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Research article

Soft strong θ-continuity and soft almost strong θ-continuity

Dina Abuzaid[1] **and Samer Al-Ghour**[2,*]

[1] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

* **Correspondence:** Email: algore@just.edu.jo.

Abstract: We continued the study of "soft strong θ-continuity" and defined and investigated "soft almost strong θ-continuity" which is a generalization of soft strong θ-continuity. We gave characterizations and examined soft composition concerning these two concepts. Furthermore, we derived several soft mapping theorems. We provided several links between these two ideas and their related concepts through examples. Lastly, we looked at the symmetry between them and their topological counterparts.

Keywords: soft θ-openness; soft δ-openness; soft strong θ-continuity; generated soft topology
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1. Introduction and preliminaries

In the social sciences, engineering, health, environmental science, and economics, mathematical modeling of uncertainty is essential for finding solutions to challenging issues. Although they have limitations, other theories like probability theory, fuzzy set theory [1], and rough set theory [2] might be helpful in handling ambiguity and uncertainty. The absence of parametrization tools is one of these mathematical methodology's main drawbacks.

The soft set theory was created in 1999 by Molodtsov [3] in response to criticisms of the previously described uncertainty management strategies. It was suggested to use soft sets, or parametrized universe possibilities. Set modeling uncertainty was introduced in [4] and refined in [5]. This standardized structure also has a lot of practical uses. Set interpretation has been successfully applied for modeling uncertainty in a range of real-world scenarios by several studies (e.g., [6–12]). These practical uses have shown the framework's ability to solve problems and have further substantiated its usefulness and efficacy. Several researchers have examined and explored the key concepts and tenets of soft set theory [12–14].

In order to create a soft topology for a certain set of parameters, Shabir and Naz [15] created one over a family of soft sets. Their work encouraged more research in this field by demonstrating the similarities between concepts in soft topology and classical topology. Many contributions have been made to the study of topological concepts in soft contexts since the beginning of soft topology, such as [16–28].

Mappings on soft sets were investigated by Majumdar and Samanta [29], along with their uses in medical diagnosis. The notion of soft mapping with characteristics was first presented by Kharal and Ahmed [30], who also suggested soft continuity for soft mappings [31].

The notion of soft continuity and its many characterizations are extensively explored in the literature reviews that were provided in numerous publications, such as [32–40]. This mathematical notion, which describes the smooth transition of a function between its values at adjacent places, is examined in these works in all of its complexity.

Soft continuity has been the subject of extensive research in soft topology and other areas of mathematics. Many disciplines, including soft topological models, data modeling, engineering, science, economics, and business, make extensive use of soft continuity. This field has drawn the attention of scientists. This motivated us to write this paper.

In this paper, we continue the study of "soft strong θ-continuity" and define and investigate "soft almost strong θ-continuity" which is a generalization of soft strong θ-continuity. We give characterizations and examine soft composition concerning these two concepts. Furthermore, we derive several soft mapping theorems. We provide several links between these two ideas and their related concepts through examples. Lastly, we look at the symmetry between them and their topological counterparts.

This article is organized as follows: In Section 2, we continue the study of soft strongly θ-continuous functions. In particular, we investigate the correspondence between them and their analog concept in general topology. Also, we show that this type of soft function is strictly stronger than soft δ-continuity. Moreover, we provide two new characterizations of them. Furthermore, we provide several results on soft preservation, composition, and products related to soft strong θ-continuity. Besides these, we give several suitable conditions under which a certain kind of soft continuous function is soft strongly θ-continuous. In Section 3, we define "soft almost strongly θ-continuous functions". We present many characterizations of them, and we investigate the correspondence between them and their analog concept in general topology. Also, we show that this class of soft functions lies strictly between the classes of soft strongly θ-continuous functions and soft δ-continuous functions. Moreover, we provide several results on soft preservation and composition related to almost soft strong θ-continuity. In addition to these, we give several suitable conditions under which a certain kind of soft continuous function is soft almost strongly θ-continuous.

Let L be an initial universe and Δ be a set of parameters. A soft set over L relative to Δ is a function $K : \Delta \longrightarrow \mathcal{P}(L)$, where $\mathcal{P}(L)$ is the power set of L. The collection of soft sets over L relative to Δ is denoted by $S(L, \Delta)$. Let $H \in S(L, \Delta)$. If $H(a) = \emptyset$ for each $a \in \Delta$, then H is called the null soft set over L relative to Δ and denoted by 0_Δ. If $H(a) = L$ for all $a \in \Delta$, then H is called the absolute soft set over L relative to Δ and denoted by 1_Δ. If there exist $b \in \Delta$ and $y \in L$ such that $H(b) = \{y\}$ and $H(a) = \emptyset$ for all $a \in \Delta - \{b\}$, then H is called a soft point over L relative to Δ and denoted by b_y. The collection of all soft points over L relative to Δ is denoted by $P(L,\Delta)$. If for some $b \in \Delta$ and $X \subseteq L$, $H(b) = X$ and $H(a) = \emptyset$ for all $a \in \Delta - \{b\}$, then H will be denoted by b_X. If for some $X \subseteq L$, $H(a) = X$ for all $a \in \Delta$, then H will be denoted by C_X. If $H \in S(L,\Delta)$ and $a_x \in P(L,\Delta)$, then a_x is said to belong to H (notation: $a_x \widetilde{\in} H$) if $x \in H(a)$. Soft topological spaces were defined in [15] as follows: A triplet $(L,\wp,\Delta)$, where $\wp \subseteq S(L,\Delta)$, is called a soft topological space if 0_Δ, $1_\Delta \in \wp$, and $\wp$ is closed under finite soft intersections and arbitrary soft unions.

Throughout this paper, we follow the notions and terminologies as they appear in [41, 42].

Let $(L,\wp,\Delta)$ and (L,α) be a soft topological space and a topological space, respectively. Let $H \in S(L,\Delta)$ and $W \subseteq L$. $Cl_\wp(H)$, $Int_\wp(H)$, $Bd_\wp(H)$, $Cl_\alpha(W)$, $Int_\alpha(W)$, and $\wp^c$ will denote the soft closure of H, the soft interior of H, the soft boundary of H, the closure of W, and the interior of W, and the family of all soft closed sets in $(L,\wp,\Delta)$, respectively.

We will now go over some of the notions that will be used in the remainder of this work.

Definition 1.1. [43] Let (L,α) be a topological space, and let $V \subseteq L$. Then V is called a θ-open set in (L,α) if for every $x \in V$, there exists $U \in \alpha$ such that $x \in U \subseteq Cl_\alpha(U) \subseteq V$. The collection of all θ-open sets in (L,α) is denoted by α_θ.

It is well-known that α_θ is a topology, $\alpha_\theta \subseteq \alpha$, and $\alpha_\theta \neq \alpha$ in general.

Definition 1.2. A function $g : (L,\alpha) \longrightarrow (M,\gamma)$ is called θ-continuous (θ-C) [44], (resp., strongly θ-continuous (S-θ-C) [45], almost strongly θ-continuous (A-S-θ-C) [46]) if for every $x \in L$ and every $U \in \gamma$ such that $g(x) \in U$, we find $V \in \alpha$ such that $x \in V$ and $g(Cl_\alpha(V)) \subseteq Cl_\gamma(U)$ (resp. $g(Cl_\alpha(V)) \subseteq U$, $g(Cl_\alpha(V)) \subseteq Int_\gamma\left(Cl_\gamma(U)\right)$

Definition 1.3. Let $(L,\wp,\Delta)$ be a soft topological space, and let $K \in S(L,\Delta)$. Then K is said to be a

(a) [47] Soft θ-open set in $(L,\wp,\Delta)$ if for every $a_x \widetilde{\in} K$, we find $H \in \wp$ such that $a_x \widetilde{\in} H \widetilde{\subseteq} Cl_\wp(H) \widetilde{\subseteq} K$. $\wp_\theta$ will denote the collection of all soft θ-open sets in $(L,\wp,\Delta)$.

(b) [48] Soft regular-open set in $(L,\wp,\Delta)$ if $K = Int_\wp(Cl_\wp(K))$. Soft complements of soft regular-open sets in $(L,\wp,\Delta)$ are called soft regular-closed sets. The families of all soft regular-open sets in $(L,\wp,\Delta)$ and soft regular-open sets in $(L,\wp,\Delta)$ are denoted by $RO(\wp)$ and $RC(\wp)$, respectively.

(c) [49] Soft δ-open set in $(L,\wp,\Delta)$ if for every $a_x \widetilde{\in} K$, we find $H \in RO(\wp)$ such that $a_x \widetilde{\in} H \widetilde{\subseteq} K$. $\wp_\delta$ will denote the collection of all soft δ-open sets in $(L,\wp,\Delta)$. It is known that $\wp_\theta$ and $\wp_\delta$ are soft topologies, $\wp_\theta \subseteq \wp_\delta \subseteq \wp$, $\wp_\theta \neq \wp_\delta$ in general, and $\wp_\delta \neq \wp$ in general.

Definition 1.4. A soft topological space $(L,\wp,\Delta)$ is called

(a) [50] Soft Hausdorff if for every $a_x, b_y \in P(L,\Delta)$ such that $a_x \neq b_y$, there exist $G, H \in \wp$ such that $a_x \widetilde{\in} G$, $b_y \widetilde{\in} H$, and $G \widetilde{\cap} H = 0_\Delta$.

(b) [50] Soft T_0 if for every $a_x, b_y \in P(L,\wp,\Delta)$ such that $a_x \neq b_y$, there exist $G \in \wp$ such that ($a_x \widetilde{\in} G$ and $b_y \widetilde{\notin} G$) or ($a_x \widetilde{\notin} G$ and $b_y \widetilde{\in} G$).

(c) [50] Soft regular if for every $a_x \in P(L,\Delta)$ and every $G \in \wp$ such that $a_x \widetilde{\in} G$, there exists $K \in \wp$ such that $a_x \widetilde{\in} K \widetilde{\subseteq} Cl_\wp(K) \widetilde{\subseteq} G$.

(d) [51] Soft Urysohn if for every $a_x, b_y \in P(L,\Delta)$ such that $a_x \neq b_y$, there exist $G, H \in \wp$ such that $a_x \widetilde{\in} G$, $b_y \widetilde{\in} H$, and $Cl_\wp(G) \widetilde{\cap} Cl_\wp(H) = 0_\Delta$.

(e) [52] Soft almost regular if for every $a_x \in P(L,\Delta)$ and every $G \in RO(\wp)$ such that $a_x \widetilde{\in} G$, there exists $K \in \wp$ such that $a_x \widetilde{\in} K \widetilde{\subseteq} Cl_\wp(K) \widetilde{\subseteq} G$.

(f) [53] Soft semi-regular if for every $a_x \in P(L,\Delta)$ and every $G \in \wp$ such that $a_x \widetilde{\in} G$, there exists $K \in \wp$ such that $a_x \widetilde{\in} K \widetilde{\subseteq} Int_\wp(Cl_\wp(K)) \widetilde{\subseteq} G$ or equivalently if $\wp_\delta = \wp$.

(g) [31] Soft compact if for any $\mathcal{G} \subseteq \wp$ such that $1_\Delta = \widetilde{\cup}_{G \in \mathcal{G}} G$, there exists a finite subcollection

$\mathcal{G}_1 \subseteq \mathcal{G}$ such that $1_\Delta = \widetilde{\cup}_{G\in\mathcal{G}_1} G$.

Definition 1.5. A soft function $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is called soft almost continuous [54], (resp., soft θ-continuous (soft θ-C) [55], soft δ-continuous [56], soft strongly θ-continuous (soft S-θ-C) [57]) if for every $a_x \in P(L, \Delta)$ and $G \in \aleph$ such that $f_{pu}(a_x) \widetilde{\in} G$, we find $H \in \wp$ such that $a_x \widetilde{\in} H$ and $f_{pu}(H) \widetilde{\subseteq} Int_\aleph(Cl_\aleph(G))$ (resp., $f_{pu}(Cl_\wp(H)) \widetilde{\subseteq} Cl_\aleph(G)$, $f_{pu}(Int_\wp(Cl_\wp(H))) \widetilde{\subseteq} Int_\aleph(Cl_\aleph(G))$, $f_{pu}(Cl_\wp(H)) \widetilde{\subseteq} G)$.

