_S$	6.3112	8.0179	1.7067	6.3904	7.4672	1.0768
	$\mathcal{E}_R(\xi = 0.4)$	7.2292	10.252	3.0231	7.5203	9.0366	1.5163
	$\mathcal{E}_R(\xi = 0.8)$	6.5234	8.4136	1.8903	6.6253	7.7702	1.1449
	$\mathcal{E}_T(\zeta = 0.4)$	21.746	63.362	41.617	28.486	46.813	18.327
	$\mathcal{E}_T(\zeta = 0.8)$	9.1089	13.098	3.9891	9.3970	11.750	2.3528

Figure 6 illustrates the profile log-likelihood functions of α and γ based on all datasets presented in Table 13. It indicates that the proposed estimates of $\mu_i,\ i = 1, 2$ based on samples Si for $i = 1, 2, ..., 6$, are existent and are unique, as well as supports the point estimation results of α and γ listed in Table 14.

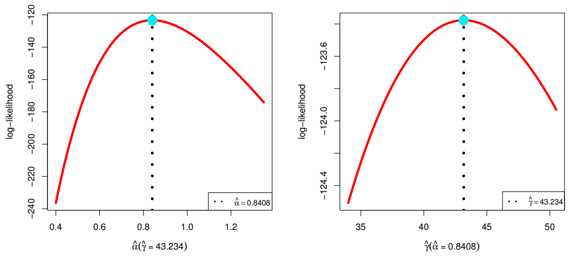
All Bayes' iterations developed from HNCPs data are implemented using the same scenario described in Subsection 5.1. As a consequence, from Tables 14 and 15, the offered estimates suggested by likelihood and Bayes methods of the IW model parameters γ and α as well as of IW entropies $\mathcal{E}_R$, $\mathcal{E}_T$, and $\mathcal{E}_S$ behave similarly. Moreover, the MCMC and 95& BCI estimates outperform compared to others in terms of lowest standard error and interval width. For example, using Sample 1 collected from the HNCPs data, trace and density plots of the simulated Markov variables for all unknown parameters are depicted in Figure 7. It is also clear that the MCMC iterations of $\mathcal{E}_T(\zeta = 0.8)$ are positively skewed while those of the other parameters are close to being symmetrical.