Definition 1.6. [31] For any two soft topological spaces $(L, \wp, \Delta)$ and $(M, \aleph, \Omega)$, the soft topology on $L \times M$ relative to $\Delta \times \Omega$ having $\{H \times K : H \in \wp$ and $K \in \aleph\}$ as a soft base is denoted by $pr(\wp \times \aleph)$.

Definition 1.7. [41] For any topological space (L, α), the soft topology $\{H \in S(L, \Delta) : H(a) \in \alpha$ for every $a \in \Delta\}$ on L relative to α will be denoted by $\tau(\alpha)$.

Definition 1.8. [41] For any collection of topological spaces $\{(L, \alpha_a) : a \in \Delta\}$, the soft topology $\{H \in S(L, \Delta) : H(a) \in \alpha_a$ for every $a \in \Delta\}$ is denoted by $\oplus_{a\in\Delta}\alpha_a$.

2. Soft strongly θ-continuity

In this section, we continue the study of soft strongly θ-continuous functions. In particular, we investigate the correspondence between them and their analog concept in general topology. Also, we show that this type of soft function is strictly stronger than soft δ-continuity. Moreover, we provide two new characterizations of them. Furthermore, we provide several results on soft preservation, composition, and products related to soft strong θ-continuity. Besides these, we give a number of suitable conditions under which a certain kind of soft continuous function is a soft strongly θ-continuous.

Theorem 2.1. If $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft S-θ-C, then $p : (L, \wp_b) \longrightarrow (M, \aleph_{v(b)})$ is S-θ-C for all $b \in \Delta$.

Proof. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft S-θ-C and let $b \in \Delta$. By Proposition 5.2 of [55], $f_{pv} : (L, \wp_\theta, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft continuous. Thus, by Corollary 5.13 of [41], $p : (L, (\wp_\theta)_b) \longrightarrow (M, \aleph_{v(b)})$ is continuous. Since by Theorem 2.20 of [58], $(\wp_\theta)_b \subseteq (\wp_b)_\theta$, then $p : (Z, (\wp_b)_\theta) \longrightarrow (Y, \aleph_{v(b)})$ is continuous. Therefore, $p : (L, \wp_b) \longrightarrow (M, \aleph_{v(b)})$ is S-θ-C.

Theorem 2.2. Let $\{(L, \lambda_a) : a \in \Delta\}$ and $\{(M, \psi_b) : b \in \Omega\}$ be two collections of topological spaces. Let $p : L \longrightarrow M$ and $v : \Delta \longrightarrow \Omega$ be functions where v is bijective. Then, $f_{pv} : (L, \oplus_{a\in\Delta}\lambda_a, \Delta) \longrightarrow (M, \oplus_{b\in\Omega}\psi_b, \Omega)$ is soft S-θ-C if and only if $p : (L, \lambda_a) \longrightarrow (M, \psi_{v(a)})$ is S-θ-C for all $a \in \Delta$.

Proof. Necessity. Let $f_{pv} : (L, \oplus_{a\in\Delta}\lambda_a, \Delta) \longrightarrow (M, \oplus_{b\in\Omega}\psi_b, \Omega)$ be soft S-θ-C. Let $a \in \Delta$. Then by Theorem 2.1, $p : (L, (\oplus_{a\in\Delta}\lambda_a)_a) \longrightarrow \left(M, (\oplus_{b\in\Omega}\psi_b)_{v(a)}\right)$ is S-θ-C. However, by Theorem 3.11 of [41], $(\oplus_{a\in\Delta}\lambda_a)_a = \lambda_a$ and $(\oplus_{b\in\Omega}\psi_b)_{v(a)} = \psi_{v(a)}$. Hence, $p : (L, \lambda_a) \longrightarrow (M, \psi_{v(a)})$ is S-θ-C.

Sufficiency. Let $p : (L, \lambda_a) \longrightarrow (M, \psi_{v(a)})$ be S-θ-C for all $a \in \Delta$. Let $H \in \oplus_{b\in\Omega}\psi_b$. By Theorem 2.21 of [58], we need only to show that $\left(f_{pv}^{-1}(H)\right)(a) \in (\lambda_a)_\theta$ for all $a \in \Delta$. Let $a \in \Delta$. Since $p : (L, \lambda_a) \longrightarrow (M, \psi_{v(a)})$ is S-θ-C and $H(v(a)) \in \psi_{v(a)}$, then $\left(f_{pv}^{-1}(H)\right)(a) = q^{-1}(H(v(a))) \in (\lambda_a)_\theta$.

Corollary 2.3. Let $p : (L, \alpha) \longrightarrow (M, \gamma)$ and $v : \Delta \longrightarrow \Omega$ be two functions where v is bijective. Then, $p : (L, \alpha) \longrightarrow (M, \gamma)$ is S-θ-C if and only if $f_{pv} : (L, \tau(\alpha), \Delta) \longrightarrow (M, \tau(\gamma), \Omega)$ is soft S-θ-C.

Proof. For every $a \in \Delta$ and $b \in \Omega$, let $\lambda_a = \alpha$ and $\psi_b = \gamma$. Then, $\tau(\alpha) = \oplus_{a\in\Delta}\lambda_a$ and $\tau(\gamma) = \oplus_{b\in\Omega}\psi_b$. By Theorem 2.2, we get the result.

In the following three examples, we apply Corollary 2.3:

Example 2.4. Let $L = \mathbb{Z}$, $\alpha = \{\emptyset, L, \mathbb{N}\}$, and $\Delta = \mathbb{R}$. Suppose that $Int_{\alpha_\theta}(\mathbb{N}) \neq \emptyset$. Then, there exists $x \in Int_{\alpha_\theta}(\mathbb{N})$. So, we find $V \in \alpha$ such that $x \in V \subseteq Cl_\alpha(V) \subseteq \mathbb{N}$. Thus, $V = \mathbb{N}$ and, hence, $Cl_\alpha(V) = L \subseteq \mathbb{N}$. Therefore, $Int_{\alpha_\theta}(\mathbb{N}) = \emptyset$. Let $p : (L, \alpha) \longrightarrow (L, \alpha)$ and $v : \Delta \longrightarrow \Delta$ be the identity functions. Since $p^{-1}(\mathbb{N}) = \mathbb{N} \notin \alpha_\theta$, then p is not S-θ-C. Therefore, by Corollary 2.3, $f_{pv} : (L, \tau(\alpha), \Delta) \longrightarrow (L, \tau(\alpha), \Delta)$ not soft S-θ-C.

Example 2.5. Let $L = \mathbb{R}$, $\alpha = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, t) : t \in \mathbb{R}\}$, and $\Delta = \mathbb{Z}$. Consider the identity functions $p : (L, \alpha) \longrightarrow (L, \alpha)$ and $v : \Delta \longrightarrow \Delta$. Then, p is θ-C but not S-θ-C. Therefore, by Theorem 3.2 of [58] and Corollary 2.3, $f_{pv} : (L, \tau(\alpha), \Delta) \longrightarrow (L, \tau(\alpha), \Delta)$ is soft θ-C but not soft S-θ-C.

Example 2.6. Let $L = \mathbb{R}$, α be the usual topology on L, and $\Delta = \mathbb{N}$. Consider the identity functions $p : (L, \alpha) \longrightarrow (L, \alpha)$ and $v : \Delta \longrightarrow \Delta$. Then, $\alpha_\theta = \alpha$ and, thus, $p : (L, \alpha) \longrightarrow (L, \alpha)$ is S-θ-C. Therefore, by Corollary 2.3, $f_{pv} : (L, \tau(\alpha), \Delta) \longrightarrow (L, \tau(\alpha), \Delta)$ is soft S-θ-C.

Theorem 2.7. Every soft S-θ-C function is soft δ-continuous.

Proof. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft S-θ-C. Let $H \in \aleph_\delta$. Since $\aleph_\delta \subseteq \aleph$, then $H \in \aleph$. Thus, $f_{pv}^{-1}(H) \in \wp_\theta \subseteq \wp_\delta$. It follows that $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft δ-continuous.

One cannot reverse Theorem 2.7:

Example 2.8. Let $L = \mathbb{R}$, α be the usual topology on L, $\gamma = \{\emptyset\} \cup \{U \subseteq L : L - U \text{ is countable}\}$, and $\Delta = \{a, b\}$. Consider the identity functions $p : (L, \alpha) \longrightarrow (L, \gamma)$ and $v : \Delta \longrightarrow \Delta$. Then, p is θ-C but not S-θ-C. Thus, $f_{pv} : (L, \tau(\alpha), \Delta) \longrightarrow (L, \tau(\alpha), \Delta)$ is soft δ-continuous but not soft S-θ-C.

Theorem 2.9. Let $f_{pv}, f_{qu} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft S-θ-C functions such that $(M, \aleph, \Omega)$ is soft Hausdorff, and let $H = \widetilde{\cup}\left\{a_x : f_{pv}(a_x) = f_{qu}(a_x)\right\}$. Then, $H \in (\wp_\theta)^c$.

Proof. Let $b_y \widetilde{\in} 1_\Delta - H$. Then, $f_{pv}\left(b_y\right) \neq f_{qu}\left(b_y\right)$. Since $(M, \aleph, \Omega)$ is soft Hausdorff, then there exist $T, S \in \aleph$ such that $f_{pv}\left(b_y\right) \widetilde{\in} T$, $f_{pu}\left(b_y\right) \widetilde{\in} S$, and $T \widetilde{\cap} S = 0_\Omega$. Since f_{pv}, f_{qu} are soft S-θ-C, then $f_{pv}^{-1}(T), f_{qu}^{-1}(S) \in \wp_\theta$.

Claim. $f_{pv}^{-1}(T) \widetilde{\cap} f_{qu}^{-1}(S) \widetilde{\cap} H = 0_\Delta$.

Proof of Claim. Assume, however, that in contrast, there exists d_z such that $f_{pv}(d_z) \widetilde{\in} T$, $f_{qu}(d_z) \widetilde{\in} S$, and $f_{pv}(d_z) = f_{qu}(d_z)$. Then, $f_{pv}(d_z) \widetilde{\in} T \widetilde{\cap} S = 0_\Omega$, which is a contradiction.

Therefore, we have $b_y \widetilde{\in} f_{pv}^{-1}(T) \widetilde{\cap} f_{qu}^{-1}(S) \in \wp_\theta$ and by the above claim, $f_{pv}^{-1}(T) \widetilde{\cap} f_{qu}^{-1}(S) \widetilde{\subseteq} 1_\Delta - H$. This shows that $1_\Delta - H \in \wp_\theta$. Hence, $H \in (\wp_\theta)^c$.

Theorem 2.10. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft S-θ-C injective and $(M, \aleph, \Omega)$ be soft Hausdorff. Then, $(L, \wp, \Delta)$ is soft Urysohn.

Proof. Let $a_x, b_y \in P(L, \Delta)$ such that $a_x \neq b_y$. Since f_{pv} is injective, then $f_{pv}(a_x) \neq f_{pv}\left(b_y\right)$. Since $(M, \aleph, \Omega)$ is soft Hausdorff, then there exist $T, S \in \aleph$ such that $f_{pv}(a_x) \widetilde{\in} T$, $f_{pv}\left(b_y\right) \widetilde{\in} S$, and $T \widetilde{\cap} S = 0_\Omega$. Since f_{pv} is soft S-θ-C, then there exist $U, V \in \wp$ such that $a_x \widetilde{\in} U$, $b_y \widetilde{\in} V$, $f_{pv}(Cl_\wp(U)) \widetilde{\subseteq} T$, and $f_{pv}(Cl_\wp(V)) \widetilde{\subseteq} S$.

Therefore, $Cl_\wp(U) \widetilde{\cap} Cl_\wp(V) \widetilde{\subseteq} f_{pv}^{-1}\left(f_{pv}(Cl_\wp(U))\right) \widetilde{\cap} f_{pv}^{-1}\left(f_{pv}(Cl_\wp(V))\right) \widetilde{\subseteq} f_{pv}^{-1}(T) \widetilde{\cap} f_{pv}^{-1}(S) = f_{pv}^{-1}\left(T \widetilde{\cap} S\right) = f_{pv}^{-1}(0_\Omega) = 0_\Delta$.

Theorem 2.11. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft S-θ-C and injective. If $(M, \aleph, \Omega)$ is soft T_0, then $(L, \wp, \Delta)$ is soft Hausdorff.

Proof. Let $a_x, b_y \in P(L, \Delta)$ such that $a_x \neq b_y$. Since f_{pv} is injective, then $f_{pv}(a_x) \neq f_{pv}\left(b_y\right)$. Since $(M, \aleph, \Omega)$ is soft T_0, then there exists $T \in \aleph$ such that $(f_{pv}(a_x) \widetilde{\in} T$ and $f_{pv}\left(b_y\right) \widetilde{\in} 1_\Omega - T)$ or $(f_{pv}\left(b_y\right) \widetilde{\in} T$ and $f_{pv}(a_x) \widetilde{\in} 1_\Omega - T)$. Without loss of generality, we may assume that $f_{pv}(a_x) \widetilde{\in} T$ and $f_{pv}\left(b_y\right) \widetilde{\in} 1_\Omega - T$. Since f_{pv} is soft S-θ-C, then there exists $U \in \wp$ such that $a_x \widetilde{\in} U$ and $f_{pv}(Cl_\wp(U)) \widetilde{\subseteq} T$. Therefore, we have $a_x \widetilde{\in} U \in \wp$, $b_y \widetilde{\in} (1_\Delta - Cl_\wp(U)) \in \wp$, and $U \widetilde{\cap} (1_\Delta - Cl_\wp(U)) = 0_\Delta$. Hence, $(L, \wp, \Delta)$ is soft Hausdorff.

Theorem 2.12. If $f_{p_1 v_1} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft S-θ-C and $f_{p_2 v_2} : (M, \aleph, \Omega) \longrightarrow (N, \Im, \Pi)$ is soft continuous, then $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft S-θ-C.

Proof. Let $f_{p_1 v_1}$ be soft S-θ-C and $f_{p_2 v_2}$ be soft continuous. Let $H \in \Im$. Since $f_{p_2 v_2}$ is soft continuous, $f^{-1}_{p_2 v_2}(H) \in \aleph$. Since $f_{p_1 v_1}$ is soft S-θ-C, then, $f^{-1}_{p_1 v_1}\left(f^{-1}_{p_2 v_2}(H)\right) = f^{-1}_{(p_2 \circ p_1)(v_2 \circ v_1)}(H) \in \wp_\theta$. This shows that $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft S-θ-C.

Corollary 2.13. If $f_{p_1 v_1} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ and $f_{p_2 v_2} : (M, \aleph, \Omega) \longrightarrow (N, \Im, \Pi)$ are soft S-θ-C functions, then $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft S-θ-C.

Proof. The proof follows from Proposition 5.3 of [57] and Theorem 2.12. For any two nonempty sets X and Y, the projection functions $g : X \times Y \longrightarrow X$ and $h : X \times Y \longrightarrow Y$ defined by $g(x, y) = x$ and $h(x, y) = y$ for all $(x, y) \in X \times Y$ will be denoted by π_X and π_Y, respectively.

Theorem 2.14. Let $(L, \wp, \Delta)$, $(M, \aleph, \Omega)$, and $(N, \Im, \Pi)$ be three soft topological spaces, and let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M \times N, pr(\aleph \times \Im), \Omega \times \Pi)$ be a soft function. Then, f_{pv} is soft S-θ-C if and only if $f_{(p \circ \pi_M)(v \circ \pi_\Omega)} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ and $f_{(p \circ \pi_N)(v \circ \pi_\Pi)} : (L, \wp, \Delta) \longrightarrow (N, \Im, \Pi)$ are soft S-θ-C.

Proof. Necessity. Let f_{pv} be soft S-θ-C. Since $f_{(\pi_M)(\pi_\Omega)} : (M \times N, pr(\aleph \times \Im), \Omega \times \Pi) \longrightarrow (N, \Im, \Pi)$ and $f_{(\pi_N)(\pi_\Pi)} : (M \times N, pr(\aleph \times \Im), \Omega \times \Pi) \longrightarrow (N, \Im, \Pi)$ are always soft continuous, then by Theorem 2.12, $f_{(p \circ \pi_M)(v \circ \pi_\Omega)}$ and $f_{(p \circ \pi_N)(v \circ \pi_\Pi)}$ are soft S-θ-C.

Sufficiency. We will apply Proposition 5.5 of [57]. Consider the soft sub-base $\{H \times 1_\Pi : H \in \aleph\} \cup \{1_\Omega \times K : K \in \Im\}$ of $(M \times N, pr(\aleph \times \Im), \Omega \times \Pi)$. Since $f_{(p \circ \pi_M)(v \circ \pi_\Omega)}$ and $f_{(p \circ \pi_N)(v \circ \pi_\Pi)}$ are soft S-θ-C, then for any $H \in \aleph$ and $K \in \Im$, $f^{-1}_{p \circ v}(H \times 1_\Pi) = f^{-1}_{(p \circ \pi_M)(v \circ \pi_\Omega)}(H) \in \wp_\theta$ and $f^{-1}_{p \circ v}(1_\Omega \times K) = f^{-1}_{(p \circ \pi_N)(v \circ \pi_\Pi)}(K) \in \wp_\theta$. Therefore, f_{pv} is soft S-θ-C.

For any function $g : X \longrightarrow Y$, the function $h : X \longrightarrow X \times Y$ defined by $h(x) = (x, g(x))$ will be denoted by $g^\#$.

Theorem 2.15. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be a soft function. Then, $f_{p^\# v^\#} : (L, \wp, \Delta) \longrightarrow (L \times M, pr(\wp \times \aleph), \Delta \times \Omega)$ is soft S-θ-C if and only if f_{pv} is soft S-θ-C and $(L, \wp, \Delta)$ is soft regular.

Proof. Necessity. Let $f_{p^\# v^\#}$ be soft S-θ-C. Then, by Theorem 2.14, $f_{pv} = f_{(p^\# \circ \pi_N)(v^\# \circ \pi_\Pi)} : (L, \wp, \Delta) \longrightarrow (N, \Im, \Pi)$ is soft S-θ-C. To show that $(L, \wp, \Delta)$ is soft regular, it is sufficient to see that $\wp \subseteq \wp_\theta$. Let $K \in \wp$. Since $f_{p^\# v^\#}$ is soft S-θ-C, then $K \times 1_\Omega \in pr(\wp \times \aleph)$, and, thus, $f^{-1}_{p^\# v^\#}(K \times 1_\Omega) = K \in \wp_\theta$.

Sufficiency. Let f_{pv} be soft S-θ-C and $(L, \wp, \Delta)$ be soft regular. Since f_{pv} is soft S-θ-C, then by Proposition 5.3 of [57], f_{pv} is soft continuous. Thus, $f_{p^\# v^\#}$ is soft continuous. Since $(L, \wp, \Delta)$ is soft regular, then $\wp_\theta = \wp$. Therefore, by Proposition 5.2 of [57], $f_{q^\# v^\#}$ is soft S-θ-C.

Lemma 2.16. Let $(L, \wp, \Delta)$ and $(M, \aleph, \Omega)$ be two soft topological spaces. Then for any $T \in S(L, \Delta)$ and $S \in S(M, \Omega)$, $T \times S \in (pr(\wp \times \aleph))_\theta$ if and only if $T \in \wp_\theta$ and $S \in \aleph_\theta$.

Proof. Necessity. Let $T \times S \in (pr(\wp \times \aleph))_\theta$. Let $a_x \widetilde{\in} T$ and $b_y \widetilde{\in} S$. Then, $(a, b)_{(x,y)} \widetilde{\in} T \times S \in (pr(\wp \times \aleph))_\theta$. Thus, there exists $G \in pr(\wp \times \aleph)$ such that $(a, b)_{(x,y)} \widetilde{\in} G \widetilde{\subseteq} Cl_{pr(\wp \times \aleph)}(G) \widetilde{\subseteq} T \times S$. Choose $U \in \wp$ and $V \in \aleph$ such that $(a, b)_{(x,y)} \widetilde{\in} U \times V \widetilde{\subseteq} G$. Since $U \times V \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{pr(\wp \times \aleph)}(G) \widetilde{\subseteq} T \times S$, then $Cl_\wp(U) \times Cl_\aleph(V) = Cl_{pr(\wp \times \aleph)}(U \times V) \widetilde{\subseteq} Cl_{pr(\wp \times \aleph)}(G) \widetilde{\subseteq} T \times S$. Therefore, we have $a_x \widetilde{\in} U \widetilde{\subseteq} Cl_\wp(U) \widetilde{\subseteq} T$ and $b_y \widetilde{\in} V \widetilde{\subseteq} Cl_\aleph(V) \widetilde{\subseteq} S$. Hence, $T \in \wp_\theta$ and $S \in \aleph_\theta$.

Sufficiency. Let $T \in \wp_\theta$ and $S \in \aleph_\theta$. Let $(a, b)_{(x,y)} \widetilde{\in} T \times S$. Then, $a_x \widetilde{\in} T \in \wp_\theta$ and $b_y \widetilde{\in} S \in \aleph_\theta$. So, there exist $U \in \wp$ and $V \in \aleph$ such that $a_x \widetilde{\in} U \widetilde{\subseteq} Cl_\wp(U) \widetilde{\subseteq} T$ and $b_y \widetilde{\in} V \widetilde{\subseteq} Cl_\aleph(V) \widetilde{\subseteq} S$. Therefore, we have

$U \times V \in pr(\wp \times \aleph)$ and $(a,b)_{(x,y)} \widetilde{\in} U \times V \widetilde{\subseteq} Cl_{\wp}(U) \times Cl_{\aleph}(V) = Cl_{pr(\wp \times \aleph)}(U \times V) \widetilde{\subseteq} T \times S$. This shows that $T \times S \in (pr(\wp \times \aleph))_{\theta}$.

Theorem 2.17. Let $f_{p_1 v_1} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ and $f_{p_2 v_2} : (N, \Im, \Pi) \longrightarrow (O, \Re, \Phi)$ be two soft functions. Let $p^* : L \times N \longrightarrow M \times O$ and $v^* : \Delta \times \Pi \longrightarrow \Omega \times \Phi$ be the functions defined by $p^*(x,y) = (p_1(x), p_2(y))$ and $v^*(a,b) = (v_1(a), v_2(b))$. Then, $f_{p^* v^*} : (L \times N, pr(\wp \times \Im), \Delta \times \Pi) \longrightarrow (M \times O, pr(\aleph \times \Re), \Omega \times \Phi)$ is soft S-θ-C if and only if $f_{p_1 v_1}$ and $f_{p_2 v_2}$ are soft S-θ-C.

Proof. Necessity. Let $f_{p^* v^*}$ be soft S-θ-C. Let $H \in \aleph$ and $K \in \Re$. Then, $H \times K \in pr(\aleph \times \Re)$. Thus, $f^{-1}_{p^* v^*}(H \times K) = f^{-1}_{p_1 v_1}(H) \times f^{-1}_{p_2 v_2}(K) \in (pr(\wp \times \Im))_{\theta}$. So, by Lemma 2.16, $f^{-1}_{p_1 v_1}(H) \in \wp_{\theta}$ and $f^{-1}_{p_2 v_2}(K) \in \Im_{\theta}$. This shows that $f_{p_1 v_1}$ and $f_{p_2 v_2}$ are soft S-θ-C.