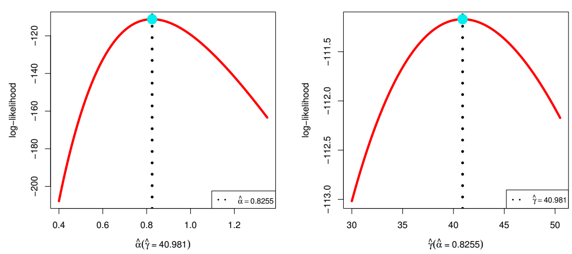
(a) Sample 1



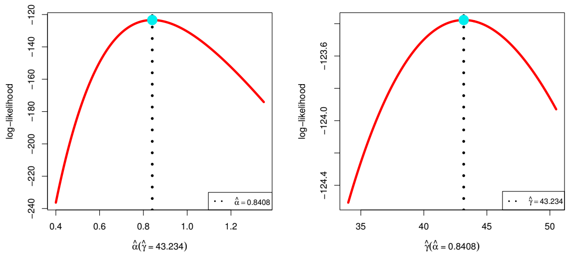
(b) Sample 2



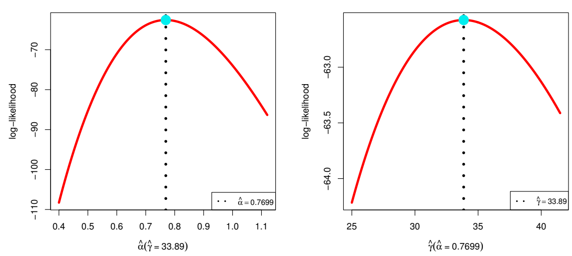
(c) Sample 3



(d) Sample 4

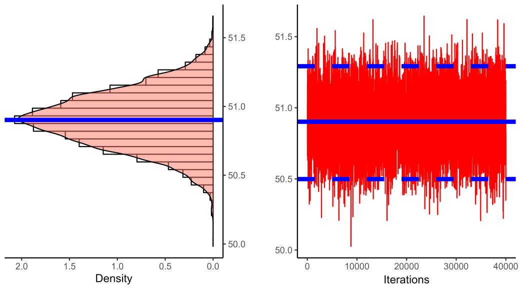


(e) Sample 5

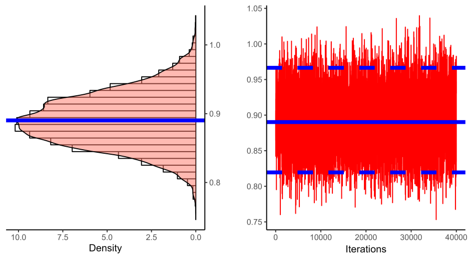


(f) Sample 6

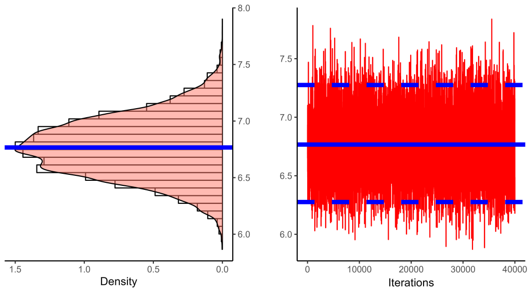
Figure 6. Profile log-likelihoods for α (left) and γ (right) from HNCPs data.



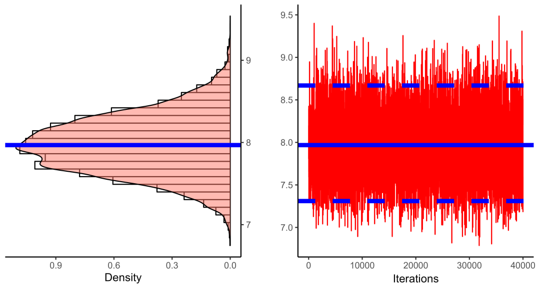
(a) γ



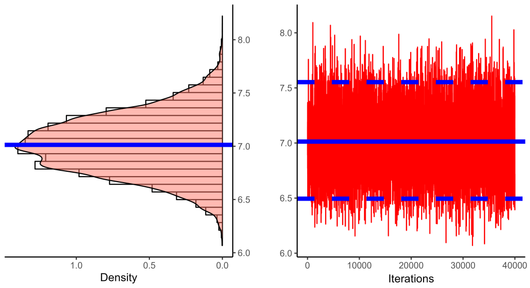
(b) α



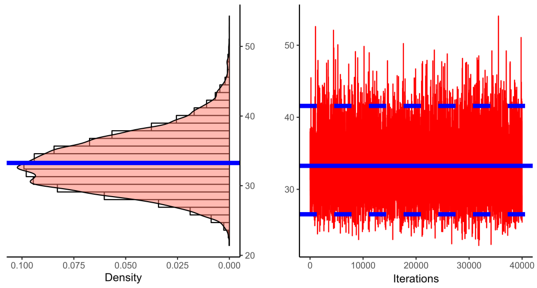
(c) $\mathcal{E}_S$



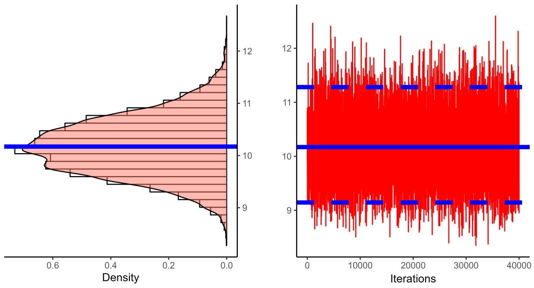
(d) $\mathcal{E}_R(\xi = 0.4)$



(e) $\mathcal{E}_R(\xi = 0.8)$



(f) $\mathcal{E}_T(\zeta = 0.4)$



(g) $\mathcal{E}_Q(\zeta = 0.8)$

Figure 7. The MCMC plots for parameters and entropies of IW model from HNCPs data.

It is evident from the obtained numerical results of the two proposed applications, under the usage of censored samples, that the proposed estimators perform effectively and satisfactorily while saving time, cost, and effort compared to complete samples. Using censored samples, we achieve abundant, accurate, improved, and efficient results while optimizing time and cost. Consequently, the scope of the proposed applications can be expanded to other practical real-life scenarios.