Sufficiency. Let $f_{p_1 v_1}$ and $f_{p_2 v_2}$ be soft S-θ-C. We will apply Proposition 5.5 of [57]. Consider the soft base (and, hence, soft sub-base) $\{H \times K : H \in \aleph \text{ and } K \in \Re\}$ of $(M \times O, pr(\aleph \times \Re), \Omega \times \Phi)$. Let $H \in \aleph$ and $K \in \Re$. Then, $f^{-1}_{p_1 v_1}(H) \in \wp_{\theta}$ and $f^{-1}_{p_2 v_2}(K) \in \Im_{\theta}$. Since $f^{-1}_{q^* v^*}(H \times K) = f^{-1}_{p_1 v_1}(H) \times f^{-1}_{p_2 v_2}(K)$, then by Lemma 2.16, $f^{-1}_{q^* v^*}(H \times K) \in (pr(\wp \times \Im))_{\theta}$. Therefore, $f_{q^* v^*}$ is soft S-θ-C.

Theorem 2.18. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft continuous such that $(M, \aleph, \Omega)$ is soft regular. Then f_{pv} is soft S-θ-C.

Proof. Let f_{pv} be soft continuous such that $(M, \aleph, \Omega)$ is soft regular. Let $a_x \in P(L, \Delta)$ and let $H \in \aleph$ such that $f_{pv}(a_x) \widetilde{\in} H$. Since $(M, \aleph, \Omega)$ is soft regular, then there exists $K \in \aleph$ such that $f_{pv}(a_x) \widetilde{\in} K \widetilde{\subseteq} Cl_{\aleph}(K) \widetilde{\subseteq} H$. Since f_{pv} is soft continuous, then $f^{-1}_{pv}(K) \in \wp$ and $Cl_{\wp}\left(f^{-1}_{pv}(K)\right) \widetilde{\subseteq} f^{-1}_{pv}(Cl_{\aleph}(K))$. Therefore, we have $a_x \widetilde{\in} f^{-1}_{pv}(K) \in \wp$ and $f_{pv}\left(Cl_{\wp}\left(f^{-1}_{pv}(K)\right)\right) \widetilde{\subseteq} f_{pv}\left(f^{-1}_{pv}(Cl_{\aleph}(K))\right) \widetilde{\subseteq} Cl_{\aleph}(K) \widetilde{\subseteq} H$. This shows that f_{pv} is soft S-θ-C.

For a given soft function $f_{pv} : S(L, \Delta) \longrightarrow S(M, \Omega)$, the soft set $\widetilde{\cup}\left\{(a, v(a))_{(x,p(x))} : a \in \Delta \text{ and } x \in L\right\}$ is called the soft graph of f_{pv} and is denoted by $G\left(f_{pv}\right)$. So, $(a,b)_{(x,y)} \widetilde{\in} G\left(f_{pv}\right)$ if and only if $f_{pv}(a_x) = b_y$ if and only if $p(x) = y$ and $v(a) = b$.

Definition 2.19. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$. Then, $G\left(f_{pv}\right)$ is said to be θ-closed with respect to $(L \times M, pr(\wp \times \aleph), \Delta \times \Omega)$ if for each $(a,b)_{(x,y)} \widetilde{\in} 1_{\Delta \times \Omega} - G\left(f_{pv}\right)$, there exist $T \in \Im$ and $S \in \aleph$ such that $a_x \widetilde{\in} T$, $b_y \widetilde{\in} S$, and $(Cl_{\wp}(T) \times Cl_{\aleph}(S)) \widetilde{\cap} G\left(f_{pv}\right) = 0_{\Delta \times \Omega}$.

Definition 2.20. Let $(L, \wp, \Delta)$ be a soft topological space, and $K \in S(L, \Delta)$. Then, K is said to be a soft H-set if for every $\mathcal{A} \subseteq \wp$ such that $K \widetilde{\subseteq} \widetilde{\cup}_{A \in \mathcal{A}} A$, there exists a finite subcollection $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $K \widetilde{\subseteq} \widetilde{\cup}_{A \in \mathcal{A}_1} Cl_{\wp}(A)$.

Theorem 2.21. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be a soft function such that $G\left(f_{pv}\right)$ is θ-closed with respect to $(L \times M, pr(\wp \times \aleph), \Delta \times \Omega)$. If $(M, \aleph, \Omega)$ is soft semi-regular and every soft regular-closed set in $(M, \aleph, \Omega)$ is a soft H-set, then f_{pv} is soft S-θ-C.

Proof. Let $a_x \in P(L, \Delta)$ and let $G \in \aleph$ such that $f_{pv}(a_x) \widetilde{\in} G$. Since $(M, \aleph, \Omega)$ is soft semi-regular, there exists $K \in RO(\aleph)$ such that $f_{pv}(a_x) \widetilde{\in} K \widetilde{\subseteq} G$. For each $b_y \widetilde{\in} 1_{\Omega} - K$, $(a,b)_{(x,y)} \widetilde{\in} 1_{\Delta \times \Omega} - G\left(f_{pv}\right)$ and by assumption, we find $T\left(b_y\right) \in \wp$ and $S\left(b_y\right) \in \aleph$ such that $a_x \widetilde{\in} T\left(b_y\right)$, $b_y \widetilde{\in} S\left(b_y\right)$, and $\left(Cl_{\wp}\left(T\left(b_y\right)\right) \times Cl_{\aleph}\left(S\left(b_y\right)\right)\right) \widetilde{\cap} G\left(f_{pv}\right) = 0_{\Delta \times \Omega}$ or that $f_{pv}\left(Cl_{\wp}\left(T\left(b_y\right)\right)\right) \widetilde{\cap} Cl_{\aleph}\left(S\left(b_y\right)\right) = 0_{\Omega}$. Since $1_{\Omega} - K \in RO(\aleph)$ and $1_{\Omega} - K \widetilde{\subseteq} \widetilde{\cup}_{b_y \widetilde{\in} 1_{\Omega} - K} S\left(b_y\right)$, then by assumption, there exists a finite subset $F \subseteq P(M, \Omega)$ such that $d_z \widetilde{\in} 1_{\Omega} - K$ for every $d_z \in F$ and $1_{\Omega} - K \widetilde{\subseteq} \widetilde{\cup}_{d_z \in F} Cl_{\aleph}\left(S\left(b_y\right)\right)$. Let $N = \widetilde{\cap}_{d_z \in F} T\left(b_y\right)$. Then, we have $a_x \widetilde{\in} N \in \wp$ and

$$\begin{aligned} f_{pv}\left(Cl_{\wp}(N)\right) &= f_{pv}\left(Cl_{\wp}\left(\widetilde{\cap}_{d_z\in F}T\left(b_y\right)\right)\right) \\ &\widetilde{\subseteq} f_{pv}\left(\widetilde{\cap}_{d_z\in F}Cl_{\wp}\left(T\left(b_y\right)\right)\right) \\ &\widetilde{\subseteq} \widetilde{\cap}_{d_z\in F}f_{pv}\left(Cl_{\wp}\left(T\left(b_y\right)\right)\right) \\ &\widetilde{\subseteq} \widetilde{\cap}_{d_z\in F}\left(1_\Omega - Cl_{\aleph}\left(S\left(b_y\right)\right)\right) \\ &= 1_\Omega - \left(\widetilde{\cup}_{d_z\in F}Cl_{\aleph}\left(S\left(b_y\right)\right)\right) \\ &\widetilde{\subseteq} K \\ &\widetilde{\subseteq} G. \end{aligned}$$

This shows that f_{pv} is soft S-θ-C.

Definition 2.22. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$. Then, $G\left(f_{pv}\right)$ is said to be θ-closed with respect to $(L, \wp, \Delta)$ if for each $(a, b)_{(x,y)}\widetilde{\in}1_{\Delta\times\Omega} - G\left(f_{pv}\right)$, there exist $T \in \Im$ and $S \in \aleph$ such that $a_x\widetilde{\in}T$, $b_y\widetilde{\in}S$, and $\left(Cl_{\wp}(T) \times S\right)\widetilde{\cap}G\left(f_{pv}\right) = 0_{\Delta\times\Omega}$.

Definition 2.23. A soft topological space $(L, \wp, \Delta)$ is said to be soft rim-compact if $(L, \wp, \Delta)$ has a soft base $\mathcal{K}$ such that $Bd_{\wp}(K)$ is soft compact for every $K \in \mathcal{K}$.

Theorem 2.24. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be a soft θ-C function such that $(M, \aleph, \Omega)$ is soft rim-compact and $G\left(f_{pv}\right)$ is θ-closed with respect to $(L, \wp, \Delta)$. Then, f_{pv} is soft S-θ-C.

Proof. Let $a_x \in P(L, \Delta)$ and let $G \in \aleph$ such that $f_{pv}(a_x)\widetilde{\in}G$. Since $(M, \aleph, \Omega)$ is soft rim-compact, there exists $K \in \aleph$ such that $f_{pv}(a_x)\widetilde{\in}K\widetilde{\subseteq}G$ and $Bd_{\wp}(K)$ is soft compact. For each $b_y\widetilde{\in}Bd_{\wp}(K)$, $(a, b)_{(x,y)}\widetilde{\in}1_{\Delta\times\Omega} - G\left(f_{pv}\right)$ and by assumption, we find $T\left(b_y\right) \in \wp$ and $S\left(b_y\right) \in \aleph$ such that $a_x\widetilde{\in}T\left(b_y\right)$, $b_y\widetilde{\in}S\left(b_y\right)$, and $\left(Cl_{\wp}\left(T\left(b_y\right)\right) \times S\left(b_y\right)\right)\widetilde{\cap}G\left(f_{pv}\right) = 0_{\Delta\times\Omega}$ or that $f_{pv}\left(Cl_{\wp}\left(T\left(b_y\right)\right)\right)\widetilde{\cap}S\left(b_y\right) = 0_\Omega$. Since $Bd_{\wp}(K)$ is soft compact and $Bd_{\wp}(K)\widetilde{\subseteq}\widetilde{\cup}_{b_y\widetilde{\in}Bd_{\wp}(K)}S\left(b_y\right)$, then by assumption, there exists a finite subset $F \subseteq P(M, \Omega)$ such that $d_z\widetilde{\in}Bd_{\wp}(K)$ for every $d_z \in F$ and $Bd_{\wp}(K)\widetilde{\subseteq}\widetilde{\cup}_{d_z\in F}S\left(b_y\right)$. Since f_{pv} is soft θ-C, there exists $N \in \wp$ such that $a_x\widetilde{\in}N$ and $f_{pv}\left(Cl_{\wp}(N)\right)\widetilde{\subseteq}Cl_{\aleph}(K)$. Let $R = N\widetilde{\cap}\left(\widetilde{\cap}_{d_z\in F}T\left(b_y\right)\right)$. Then, we have $a_x\widetilde{\in}N \in \wp$ and $Cl_{\wp}(R) = Cl_{\wp}\left(N\widetilde{\cap}\left(\widetilde{\cap}_{d_z\in F}T\left(b_y\right)\right)\right)\widetilde{\subseteq}Cl_{\wp}(N)\widetilde{\cap}\left(\widetilde{\cap}_{d_z\in F}Cl_{\wp}\left(T\left(b_y\right)\right)\right)$. Thus,

$$\begin{aligned} f_{pv}\left(Cl_{\wp}(R)\right)\widetilde{\cap}\left(1_\Omega - K\right) &= f_{pv}\left(Cl_{\wp}(R)\right)\widetilde{\cap}Bd_{\wp}(K) \\ &\widetilde{\subseteq} \widetilde{\cup}_{d_z\in F}\left(f_{pv}\left(Cl_{\wp}(R)\right)\widetilde{\cap}S\left(b_y\right)\right) \\ &\widetilde{\subseteq} \widetilde{\cup}_{d_z\in F}\left(f_{pv}\left(Cl_{\wp}\left(T\left(b_y\right)\right)\right)\widetilde{\cap}S\left(b_y\right)\right) \\ &= 0_\Omega. \end{aligned}$$

Therefore, $f_{pv}\left(Cl_{\wp}(R)\right)\widetilde{\subseteq}K\widetilde{\subseteq}G$. This shows that f_{pv} is soft S-θ-C.

Theorem 2.25. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be a soft function such that $(M, \aleph, \Omega)$ is soft compact and $G\left(f_{pv}\right)$ is θ-closed with respect to $(L, \wp, \Delta)$. Then, f_{pv} is soft S-θ-C.

Proof. Let $a_x \in P(L, \Delta)$ and let $G \in \aleph$ such that $f_{pv}(a_x)\widetilde{\in}G$. For each $b_y\widetilde{\in}1_\Omega - G$, $(a, b)_{(x,y)}\widetilde{\in}1_{\Delta\times\Omega} - G\left(f_{pv}\right)$ and by assumption, we find $T\left(b_y\right) \in \Im$ and $S\left(b_y\right) \in \aleph$ such that $a_x\widetilde{\in}T\left(b_y\right)$, $b_y\widetilde{\in}S\left(b_y\right)$, and $\left(Cl_{\wp}\left(T\left(b_y\right)\right) \times S\left(b_y\right)\right)\widetilde{\cap}G\left(f_{pv}\right) = 0_{\Delta\times\Omega}$ or that $f_{pv}\left(Cl_{\wp}\left(T\left(b_y\right)\right)\right)\widetilde{\cap}S\left(b_y\right) = 0_\Omega$. Since $(M, \aleph, \Omega)$ is soft compact and $1_\Omega - G \in \aleph^c$, then $1_\Omega - G$ is soft compact. Since $1_\Omega - G\widetilde{\subseteq}\widetilde{\cup}_{b_y\widetilde{\in}Bd_{\wp}(K)}S\left(b_y\right)$, then there exists a finite subset $F \subseteq P(M, \Omega)$ such that $d_z\widetilde{\in}1_\Omega - G$ for every $d_z \in F$ and $1_\Omega - G\widetilde{\subseteq}\widetilde{\cup}_{d_z\in F}S\left(b_y\right)$. Now, we have $f_{pv}\left(\widetilde{\cap}_{d_z\in F}Cl_{\wp}\left(T\left(b_y\right)\right)\right)\widetilde{\cap}\left(\widetilde{\cup}_{d_z\in F}S\left(b_y\right)\right) = 0_\Omega$. Let $N = \widetilde{\cap}_{d_z\in F}T\left(b_y\right)$. Then, we have $a_x\widetilde{\in}N \in \wp$ and $f_{pv}\left(Cl_{\wp}(N)\right)\widetilde{\subseteq}G$. This shows that f_{pv} is soft S-θ-C.

Theorem 2.26. If $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft S-θ-C and $(M, \aleph, \Omega)$ is soft Hausdorff, then $G\left(f_{pv}\right)$ is θ-closed with respect to $(L, \wp, \Delta)$.

Proof. Let $(a,b)_{(x,y)}\widetilde{\in}1_{\Delta\times\Omega}-G\left(f_{pv}\right)$. Then, $f_{pv}\left(a_x\right)\neq b_y$. Since $(M,\aleph,\Omega)$ is soft Hausdorff, then there exist $T,S\in\aleph$ such that $f_{pv}\left(a_x\right)\widetilde{\in}T$, $b_y\widetilde{\in}S$, and $T\widetilde{\cap}S=0_\Omega$. Since f_{pv} is soft S-θ-C, then there exists $N\in\wp$ such that $a_x\widetilde{\in}N$ and $f_{pv}\left(Cl_\wp\left(N\right)\right)\widetilde{\subseteq}T$. Therefore, $f_{pv}\left(Cl_\wp\left(N\right)\right)\widetilde{\cap}S=0_\Omega$ and, hence, $\left(Cl_\wp\left(N\right)\times S\right)\widetilde{\cap}G\left(f_{pv}\right)=0_{\Delta\times\Omega}$. This shows that $G\left(f_{pv}\right)$ is θ-closed with respect to $(L,\wp,\Delta)$.

Theorem 2.27. Let $(M,\aleph,\Omega)$ be soft Hausdorff and soft compact. Then, $f_{pv}:(L,\wp,\Delta)\longrightarrow(M,\aleph,\Omega)$ is soft S-θ-C if and only if $G\left(f_{pv}\right)$ is θ-closed with respect to $(L,\wp,\Delta)$.

Proof. The proof follows from Theorems 2.25 and 2.26.

Theorem 2.28. Let $f_{pv}:(L,\wp,\Delta)\longrightarrow(M,\aleph,\Omega)$ be a soft strongly θ-C function. If $K\in S\left(L,\Delta\right)$ such that K is a soft H-set, then $f_{pv}(K)$ is soft compact.

Proof. Let $\mathcal{A}\subseteq\aleph$ such that $f_{pv}(K)\widetilde{\subseteq}\widetilde{\cup}_{A\in\mathcal{A}}A$. For each $d_x\widetilde{\in}K$, there is $A\left(d_x\right)\in\mathcal{A}$ such that $f_{pv}\left(d_x\right)\widetilde{\in}A\left(d_x\right)$. Since f_{pv} is soft S-θ-C, then there exists $N\left(d_x\right)$ such that $f_{pv}(Cl_\wp\left(N\left(d_x\right)\right))\widetilde{\subseteq}A\left(d_x\right)$. Since K is a soft H-set and $K\widetilde{\subseteq}\widetilde{\cup}_{d_x\widetilde{\in}K}N\left(d_x\right)$, then there exists a finite subset $F\subseteq P\left(L,\Delta\right)$ such that $d_z\widetilde{\in}K$ for every $d_z\in F$ and $K\widetilde{\subseteq}\widetilde{\cup}_{d_z\in F}Cl_\wp\left(N\left(d_x\right)\right)$. Thus, $f_{pv}(K)\widetilde{\subseteq}f_{pv}\left(\widetilde{\cup}_{d_z\in F}Cl_\wp\left(N\left(d_x\right)\right)\right)=\widetilde{\cup}_{d_z\in F}f_{pv}\left(Cl_\wp\left(N\left(d_x\right)\right)\right)\widetilde{\subseteq}\widetilde{\cup}_{d_z\in F}f_{pv}\left(A\left(d_x\right)\right)$. Consequently, $f_{pv}(K)$ is soft compact.

3. Soft almost strongly θ-continuity

In this section, we define "soft almost strongly θ-continuous functions". We present many characterizations of them, and we investigate the correspondence between them and their analog concept in general topology. Also, we show that this class of soft functions lies strictly between the classes of soft strongly θ-continuous functions and soft δ-continuous functions. Moreover, we provide several results on soft preservation and composition related to almost soft strong θ-continuity. In addition to these, we give several suitable conditions under which a certain kind of soft continuous function is soft almost strongly θ-continuous.

Definition 3.1. A soft function $f_{pv}:(L,\wp,\Delta)\longrightarrow(M,\aleph,\Omega)$ is called soft almost strongly θ-continuous (soft A-S-θ-C) if for $a_x\in P(L,\Delta)$ and each $H\in\aleph$ such that $f_{pv}\left(a_x\right)\in H$, we find $K\in\wp$ such that $a_x\widetilde{\in}K$ and $f_{pv}(Cl_\wp\left(K\right))\widetilde{\subseteq}Int_\aleph\left(Cl_\aleph\left(H\right)\right)$.

Theorem 3.2. The following are equivalent for the soft function $f_{pv}:(L,\wp,\Delta)\longrightarrow(M,\aleph,\Omega)$:

(a) f_{pv} is soft A-S-θ-C.

(b) For every $H\in RO\left(\aleph\right)$, $f_{pv}^{-1}(H)\in\wp_\theta$.

(c) For every $G\in RC\left(\aleph\right)$, $f_{pv}^{-1}(G)\in\left(\wp_\theta\right)^c$.

(d) For each $a_x\in P(L,\Delta)$ and each $H\in RO\left(\aleph\right)$ such that $f_{pv}\left(a_x\right)\widetilde{\in}H$, we find $K\in\wp$ such that $a_x\widetilde{\in}K$ and $f_{pv}(Cl_\wp\left(K\right))\widetilde{\subseteq}H$.

(e) For every $N\in\aleph_\delta$, $f_{pv}^{-1}(N)\in\wp_\theta$.

(f) For every $C\in\left(\aleph_\delta\right)^c$, $f_{pv}^{-1}(C)\in\left(\wp_\theta\right)^c$.

(g) For every $T\in S\left(L,\Delta\right)$, $f_{pv}(Cl_{\wp_\theta}(T))\widetilde{\subseteq}Cl_{\aleph_\delta}\left(f_{pv}(T)\right)$.

(h) For every $N\in S\left(M,\Omega\right)$, $Cl_{\wp_\theta}\left(f_{pv}^{-1}(N)\right)\widetilde{\subseteq}f_{pv}^{-1}(Cl_{\aleph_\delta}\left(N\right))$.

Proof. (a) $\longrightarrow$ (b): Let $H\in RO\left(\aleph\right)$ and let $a_x\widetilde{\in}f_{pv}^{-1}(H)$. Then, $f_{pv}(a_x)\widetilde{\in}H$. So by (a), there exists $K\in\wp$ such that $a_x\widetilde{\in}K$ and $f_{pv}(Cl_\wp\left(K\right))\widetilde{\subseteq}Int_\aleph\left(Cl_\aleph\left(H\right)\right)=H$. Thus, $a_x\widetilde{\in}K\widetilde{\subseteq}Cl_\wp\left(K\right)\widetilde{\subseteq}f_{pv}^{-1}(H)$. This shows that $f_{pv}^{-1}(H)\in\wp_\theta$.

(b) $\longrightarrow$ (c): Let $G\in RC\left(\aleph\right)$. Then, $1_\Omega-G\in RO\left(\aleph\right)$. So, by (b), $f_{pv}^{-1}(1_\Omega-G)=1_\Delta-f_{pv}^{-1}(G)\in\wp_\theta$.

Hence, $f_{pv}^{-1}(G) \in (\wp_\theta)^c$.

(c) $\longrightarrow$ (d): Let $a_x \in P(L,\Delta)$ and let $H \in RO(\aleph)$ such that $f_{pv}(a_x) \widetilde{\in} H$. Then, $1_\Omega - H \in RC(\aleph)$. So, by (c), $f_{pv}^{-1}(1_\Omega - H) = 1_\Delta - f_{pv}^{-1}(H) \in (\wp_\theta)^c$. Thus, $f_{pv}^{-1}(H) \in \wp_\theta$. Since $a_x \widetilde{\in} f_{pv}^{-1}(H) \in \wp_\theta$, there exists $K \in \wp$ such that $a_x \widetilde{\in} K \widetilde{\subseteq} Cl_\wp(K) \widetilde{\subseteq} f_{pv}^{-1}(H)$. Hence, $f_{pv}(Cl_\wp(K)) \widetilde{\subseteq} f_{pv}\left(f_{pv}^{-1}(H)\right) \widetilde{\subseteq} H$.

(d) $\longrightarrow$ (e): Let $N \in \aleph_\delta$ and let $a_x \widetilde{\in} f_{pv}^{-1}(N)$. Since $f_{pv}(a_x) \widetilde{\in} N \in \aleph_\delta$, there exists $H \in RO(\aleph)$ such that $f_{pv}(a_x) \widetilde{\in} H \widetilde{\subseteq} N$. So, by (d), there exists $K \in \wp$ such that $a_x \widetilde{\in} K$ and $f_{pv}(Cl_\wp(K)) \widetilde{\subseteq} H$. Thus, $a_x \widetilde{\in} K \widetilde{\subseteq} Cl_\wp(K) \widetilde{\subseteq} f_{pv}^{-1}(H) \widetilde{\subseteq} N$. This shows that $f_{pv}^{-1}(N) \in \wp_\theta$.