6. Concluding remarks

We examine two approaches for estimating Rényi, Tsallis, and Shannon entropies for the inverse Weibull distribution using unified hybrid censored data. The estimations are achieved based on two approaches: Maximum likelihood and Bayesian methods. The point estimators are derived using the invariance property, and the asymptotic confidence intervals are calculated where the variances are approximated using the delta methods. The Bayesian method offers Bayesian estimates using the Markov Chain Monte Carlo sampling methodology. This is accomplished using two loss functions: Squared error and general entropy loss functions. Additionally, we investigate Bayes credible intervals. Monte Carlo simulations are performed to assess the estimation efficiency of the proposed inferential approaches. Furthermore, two applications are examined for the same objective. The numerical research shows that for all three entropy metrics, the Bayesian estimation method produces adequate point and interval estimates based on some statistical criteria. Using the same entropy measures for the inverse Weibull distribution and unified hybrid censored samples, some potential future work includes: (1) Investigating the estimation of entropy measures within a competing risks model; (2) examining estimation issues for entropy measures in the context of accelerated life tests; and (3) exploring classical estimation methods other than maximum likelihood, such as least squares, weighted least squares, and maximum product of spacings methods, among others.

Author contributions

Refah Alotaibi: Conceptualization, Methodology, Investigation, Funding acquisition, Writing-original draft; Mazen Nassar: Conceptualization, Methodology, Investigation, Writing-review & editing; Zareen A. Khan: Conceptualization, Investigation; Wejdan Ali Alajlan: Methodology, Writing-eview & editing; Ahmed Elshahhat: Methodology, Writing-original draft. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

The authors extend their appreciation to the Deanship of Scientific Research and Libraries in Princess Nourah bint Abdulrahman University for funding this research work through the Research Group project, Grant No. (RG-1445-0011).

Acknowledgments

The authors would like to express their thanks to the editor and the three anonymous referees for valuable comments and helpful observations. Princess Nourah bint Abdulrahman University Researchers Supporting Project number (RG-1445-0011), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflict of interest.

References

1. R. Clausius, *The mechanical theory of heat*, London, UK: McMillan and Co., 1865.
2. C. E. Shannon, A mathematical theory of communication, *The Bell System Technical Journal*, **27** (1948), 379–423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x
3. J. M. Amigó, S. G. Balogh, S. Hernández, A brief review of generalized entropies, *Entropy*, **20** (2018), 813. https://doi.org/10.3390/e20110813
4. A. Rényi, On measures of entropy and information, In: *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics*, 1961, 547–561.
5. C. Tsallis, Possible generalization of Boltzmann–Gibbs statistics, *J. Stat. Phys.*, **52** (1988), 479–487. https://doi.org/10.1007/BF01016429
6. R. A. Bantan, M. Elgarhy, C. Chesneau, F. Jamal, Estimation of entropy for inverse Lomax distribution under multiple censored data, *Entropy*, **22** (2020), 601. https://doi.org/10.3390/e22060601
7. A. A. Al-Babtain, I. Elbatal, C. Chesneau, M. Elgarhy, Estimation of different types of entropies for the Kumaraswamy distribution, *PLoS ONE*, **16** (2021), e0249027. https://doi.org/10.1371/journal.pone.0249027
8. T. Brochet, J. Lapuyade-Lahorgue, A. Huat, S. Thureau, D. Pasquier, I. Gardin, et al., A quantitative comparison between Shannon and Tsallis–Havrda–Charvat entropies applied to cancer outcome prediction, *Entropy*, **24** (2022), 436. https://doi.org/10.3390/e24040436
9. H. Okasha, M. Nassar, Product of spacing estimation of entropy for inverse Weibull distribution under progressive type-II censored data with applications, *J. Taibah Univ. Sci.*, **16** (2022), 259–269. https://doi.org/10.1080/16583655.2022.2046945
10. H. Ren, X. Hu, Bayesian estimations of Shannon entropy and Rényi entropy of inverse Weibull distribution, *Mathematics*, **11** (2023), 2483. https://doi.org/10.3390/math11112483
11. F. M. A. Alam, M. Nassar, On entropy estimation of inverse Weibull distribution under improved adaptive progressively Type-II censoring with applications, *Axioms*, **12** (2023), 751. https://doi.org/10.3390/axioms12080751
12. E. A. Ahmed, M. El-Morshedy, L. A. Al-Essa, M. S. Eliwa, Statistical inference on the entropy measures of gamma distribution under progressive censoring: EM and MCMC algorithms, *Mathematics*, **11** (2023), 2298. https://doi.org/10.3390/math11102298
13. A. Feutrill, M. Roughan, A review of Shannon and differential entropy rate estimation, *Entropy*, **23** (2021), 1046. https://doi.org/10.3390/e23081046
14. A. Z. Keller, M. T. Goblin, N. R. Farnworth, Reliability analysis of commercial vehicle engines, *Reliability Engineering*, **10** (1985), 15–25. https://doi.org/10.1016/0143-8174(85)90039-3

15. P. Xiuyun, Y. Zaizai, Bayesian estimation and prediction for the inverse Weibull distribution under general progressive censoring, *Commun. Stat.-Theor. Meth.*, **45** (2016), 621–635. https://doi.org/10.1080/03610926.2013.834452

16. K. Lee, Estimation of entropy of the inverse Weibull distribution under generalized progressive hybrid censored data, *Journal of the Korean Data and Information Science Society*, **28** (2017), 659–668.

17. A. S. Hassan, A. N. Zaky, Estimation of entropy for inverse Weibull distribution under multiple censored data, *J. Taibah Univ. Sci.*, **13** (2019), 331–337. https://doi.org/10.1080/16583655.2019.1576493

18. N. Jana, S. Bera, Interval estimation of multicomponent stress–strength reliability based on inverse Weibull distribution, *Math. Comput. Simulat.*, **191** (2022), 95–119. https://doi.org/10.1016/j.matcom.2021.07.026

19. M. Maswadah, An optimal point estimation method for the inverse Weibull model parameters using the Runge-Kutta method, *Aligarh Journal of Statistics*, **41** (2021), 1–22.

20. M. Kayid, M. A. Alshehri, Modified maximum likelihood estimation of the inverse Weibull model, *Axioms*, **12** (2023), 961. https://doi.org/10.3390/axioms12100961

21. Z. Mou, G. Liu, J.-Y. Chiang, S. Chen, Inference on the reliability of inverse Weibull with multiply Type-I censored data, *J. Stat. Comput. Sim.*, **94** (2024), 2189–2209. https://doi.org/10.1080/00949655.2024.2324366

22. B. Epstein, Truncated life tests in the exponential case, *Ann. Math. Statist.*, **25** (1954), 555–564. https://doi.org/10.1214/aoms/1177728723

23. A. Childs, B. Chandrasekar, N. Balakrishnan, D. Kundu, Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution, *Ann. Inst. Stat. Math.*, **55** (2003), 319–330. https://doi.org/10.1007/BF02530502

24. B. Chandrasekar, A. Childs, N. Balakrishnan, Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring, *Nav. Res. Log.*, **51** (2004), 994–1004. https://doi.org/10.1002/nav.20038

25. N. Balakrishnan, A. Rasouli, N. Sanjari Farsipour, Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution, *J. Stat. Comput. Sim.*, **78** (2008), 475–488. https://doi.org/10.1080/00949650601158336

26. J. Yu, W. Gui, Y. Shan, Statistical inference on the Shannon entropy of inverse Weibull distribution under the progressive first-failure censoring, *Entropy*, **21** (2019), 1209. https://doi.org/10.3390/e21121209

27. R. Xu, W. Gui, Entropy estimation of inverse Weibull distribution under adaptive type-II progressive hybrid censoring schemes, *Symmetry*, **11** (2019), 1463. https://doi.org/10.3390/sym11121463

28. K. Lee, Bayesian and maximum likelihood estimation of entropy of the inverse Weibull distribution under generalized type I progressive hybrid censoring, *Commun. Stat. Appl. Meth.*, **27** (2020), 469–486. https://doi.org/10.29220/CSAM.2020.27.4.469

29. S. F. Ateya, Estimation under inverse Weibull distribution based on Balakrishnan's unified hybrid censored scheme, *Commun. Stat.-Simul. Comput.*, **46** (2017), 3645–3666. https://doi.org/10.1080/03610918.2015.1099666

30. Y. E. Jeon, S. B. Kang, Estimation of the Rayleigh distribution under unified hybrid censoring, *Aust. J. Stat.*, **50** (2021), 59–73. https://doi.org/10.17713/AJS.V50I1.990