(e) $\longrightarrow$ (f): Let $C \in (\aleph_\delta)^c$. Then, $1_\Omega - C \in \aleph_\delta$. So, by (e), $f_{pv}^{-1}(1_\Omega - C) = 1_\Delta - f_{pv}^{-1}(C) \in \wp_\theta$. Hence, $f_{pv}^{-1}(C) \in (\wp_\theta)^c$.

(f) $\longrightarrow$ (g): Let $T \in S(L,\Delta)$. Then, $Cl_{\aleph_\delta}\left(f_{pv}(T)\right) \in (\aleph_\delta)^c$. So, by (f), $f_{pv}^{-1}(Cl_{\aleph_\delta}\left(f_{pv}(T)\right)) \in (\wp_\theta)^c$. Since $T \widetilde{\subseteq} f_{pv}^{-1}(Cl_{\aleph_\delta}\left(f_{pv}(T)\right))$, $Cl_{\wp_\theta}(T) \widetilde{\subseteq} f_{pv}^{-1}(Cl_{\aleph_\delta}\left(f_{pv}(T)\right))$ and, hence, $f_{pv}(Cl_{\wp_\theta}(T)) \widetilde{\subseteq} Cl_{\aleph_\delta}\left(f_{pv}(T)\right)$.

(g) $\longrightarrow$ (h): Let $N \in S(M,\Omega)$. Then, by (f), $f_{pv}(Cl_{\wp_\theta}(f_{pv}^{-1}(N))) \widetilde{\subseteq} Cl_{\aleph_\delta}\left(f_{pv}(f_{pv}^{-1}(N))\right) \widetilde{\subseteq} Cl_{\aleph_\delta}(N)$. Thus, $Cl_{\wp_\theta}\left(f_{pv}^{-1}(N)\right) \widetilde{\subseteq} f_{pv}^{-1}\left(f_{pv}(Cl_{\wp_\theta}(f_{pv}^{-1}(N)))\right) \widetilde{\subseteq} f_{pv}^{-1}(Cl_{\aleph_\delta}(N))$.

(h) $\longrightarrow$ (a): Let $a_x \in P(L,\Delta)$ and let $H \in \aleph$ such that $f_{pv}(a_x) \in H$. Let $C = 1_\Omega - Int_\aleph(Cl_\aleph(H))$. Then, $C \in RC(\aleph) \subseteq (\aleph_\delta)^c$. By (h), $Cl_{\wp_\theta}\left(f_{pv}^{-1}(C)\right) \widetilde{\subseteq} f_{pv}^{-1}(Cl_{\aleph_\delta}(C)) = f_{pv}^{-1}(C)$. Thus, $f_{pv}^{-1}(C) \in (\wp_\theta)^c$. Since $a_x \widetilde{\in} f_{pv}^{-1}(Int_\aleph(Cl_\aleph(H))) \in \wp_\theta$, there exists $K \in \wp$ such that $a_x \widetilde{\in} K \widetilde{\subseteq} Cl_\wp(K) \widetilde{\subseteq} f_{pv}^{-1}(Int_\aleph(Cl_\aleph(H)))$. Hence, $f_{pv}(Cl_\wp(K)) \widetilde{\subseteq} f_{pv}\left(f_{pv}^{-1}(Int_\aleph(Cl_\aleph(H)))\right) \widetilde{\subseteq} Int_\aleph(Cl_\aleph(H))$. This shows that f_{pv} is soft A-S-θ-C.

Theorem 3.3. For a soft function $f_{pv} : (L,\wp,\Delta) \longrightarrow (M,\aleph,\Omega)$, the following are equivalent:

(a) $f_{pv} : (L,\wp,\Delta) \longrightarrow (M,\aleph,\Omega)$ is soft A-S-θ-C.

(b) $f_{pv} : (L,\wp_\theta,\Delta) \longrightarrow (M,\aleph,\Omega)$ is soft almost continuous.

(c) $f_{pv} : (L,\wp_\theta,\Delta) \longrightarrow (M,\aleph_\delta,\Omega)$ is soft continuous.

Proof. (a) $\longrightarrow$ (b): Let $H \in RO(\aleph)$. Then, by (a) and Theorem 3.2 (b), $f_{pv}^{-1}(H) \in \wp_\theta$. Thus, by Theorem 3.8 (b) of [54], $f_{pv} : (L,\wp_\theta,\Delta) \longrightarrow (M,\aleph,\Omega)$ is soft almost continuous.

(b) $\longrightarrow$ (c): Let $G \in \aleph_\delta$. Then, there exists $\mathcal{H} \subseteq RO(\aleph)$ such that $G = \widetilde{\cup}_{H\in\mathcal{H}} H$. By (a) and Theorem 3.8 (b) of [54], $f_{pv}^{-1}(H) \in \wp_\theta$ for all $H \in \mathcal{H}$. Thus, $f_{pv}^{-1}(G) = f_{pv}^{-1}\left(\widetilde{\cup}_{H\in\mathcal{H}} H\right) = \widetilde{\cup}_{H\in\mathcal{H}} f_{pv}^{-1}(H) \in \wp_\theta$. This shows that $f_{pv} : (L,\wp_\theta,\Delta) \longrightarrow (M,\aleph_\delta,\Omega)$ is soft continuous.

(c) $\longrightarrow$ (a): The proof follows from Theorem 3.2 (e).

Theorem 3.4. Let $\{(L,\lambda_a) : a \in \Delta\}$ and $\{(M,\psi_b) : b \in \Omega\}$ be two collections of topological spaces. Let $p : L \longrightarrow M$ and $v : \Delta \longrightarrow \Omega$ be functions where v is bijective. Then, $f_{pv} : (L,\oplus_{a\in\Delta}\lambda_a,\Delta) \longrightarrow (M,\oplus_{b\in\Omega}\psi_b,\Omega)$ is soft A-S-θ-C if and only if $p : (L,\lambda_a) \longrightarrow (M,\psi_{v(a)})$ is A-S-θ-C for all $a \in \Delta$.

Proof. Necessity. Let $f_{pv} : (L,\oplus_{a\in\Delta}\lambda_a,\Delta) \longrightarrow (M,\oplus_{b\in\Omega}\psi_b,\Omega)$ be soft A-S-θ-C. Let $a \in \Delta$. Let $U \in RO(\psi_{v(a)})$. Then, by Theorem 14 of [59], $(v(a))_U \in RO(\oplus_{b\in\Omega}\psi_b)$. So, by Theorem 3.2 (b), $f_{pv}^{-1}((v(a))_U) \in (\oplus_{a\in\Delta}\lambda_a)_\theta$. By Theorem 2.21 of [58], $\left(f_{pv}^{-1}((v(a))_U)\right)(a) = p^{-1}(U) \in (\lambda_a)_\theta$. Thus, by Theorem 3.1 (b) of [46], $p : (L,\lambda_a) \longrightarrow (M,\psi_{v(a)})$ is A-S-θ-C.

Sufficiency. Let $p : (L,\lambda_a) \longrightarrow (M,\psi_{v(a)})$ be A-S-θ-C for all $a \in \Delta$. Let $H \in RO(\oplus_{b\in\Omega}\psi_b)$. By Theorem 2.21 of [58], we need only to show that $\left(f_{pv}^{-1}(H)\right)(a) \in (\lambda_a)_\theta$ for all $a \in \Delta$. Let $a \in \Delta$. Since $H \in RO(\oplus_{b\in\Omega}\psi_b)$, by Theorem 14 of [59], $H(v(a)) \in RO(\psi_{v(a)})$. Since $p : (L,\lambda_a) \longrightarrow (M,\psi_{v(a)})$ is A-S-θ-C and $H(v(a)) \in RO(\psi_{v(a)})$, by Theorem 3.1 (b) of [46], $\left(f_{pv}^{-1}(H)\right)(a) = p^{-1}(H(v(a))) \in (\lambda_a)_\theta$.

Corollary 3.5. Let $p : (L,\alpha) \longrightarrow (M,\gamma)$ and $v : \Delta \longrightarrow \Omega$ be two functions where v is bijective. Then, $p : (L,\alpha) \longrightarrow (M,\gamma)$ is A-S-θ-C if and only if $f_{pv} : (L,\tau(\alpha),\Delta) \longrightarrow (M,\tau(\gamma),\Omega)$ is soft A-S-θ-C.

Proof. For every $a \in \Delta$ and $b \in \Omega$, let $\lambda_a = \alpha$ and $\psi_b = \gamma$. Then, $\tau(\alpha) = \oplus_{a\in\Delta}\lambda_a$ and $\tau(\gamma) = \oplus_{b\in\Omega}\psi_b$. By Theorem 3.4, we get the result.

Theorem 3.6. Every soft S-θ-C is soft A-S-θ-C.

Proof. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft S-θ-C. Let $G \in RO(\wp) \subseteq \wp$. Then, by Theorem 5.2 of [57], $f_{pv}^{-1}(G) \in \wp_\theta$. Therefore, by Theorem 3.2 (b), f_{pv} is soft A-S-θ-C.

The example below shows that the inverse of Theorem 3.6 is not always true in general.

Example 3.7. Let $L = \{1,2,3,4\}$, $M = \{5,6,7,8\}$, $\alpha = \{\emptyset, L, \{3\}, \{1,2\}, \{1,2,3\}\}$, $\gamma = \{\emptyset, M, \{5\}, \{7\}, \{5,6\}, \{5,7\}, \{5,6,7\}, \{5,7,8\}\}$, and $\Delta = \mathbb{Q}$. Define $p : (L,\alpha) \longrightarrow (M,\gamma)$ and $v : \Delta \longrightarrow \Delta$ by $p(1) = p(2) = 6$, $p(3) = p(4) = 5$, $v(a) = b$, and $v(b) = a$. Then, f_{pv} is soft A-S-θ-C, but it is not soft S-θ-C.

Theorem 3.8. Every soft A-S-θ-C is soft δ-continuous.

Proof. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft A-S-θ-C. Let $G \in RO(\wp)$. Then, by Theorem 3.2 (b), $f_{pv}^{-1}(G) \in \aleph_\theta \subseteq \aleph_\delta$. Therefore, by Theorem 6.2 (7) of [49], f_{pv} is soft δ-continuous.

The example below shows that the inverse of Theorem 3.8 is not always true in general.

Example 3.9. Let L, α, Δ, and $v : \Delta \longrightarrow \Delta$ be as in Example 3.7. Let $M = \{5,6,7\}$ and $\gamma = \{\emptyset, M, \{5\}, \{6\}, \{5,6\}\}$. Define $p : (L,\alpha) \longrightarrow (M,\gamma)$ by $p(1) = p(2) = 6$, $p(3) = p(4) = 7$. Then, f_{pv} is soft δ-continuous, but it is not soft A-S-θ-C.

Theorem 3.10. Every soft δ-continuous is soft θ-C.

Proof. The proof follows directly from Theorems 7 and 8 of [19].

The following corollary follows from Theorems 3.8 and 3.10.

Corollary 3.11. Every soft A-S-θ-C is soft θ-C.

The example below shows that the inverse of Theorem 3.10 is not always true in general.

Example 3.12. Let $L = \{1,2,3\}$, $M = \{5,6,7,8\}$, $\alpha = \{\emptyset, L, \{1\}, \{3\}, \{1,2\}, \{1,3\}\}$, $\gamma = \{\emptyset, L, \{1\}, \{3\}, \{1,3\}\}$, and $\Delta = \mathbb{R}$. Consider the identity functions $p : (L,\alpha) \longrightarrow (L,\gamma)$ and $v : \Delta \longrightarrow \Delta$. Then, f_{pv} is soft θ-C, but it is not soft δ-continuous.

Theorem 3.13. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft continuous such that $(L, \wp, \Delta)$ is soft regular. Then, f_{pv} is soft S-θ-C.

Proof. Let f_{pv} be soft continuous such that $(M, \aleph, \Omega)$ is soft regular. Let $a_x \in P(L, \Delta)$ and let $H \in \aleph$ such that $f_{pv}(a_x) \widetilde{\in} H$. Since f_{pv} is soft continuous, there exists $K \in \aleph$ such that $a_x \widetilde{\in} K$ and $f_{pv}(K) \widetilde{\subseteq} H$. Since $(M, \aleph, \Omega)$ is soft regular, there exists $G \in \wp$ such that $a_x \widetilde{\in} G \widetilde{\subseteq} Cl_\aleph(G) \widetilde{\subseteq} K$. Thus, we have $a_x \widetilde{\in} G \in \wp$ and $f_{pv}(G) \widetilde{\subseteq} f_{pv}(Cl_\aleph(G)) \widetilde{\subseteq} H$. Therefore, f_{pv} is soft S-θ-C.

Theorem 3.14. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft δ-continuous such that $(L, \wp, \Delta)$ is soft almost regular. Then, f_{pv} is soft A-S-θ-C.

Proof. Let f_{pv} be soft δ-continuous such that $(M, \aleph, \Omega)$ is soft almost regular. Let $a_x \in P(L, \Delta)$ and let $H \in RO(\aleph)$ such that $f_{pv}(a_x) \widetilde{\in} H$. Since f_{pv} is soft δ-continuous, by Theorem 6.2 (2) of [49], there exists $K \in RO(\aleph)$ such that $a_x \widetilde{\in} K$ and $f_{pv}(K) \widetilde{\subseteq} H$. Since $(M, \aleph, \Omega)$ is soft almost regular, by Theorem 3.4 (2) of [52], there exists $G \in RO(\wp) \subseteq \wp$ such that $a_x \widetilde{\in} G \widetilde{\subseteq} Cl_\aleph(G) \widetilde{\subseteq} K$. Thus, we have $a_x \widetilde{\in} G \in \wp$ and $f_{pv}(Cl_\aleph(G)) \widetilde{\subseteq} f_{pv}(K) \widetilde{\subseteq} H$. Therefore, by Theorem 3.2 (d), f_{pv} is soft A-S-θ-C.

Theorem 3.15. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft θ-C such that $(M, \aleph, \Omega)$ is soft almost regular. Then, f_{pv} is soft A-S-θ-C.

Proof. Let f_{pv} be soft θ-C such that $(M, \aleph, \Omega)$ is soft almost regular. To show that f_{pv} is soft A-S-θ-C, we will apply Theorem 3.2 (d). Let $a_x \in P(L, \Delta)$ and let $H \in RO(\aleph)$ such that $f_{pv}(a_x) \widetilde{\in} H$. Since $(M, \aleph, \Omega)$ is soft almost regular, there exists $K \in RO(\aleph)$ such that $f_{pv}(a_x) \widetilde{\in} K \widetilde{\subseteq} Cl_\aleph(K) \widetilde{\subseteq} H$. Since f_{pv} is soft θ-C, there exists $G \in \wp$ such that $a_x \widetilde{\in} G$ and $f_{pv}(Cl_\wp(G)) \widetilde{\subseteq} Cl_\aleph(K) \widetilde{\subseteq} H$. This shows that f_{pv} is soft A-S-θ-C.

The following result follows from Theorems 3.10 and 3.15:

Corollary 3.16. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft δ-continuous such that $(M, \aleph, \Omega)$ is soft almost regular. Then, f_{pv} is soft A-S-θ-C.

Theorem 3.17. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft A-S-θ-C such that $(M, \aleph, \Omega)$ is soft semi-regular. Then, f_{pv} is soft S-θ-C.

Proof. Let f_{pv} be soft A-S-θ-C such that $(M, \aleph, \Omega)$ is soft semi-regular. Let $H \in \aleph$. Since $(M, \aleph, \Omega)$ is soft semi-regular, $\aleph_\delta = \aleph$. So, $H \in \aleph_\delta$. Since f_{pv} be soft A-S-θ-C, by Theorem 3.2 (b), $f_{pv}^{-1}(H) \in \wp_\theta$. Therefore, by Proposition 5.2 of [57], f_{pv} is soft S-θ-C.

Corollary 3.18. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft A-S-θ-C such that $(M, \aleph, \Omega)$ is soft regular. Then, f_{pv} is soft S-θ-C.

Theorem 3.19. If $f_{p_1v_1} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft S-θ-C and $f_{p_2v_2} : (M, \aleph, \Omega) \longrightarrow (N, \Im, \Pi)$ is soft almost continuous, then $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft A-S-θ-C.

Proof. Let $f_{p_1v_1}$ be soft S-θ-C and $f_{p_2v_2}$ be soft almost continuous. Let $H \in RO(\Im)$. Since $f_{p_2v_2}$ is soft almost continuous, by Theorem 3.8 (b) of [54], $f_{p_2v_2}^{-1}(H) \in \aleph$. Since $f_{p_1v_1}$ is soft S-θ-C, $f_{p_1v_1}^{-1}\left(f_{p_2v_2}^{-1}(H)\right) = f_{(p_2 \circ p_1)(v_2 \circ v_1)}^{-1}(H) \in \wp_\theta$. This shows that $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft A-S-θ-C.

Theorem 3.20. If $f_{p_1v_1} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft A-S-θ-C and $f_{p_2v_2} : (M, \aleph, \Omega) \longrightarrow (N, \Im, \Pi)$ is soft δ-continuous, then $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft A-S-θ-C.

Proof. Let $f_{p_1v_1}$ be soft A-S-θ-C and $f_{p_2v_2}$ be soft δ-continuous. Let $H \in RO(\Im)$. Since $f_{p_2v_2}$ is soft δ-continuous, by Theorem 6.2 (7) of [49], $f_{p_2v_2}^{-1}(H) \in \aleph_\delta$. Since $f_{p_1v_1}$ is soft A-S-θ-C, by Theorem 3.2 (e), $f_{p_1v_1}^{-1}\left(f_{p_2v_2}^{-1}(H)\right) = f_{(p_2 \circ p_1)(v_2 \circ v_1)}^{-1}(H) \in \wp_\theta$. This shows that $f_{(p_2 \circ p_1)(v_2 \circ v_1)}$ is soft A-S-θ-C.

The following result follows from Theorems 3.8 and 3.20:

Corollary 3.21. The soft composition of two soft A-S-θ-C functions is soft A-S-θ-C.

Theorem 3.22. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft A-S-θ-C injective and $(M, \aleph, \Omega)$ be soft Hausdorff. Then, $(L, \wp, \Delta)$ is soft Urysohn.

Proof. Let $a_x, b_y \in P(L, \Delta)$ such that $a_x \neq b_y$. Since f_{pv} is injective, then $f_{pv}(a_x) \neq f_{pv}\left(b_y\right)$. Since $(M, \aleph, \Omega)$ is soft Hausdorff, then there exist $T, S \in \aleph$ such that $f_{pv}(a_x) \widetilde{\in} T$, $f_{pv}\left(b_y\right) \widetilde{\in} S$, and $T \widetilde{\cap} S = 0_\Omega$. It is not difficult to check that $Int_\aleph(Cl_\aleph(T)) \widetilde{\cap} Int_\aleph(Cl_\aleph(S)) = 0_\Omega$. Since $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ is soft A-S-θ-C, there exist $H, K \in \wp$ such that $a_x \widetilde{\in} U$, $b_y \widetilde{\in} V$, $f_{pv}(Cl_\wp(U)) \widetilde{\subseteq} Int_\aleph(Cl_\aleph(T))$, and $f_{pv}(Cl_\wp(V)) \widetilde{\subseteq} Int_\aleph(Cl_\aleph(S))$. Therefore, $Cl_\wp(U) \widetilde{\cap} Cl_\wp(V) \widetilde{\subseteq} f_{pv}^{-1}\left(f_{pv}(Cl_\wp(U))\right) \widetilde{\cap} f_{pv}^{-1}\left(f_{pv}(Cl_\wp(V))\right) \widetilde{\subseteq} f_{pv}^{-1}(Int_\aleph(Cl_\aleph(T))) \widetilde{\cap} f_{pv}^{-1}(Int_\aleph(Cl_\aleph(S))) = f_{pv}^{-1}\left(Int_\aleph(Cl_\aleph(T)) \widetilde{\cap} Int_\aleph(Cl_\aleph(S))\right) = f_{pv}^{-1}(0_\Omega) = 0_\Delta$. This shows that $(L, \wp, \Delta)$ is soft Urysohn.

Definition 3.23. A soft topological space $(L, \wp, \Delta)$ is called soft weakly Hausdorff if for each $a_x \in P(L, \Delta)$, $a_x = \widetilde{\cap}\left\{G : G \in RC(\wp) \text{ and } a_x \widetilde{\in} G\right\}$.

Theorem 3.24. Let $f_{pv} : (L, \wp, \Delta) \longrightarrow (M, \aleph, \Omega)$ be soft A-S-θ-C injective and $(M, \aleph, \Omega)$ be soft weakly Hausdorff. Then, $(L, \wp, \Delta)$ is soft Hausdorff.

Proof. Let $a_x, b_y \in P(L, \Delta)$ such that $a_x \neq b_y$. Since f_{pv} is injective, then $f_{pv}(a_x) \neq f_{pv}\left(b_y\right)$. Since $(M, \aleph, \Omega)$ is soft weakly Hausdorff, then there exist $T \in RC(\aleph)$ such that $f_{pv}\left(b_y\right) \widetilde{\in} T$ and $f_{pv}(a_x) \widetilde{\in} 1_\Omega - T \in RO(\aleph)$. Since f_{pv} soft A-S-θ-C, by Theorem 3.2 (d), there exists $K \in \wp$ such that $a_x \widetilde{\in} K$ and $f_{pv}(Cl_\wp(K)) \widetilde{\subseteq} 1_\Omega - T$. So, we have $b_y \widetilde{\in} f_{pv}^{-1}(T) \widetilde{\subseteq} 1_\Omega - Cl_\wp(K)$. This shows that $(L, \wp, \Delta)$ is soft Hausdorff.

4. Conclusions

Uncertainty is a part of many of the things we deal with every day. Soft set theory and its related concepts are among the important ideas developed to deal with uncertainty. Soft topology is among the frameworks to emerge from soft set theory. This work addresses the concept of soft continuity, which is one of the most important concepts in soft topology.

In this paper, we investigate the correspondence between soft strongly θ-continuous functions and their analog concept in general topology (Theorems 2.1, 2.2, and Corollary 2.3). Also, we show that soft strong θ-continuity is strictly stronger than soft δ-continuity (Theorem 2.7 and Example 2.8). Moreover, we provide two new characterizations of soft strong θ-continuity (Theorems 2.15 and 2.27). Furthermore, we give several results on soft preservation (Theorems 2.10, 2.11 and 2.28), composition (Theorems 2.12, 2.14, and Corollary 2.13), and products (Theorem 2.17) related to soft strong θ-continuity. Furthermore, we give several suitable conditions under which a certain kind of soft continuous function is soft strongly θ-continuous (Theorems 2.18, 2.21, 2.24, 2.25, 3.13 and 3.17). On the other hand, we define and explore soft almost strongly θ-continuous functions (Definition 3.1). We present many characterizations of them (Theorems 3.2 and 3.3), and we investigate the correspondence between them and their analog concept in general topology (Theorem 3.4 and Corollary 3.5). Also, we show that this class of soft functions lies strictly between the classes of soft strongly θ-continuous functions and soft δ-continuous functions (Theorems 3.6, 3.8 and Examples 3.7, 3.9). Moreover, we give several suitable conditions under which a certain kind of soft continuous function is soft almost strongly θ-continuous (Theorems 3.14 and 3.15). In addition to these, we provide several results on soft composition (Theorems 3.19 and 3.20) and preservation (Theorems 3.22 and 3.24) related to almost soft strong θ-continuity.