31. A. Alrashidi, A. Rabie, A. A. Mahmoud, S. G. Nasr, M. S. Mustafa, A. Al Mutairi, et al., Exponentiated gamma constant-stress partially accelerated life tests with unified hybrid censored data: statistical inferences, *Alex. Eng. J.*, **88** (2024), 268–275. https://doi.org/10.1016/j.aej.2023.12.066

32. W. H. Greene, *Econometric analysis*, 7 Eds., Upper Saddle River, New Jersey: Pearson Prentice-Hall, 2012.

33. A. Elshahhat, W. S. Abu El Azm, Statistical reliability analysis of electronic devices using generalized progressively hybrid censoring plan, *Qual. Reliab. Eng. Int.*, **38** (2022), 1112–1130. https://doi.org/10.1002/qre.3058

34. S. Zhou, A. Xu, Y. Tang, L. Shen, Fast Bayesian inference of reparameterized gamma process with random effects, *IEEE Trans. Reliab.*, **73** (2024), 399–412. https://doi.org/10.1109/TR.2023.3263940

35. A. Henningsen, O. Toomet, maxLik: A package for maximum likelihood estimation in R, *Comput. Stat.*, **26** (2011), 443–458. https://doi.org/10.1007/s00180-010-0217-1

36. M. Plummer, N. Best, K. Cowles, K. Vines, CODA: convergence diagnosis and output analysis for MCMC, *R News*, **6** (2006), 7–11.

37. A. Elshahhat, B. R. Elemary, Analysis for Xgamma parameters of life under Type-II adaptive progressively hybrid censoring with applications in engineering and chemistry, *Symmetry*, **13** (2021), 2112. https://doi.org/10.3390/sym13112112

38. R. Alotaibi, M. Nassar, Z. A. Khan, A. Elshahhat, Statistical analysis of stress–strength in a newly inverted Chen model from adaptive progressive Type-II censoring and modelling on light-emitting diodes and pump motors, *AIMS Math.*, **9** (2024), 34311–34355. https://doi.org/10.3934/math.20241635

39. M. Nassar, R. Alotaibi, A. Elshahhat, E-Bayesian estimation using spacing function for inverse Lindley adaptive Type-I progressively censored samples: comparative study with applications, *Appl. Bionics Biomech.*, **2024** (2024), 5567457. https://doi.org/10.1155/2024/5567457

40. R. Alotaibi, M. Nassar, Z. A. Khan, A. Elshahhat, Analysis of Weibull progressively first-failure censored data with beta-binomial removals, *AIMS Math.*, **9** (2024), 24109–24142. https://doi.org/10.3934/math.20241172

41. I. Elbatal, M. Nassar, A. B. Ghorbal, L. S. G. Diab, A. Elshahhat, Bayesian and E-Bayesian reliability analysis of improved adaptive Type-II progressive censored inverted Lindley data, *IEEE Access*, **12** (2024), 101829–101841. https://doi.org/10.1109/ACCESS.2024.3408042

42. R. Alotaibi, M. Nassar, A. Elshahhat, Analysis of reliability indicators for inverted Lomax model via improved adaptive Type-II progressive censoring plan with applications, *Complexity*, **2024** (2024), 4848673. https://doi.org/10.1155/2024/4848673

43. M. Nassar, A. Elshahhat, Estimation procedures and optimal censoring schemes for an improved adaptive progressively Type-II censored Weibull distribution, *J. Appl. Stat.*, **51** (2024), 1664–1688. https://doi.org/10.1080/02664763.2023.2230536

44. D. N. P. Murthy, M. Xie, R. Jiang, *Weibull models*, Hoboken, NJ, USA: Wiley, 2004. https://doi.org/10.1002/047147326X

45. H. S. Mohammed, A. Elshahhat, R. Alotaibi, Inferences of inverted gompertz parameters from an adaptive Type-II progressive hybrid censoring, *Phys. Scr.*, **98** (2023), 105222. https://doi.org/10.1088/1402-4896/acf5ad

46. B. Efron, Logistic regression, survival analysis, and the Kaplan-Meier curve, *J. Amer. Stat. Assoc.*, **83** (1988), 414–425. https://doi.org/10.1080/01621459.1988.10478612

47. A. Elshahhat, M. Nassar, Analysis of adaptive Type-II progressively hybrid censoring with binomial removals, *J. Stat. Comput. Sim.*, **93** (2023), 1077–1103. https://doi.org/10.1080/00949655.2022.2127149