Future studies might examine the following subjects: (1) Defining soft strongly δ-continuous functions; (2) defining strongly θ-semicontinuous functions; (3) applying our recently developed concepts of soft continuity to a "decision-making problem".

Author contributions

Dina Abuzaid and Samer Al-Ghour: Conceptualization, methodology, formal analysis, writing–original draft, writing–review and editing, and funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. Control,* **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
3. D. Molodtsov, Soft set theory-First results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
4. J. L. Yang, Y. Y. Yao, Semantics of soft sets and three-way decision with soft sets, *Knowl.-Based Syst.*, **194** (2020), 105538. https://doi.org/10.1016/j.knosys.2020.105538
5. J. C. R. Alcantud, The semantics of N-soft sets, their applications, and a coda about three-way decision, *Inf. Sci.*, **606** (2022), 837–852. https://doi.org/10.1016/j.ins.2022.05.084
6. J. Gwak, H. Garg, N. Jan, Hybrid integrated decision-making algorithm for clustering analysis based on a bipolar complex fuzzy and soft sets, *Alex. Eng. J.*, **67** (2023), 473–487. https://doi.org/10.1016/j.aej.2022.12.003
7. O. Kguller, A soft set theoretic approach to network complexity and a case study for Turkish Twitter users, *Appl. Soft Comput.*, **143** (2023), 110344. https://doi.org/10.1016/j.asoc.2023.110344
8. O. Dalkılıc, N. Demirtas, Algorithms for covid-19 outbreak using soft set theory: Estimation and application, *Soft Comput.*, **27** (2023), 3203–3211. https://doi.org/10.1007/s00500-022-07519-5
9. M. A. Balci, L. M. Batrancea, O. Akguller, Network-induced soft sets and stock market applications, *Mathematics*, **10** (2022), 3964. https://doi.org/10.3390/math10213964
10. H. W. Qin, Q. H. Fei, X. Q. Ma, W. H. Chen, A new parameter reduction algorithm for soft sets based on chi-square test, *Appl. Intell.*, **51** (2021), 7960–7972. https://doi.org/10.1007/s10489-021-02265-x
11. X. Q. Ma, H. W. Qin, Soft set based parameter value reduction for decision making application, *IEEE Access*, **7** (2019), 35499–35511. https://doi.org/10.1109/ACCESS.2019.2905140
12. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077–1083. https://doi.org/10.1016/S0898-1221(02)00216-X
13. J. C. R. Alcantud, A. Z. Khameneh, G. Santos-Garcıa, M. Akram, A systematic literature review of soft set theory, *Neural Comput. Appl.*, 2024. https://doi.org/10.1007/s00521-024-09552-x
14. F. Feng, C. X. Li, B. Davvaz, M. I. Ali, Soft sets combined with fuzzy sets and rough sets: A tentative approach, *Soft Comput.*, **14** (2010), 899–911. https://doi.org/10.1007/s00500-009-0465-6
15. M. Shabir, M. Naz, On soft topological spaces, *Comput. Math. Appl.*, **61** (2011), 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006
16. D. Abuzaid, S. A. Ghour, M. Naghi, Soft super-continuity and soft delta-closed graphs, *Plos One*, **19** (2024), e0301705. https://doi.org/10.1371/journal.pone.0301705
17. T. M. Al-shami, M. E. El-Shafei, M. Abo-Elhamayel, Seven generalized types of soft semi-compact spaces, *Korean J. Math.*, **27** (2019), 661–690. https://doi.org/10.11568/kjm.2019.27.3.661

18. M. H. Alqahtani, Z. A. Ameen, Soft nodec spaces, *AIMS Math.*, **9** (2024), 3289–3302. https://doi.org/10.3934/math.2024160
19. D. Abuzaid, M. Naghi, S. A. Ghour, Soft faint continuity and soft faint theta omega continuity between soft topological spaces, *Symmetry*, **16** (2024), 268. https://doi.org/10.3390/sym16030268
20. Z. A. Ameen, M. H. Alqahtani, Baire category soft sets and their symmetric local properties, *Symmetry*, **15** (2023), 1810. https://doi.org/10.3390/sym15101810
21. T. M. Al-shami, A. Mhemdi, R. Abu-Gdairi, A Novel framework for generalizations of soft open sets and its applications via soft topologies, *Mathematics*, **11** (2023), 840. https://doi.org/10.3390/math11040840
22. A. Mhemdi, Novel types of soft compact and connected spaces inspired by soft Q-sets, *Filomat*, **37** (2023), 9617–9626.
23. X. C. Guan, Comparison of two types of separation axioms in soft topological spaces, *J. Intell. Fuzzy Syst.*, **44** (2023), 2163–2171. https://doi.org/10.3233/JIFS-212432
24. T. M. Al-shami, Soft somewhat open sets: Soft separation axioms and medical application to nutrition, *Comput. Appl. Math.*, **41** (2022). https://doi.org/10.1007/s40314-022-01919-x
25. T. M. Al-shami, On soft separation axioms and their applications on decision-making problem, *Math. Probl. Eng.*, **2021** (2021), 1–12. https://doi.org/10.1155/2021/8876978
26. T. M. Al-shami, L. D. R. Kocinac, B. A. Asaad, Sum of soft topological spaces, *Mathematics*, **8** (2020), 990. https://doi.org/10.3390/math8060990
27. T. M. Al-shami, M. E. El-Shafei, Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone, *Soft Comput.*, **24** (2020), 5377–5387. https://doi.org/10.1007/s00500-019-04295-7
28. J. C. R. Alcantud, Soft open bases and a novel construction of soft topologies from bases for topologies, *Mathematics*, **8** (2020), 672. https://doi.org/10.3390/math8050672
29. P. Majumdar, S. K. Samanta, On soft mappings, *Comput. Math. Appl.*, **60** (2010), 2666–2672. https://doi.org/10.1016/j.camwa.2010.09.004
30. A. Kharal, B. Ahmad, Mappings on soft classes, *New Math. Nat. Comput.*, **7** (2011), 471–481. https://doi.org/10.1142/S1793005711002025
31. A. Aygunoglu, H. Aygun, Some notes on soft topological spaces, *Neural Comput. Applic.*, **21** (2012), 113–119. https://doi.org/10.1007/s00521-011-0722-3
32. T. M. Al-shami, A. Mhemdi, A weak form of soft α-open sets and its applications via soft topologies, *AIMS Math.*, **8** (2023), 11373–11396. https://doi.org/10.3934/math.2023576
33. T. M. Al-shami, Z. A. Ameen, B. A. Asaad, A. Mhemdi, Soft bi-continuity and related soft functions, *J. Math. Comput. Sci.*, **30** (2023), 19–29. https://doi.org/10.22436/jmcs.030.01.03
34. S. A. Ghour, Soft θ_ω-open sets and soft θ_ω-continuity, *Int. J. Fuzzy Log. Intel. Syst.* **22** (2022), 89–99. https://doi.org/10.5391/IJFIS.2022.22.1.89
35. S. A. Ghour, Soft ω_p-open sets and soft ω_p-continuity in soft topological spaces, *Mathematics*, **9** (2021), 2632. https://doi.org/10.3390/math9202632

36. T. M. Al-shami, I. Alshammari, B. A. Asaad, Soft maps via soft somewhere dense sets, *Filomat*, **34** (2020), 3429–3440. https://doi.org/10.2298/FIL2010429A
37. T. Y. Ozturk, S. Bayramov, Topology on soft continuous function spaces, *Math. Comput. Appl.*, **22** (2017), 32. https://doi.org/10.3390/mca22020032
38. I. Zorlutuna, H. Cakir, On continuity of soft mappings, *Appl. Math. Inform. Sci.*, **9** (2015), 403–409. https://doi.org/10.12785/amis/090147
39. M. Akdag, A. Ozkan, On soft β-open sets and soft β-continuous functions, *The Scientific World J.*, **2014** (2014), 843456. https://doi.org/10.1155/2014/843456
40. M. Akdag, A. Ozkan, Soft α-open sets and soft α-continuous functions, *Abstr. Appl. Anal.*, **2014** (2014), 891341. https://doi.org/10.1155/2014/891341
41. S. A. Ghour, A. Bin-Saadon, On some generated soft topological spaces and soft homogeneity, *Heliyon*, **5** (2019), E02061. https://doi.org/10.1016/j.heliyon.2019.e02061
42. S. A. Ghour, W. Hamed, On two classes of soft sets in soft topological spaces, *Symmetry*, **12** (2020), 265. https://doi.org/10.3390/sym12020265
43. N. V. Velicko, H-closed topological spaces, *Mat. Sb.*, **70** (1966), 98–112.
44. S. Fomin, Extensions of topological spaces, *Ann. Math.*, **44** (1943), 471–480. https://doi.org/10.2307/1968976
45. T. Noiri, On δ-continuous functions, *J. Korean Math. Soc.*, **16** (1980), 161–166.
46. T. Noiri, S. M. Kang, On almost strongly θ-continuous functions, *Indian J. Pure Appl. Math.*, **15** (1984), 1–8.
47. M. Shabir, M. Naz, On soft topological spaces, *Comput. Math. Appl.*, **61** (2011), 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006
48. S. Yuksel, N. Tozlu, Z. G. Ergul, Soft regular generalized closed sets in soft topological spaces, *Int. J. Math. Anal.*, **8** (2014), 355–367. https://doi.org/10.12988/ijma.2014.4125
49. R. A. Mohammed, O. R. Sayed, A. Eliow, Some properties of soft delta-topology, *Acad. J. Nawroz Univ.*, **8** (2019), 352–361. https://doi.org/10.25007/ajnu.v8n4a481
50. S. Hussain, B. Ahmad, Soft separation axioms in soft topological spaces, *Hacet. J. Math. Stat.*, **44** (2015), 559–568.
51. S. Ramkumar, V. Subbiah, Soft separation axioms and soft product of soft topological spaces, *Maltepe J. Math.*, **2** (2020), 61–75.
52. A. K. Prasad, S. S. Thakur, Soft almost regular spaces, *Malaya J. Mat.*, **7** (2019), 408–411. https://doi.org/10.26637/MJM0703/0007
53. S. M. A. Salem, Soft regular generalized b-closed sets in soft topological spaces, *J. Linear Topol. Algebra*, **3** (2014), 195–204.
54. S. S. Thakur, A. S. Rajput, Soft almost continuous mappings, *Int. J. Adv. Math.*, **2017** (2017), 23–29.
55. D. N. Georgiou, A. C. Megaritis, V. I. Petropoulos, On soft topological spaces, *Appl. Math. Inform. Sci.*, **7** (2013), 1889–1901. https://doi.org/10.12785/amis/070527

56. S. Jeyashri, S. Tharmar, G. Ramkumar, On soft δ-continuous functions, *Int. J. Math. Trends Tech.*, **45** (2017), 28–34. https://doi.org/10.14445/22315373/IJMTT-V45P505

57. O. R. Sayed, A. M. Khalil, Some properties of soft θ-topology, *Hacet. J. Math. Stat.*, **44** (2015), 1133–1145.

58. S. A. Ghour, Soft θ_ω-open sets and soft θ_ω-continuity, *Int. J. Fuzzy Log. Inte.*, **22** (2022), 89–99. https://doi.org/10.5391/IJFIS.2022.22.1.89

59. S. A. Ghour, Soft R_ω-open sets and the soft topology of soft δ_ω-open sets, *Axioms*, **11** (2022), 177. https://doi.org/10.3390/axioms11040177

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